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CORRECTION

Correction: Non-classical probabilities invariant under symmetries

Alexander R. Pruss[1]

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Let $\mathcal{F}$ be an algebra of subsets of Ω. A full conditional probability on $\mathcal{F}$ is a real-valued function on $\mathcal{F} \times (\mathcal{F} - \{\varnothing\})$ such that:

(C1) $P(\cdot \mid B)$ is a finitely additive probability function
(C2) $P(A \cap B \mid C) = P(A \mid C)P(B \mid A \cap C)$.[1]

If G is a group (intuitively, a group of symmetries, such as rigid motions on $\mathbb{R}^n$) acting on a set Ω^* containing Ω, we say that P is G-invariant provided that $P(gA \mid B) = P(A \mid B)$ whenever A, gA and B are all in $\mathcal{F}$ with B nonempty, $g \in G$, and $A \cup gA \subseteq B$. (There is no assumption here that $\mathcal{F}$ is itself G-invariant.)

One of the main theorems in Pruss (2021) characterized when exactly a G-invariant full conditional probability on the powerset $\mathcal{P}\Omega$ exists. Unfortunately, the proof of Lemma 2 was erroneous. The proof used the claim

$$\frac{\sum_{\mu \in \mathcal{B}_B} \mu(A)}{\sum_{\mu \in \mathcal{B}_B} \mu(B)} \cdot \frac{\sum_{\mu \in \mathcal{B}_C} \mu(B)}{\sum_{\mu \in \mathcal{B}_C} \mu(C)} = \frac{\sum_{\mu \in \mathcal{B}_C} \mu(A)}{\sum_{\mu \in \mathcal{B}_C} \mu(C)},$$

which it was erroneously said "follows" from the identity $\frac{\alpha}{\beta} \cdot \frac{\beta}{\gamma} = \frac{\alpha}{\gamma}$.

There does not seem to be a simple fix for this, but there is a new proof using the Rényi order in a way inspired by ideas in Armstrong (1989).[2]

[1] Pruss (2021) also includes the condition that if $P(A \mid B) = P(B \mid A) = 1$, then $P(C \mid A) = P(C \mid B)$, but that follows from (C1) and (C2).
[2] I am grateful to Grzegorz Tomkowicz for comments on the proof.

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✉ Alexander R. Pruss
Alexander_Pruss@baylor.edu

1 Baylor University, One Bear Place #97273, Waco, TX 76798-7273, USA

Lemma 1 *Let G act on $\Omega^* \supseteq \Omega$. Suppose that for every nonempty subset E of Ω, there is a G-invariant finitely additive measure $\mu : \mathcal{P}\Omega \to [0, \infty]$ with $\mu(E) = 1$. Let $\mathcal{F}$ be a finite algebra on Ω. Then there is a G-invariant full conditional probability on $\mathcal{F}$.*

Proof All the measures in the proof will be finitely additive. If μ and ν are measures on the same algebra, say that $\mu \prec \nu$ provided that for all $A \in \mathcal{F}$, if $\nu(A) > 0$, then $\mu(A) = \infty$. Say that a measure μ is non-degenerate provided that $0 < \mu(A) < \infty$ for some A. Then $\prec$ is known as the Rényi order (Armstrong, 1989; Rényi, 1956) and is a strict partial order on non-degenerate measures.

Choose a G-invariant probability measure μ_1 on $\mathcal{F}$ (there is one on $\mathcal{P}\Omega$, so restrict it to $\mathcal{F}$).

For $n \geq 1$, supposing we have chosen a G-invariant measure μ_n on $\mathcal{P}\Omega$, let

$$E_{n+1} = \bigcup \{B \in \mathcal{F} : \mu_n(B) = 0\}.$$

Note that $\mu_n(E_{n+1}) = 0$ since $\mathcal{F}$ is finite, so E_{n+1} is the largest μ_n-null member of $\mathcal{F}$. If $E_{n+1} = \varnothing$, let $N = n$, and our construction of $\mu_1, \ldots, \mu_N$ is complete.

If E_{n+1} is nonempty, choose a G-invariant measure ν on $\mathcal{P}\Omega$ with $\nu(E_{n+1}) = 1$. For $A \in \mathcal{F}$, let $\mu_{n+1}(A) = \nu(A)$ if $A \subseteq E_{n+1}$ and $\mu_{n+1}(A) = \infty$ otherwise.

I claim that μ_{n+1} is a G-invariant measure on $\mathcal{F}$. To check finite additivity, suppose A and B are disjoint members of $\mathcal{F}$. Then if A or B fails to be a subset of E_{n+1}, so does $A \cup B$, and so $\mu_{n+1}(A) + \mu_{n+1}(B) = \infty = \mu_{n+1}(A \cup B)$, and if $A \cup B$ fails to be a subset of E_{n+1}, so does at least one of A and B. But if A, B and $A \cup B$ are all subsets of E_{n+1}, then μ_{n+1} agrees with ν as applied to these sets, and ν is finitely additive.

It remains to check G-invariance. Suppose that $A, gA \in \mathcal{F}$. If both A and gA are subsets of E_{n+1}, the identity $\mu_{n+1}(A) = \mu_{n+1}(gA)$ follows from the G-invariance of ν. If neither is a subset of E_{n+1}, then $\mu_{n+1}(A) = \infty = \mu_{n+1}(gA)$. It remains to consider the case where one of A and gA is a subset of E_{n+1} and the other is not. Without loss of generality, suppose that A is a subset of E_{n+1} and gA is not (in the other case, let $A' = gA$ and $g' = g^{-1}$, so A' is a subset of E_{n+1} and $g'A'$ is not). Since $A \subseteq E_{n+1}$, we have $\mu_n(A) = 0$. By G-invariance, $\mu_n(gA) = 0$, and so $gA \subseteq E_{n+1}$, and thus the case is impossible.

