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ORIGINAL RESEARCH

Fichte's formal logic

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Abstract

Fichte's *Foundations of the Entire Wissenschaftslehre 1794* is one of the most fundamental books in classical German philosophy. The use of laws of thought to establish foundational principles of transcendental philosophy was groundbreaking in the late eighteenth and early nineteenth century and is still crucial for many areas of theoretical philosophy and logic in general today. Nevertheless, contemporaries have already noted that Fichte's derivation of foundational principles from the law of identity is problematic, since Fichte lacked the tools to correctly present the formal parts of *Foundations*. In this paper, however, we argue that Fichte's approach intuitively offers an important contribution to transcendental philosophy, and especially to philosophy of logic. We first point out the difficulties of Fichte's logic in the *Foundations* and improve it in a second part on the basis of a formal system in which both propositional logic and syllogistic are combined.

Keywords German idealism · Formal logic · Non-well-founded set theory · Lviv-Warsaw school · Johann Gottlieb Fichte · Transcendental philosophy

1 Introduction

In most logical-philosophical pamphlets of the early twentieth century, logic before Boole, Frege, and Peirce was dismissed as defective and associated only with logical textbooks that included, first of all, Aristotelian syllogistic, and such estimations

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remained standard until the second half of the twentieth century: “when writers refer to ‘traditional logic’ they usually have this degenerate textbook tradition in mind. Since the beginning of the modern era most of the contributions to logic have been made by mathematicians” (King & Shapiro, 1995, p. 3). In the meantime, these confessions to the formal logic of the twentieth century are no longer necessary, since everyone has accepted that there is no way back (for example, to the purely Aristotelian logic). For this very reason, however, many logicians have dared to look into the past again in recent decades and have made astonishing discoveries in ancient, medieval and early modern logic with a current level of knowledge. For instance, in Judaic and Islamic logics (Schumann, 2013), consequence relations are discussed as alternatives to contemporary approaches and in Jaina logic, there can be found some remarkable ideas of paraconsistency and dialetheism (Schang, 2009).

Logics in the so-called “long nineteenth century” have also attracted particular interest in recent years, especially in Kantianism. Recently, researchers have shown that in Kant’s transcendental logic, the laws of thought play an important role, and it has been shown by comparison with K. Gödel that Kant’s general logic can be extended to a formal system that is sound and complete (Achourioti & van Lambalgen, 2011; Kovač, 2020). Also dialectical logic proposed by Hegel has been associated with inferentialism in the vein of R. Brandom and with paraconsistent logic in the vein of G. Priest (Berto, 2007; Ficara, 2020). The long-forgotten logic lectures by Schopenhauer have recently been analysed especially for their diagrammatic innovations with regard to S.-J. Shin (Lemanski, 2021). Furthermore, the tradition of transcendental philosophy as such from Kant to Habermas may be regarded as an especial tradition of the so-called content-genetic logic (Schumann, 2002).

One author of this period, whose logic, however, has received little international attention compared to those previously mentioned, is Johann Gottlieb Fichte. Fichte has dealt intensively with logic since his early writings and, even in his late years, still designed very idiosyncratic approaches to syllogistic. A survey of these writings is given by several researchers, e.g., Paimann (2008) and Martin (2010), but no one seems to have seen in them so far a point of connection to modern formal logic. This is not surprising, however, since contemporaries have already discovered logical problems in Fichte’s best-known work, the *Foundations of the Entire Wissenschaftslehre* (*Grundlage der gesamten Wissenschaftslehre* = GWL) (Fichte, 1962–2012), that have not been remedied to this day.

In this paper we would like to pioneer and show that a logic in Fichte’s GWL is consonant with a non-well-founded set theory (i.e., the set theory without the axiom of regularity), see Aczel (1988), hence, it is possible to correct the understanding of GWL by the means of modern formal logic, in particular by modern set theory. However, our approach is not oriented to Gödel, Brandom, Priest or Shin, but to the Lviv-Warsaw school, in particular to J. Łukasiewicz, and to P. Aczel’s new set theory. Łukasiewicz formalised Aristotle’s syllogistic, which had been used for many centuries as naïve set theory. Following Łukasiewicz’s formal language, we have formalised another syllogistic that can be used as a naïve theory of non-well-founded sets, consonant with Aczel’s approach to circular phenomena. It is in this syllogistic that we can develop Fichte’s ideas of dialetheism from GWL.

To this end, we will first (Sect. 2) analyse Fichte's original approach in GWL, illustrating in particular the problems using Fichte's 'proof' of the third foundational principle. Then, in a second step (Sect. 3), we will present a propositional logic and syllogistic based on Łukasiewicz's formalisation to show that Aristotle's syllogistic played the role of naïve set theory at the time of Fichte. Then we propose a non-Łukasiewicz formalisation of syllogistic in which we can regard a fragment of naïve non-well-founded set theory in the meaning of Aczel, which transforms the cloudy logic of GWL into a formal one. As a result, from Łukasiewicz, we use the very idea of formalising syllogistic, but we formalise the syllogistic that corresponds to Fichte's intuitions, and not to Aristotle's logic. It is to show that Fichte's foundational principles of GWL can be reformulated in a formal system that is sound and complete in some models.

With the results presented here, we believe we offer a contribution to three areas of research: First, to the history of logic, by showing that traditional books including logic once branded as obsolete can still offer many ideas for modern logic; second, to the philosophy of German idealism and Kantianism, by showing that Fichte's formal logic can be ranked with Kant, Hegel, and Schopenhauer; and third, to the philosophy of logic, since Fichte discusses in GWL the laws of thought that are still important today, e.g., for the difference between classical and non-classical logic.

2 Overview of Fichte's logic

In this section, we give an overview of Fichte's early writings around 1794 in which logic plays an important role. In particular, the GWL is one of the most important documents of post-Kantian philosophy. In terms of philosophical history, it represents the transition between Kantian transcendental philosophy and the idealistic philosophy continued by, e.g., Reinhold, Schad, Krause, Hegel and Schelling, but in terms of cultural and artistic history, it also represents the transition from the period of the Enlightenment to Romanticism. Furthermore, the book can be understood as a theoretic guideline on which not only Fichte and other philosophers of the time built, but which also served as the basis for disciplines such as jurisprudence, political theory, finance, ethics and many others.

We will first make a few very general remarks on Fichte's logic and on his role in the history of logic (Sect. 2.1). Then, we will introduce the foundational principles of *Wissenschaftslehre* and show how these correspond to the laws of thought (Sect. 2.2). Furthermore, we will state several logical problems of the GWL and demonstrate them by a proof (Sect. 2.3). The logical form of the foundational principle are then the starting point for Sect. 3, in which we fix the logical problems and set up a formal system that is sound and complete in some models.

2.1 Fichte and logic

Fichte's role in the history of logic has hardly been examined. On the one hand, this is due to the fact that twentieth-century historiography of logic has presented almost all authors in Kant's wake as a low point in the history of logic and never examined it more closely. On the other hand, one can see the reason in the fact that even twentieth-century Fichte scholarship barely examined Fichte's formal logic (but rather transcendental logic). Only in recent years has this picture changed: Due to new developments in the field of non-classical as well as diagrammatic logic, logical texts of the age of Kantianism have again slipped into the focus of interest. In Fichte research, too, an increased focus on formal logic has become apparent in the last 20 years, although a conclusive or unified research opinion has not yet emerged.

That Fichte was intensively concerned with formal logic can already be inferred from his curriculum vitae. Already in his youth, Fichte makes intensive contact with the syllogistic steeped in Cicero and the Stoics by Johann August Ernesti's textbook *Initia doctrinae solidioris*. From 1794 onwards, Fichte used Ernst Platner's *Philosophische Aphorismen* for his general lectures, covering traditional syllogistic, Stoic logic and, in notes, almost the entire history of formal logic up to the most recent Leibnizians and Kantians. With this textbook knowledge alone, Fichte has the best prerequisites for making a decisive contribution to the philosophy of logic for his time.

Throughout his career as a full professor, Fichte not only deals with textbook logic, but also with the philosophy of logic. He thus not only teaches formal logic, as the syllabus stipulates, but also incorporates the philosophical questions of what logic is and how it works into his specialist philosophy lectures. During this period, Fichte repeatedly refers back to Kant's distinction between general or formal and transcendental logic. As Paimann has convincingly established, however, this relationship is constantly changing (Paimann, 2008). A generally valid definition of what the individual kinds of logics are for Fichte, i.e., formal logic, transcendental logic etc., and how they relate to each other can therefore not be given. They can only be derived from the context yet to be shown.

Only as a rule of thumb, however, one can say that Fichte's formal logic is topic-neutral, non-empirical, rule-guided and can be expressed in formulae, whereas transcendental logic (or Wissenschaftslehre) represents the mental conditions of logic in the subject. As far as we know today, the GWL of 1794, the *Erlanger Logik* of 1805 and the unfinished *Transzendentale Logik* of 1812 are of particular importance for the study of formal logic. In the course of this time, an increasingly strong tendency from the philosophy of logic to formal logic becomes apparent. In the following, we adopt Tschirner's assessment that formal logic cannot determine itself in the GWL, but that it provides a foundational role with respect to the philosophical discussion on the transcendental laws of thought (Tschirner, 2017, pp. 203ff.).

