Synthese (2021) 199 (Suppl 3):S493–S525
https://doi.org/10.1007/s11229-021-03415-x

Substructural approaches to paradox: an introduction to the special issue

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Accepted: 30 July 2021 / Published online: 23 December 2021

1 Volume topic

The idea of a "*substructural approach to a paradox*" is, naturally enough, the idea of an *approach to a paradox that uses a substructural logic*. While, contrary to what some might perhaps expect given also the formal sophistication of many such approaches, as it turns out no component of this complex idea lends itself to a very formal definition, it will be useful first to go through some informal elucidations of the idea's two main components: paradoxicality (Sect. 2) and substructurality (Sect. 3). Against that background, I'll then proceed to expound the philosophical significance of substructural approaches to paradox (Sect. 4), which will in turn serve as a springboard to introducing the papers of this volume (Sect. 5).

Thanks to all who taught me something about paradoxes, to all who taught me something about substructural logics and to all with whom I discussed substructural approaches to paradox. Thanks also to Gila Sher, Catarina Dutilh Novaes and Otávio Bueno for precious advice along the way as well as to the audiences at the 14th SIFA Conference *Shaping the Future: What Is the Role of Philosophy?* in Noto (University of Messina) and at the LanCog Seminar Series in Analytic Philosophy (University of Lisbon), where parts of the material in this introduction have been presented in 2021. Work on this introduction was supported by the Ramón y Cajal Research Fellowship RYC-2017-22883. Additionally, support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged. I also benefited from the FCT Project PTDC/FER-FIL/28442/2017 *Companion to Analytic Philosophy 2*, from the Project PID2019-105746GB-I00 of the Spanish Ministry of Science and Innovation *Linguistic Relativity and Experimental Philosophy* and from the Project 2019PIDPID-107667GB-I00 of the Spanish Ministry of Science and Innovation *Worlds and Truth Values: Challenges to Formal Semantics*.

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2 Paradoxicality

According to the traditional definition of paradoxicality (e.g. Sainsbury, 2009, p. 1), a *paradox*[1] is a situation where *apparently*[2] *true premises apparently entail an apparently false conclusion*.[3] Let's see some paradigmatic cases of paradox that typically motivate this definition (and that are extensively discussed in the papers of this volume):

The Liar paradox. For every circumstance that can hold or can fail to hold, there is *a sentence (of some language or other) expressing that that circumstance holds*, a sentence which is thereby *true* iff the circumstance holds. Letting T express truth, this entails the principle of *correlation* according to which, for every sentence φ, there is a sentence s such that $T(s) \leftrightarrow \varphi$ holds. Letting $\ulcorner\varphi\urcorner$ be a name for φ, for every φ an optimal candidate for satisfying correlation is φ itself, so that $T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$ holds. Moreover, there is a sentence λ identical with $\neg T(\ulcorner\lambda\urcorner)$, so that $T(\ulcorner\lambda\urcorner) \leftrightarrow \lambda$ holds. However, suppose that $T(\ulcorner\lambda\urcorner)$ holds. Then, by correlation and the entailment of

[1] Throughout, 'paradox' and its relatives are understood in the usual rather specific philosophical sense, as referring to, roughly, *a puzzle where something happens that goes against deeply rooted and well-grounded patterns of thinking*. In order to facilitate grasp of the intended notion, by way of *ostension* I'll presently offer some *paradigmatic cases*, and, by way of *definition*, I'll discuss throughout this section several *candidate characterisations*. By way of contrast, the intended notion is *not* at stake in many *ordinary* uses of 'paradox'-talk (witness "It's a paradox that a minute minority of global population owns the vast majority of global wealth") as well as in a few *philosophical* ones (witness "the sadness paradox"). Unsurprisingly, the intended notion might have indeterminate cases, but, given the good heuristic services it has lent to philosophy over the centuries, it is plausible to assume that it does mark a natural kind of philosophical inquiry. Thanks to Federico Lauria for questions that led to these clarifications.

[2] Throughout, by 'Apparently, φ' and its relatives, I understand the same as I do by 'There is a *prima facie* justification for believing that φ' and its relatives. Such understanding does need to be fine-tuned in several directions, but it works well enough for the purposes of this introduction.