Next note that that $\mu_{n+1} \prec \mu_n$. For if $\mu_n(A) > 0$, then A is not a subset of E_{n+1} and so $\mu_{n+1}(A) = \infty$.

The finiteness of $\mathcal{F}$ guarantees that the construction must terminate in a finite number N of steps, since we cannot have an infinite sequence of non-degenerate measures on a finite algebra $\mathcal{F}$ that are totally ordered by $\prec$.

We have thus constructed a sequence of G-invariant measures $\mu_1, \ldots, \mu_N$ such that $\mu_N \prec \cdots \prec \mu_1$. I claim that for any nonempty $A \in \mathcal{F}$, there is a unique $n = n_A$ such that $0 < \mu_n(A) < \infty$. Uniqueness follows immediately from the ordering $\mu_N \prec \cdots \prec \mu_1$, so only existence needs to be shown. By our construction, the only μ_N-null set is $\varnothing$, so $\mu_N(A) > 0$. Let n be the smallest index such that $\mu_n(A) > 0$. If $\mu_n(A) < \infty$, we are done. So suppose $\mu_n(A) = \infty$. We cannot have $n = 1$, since

μ_1 is a probability measure on $\mathcal{F}$. Thus, $n > 1$. By minimality of n, we must have $\mu_{n-1}(A) = 0$. Thus, $A \subseteq E_n$, and so $\mu_n(A) \le \mu_n(E_n) = 1$, a contradiction.

Now, for any $(A, B) \in \mathcal{F} \times (\mathcal{F} - \{\varnothing\})$, let $P(A \mid B) = \mu_{n(B)}(A \cap B)/\mu_{n(B)}(B)$. Then $P(\cdot \mid B)$ is finitely additive since $\mu_{n(B)}$ is.

Next, suppose we have A, B and C with $A \cap C$ nonempty. If $n(A \cap C) = n(C)$, then let $\mu = \mu_{n(C)} = \mu_{n(A\cap C)}$, so we have

$$\begin{aligned} P(A \mid C)P(B \mid A \cap C) &= \frac{\mu(A \cap C)}{\mu(C)} \cdot \frac{\mu(B \cap A \cap C)}{\mu(A \cap C)} \\ &= \frac{\mu(A \cap B \cap C)}{\mu(C)} = P(A \cap B \mid C). \end{aligned}$$

Now suppose that $n(A \cap C) \ne n(C)$ so $\mu_{n(C)}(A \cap C) \notin (0, \infty)$. Since $\mu_{n(C)}(A \cap C) \le \mu_{n(C)}(C) < \infty$, we must have $\mu_{n(C)}(A \cap C) = 0$. But then $P(A \mid C) = \mu_{n(C)}(A \cap C)/\mu_{n(C)}(C) = 0$ and $P(A \cap B \mid C) = \mu_{n(C)}(A \cap B \cap C)/\mu_{n(C)}(C) = 0$, and so both sides of (C2) are zero.

Finally, G-invariance of P follows immediately from G-invariance of the μ_n. □

We then get the following which is the same as the Lemma 2 in Pruss (2021) whose proof was flawed.

Corollary 1 *Let G act on $\Omega^* \supseteq \Omega$. There is a G-invariant full conditional probability on $\mathcal{P}\Omega$ if and only if for every nonempty subset E of Ω there is a G-invariant finitely additive measure $\mu : \mathcal{P}\Omega \to [0, \infty]$ with $\mu(E) = 1$.*

Proof First suppose there is a G-invariant full conditional probability P on $\mathcal{P}\Omega$. Then if E were a nonempty paradoxical subset of Ω^*, we could partition E into disjoint subsets A and B that could be decomposed under the action of G to form all of E, so that $1 = P(E \mid E) = P(A \mid E) + P(B \mid E) = P(E \mid E) + P(E \mid E) = 2$ by the finite additivity and G-invariance of $P(\cdot \mid E)$. But if E is not a paradoxical subset, then by Tarski's Theorem (Tomkowicz and Wagon 2016, Cor 11.2) there is a G-invariant finitely additive measure μ on $\mathcal{P}\Omega^*$ with $\mu(E) = 1$, and we can then restrict μ to $\mathcal{P}\Omega$.

Conversely, suppose for every nonempty E there is a μ as in the statement of the Corollary. For a finite algebra $\mathcal{F}$ on Ω, let P be a G-invariant full conditional probability on $\mathcal{F}$ by Lemma 1. Let $P_{\mathcal{F}}(A \mid B) = P(A \mid B)$ for $(A, B) \in \mathcal{F} \times (\mathcal{F} - \{\varnothing\})$ and $P_{\mathcal{F}}(A \mid B) = 0$ for all other $(A, B) \in \mathcal{P}\Omega \times (\mathcal{P}\Omega - \{\varnothing\})$. The set F of all finite algebras $\mathcal{F}$ on Ω, ordered by inclusion, is a directed set. Since $[0, 1]^{\mathcal{P}\Omega\times(\mathcal{P}\Omega-\{\varnothing\})}$ is a compact set by the Tychonoff Theorem, there will be a convergent subnet of the net $(P_{\mathcal{F}})_{\mathcal{F}\in F}$, and the limit of that subnet then satisfies the conditions for a G-invariant full conditional probability. □

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