2.2 The foundational principles and laws of thought

Fichte's philosophy is known to bear the title *Wissenschaftslehre*, often translated by "science of knowledge" or "science of knowing." This title he introduced into German philosophy has been adopted by numerous authors such as Arthur Schopenhauer, Bernard Bolzano or Kuno Fischer, maybe because of the title's reference to Aristotle's *Analytica Posteriora* (Aristotle, 1975). Furthermore, Fichte also distinguishes between the plan or concept of Wissenschaftslehre and its realisation, which is then simply called *Wissenschaftslehre* or foundations of the same. We analyse first the concept (Sect. 2.2.1), then the implementation (Sect. 2.2.2).

2.2.1 The concept of the Wissenschaftslehre

Fichte already explained the close connection between logic and the Wissenschaftslehre in his treatise *Concerning the Concept of the Wissenschaftslehre* (*Ueber den Begriff der Wissenschaftslehre*, = BWL), published in 1794: Philosophy is a science, and every science is based on at least one foundational principle or Grundsatz (Fichte, 1962–2012, [I/2, 112ff., 121]),[1]—today we would rather speak of axioms. All foundational principles or axioms are immediately certain or evident, so we do not have to justify their truth content (Fichte, 1962–2012, [I/2, 114ff.]). Logic also has such immediately certain propositions, which one usually identifies with the laws of thought: the foundational principle of identity (PI), the foundational principle of (non-) contradiction (PNC), the foundational principle of sufficient reason (PSR) and so on.

For Fichte, however, logic is only a formal science that provides all other applied sciences with a mean to derive valid propositions in a formally correct way (Fichte, 1962–2012, [1/2, 137ff.]). Nevertheless, logic cannot be a purely fundamental science either, since it has abstracted from all content and is only concerned with form. Logic thus stands in a non-exclusive relationship to all other sciences. Whereas the other applied sciences only investigate the content and take over the form from logic, logic has abstracted from this content and only made the form the object of its investigation. But so that logic and the other sciences do not end up in a contrary or contradictory relationship, there must be something that is both form and content, that is, that integrates the content of applied science on the one hand and the form of logic on the other. Fichte calls this science 'Wissenschaftslehre', i.e., science of knowing (Fichte, 1962–2012, [I/2, 119ff.]).

The Wissenschaftslehre now combines content and form. It is a science that examines what a science is in general. As a science, it has foundational principles and axioms that are determined according to their content and form. This results in 2^2 possible combinations: A proposition has (A1) content and form, (A2) content but no form, (A3) form but no content, (A4) neither form nor content. Since (A4) states that neither form nor content have certainty, (A4) cannot be a foundational principle. The number of foundational principles or laws in the Wissenschaftslehre is thus limited to three (Fichte, 1962–2012, [I/2, 121]).

2.2.2 The foundation of the Wissenschaftslehre

In the GWL, Fichte tries to implement this programme of BWL. The Wissenschaftslehre, he claims, should have the same certainty and rigor as mathematics (Tschirner, 2017, p. 63f.). But in order to see the evidence of the foundational principles or axioms at all, Fichte first of all uses transcendental arguments (Schüz, 2022), i.e., he

[1] Here, as in the following, citations are made according to the complete edition (Fichte, 1962–2012) the pagination of which is also included in the English edition (Fichte, 2021). Quotations and use of language are based on Fichte (2021).

asks about the condition of the possibility of a science and thus of an immediately certain foundational principle. In order to find the first foundational principle of the Wissenschaftslehre in §1 of the GWL (and also in BWL §3), Fichte starts from logic. His hypothesis is as follows: We assume that the logical laws of thought are valid. If, assuming the validity of the logical laws of thought, we can explain the foundational principles of the Wissenschaftslehre, we must also be able to deduce the logical laws of thought from them. These are assumed to be valid, since they probably depend on the axioms of the Wissenschaftslehre. What we introduce into the reasoning bottom-up, we must later explain again in a top-down manner. This is a necessary circular reasoning, since it corresponds precisely to the structure of knowledge in the human mind (Tschirner, 2017, p. 67f.).

As the logical basis for his transcendental argument, Fichte first assumes the tautology of PI, for example in the categorical form $A = A$ (Fichte, 1962–2012, [I/2, 256]) or in the conditional form $\texttt{If } A\texttt{, then } A$ (Fichte, 1962–2012, [I/2, 257]). In the sense of the transcendental argument, he asks (Fichte, 1962–2012, [I/2, 257]), under what condition does A exists? This is the self (‘I’, ‘me’; German: 𝔍𝔠𝔥) or the subject S, which in the categorical form assumes the subject, the form of the copula ($=$) and the predicate, or in the conditional form assumes the A in the antecedent, the A in the consequent or the form of the conditional ($\texttt{If} \ldots\texttt{, then} \ldots$). In this way, a content is immediately established, since the self or I recognises itself in the examined form as each time the instance that assumes the certain components of the tautology (Fichte, 1962–2012, [I/2, 257f.]). In formal logic, the condition of the possibility of making tautological assumptions is thus the subject or self. Therefore, one can also put the ‘I’, i.e., 𝔍𝔠𝔥 or the ‘S’, i.e., the subject, in the place of the variable and fill the formal logic with content: $I = I$ or $\texttt{If I, then I}$ or in short $\texttt{I am}$. In terms of subjectivity, $S = S$ or $\texttt{If } S\texttt{, then } S$ or in short $S \texttt{ is}$. In this way, Fichte wants to connect to the Kantian and Cartesian theory of subjectivity and distance himself from the objectivism of Spinozism and Leibnizianism (Schäfer, 2006, Chap. I.4).

In §2 of the GWL, Fichte also starts from a logical principle that emphasises the difference, negation and contradiction between A and $\texttt{not-}A$, i.e., $-A \texttt{ not} = A$ or $-A \neq A$ (Fichte, 1962–2012, [I/2, 264]). Fichte again forms a transcendental argument to this law of thought. He asks what the condition of the proposition $-A \neq A$ is and comes to the conclusion that the PI discussed in §1 is a presupposition (Fichte, 1962–2012, [I/2, 265]). If $A = A$ were not known, one would not be able to judge whether the negation of A, i.e., $-A$, is identical with A or not. But the one who makes this judgement must be the same subject or I as in the judgement of the proposition $A = A$ (Fichte, 1962–2012, [I/2, 266]). Because of these presuppositions, the second foundational principle of the Wissenschaftslehre must be: $\texttt{The contradiction of } I = \texttt{not-}I$ (Fichte, 1962–2012, [I/2, 266f.]). Fichte identifies this law with the PNC, which, however, must be critically discussed again in Sect. 2.2.

Fichte’s aim in §3 of the GWL is now to show that A is after all—at least partially—identical with $\texttt{not-}A$, since otherwise the Wissenschaftslehre would lack all objectivity and would remain a pure philosophy of the self. Many researchers have pointed out in recent years that Fichte nowhere explicitly formulates the third foundational principle and that it can therefore only be read between the lines (Schäfer, 2006, p. 72). But in order for the self to have any reference to objectivity and to the other at all, the principle can only be $A = -A$ (Fichte, 1962–2012, [I/2, 132, 269, 272]) or as a conditional according to the PSR: `If` A, `then` $-A$. Fichte, however, realised in the course of the paragraph that this proposition can be justified in a reasonably meaningful way only by its weakened mereological form, namely, A `partially` $= -A$.[2] Let us take a closer look at the argumentation process of GWL §3, since its logical structure is highly controversial, as we will see below.

Fichte reflects at the beginning of GWL §3 on the difference between the two axioms already found and the third that is now to be established: The first foundational principle was unconditional; the second foundational principle, although it presupposed the first foundational principle, was also unconditional in form, since contradiction is an activity of its own and cannot be derived from identity. The third foundational principle, we read at the beginning of GWL §3, "is almost completely susceptible of proof" (Fichte, 1962–2012, [I/2, 268]).

The rest of the paragraph is organised differently, in contrast to the first two paragraphs. Fichte announces a deduction that is to lead up to a "decree of reason" (unbedingten Machtspruch der Vernunft) (Fichte, 1962–2012, [I/2, 268]). Four sections follow, which are marked A, B, C and D. All of them have their own numbering (which always begins with 1). In terms of content and form, it is noticeable that there is a clear break from section D onwards. Fichte explains here that all the foundational principles have now been established, that they can be expressed in a formula and that the entire system of the human mind can be derived from these foundational principles.

The third foundational principle corresponds to divisibility and the PSR, here formulated as A `is, in part,` $= -A$, `and vice versa`. For Fichte, this means that there is a similarity between what is different and a difference between what is similar. Fichte thus integrated the fundamental laws of thought, which were strongly favoured especially in Leibnizianism, into his Wissenschaftslehre: the first foundational principle corresponds to the PI, the second to the PNC, the third to the PSR, which in turn breaks down into two forms, the ground of conjunction and the ground of distinction. Fichte knew these laws of thought, coined in the Leibniz-Wolff school, at the latest from Ernst Platner's *Philosophische Aphorismen* §§484ff., which Fichte took as the basis for his university lectures on logic around 1794 (Zöller, 1998, p. 130).

It was already noticeable at the beginning of the deduction of §3 of the GWL that Fichte reversed the direction of argumentation. Whereas in §§1 and 2 transcendental arguments were used to find the foundational principles of the Wissenschaftslehre with the help of logical laws of thought, in §3 he deduces the corresponding logical law of thought from the foundational principle of the Wissenschaftslehre. This is continued in section D of GWL §3. The PSR is now to be proved and determined on the basis of the material foundational principle of the Wissenschaftslehre. This proof seems to take on a central function, for after all, one can argue for it being the only formal derivation of a logical proposition from the Wissenschaftslehre; all other logical propositions, on the other hand, were the starting point for the transcendental arguments that led to the foundational principles of the Wissenschaftslehre. Furthermore, the foundational

[2] For this to work at all Fichte even distinguishes between different forms of the A, the $\mathfrak{Ich}$, or the S. We omit this topic here, however, and refer to Pluder (2013, pp. 190ff.).

principle of GWL §3 represents the starting point for the theorems of theoretical and practical reason that follow in Parts 2 and 3 of GWL.

2.3 Criticisms of Fichte's formal logic

In the nineteenth century, one only very occasionally finds a detailed critique of the formal logic of Fichte. But when found, these criticisms are very clear and name several problems in detail. For example, Schopenhauer criticises the imprecise terminology in many places and speaks of confusions concerning the provability (Schopenhauer, 1968, p. 48). Herbart also shares this impression, claiming that the whole event in this paragraph is puzzling (Herbart & Werke, 1851, p. 555). In more recent times, sympathetic interpreters have not even entered into discussions of Fichte's formal logic and have also left the above-mentioned important proof of Sect. 3 uncommented. For example, Schäfer (2006) does not mention the proof in his commentary. Baumanns (1972) touches on the proof in several places, but does not discuss it in detail. Seidel (1993) gives only four general sentences to the whole proof. In the commentary by Class and Soller (2004), some interesting cross-references are made, but again the significance of the proof is not clearly presented or critically examined. Therefore, we will first present some of the criticisms of Fichte's formal logic and then exemplify them in the proof.

2.3.1 Precise points of criticism

Before we turn to the above-mentioned proof, one can already identify several logical points of criticism, some of which have already been named by Fichte's contemporaries and successors, and some of which have already been named in modern research. These points are: (i) Fichte used a set-theoretic notation that seemed to be problematic; (ii) Fichte connected several logics together without justifying this connection itself; (iii) The logical terms were ambiguous; (iv) The PNC seemed to be incorrectly formalised even from the point of view of the prevailing algebraic notation of the time.

(i) The first criticism must be weighed sensibly: It is of course correct that we no longer identify the copula `is` with the equal sign (=), since, for example, particular sentences do not express a complete identity between subject and predicate. However, we can understand $A = B$ as `All` A `are` B `and all` B `are` A (or as A `is` B & B `is` A). Perhaps, Fichte shared this intuition. The representation of the negation with the minus sign also has its pitfalls. However, these are modern views and we should not overstate them because the notation used by Fichte had been introduced at the time by the best logicians such as J. Bernoulli or S. Maimon and was therefore up to date (Heinemann, 2020). Thus, if one interprets Fichte, one should treat his notation with the same benevolence that we apply to Leibniz et al. and one should only pay attention to whether the notation is consistent in itself.

(ii) Moreover, the logic on which Fichte's Wissenschaftslehre is based seems to be a mixture of syllogistic as a naïve set theory, on the one hand, and a non-well-founded reasoning expressing loops, on the other hand, as foundational principles

1 and 2 are formulated in Aristotle's syllogistic and foundational principle 3 in a naïve set theory allowing for loops. We will see the use of both systems in more detail in the following. Furthermore, the third foundational principle in its final form, i.e., A `partly` $= -A$, seems to make sense at all only by introducing e.g., mereological relations or partitions. However, Fichte does not reflect at any point on the rules of the logics used.

(iii) Fichte's technical terms of formal logic remain ambiguous and are sometimes mixed with psychological or transcendental philosophical terms. Already his contemporaries have complained about the fact that Fichte does not name `opposition` (entgegensetzen) more precisely and therefore sometimes uses it as `contradiction`, sometimes as `contrariety` and sometimes even as `subcontrariety`. The terms `positing` (setzen) and `annulled` (aufheben) are sometimes used in a syntactic sense with `affirmation` and `negation` and sometimes in a semantic sense with `true` and `false`. From a modern point of view, this is a particularly strong flaw.

(iv) The problems mentioned so far add up in the debate over the second principle, which is to correlate with the PNC. Some points are known in today's research, e.g., in Heinemann (2020). In GWL §2, Fichte seems to have recognised these problems himself, because he modifies or corrects the logical form several times in the course of the paragraph. Fichte starts first in § 2.1 with the formulations already mentioned in § 2.1. In §2.3, he comes to $-A = -A$, but in §2.10 the result of his investigation is `the opposite of I is` $=$ `not-I`. Nowhere in GWL §2 does one find a formalisation that corresponds to one of the (i) notations prevalent at the time. For example, Darjes writes that `it cannot be that` $+A$ & $-A$ `is the case` (Darjes, 1747, §18) and also Wolff gives among the numerous explanations at least one formula that is quite acceptable from the modern point of view such as `it is false that this` A `is` B & `this` A `is not` B (Wolff, 1736, §34). Nevertheless, Fichte's formulation is acceptable from today's point of view, as it corresponds to the modern formalisation of the PNC, e.g., $\neg(-S = S)$ or in another notation: $-S \neq S$, see Fichte's Foundational Principle 2 in Sect. 3.1.3.

2.3.2 The proof as an example

Let us now illustrate these problems of Fichte's formal logic (which appear at first glance) by the third foundational principle of which Fichte says that it "is almost completely susceptible of proof". The proof in GWL §3 is divided into two parts: It addresses first the ground of distinction or disjunction (Unterscheidungsgrund) and then the ground of connection or conjunction (Beziehungsgrund). For Fichte, a ground of conjunction is the question of how things are related. The ground of disjunction, on the other hand, is based on the question of what separates things from one another. Before we take a closer look at the proof, however, it makes sense to first look at the examples Fichte gives for both grounds. For only through these examples could it become clear which reasons Fichte had in mind in the first place when he divided the foundational PSR into two parts, which are then proved individually.

Fichte gives two examples in which the relation of the terms `bird`, `animal` and `plant` occur (Fichte, 1962–2012, [I/2, 276f.]). An example of an affirmative judgement (AJ) in syllogistic would be in the categorical form: `all birds are animals`; by using conditional forms as found in Stoic logic it is something like `If` x `is a bird, then` x `is an animal`. In such judgements, the reason for the relationship is reflected upon, abstracted from the reason of conjunction. Exactly the opposite is the case with negative judgements (NJ), which are in the categorical form `no plant is an animal` and in the conditional form `if` x `is a plant, then` x `is not an animal`. Here, the reason for the disjunction is abstracted, but reflected on the reason for the conjunction.

Thus, it depends solely on the judge whether she wants to emphasise the difference or the identity. If she emphasises one, she neglects the other. In AJ, the ground of conjunction is emphasised, i.e., that the set of all animals also contains the set of birds. The ground for disjunction as a specific difference, on the other hand, is neglected, namely the set of properties that separates the bird from all other animals: laying eggs, having wings, feathers, a beak, and so on. In NJ, it is precisely this specific difference, the ground of disjunction, that is emphasised, i.e., the set of distinguishing properties, e.g., that plants are autotrophic, but animals are heterotrophic living beings, etc. In return, however, the reason for the relationship is abstracted, e.g., that both have similar material properties such as cell membrane, cell plasma and cell nucleus. Thus, we cannot separate conjunction and disjunction from each other. This circumstance will play an important role in the statement $A = -A$.

By modernising the contents of the two examples somewhat, it should become clear that we may still have something to gain from Fichte's division of the PSR today. In particular, the reason for the relationship comes close to the views that are discussed today under the heading of grounding (Amijee, 2020). Furthermore, to demonstrate the logical validity of the PSR is an interesting undertaking from today's point of view, since we have had several critical discussions in logic in the twentieth and twenty-first century on the question of whether the PSR can be formalised and whether it is valid as a law of thought or not, cf. Lemanski (2021, pp. 299ff.) and Woleński (2018).

However, it remains questionable whether Fichte's proof of the third foundational principle is also valid. This proof [Beweis] goes step by step through the two grounds of the third principle. It reads as follows:

> Demonstrated [Bewiesen], because
> a) Every $= -A$ that is posited in opposition to an A, and this A is posited.
> To posit $-A$ is to annul A and at the same time not to annul it.
> Consequently, A is now annulled only in part; instead of that X which has not been annulled in A, what is posited in which $-A$ is X itself (rather than $-X$): and thus, in X, $A = -A$. This was the first point.
> b) Everything that is posited as the same ($= A$, $= B$) is the same as itself, by virtue of its being-posited in the I. $A = A.B = B$.
> But B is now posited as $= A$, and thus B is not posited by means of A; for, if that were that the case, then B would $= A$ and not $= B$. (Two terms would not have been posited, but only one.)
> If, however, B is not posited by positing A, then, to this extent, $B = -A$, and,

> by positing both as the same, neither A nor B is posited; what is posited instead is some X that is $= X$ and $= A$ and $= B$. This was the second point.
> This shows how the proposition $A = B$ can be valid, even though, in itself, it contradicts the proposition $A = A.X = X$, $A = X$, and $B = X$. Therefore $A = B$, insofar as each $= X$; but $A = -B$ insofar as each $= -X$ (Fichte, 1962–2012, [I/2, 272f.]).

The quotation suggests proof by exhaustion. Proof step a refers to the "position in opposition" (Entgegensetzung), i.e., the ground of disjunction, proof step b to the "position as the same" (Gleichsetzung), i.e., the ground of conjunction. If both grounds are proven, the PSR must also be considered proven, since it is also composed of both grounds and formally corresponds to the third foundational principle of the Wissenschaftslehre, $A = -A$ (a) or (by substituting $-A$ with B) $A = B$ (b). But to make the problems with the proof clear, a look at step a is enough.

Since proof step a also refers to the ground of conjunction, it should be proved here that opposites can be related to each other, such as `animal` and `plant`. In the example given above, this was done by a third term that belongs to both opposing terms or contains both opposing terms under itself, e.g., `matter`. In fact, step a seems to describe this as well, since one can substitute A as `animal`, $-A$ as `plant` and X as `matter`. Fichte wants to prove that there is a unit ($= X$) to which a pair of oppositional terms can be referred. This intuition developed by Fichte can be formalised in the set theory without axiom of foundation (or regularity) (Aczel, 1988). We have two different notions A and B which belong to the same loop denoted by X. Then in this loop, $A = -A$, see Fichte's Foundational Principle 3 in Sect. 3.2.2.

Here, however, several criticisms can be stated which correspond to the problems mentioned above:

(i) The algebraic notation is unclear. As a rule, $=$ is read as `is` by Fichte and his contemporaries. To avoid this problem, we understood $A = B$ as `All` A `are` B `and all` B `are` A. The expression "in X, $A = -A$" makes more sense if $=$ is read here as a loop connecting A with $-A$ in X.

(ii) The connection of different logics was already mentioned above and illustrated by the example with `bird` etc. Which rules of logics and which theory of proof Fichte has in mind seems to be unexplained. The main problem is that in the logic of the nineteenth century, logical reasoning was not distinguished from the algebraic model of its validation, as is done now. As a consequence, reasoning could be speculative (as in the case of Fichte) without explication of the models on which they are implemented through verification procedures. Aristotle's arguments like $A = A$ are realised on the denotations of concepts, as we know, therefore some set-theoretic models such as Venn diagrams can be applied. Fichte explicitly says that we cannot limit our reasoning to denotations only, we also refer to the general condition X, in which we distinguish between A and B, and also identify them. These are conditions outside models with denotations. To avoid this problem, we introduce two different set theories for explicating Fichte's intuition with two different models: Aristotle's syllogistic as a naïve set theory realised on the models of standard sets and a non-Aristotelian system without foundation axiom—the so-called non-well-founded syllogistic as a naïve set theory realised on the models of non-well-founded sets.

(iii) Some terms used seem to be logically unclear. For example, if one substitutes the term `contradiction` for `opposition` and the term `true` for `posited`, the statements no longer make sense. Also the term "in part" in the third sentence of step *a* remains vague. Nevertheless, we claim that Fichte's reasoning can be formalized if we appeal to Aczel's set theory with loops, but we then have to distinguish between reasoning and its models—something that was not done in nineteenth century logic.
In sum, Fichte's principles are rather convincing in that they invoke a symmetry between Kant's categories (reality, negation, limitation), the laws of thought (PI, PNC and PSR) and some 'psychological' acts (pose, oppose, divide) that seems plausible at first sight. However, if the reader does not want to rely on Fichte's genius or is not convinced by the symmetry, one cannot avoid looking for logical reasons for the validity of Fichte's Wissenschaftslehre.

3 Improvement of Fichte's cloudy logic

As shown in Sect. 2.1, Fichte had numerous references to formal logic in his early writings, although he did not accord logic the same fundamental status as the Wissenschaftslehre. Fichte had claimed that only the third principle was capable of almost universal proof. As was shown in Sect. 2.2, however, this proof is not convincing for modern logicians (because we do not have, for example, appropriate explicated algebraic models here and we have to ask on which sets is this proof valid), nor are many other references to logic found in the GWL. The mathematical rigor that Fichte thus desired was therefore already rightly questioned by his contemporaries.

However, we believe that Fichte was aware of these problems of formal logic and that he therefore subordinated formal logic to Wissenschaftslehre because he believed that the two main principles of formal logic $A = A$ and $A \neq -A$ can be supplemented by the new principle $A = -A$ of Wissenschaftslehre. As we have seen, Fichte certainly had the logical competence to notice such problems and his multiple revisions of the Wissenschaftslehre are an expression of this.

We claim, therefore, that Fichte's logic can be substantiated by the means of modern resources, e.g., by applying Aczel's set theory. We will for this purpose in a first step combine propositional logic with syllogistic on the basis of the work of Łukasiewicz and thus prove Fichte's first two foundational principles corresponding to PI and PNC: $A = A$ and $A \neq -A$ (Sect. 3.1). In a second step we will prove the problematic third foundational principle $A = -A$ corresponding to the PSR and show that it too is part of a formal system of non-well-founded syllogistic which is sound and complete in cycle graphs (Sect. 3.2). The second syllogistic has other axioms than Łukasiewicz's one and it is realisable on non-well-founded sets.

3.1 Well-founded or Aristotelian syllogistic

In the following, we will first define propositional logic and build Aristotle's syllogistic as an extension of this logic. On this basis, Fichte's first two principles $A = A$ and $A \neq -A$ can be proved as theorems of Aristotle's syllogistic, formalised by Łukasiewicz.

3.1.1 Propositional logic

According to Łukasiewicz (1957), syllogistic is an extension of propositional logic. It means that all the axioms/theorems of propositional logic are also axioms/theorems of syllogistic, but not vice versa. And if some composite formulas of syllogistic contain propositional connectives such as conjunction or disjunction we should appeal to propositional calculus. Let us remember that the alphabet $\mathcal{A}$ of propositional logic consists of

- the set V of propositional variables $p, q, r, \ldots$;
- the set of propositional connectives consisting of $\neg$ (negation),[3] $\wedge$ (conjunction), $\vee$ (disjunction), $\Rightarrow$ (implication);
- the set of auxiliary symbols containing two brackets: (,).

The language of propositional logic is the ordered system $\mathcal{L} = \langle \mathcal{A}, \mathcal{F} \rangle$, where

- $\mathcal{A}$ is the alphabet of propositional logic;
- $\mathcal{F}$ is the set of all formulas that are formed by means of symbols in $\mathcal{A}$.

The elements of $\mathcal{F}$ are defined by induction:

- every propositional variable $p, q, r, \ldots$is a formula of propositional logic;
- if α, β are formulas, then expressions $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \Rightarrow \beta$ are formulas of propositional logic;
- a finite sequence of symbols is called a formula of propositional logic if that sequence satisfies two above mentioned conditions.

Let us take the set of axioms of *Łukasiewicz's propositional calculus* (see Łukasiewicz, 1957):

$$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)), \tag{1}$$

$$(\neg p \Rightarrow p) \Rightarrow p, \tag{2}$$

$$p \Rightarrow (\neg p \Rightarrow q). \tag{3}$$

As we see, implication and negation are given here as basic connectives. Other connectives are derivable, e.g., conjunction and disjunction are defined thus:

$$p \wedge q := \neg(p \Rightarrow \neg q), \tag{4}$$

$$p \vee q := \neg p \Rightarrow q. \tag{5}$$

[3] Instead of Fichte's sign $-$ for negation, we will use the standard sign $\neg$ of modern logic to denote negation of propositions.

Note that we can use any other axiomatization of propositional logic, such as Hilbert's propositional calculus. All these systems differ in the choice of initial axioms, but contain the same set of theorems.

Inference rules of propositional logic are as follows:

- *substitution rule*, we replace a propositional variable p_j of formula $\alpha(p_1, \ldots, p_n)$, containing propositional variables $p_1, \ldots, p_n$, by a formula $\beta(q_1, \ldots, q_k)$, containing propositional variables $q_1, \ldots, q_k$, and we obtain a new formula $\alpha'(p_1, \ldots, p_{j-1}, \beta(q_1, \ldots, q_k), p_{j+1}, \ldots, p_n)$:

$$\frac{\alpha(p_1, \ldots, p_j, \ldots, p_n)}{\alpha'(p_1, \ldots, p_{j-1}, \beta(q_1, \ldots, q_k), p_{j+1}, \ldots, p_n)};$$

- *modus ponens*, if two formulas α and $\alpha \Rightarrow \beta$ hold, then we deduce a formula β:

$$\frac{\alpha, \alpha \Rightarrow \beta}{\beta}.$$

Applying the inference rules to axioms (1)–(3) and definitions (4), (5), we obtain all other theorems of propositional logic.

3.1.2 Aristotle's syllogistic

Aristotle's syllogistic (Aristotle, 1975) can be interpreted as an extension of propositional logic (see Łukasiewicz, 1957; Patzig, 1968; Rose, 1968). Its alphabet $\mathcal{A}_{SA}$ contains:

- the set V of propositional variables $p, q, r, \ldots$;
- the set Q of syllogistic variables $S, P, M, \ldots$;
- the set of propositional connectives consisting of $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\Rightarrow$ (implication);
- the set of binary syllogistic connectives containing four elements **a**, **e**, **i**, **o**, respectively called the functors "every…is…", "no …is…", "some …is…", and "some…is not …";
- the set of unary syllogistic connectives consisting of negation $-\ldots$ for terms, which is read "non-…"[4];
- the set of auxiliary symbols containing two brackets: (,).

The language of Aristotle's syllogistic is the ordered system $\mathcal{L}_{SA} = \langle \mathcal{A}_{SA}, \mathcal{F}_{SA} \rangle$, where

- $\mathcal{A}_{SA}$ is the alphabet of Aristotle's syllogistic;
- $\mathcal{F}_{SA}$ is the set of all formulas formed by means of symbols in $\mathcal{A}_{SA}$; this set $\mathcal{F}_{SA}$ contains all formulas of $\mathcal{F}$ and additional formulas defined by the following rules:

[4] Let $-S$ be a negation of S. In modern formalizations of Aristotle's syllogistic some other notations for this negation are more convenient such as S' or $\bar{S}$. But we use $-S$ to be closer to Fichte's notation.

- if S and P are syllogistic variables, then expressions $S\mathbf{a}P$[5], $S\mathbf{e}P$,[6] $S\mathbf{i}P$[7], $S\mathbf{o}P$[8] are formulas of Aristotle's syllogistic[9], which are called atomic syllogistic formulas;
- if S and P are syllogistic variables, then expressions $-S\mathbf{x}P$, $-S\mathbf{x}-P$, $S\mathbf{x}-P$, where $\mathbf{x} \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$, are formulas of Aristotle's syllogistic, which are called atomic syllogistic formulas;
- if α and β are formulas of Aristotle's syllogistic, then expressions $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \Rightarrow \beta$ are also formulas of Aristotle's syllogistic.

Axioms of Aristotle's syllogistic are obtained by adding to the axioms of propositional logic (e.g., to formulas (1)–(3)) the following new expressions:

$$S\mathbf{a}S, \tag{6}$$

$$S\mathbf{i}S, \tag{7}$$

$$(M\mathbf{a}P \wedge S\mathbf{a}M) \Rightarrow S\mathbf{a}P, \textit{ i.e., Barbara}, \tag{8}$$

$$(M\mathbf{a}P \wedge M\mathbf{i}S) \Rightarrow S\mathbf{i}P, \textit{ i.e., Datisi}. \tag{9}$$

This axiomatic system was proposed by Jan Łukasiewicz (see Łukasiewicz, 1957). The two functors **a** and **i** are treated as basic and two others are derivative:

$$S\mathbf{e}P := \neg(S\mathbf{i}P), \tag{10}$$

$$S\mathbf{o}P := \neg(S\mathbf{a}P). \tag{11}$$

To the same extent, we can define the following rules of obversion, containing negations for syllogistic variables:

$$S\mathbf{a} - P := S\mathbf{e}P, \tag{12}$$

$$S\mathbf{e} - P := S\mathbf{a}P, \tag{13}$$

$$S\mathbf{i} - P := S\mathbf{o}P, \tag{14}$$

$$S\mathbf{o} - P := S\mathbf{i}P. \tag{15}$$

The rules of contraposition:

$$S\mathbf{a}P := -P\mathbf{e}S, \tag{16}$$

$$S\mathbf{o}P := -P\mathbf{i}S. \tag{17}$$

Inference rules of Aristotle's syllogistic are as follows:

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[5] The proposition "every S is P" has the following notation in predicate logic: $\forall x(x \in S \Rightarrow x \in P)$ or $\neg\exists x(x \in S \wedge x \notin P)$.

[6] The proposition "no S is P" has the following notation in predicate logic: $\forall x(x \in S \Rightarrow x \notin P)$ or $\neg\exists x(x \in S \wedge x \in P)$.

[7] The proposition "some S is P" has the following notation in predicate logic: $\exists x(x \in S \wedge x \in P)$.

[8] The proposition "some S is not P" has the following notation in predicate logic: $\exists x(x \in S \wedge x \notin P)$.

[9] Nominal constants that we substitute for the variable S are called a subject. Nominal constants that we substitute for the variable P are called a predicate.

1. *substitution rule*, we replace a propositional variable p_j of formula $\alpha(p_1, \ldots, p_n)$, containing propositional variables $p_1, \ldots, p_n$, by a formula $\beta(q_1, \ldots, q_k)$, containing propositional variables $q_1, \ldots, q_k$ (or by a syllogistic formula $\beta(S_l, P_m)$, containing syllogistic variables S_l, P_m), and we obtain a new propositional formula $\alpha'(p_1, \ldots, p_{j-1}, \beta(q_1, \ldots, q_k), p_{j+1}, \ldots, p_n)$ (or a new syllogistic formula $\alpha'(p_1, \ldots, p_{j-1}, \beta(S_l, P_m), p_{j+1}, \ldots, p_n)$):

$$\frac{\alpha(p_1, \ldots, p_j, \ldots, p_n)}{\alpha'(p_1, \ldots, p_{j-1}, \beta(q_1, \ldots, q_k), p_{j+1}, \ldots, p_n)}$$

or

$$\frac{\alpha(p_1, \ldots, p_j, \ldots, p_n)}{\alpha'(p_1, \ldots, p_{j-1}, \beta(S_l, P_m), p_{j+1}, \ldots, p_n)};$$

in the same way, from any syllogistic formula $\alpha(S_j, P_i)$ follows a new formula $\alpha'(S_k, P_i)$ or $\alpha'(S_j, P_l)$ if we replace a syllogistic variable S_j by a syllogistic variable S_k or P_i by P_l:

$$\frac{\alpha(S_j, P_i)}{\alpha'(S_k, P_i)}$$

or

$$\frac{\alpha(S_j, P_i)}{\alpha'(S_j, P_l)};$$

also, we can substitute a negation of syllogistic variable $-S_k$ for a syllogistic variable S_j or $-P_l$ for P_i:

$$\frac{\alpha(S_j, P_i)}{\alpha'(-S_k, P_i)}$$

or

$$\frac{\alpha(S_j, P_i)}{\alpha'(S_j, -P_l)};$$

2. *double negation*, we can delete double negation before a syllogistic variable $--S_l$ or $--P_m$:

$$\frac{\alpha(p_1, \ldots, \beta(--S_l, P_m), \ldots, p_n)}{\alpha'(p_1, \ldots, p_{j-1}, \beta(S_l, P_m), p_{j+1}, \ldots, p_n)}$$

or

$$\frac{\alpha(p_1, \ldots, \beta(S_l, --P_m), \ldots, p_n)}{\alpha'(p_1, \ldots, p_{j-1}, \beta(S_l, P_m), p_{j+1}, \ldots, p_n)};$$

3. *modus ponens*, if two formulas of Aristotle's syllogistic α and $\alpha \Rightarrow \beta$ hold, then we deduce a formula β:

$$\frac{\alpha, \alpha \Rightarrow \beta}{\beta}.$$

Applying the inference rules to axioms (1)–(3), (6)–(9) and definitions (4), (5), (10)–(17), we obtain all theorems of Aristotle's syllogistic.

3.1.3 Foundational Principle 1 and 2

Using the logic defined in Sect. 3.1.2, we can now prove the first two principles of GWL.

Fichte's Foundational Principle 1 $S = S$.

Proof Let us define the sign of equality in Aristotle's syllogistic as follows:

$$S = P := (S\mathbf{a}P \wedge P\mathbf{a}S). \tag{18}$$

Then $S = S$ has the meaning: $S\mathbf{a}S \wedge S\mathbf{a}S$. Let us take the following tautology of propositional logic:

$$p \Rightarrow (p \wedge p). \tag{19}$$

Then we replace the variable p by $S\mathbf{a}S$:

$$S\mathbf{a}S \Rightarrow (S\mathbf{a}S \wedge S\mathbf{a}S). \tag{20}$$

Thus, we infer $S = S$ by *modus ponens* from (20) and (6). It means that $S = S$ is a theorem (can be used as an axiom) of Aristotle's syllogistic.

Fichte's Foundational Principle 2 $S \neq -S$

Proof The expression $S \neq -S$ or $\neg(S = -S)$ has the following meaning in Aristotle's syllogistic:

$$\neg(-S\mathbf{a}S \wedge S\mathbf{a} - S).$$

From the commutativity of conjunction it follows that $S \neq -S$ is the same as $-S \neq S$.

From (9) we receive the following tautology through the substitution of S for M:

$$(S\mathbf{a}P \wedge S\mathbf{i}S) \Rightarrow S\mathbf{i}P. \tag{21}$$

Then we obtain $S\mathbf{a}P \Rightarrow S\mathbf{i}P$ from the tautology of propositional logic:

$$((p \wedge q) \Rightarrow r) \Rightarrow (q \Rightarrow (p \Rightarrow r)). \tag{22}$$

Indeed, we replace p by $S\mathbf{a}P$, q by $S\mathbf{i}S$, and r by $S\mathbf{i}P$:

$$((S\mathbf{a}P \wedge S\mathbf{i}S) \Rightarrow S\mathbf{i}P) \Rightarrow (S\mathbf{i}S \Rightarrow (S\mathbf{a}P \Rightarrow S\mathbf{i}P)). \tag{23}$$

Then by *modus ponens* from (23) and (21), we have

$$S\mathbf{i}S \Rightarrow (S\mathbf{a}P \Rightarrow S\mathbf{i}P). \tag{24}$$

We apply *modus ponens* again to (24) and (7):

$$S\mathbf{a}P \Rightarrow S\mathbf{i}P. \tag{25}$$

Then from definition (10):

$$S\mathbf{a}P \Rightarrow \neg(S\mathbf{e}P). \tag{26}$$

We substitute S for P:

$$S\mathbf{a}S \Rightarrow \neg(S\mathbf{e}S). \tag{27}$$

From (6) and (27) by *modus ponens*:

$$\neg(S\mathbf{e}S). \tag{28}$$

According to definition (12), the expression $S\mathbf{a} - S$ has the meaning $S\mathbf{e}S$. So, we deduce:

$$\neg(S\mathbf{a} - S). \tag{29}$$

Now, let us replace S by $-S$ in (29):

$$\neg(-S\mathbf{a} - -S). \tag{30}$$

From double negation we obtain:

$$\neg(-S\mathbf{a}S). \tag{31}$$

Let us take the following tautology of propositional logic:

$$\neg p \Rightarrow (\neg q \Rightarrow \neg(p \wedge q)). \tag{32}$$

We substitute $-S\mathbf{a}S$ for p and $S\mathbf{a} - S$ for q:

$$\neg(-S\mathbf{a}S) \Rightarrow (\neg(S\mathbf{a} - S) \Rightarrow \neg(-S\mathbf{a}S \wedge S\mathbf{a} - S)). \tag{33}$$

By *modus ponens* from (33) and (31):

$$\neg(S\mathbf{a} - S) \Rightarrow \neg(-S\mathbf{a}S \wedge S\mathbf{a} - S). \tag{34}$$

By *modus ponens* from (34) and (29):

$$\neg(-S\mathbf{a}S \wedge S\mathbf{a} - S). \tag{35}$$

Hence, we infer $\neg(S = -S)$ or $S \neq -S$ in Aristotle's syllogistic as a theorem. It means that $S \neq -S$ can be used as an axiom of Aristotle's syllogistic.

3.1.4 Semantics of Aristotle's syllogistic

There are many traditional tools for checking a validity of Aristotle's reasoning such as Venn diagrams. But we define some algebraic models which verify all the theorems of Łukasiewicz's formalization of Aristotle's syllogistic. Let $\|\alpha\|$ be an interpretation of a formula $\alpha \in \mathcal{F}$ of propositional logic in the Boolean algebra $\mathcal{B} = \langle B; \cap, \cup, \neg, 1, 0\rangle$:

- for any formula $\alpha \in \mathcal{F}$, $\|\alpha\| \in B$;
- for any formula $\neg\alpha \in \mathcal{F}$, $\|\neg\alpha\| = \neg\|\alpha\| \in B$;
- for any formula $(\alpha \wedge \beta) \in \mathcal{F}$, $\|\alpha \wedge \beta\| = (\|\alpha\| \cap \|\beta\|) \in B$;
- for any formula $(\alpha \vee \beta) \in \mathcal{F}$, $\|\alpha \vee \beta\| = (\|\alpha\| \cup \|\beta\|) \in B$;
- for any formula $(\alpha \Rightarrow \beta) \in \mathcal{F}$, $\|\alpha \Rightarrow \beta\| = (\neg\|\alpha\| \cup \|\beta\|) \in B$.

A formula α is true in $\mathcal{B}$ (symbolically $\mathcal{B} \models \alpha$) if and only if $\|\alpha\| = 1 \in B$. In this way, $\mathcal{B}$ is said to be a model of propositional logic. A formula α is valid if and only if it is true in every model. For instance, $\alpha \vee \neg\alpha$ is valid. Indeed, $\|\alpha \vee \neg\alpha\| = (\|\alpha\| \cup \neg\|\alpha\|) = 1 \in B$ for all Boolean algebras. Propositional logic is sound and complete in Boolean algebras: (i) if $\alpha \in \mathcal{F}$ is a tautology in the logical system, then it is true in every model (soundness); (ii) if $\alpha \in \mathcal{F}$ is true in every model, it is a tautology in the logical system (completeness).

Now, let us define a model for atomic syllogistic formulas. First, let us take a *finite acyclic* or *well-founded* graph (it can be a tree, but not necessary) $\mathcal{G} = \langle N, E\rangle$, where N is a set of nodes or vertices, E is a set of edges or links, and there is a reachability relation on the nodes of the graph: two nodes $a, b \in N$ are reachable for each other when there exists a path through edges from a to b in $\mathcal{G}$, i.e., when a can reach b or b is reachable from a. We can define a $\cap$-semilattice $\mathcal{B}_\cap = \langle \wp(N), \cap, \mathbf{0}\rangle$ on the powerset $\wp(N)$ of all the nodes of $\mathcal{G}$. For any $A, B \in \wp(N)$, $A \cap B$ means all the joint nodes of A and B that can reach each other. In $\mathcal{B}_\cap$ there are the following four possibilities, assuming that $\prec$ means "less than" and $\preceq$ means "less than or equal to":

- $A \cap B = A$, when $A \preceq B$; that is, all the nodes of A can reach at least some nodes of B;
- $A \cap B = \mathbf{0}$, when there are no nodes of A that can reach at least some nodes of B;
- $\mathbf{0} \prec A \cap B \preceq A$, when at least some nodes of A can reach at least some nodes of B;
- $\mathbf{0} \preceq A \cap B \prec A$, when at least some nodes of A cannot reach at least some nodes of B.

Let $X\mathbf{x}Y$, where $\mathbf{x} \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$, be a metaformula that denotes the following formulas: $S\mathbf{x}P$, $-S\mathbf{x}P$, $S\mathbf{x} - P$, $-S\mathbf{x} - P$ for $S, P \in Q$ of $\mathcal{A}_{SA}$. Then $\|X\mathbf{x}Y\|$ is an interpretation of an atomic syllogistic formula $X\mathbf{x}Y$ of Aristotle's syllogistic in $\mathcal{G}$:

- for any syllogistic variable $S \in Q$, $\|S\|$ is not empty and $\|S\| \subseteq N$; that is, each syllogistic variable is interpreted as a non-empty subset of nodes;
- for any syllogistic variable with negation $-S$, such as $S \in Q$, $\|-S\| = N \backslash \|S\| \subset N$; that is, each syllogistic variable with negation $-S$ is interpreted as a complement of $\|S\|$ to all nodes of N;
- for any atomic syllogistic formula $X\mathbf{a}Y \in \mathcal{F}_{SA}$, $X\mathbf{a}Y$ is true in $\mathcal{G}$ (symbolically $\mathcal{G} \models X\mathbf{a}Y$) if and only if all the nodes of $\|X\| \subseteq N$ can reach some or all nodes of $\|Y\| \subseteq N$ in $\mathcal{G}$; it means that $\|X\mathbf{a}Y\| = \|X\| \cap \|Y\| = \|X\|$ in $\mathcal{B}_{\cap}$, otherwise $X\mathbf{a}Y$ is false in $\mathcal{G}$;
- for any formula $X\mathbf{e}Y \in \mathcal{F}_{SA}$, $X\mathbf{e}Y$ is true in $\mathcal{G}$ (symbolically $\mathcal{G} \models X\mathbf{e}Y$) if and only if there are no nodes of $\|X\| \subseteq N$ that can reach the nodes of $\|Y\| \subseteq N$ in $\mathcal{G}$; it means that $\|X\mathbf{e}Y\| = \|X\| \cap \|Y\| = \mathbf{0}$ in $\mathcal{B}_{\cap}$, otherwise $X\mathbf{a}Y$ is false in $\mathcal{G}$;
- for any formula $X\mathbf{i}Y \in \mathcal{F}_{SA}$, $X\mathbf{i}Y$ is true in $\mathcal{G}$ (symbolically $\mathcal{G} \models X\mathbf{i}Y$) if and only if some or all nodes of $\|X\| \subseteq N$ can reach some or all nodes of $\|Y\| \subseteq N$ in $\mathcal{G}$; it means that $\mathbf{0} \prec \|X\| \cap \|Y\| = \|X\mathbf{i}Y\|$ in $\mathcal{B}_{\cap}$, otherwise $X\mathbf{i}Y$ is false in $\mathcal{G}$;
- for any formula $X\mathbf{o}Y \in \mathcal{F}_{SA}$, $X\mathbf{o}Y$ is true in $\mathcal{G}$ (symbolically $\mathcal{G} \models X\mathbf{o}Y$) if and only if there are some or all nodes of $\|X\| \subseteq N$ that cannot reach the nodes of $\|Y\| \subseteq N$ in $\mathcal{G}$; it means that $\|X\mathbf{o}Y\| = \|X\| \cap \|Y\| \prec \|X\|$ in $\mathcal{B}_{\cap}$, otherwise $X\mathbf{o}Y$ is false in $\mathcal{G}$.

From this definition we can extend the interpretation up to all the syllogistic formulas $\alpha, \beta \in \mathcal{F}_{SA}$:

- $\mathcal{G} \models p$, where $p \in V$ of $\mathcal{A}_{SA}$, if and only if $\|p\|$ is not empty and $\|p\| \subseteq N$ in $\mathcal{G}$;
- $\mathcal{G} \models \neg\alpha$ if and only if α is false in $\mathcal{G}$;
- $\mathcal{G} \models (\alpha \wedge \beta)$ if and only if $\mathcal{G} \models \alpha$ and $\mathcal{G} \models \beta$;
- $\mathcal{G} \models (\alpha \vee \beta)$ if and only if $\mathcal{G} \models \alpha$ or $\mathcal{G} \models \beta$;
- $\mathcal{G} \models (\alpha \Rightarrow \beta)$ if and only if from $\mathcal{G} \models \alpha$ it follows that $\mathcal{G} \models \beta$.

Let us consider an example of models for Aristotle's syllogistic, given in Fig. 1. We can check that in this model, (6)–(9) hold true.

A syllogistic formula $\alpha \in \mathcal{F}_{SA}$ is valid if and only if α is true in all models (in all finite acyclic graphs). Let us check that such formulas exist. The formula $S\mathbf{a}S$ (see (6)) has the following interpretation in $\mathcal{G}$: $\|S\mathbf{a}S\| = \|S\| \cap \|S\| = \|S\| \subseteq N$ that is trivial and therefore it holds true in any finite acyclic graph. Hence, $S\mathbf{a}S$ is valid. To the same extent, the formula $S\mathbf{i}S$ (see (7)) holds true in any finite acyclic graph, too. Indeed, its interpretation in any $\mathcal{G}$ means that the set of nodes N of $\mathcal{G}$ is not empty: $\|S\mathbf{i}S\| = \|S\| \cap \|S\| = \|S\| \succ \mathbf{0}$. Aristotle's syllogistic is sound and complete in finite acyclic graphs: (i) if $\alpha \in \mathcal{F}_{SA}$ is a tautology in Aristotle's system, then it is true in every model (soundness); (ii) if $\alpha \in \mathcal{F}_{SA}$ is true in every model, it is a tautology in Aristotle's system (completeness).

As we see, Aristotle's syllogistic can play the role of naïve set theory with the models on finite acyclic (well-founded) graphs. Therefore, Fichte's Foundational Principles 1 and 2 can be formalized in this syllogistic to show that each set here is understood

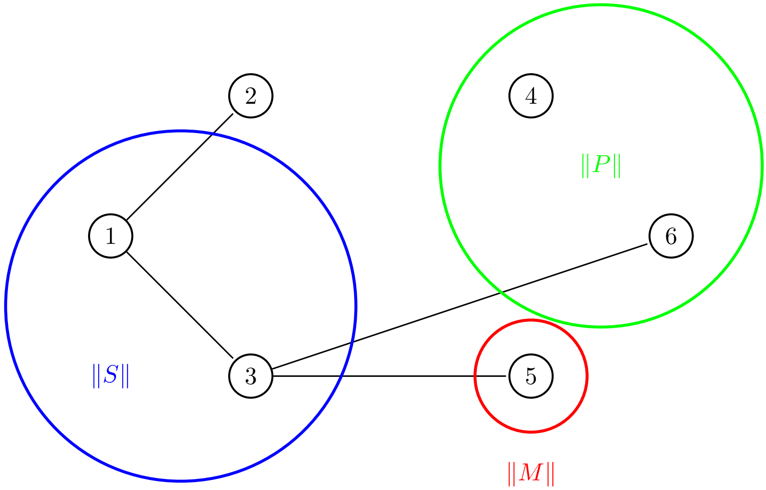

Fig. 1 Let $\|S\| = \{1, 3\}$ (blue circle), $\|M\| = \{5\}$ (red circle), and $\|P\| = \{4, 6\}$ (green circle). In this model, $S\mathbf{a}P$, $P\mathbf{i}S$, $S\mathbf{a}M$, $M\mathbf{a}S$, $M\mathbf{a}P$, $P\mathbf{i}M$ are true. (Color figure online)

without possible cycles (loops). As a consequence, $S = S$ and $S \neq -S$ are axioms of Aristotle's system as well as of any other well-founded set theory.

3.2 Non-well-founded syllogistic

Fichte draws attention to the fact that the phenomenon of the self is cyclic in nature and, therefore, suggests a different logical model, in which `I am not I` (that is, $S = -S$). In terms of transcendental logic, the I is not with itself, but with the other, with the not-I. In the following, we will first state the required axioms and definitions of non-Aristotelian syllogistic, then derive the third principle and finally show that this is also part of a formal system which is sound and complete, but this system is realizable on non-well-founded sets.

3.2.1 Syllogistic without axiom of foundation

Let us formalize the statement $S = -S$ as a tautology of a non-well-founded syllogistic, in which we have the same alphabet $\mathcal{A}_{SA}$ and the same set of well-formed formulas $\mathcal{F}_{SA}$ as in Aristotle's syllogistic.

The non-well-founded syllogistic is an extension of propositional logic, too. For instance, it can include axioms (1)–(3) and definitions (4)–(5). However, instead of syllogistic axioms (6)–(9), the following statement is a tautology:

$$S\mathbf{a}P. \tag{36}$$

Let us include definitions (10)–(15). The inference rules are the same as in Aristotle's syllogistic. Then this new syllogistic is not Łukasiewicz's one, but it is also an extension of propositional calculus.

3.2.2 Foundational Principle 3

With the syllogistic now defined, we will succeed in proving Fichte's third principle that would be absurdic in Aristotle's (Łukasiewicz's) syllogistic.

Fichte's Foundational Principle $S = -S$.

Proof The statement $S = -S$ has the meaning $-S\mathbf{a}S \wedge S\mathbf{a} - S$. Let us take (36) and replace P by $-S$:

$$S\mathbf{a} - S, \tag{37}$$

Then substitute $-S$ for S:

$$-S\mathbf{a} - -S, \tag{38}$$

From double negation:

$$-S\mathbf{a}S. \tag{39}$$

Take the following propositional tautology:

$$p \Rightarrow (q \Rightarrow (p \wedge q)) \tag{40}$$

and replace p by $-S\mathbf{a}S$ and q by $S\mathbf{a} - S$:

$$-S\mathbf{a}S \Rightarrow (S\mathbf{a} - S \Rightarrow (-S\mathbf{a}S \wedge S\mathbf{a} - S)). \tag{41}$$

From (41) and (39) by *modus ponens*:

$$S\mathbf{a} - S \Rightarrow (-S\mathbf{a}S \wedge S\mathbf{a} - S). \tag{42}$$

From (42) and (37) by *modus ponens*:

$$-S\mathbf{a}S \wedge S\mathbf{a} - S. \tag{43}$$

Thus, $S = -S$ is a tautology of non-well-founded syllogistic.

It is worth noting that we have used Łukasiewicz's syllogistic language, but changed the axioms to different ones. As a result, we have been able to prove a statement $S = -S$ that is not provable by Aristotle's syllogisms. And now we can show that there are algebraic models in which the proposition $S = -S$ is always true.

3.2.3 Semantics of non-well-founded syllogistic

Let $\mathcal{G}_{\bigcirc} = \langle N, E\rangle$ be a *cycle* or *non-well-founded* graph (it consists just of one cycle). The relation of reachability is the same as for the finite acyclic graphs: two nodes a,

$b \in N$ are reachable for each other when there exists a path through edges from a to b in $\mathcal{G}_{\bigcirc}$. The $\cap$-semilattice $\mathcal{B}_{\cap} = \langle \wp(N), \cap, \mathbf{0} \rangle$ on the powerset $\wp(N)$ does not hold true for the cycle graphs. Indeed, for any non-empty $A, B \subseteq N$, $A \cap B \neq \mathbf{0}$. Instead of $\mathcal{B}_{\cap}$, let us examine $\tilde{\mathcal{B}_{\cap}} = \langle \wp(N), \cap \rangle$, where $\cap$ is idempotent, commutative and associative.

Let us define the interpretation of syllogistic formulas of non-well-founded syllogistic in the same way as it was done for Aristotle's syllogistic, but with applying $\tilde{\mathcal{B}_{\cap}}$ instead of $\mathcal{B}_{\cap}$. Assume, $X\mathbf{x}Y$, where $\mathbf{x} \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$, is a metaformula that denotes the following formulas: $S\mathbf{x}P$, $-S\mathbf{x}P$, $S\mathbf{x} - P$, $-S\mathbf{x} - P$ for $S, P \in Q$ of $\mathcal{A}_{SA}$. Then $\|X\mathbf{x}Y\|$ is an interpretation of an atomic syllogistic formula $X\mathbf{x}Y$ of non-well-founded syllogistic in $\mathcal{G}_{\bigcirc}$:

- for any syllogistic variable $S \in Q$, $\|S\|$ is not empty and $\|S\| \subset N$ in $\mathcal{G}_{\bigcirc}$, i.e., it is a proper non-empty subset of N;
- for any syllogistic variable with negation $-S$, such as $S \in Q$, $\| - S\| = N \backslash \|S\| \subset N$ in $\mathcal{G}_{\bigcirc}$, i.e., $\| - S\|$ cannot be an empty set;
- for any atomic syllogistic formula $X\mathbf{a}Y \in \mathcal{F}_{SA}$, $X\mathbf{a}Y$ is true in $\mathcal{G}_{\bigcirc}$ (symbolically $\mathcal{G}_{\bigcirc} \models X\mathbf{a}Y$) if and only if $\|X\mathbf{a}Y\| = \|X\| \cap \|Y\| = \|X\|$ in $\tilde{\mathcal{B}_{\cap}}$, otherwise $X\mathbf{a}Y$ is false in $\mathcal{G}_{\bigcirc}$;
- for any formula $X\mathbf{i}Y \in \mathcal{F}_{SA}$, $X\mathbf{i}Y$ is true in $\mathcal{G}_{\bigcirc}$ (symbolically $\mathcal{G}_{\bigcirc} \models X\mathbf{a}Y$) if and only if some but not all nodes of $\|X\| \subset N$ can reach some nodes of $\|Y\| \subset N$ in $\mathcal{G}_{\bigcirc}$, otherwise $X\mathbf{i}Y$ is false in $\mathcal{G}_{\bigcirc}$;
- for any formula $X\mathbf{e}Y \in \mathcal{F}_{SA}$, $X\mathbf{e}Y$ is true in $\mathcal{G}_{\bigcirc}$ (symbolically $\mathcal{G}_{\bigcirc} \models X\mathbf{e}Y$) if and only if $X\mathbf{i}Y$ is false in $\mathcal{G}_{\bigcirc}$, otherwise $X\mathbf{e}Y$ is false in $\mathcal{G}_{\bigcirc}$;
- for any formula $X\mathbf{o}Y \in \mathcal{F}_{SA}$, $X\mathbf{o}Y$ is true in $\mathcal{G}_{\bigcirc}$ (symbolically $\mathcal{G}_{\bigcirc} \models X\mathbf{o}Y$) if and only if $X\mathbf{a}Y$ is false in $\mathcal{G}_{\bigcirc}$, otherwise $X\mathbf{o}Y$ is false in $\mathcal{G}_{\bigcirc}$.

Extend the interpretation of this definition up to all the syllogistic formulas $\alpha, \beta \in \mathcal{F}_{SA}$:

- $\mathcal{G}_{\bigcirc} \models p$, where $p \in V$ of $\mathcal{A}_{SA}$, if and only if $\|p\|$ is not empty and $\|p\| \subseteq N$ in $\mathcal{G}_{\bigcirc}$;
- $\mathcal{G}_{\bigcirc} \models \neg\alpha$ if and only if α is false in $\mathcal{G}_{\bigcirc}$;
- $\mathcal{G}_{\bigcirc} \models (\alpha \wedge \beta)$ if and only if $\mathcal{G}_{\bigcirc} \models \alpha$ and $\mathcal{G}_{\bigcirc} \models \beta$;
- $\mathcal{G}_{\bigcirc} \models (\alpha \vee \beta)$ if and only if $\mathcal{G}_{\bigcirc} \models \alpha$ or $\mathcal{G}_{\bigcirc} \models \beta$;
- $\mathcal{G}_{\bigcirc} \models (\alpha \Rightarrow \beta)$ if and only if from $\mathcal{G}_{\bigcirc} \models \alpha$ it follows that $\mathcal{G}_{\bigcirc} \models \beta$.

A syllogistic formula $\alpha \in \mathcal{F}_{SA}$ of non-well-founded system is valid if and only if α is true in all models (in all cycle graphs). We can prove that such formulas exist. The expression $S = -S$ has the following interpretation in $\mathcal{G}_{\bigcirc}$: $\|S = -S\| = \|S\| \cap -\|S\| = \|S\| = -\|S\| \subset N$ and it holds true in any cycle graph, see Fig. 2. Non-well-founded syllogistic is sound and complete in the cycle graphs: (i) if $\alpha \in \mathcal{F}_{SA}$ is a tautology in non-well-founded syllogistic, then it is true in every model $\mathcal{G}_{\bigcirc}$ (soundness); (ii) if $\alpha \in \mathcal{F}_{SA}$ is true in every model $\mathcal{G}_{\bigcirc}$, then it is a tautology in non-well-founded syllogistic (completeness).

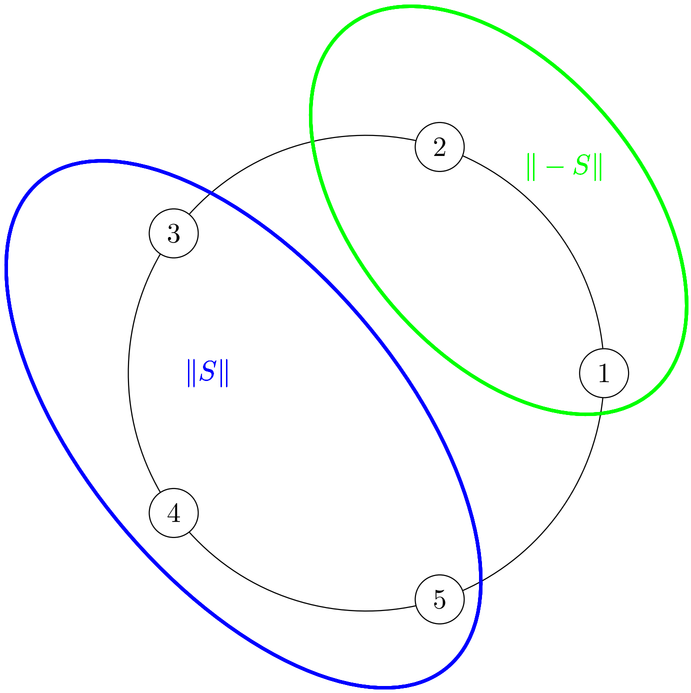

Fig. 2 Let $\|S\| = \{3, 4, 5\}$ (blue ellipse) and its complement $\| - S\| = \{1, 2\}$ (green ellipse). In this model, $S\mathbf{a} - S$, $-S\mathbf{a}S$ are true. (Color figure online)

4 Conclusion and discussion

We have seen in this paper that Fichte made a high demand on the Wissenschaftslehre. Although it is not supposed to be identical with formal logic, Fichte often refers back to formal logic in his Wissenschaftslehre, names connections between Wissenschaftslehre and formal logic, and even demands a rigor that is supposed to be equal to mathematics.

At least his early Wissenschaftslehre could not fulfil this high demand, since it showed numerous problems, explained by the level of formal logic in the nineteenth century, in which reasoning and its algebraic models were not distinguished. Nevertheless, we were able to name these problems, to recognise Fichte's intention, to concretise the aims of the Wissenschaftslehre from a logical point of view, and to demonstrate (with the help of Łukasiewicz's formalisation of Aristotle's syllogistic and Aczel's formalisation of non-well-founded sets containing some loops) that Fichte was intuitively on the right track.

In Aristotle's syllogistic with axioms (1)–(3), (6)–(9), we have proved the reflexivity of equality relation or PI as given by Fichte's Foundational Principle 1: $A = A$. Furthermore, we can also prove the symmetry of equality ($A = B$ if and only if $B = A$) and the transitivity of equality (if $A = B$ and $B = C$, then $A = C$) that were also discussed in GWL. It shows that Aristotle's syllogistic can be regarded as a naïve set theory with some basic operations on sets, such as the equality relation. In this system, we can also prove the PNC ($\neg(A = -A)$ or $A \neq -A$) as intended by Fichte's Foundational Principle 2.

Following Fichte's reasoning about reflectivity of I, assuming non-I, we can construct the non-well-founded syllogistic consisting of axioms (1)–(3), (36). In this syllogistic, on the one hand, synthetic *a priori* propositions (36) are axioms. On the other hand, we can prove the statement corresponding to the PSR $A = -A$ as aimed by Fichte's Foundational Principle 3. In the meanwhile, this system is consistent and furthermore it is sound and complete on the models consisting of cycle graphs.

The investigations presented here can be further developed in several directions, e.g.:

(i) The correspondence between the formal logic developed here and the transcendental approach can be developed much further in the future. What exactly does the transcendental side of GWL look like if its formal foundations are adequate? For example, to what extent do acts of the I now correspond to laws of thought?
(ii) The relation between the Wissenschaftslehre and the other parts of philosophy could be reconsidered. If parts of the foundational principles has been reformulated and even improved, how does this affect, for example, the philosophy of law or the doctrine of morals?
(iii) But our results are not only relevant for Fichte research or transcendental philosophy, but also for the philosophy of logic: can a transcendental philosophical approach that clarifies the order of the laws of thought perhaps give a basis for a new syllogistic? Or more modestly: Can one build a non-well-founded syllogistic on any non-well-founded graphs, including not only cycle graphs, but also cyclic and infinite graphs?

We would also like to note the high importance of Fichte's ideas for modern cognitive science. Since Descartes, self-reflection has been understood as `I = I`. Fichte, on the other hand, drew attention to the fact that self-reflection is rather `I = - I`. This corresponds, in particular, to Belzung and Chevalley's paradox (Belzung & Chevalley, 2002) that the same person can have different conscious emotional reactions to the same objects at different times. The same ambiguous behaviour is seen even at the level of single-celled organisms such as the amoeba or the polycephalic slime mould (Schumann & Adamatzky, 2015). Mathematically, this can be represented as an undecidable arithmetic function in which the number of inputs is less than the number of outputs (Schumann, 2017).

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Declarations

Conflict of interest There are no conflicts of interest.

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