[3] Please take a moment and appreciate the wisdom with which tradition tries to *multiply and distribute* appearances on separate elements. In one direction, the main alternative would be a *single undistributed* appearance that *true premises entail a false conclusion*, but the existence of *that* appearance is most doubtful. I should add however that, in this direction, such wisdom only goes so far, since a few paradoxes (for example, the second one in the list below in the main text) can naturally be represented—by assuming suitable entailments (fn 29)—as having no premises and an obviously false conclusion, in which case there would have to be an appearance that *a logical truth is false*, but also the existence of *that* appearance is most doubtful. Throughout letting an *argument* be a piece of reasoning possibly made up by more than one basic logical principle, in my view, what is essentially apparently valid in a paradox is not the *fully interpreted argument* at play in the paradox, but its corresponding *argument form*, with the bad surprise coming in when that form is instantiated with the relevant sentences at play in the paradox (when presenting my views, I'll henceforth assume this understanding of what is essentially apparently valid in a paradox). Such a pattern of *a universal generalisation's apparently holding without a certain instance's apparently doing so* is far from unusual: for example, it is apparently the case that every instance of *modus ponens* is valid, but it is not apparently the case that McGee-style instances of *modus ponens* (McGee, 1985) are valid. In the opposite direction, the main alternative would be *appearances distributed on the individual premises*, but that would count the subject of a Preface paradox (the fifth paradox in the list below in the main text) as being *herself* subject to paradox-generating appearances of *first-level propositions concerning mundane matters* (over and above the fact that *those reflecting on her situation*—who might include her!—are subject to paradox-generating appearances of *second-level propositions concerning her knowledge*), which she almost invariantly isn't (this point affects the definition of paradoxicality offered e.g. by Lycan, 2010, according to which, roughly, a paradox is a situation where *there is an inconsistent set of propositions—each premise plus the negation of the conclusion—each of which is apparently true*). Thanks to Sergi Oms for discussion of surrounding issues.

modus ponens ($\varphi, \varphi \to \psi \vdash \psi$),[4] λ (*i.e.* $\neg T(\ulcorner\lambda\urcorner)$) holds, and so it seems that, by the entailment of *adjunction* ($\varphi, \psi \vdash \varphi \,\&\, \psi$), $T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$ holds. Suppose next that $\neg T(\ulcorner\lambda\urcorner)$ (*i.e.* λ) holds. Then, by correlation and *modus ponens*, $T(\ulcorner\lambda\urcorner)$ holds, and so it seems that, by adjunction, $T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$ holds. Therefore, discharging the suppositions, by the metaentailment of *reasoning by cases* (if $\Gamma_0, \varphi \vdash \Delta_0$ holds and $\Gamma_1, \psi \vdash \Delta_1$ holds, $\Gamma_0, \Gamma_1, \varphi \vee \psi \vdash \Delta_0, \Delta_1$ holds), $T(\ulcorner\lambda\urcorner) \vee \neg T(\ulcorner\lambda\urcorner)$—which, being the conclusion of the law of *excluded middle* ($\oslash \vdash \varphi \vee \neg\varphi$), is a logical truth—entails $T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$—which, being the premise of the law of *noncontradiction* ($\varphi \,\&\, \neg\varphi \vdash \oslash$), is absurd. The argument is apparently valid, the premises (correlation, the existence of a sentence such as λ and $T(\ulcorner\lambda\urcorner) \vee \neg T(\ulcorner\lambda\urcorner)$) are apparently true, but the conclusion ($T(\ulcorner\lambda\urcorner) \,\&\, \neg T(\ulcorner\lambda\urcorner)$) is apparently false (to the best of my knowledge, the first clear version of the paradox was given by Eubulides of Miletus).

Curry's paradox. Letting χ be an absurd sentence, there is a sentence κ identical with $T(\ulcorner\kappa\urcorner) \to \chi$, so that $T(\ulcorner\kappa\urcorner) \leftrightarrow \kappa$ holds. However, suppose that $T(\ulcorner\kappa\urcorner)$ holds. Then, by correlation and *modus ponens*, κ (*i.e.* $T(\ulcorner\kappa\urcorner) \to \chi$) holds, and so it seems that, by *modus ponens*, χ holds. Therefore, discharging the supposition, by the metaentailment of *unipremise conditional proof* (if $\varphi \vdash \psi$ holds, $\oslash \vdash \varphi \to \psi$ holds), κ holds, and so, by correlation and *modus ponens*, $T(\ulcorner\kappa\urcorner)$ holds, and hence, by *modus ponens*, χ holds. The argument is apparently valid, the premises (correlation and the existence of a sentence such as κ) are apparently true, but the conclusion (χ) is apparently false (to the best of my knowledge, the first clear version of the paradox was given by Juan de Celaya, though variations thereof (fn 51) appeared earlier in medieval times).

Russell's paradox. For every way objects can be or can fail to be, there is *a set of the objects that are that way*, a set which thereby *contains* all and only those objects that are that way. Letting $\in$ express (set) containment, this is typically taken to entail the principle of *comprehension* according to which, for every formula φ, there is a set s (with 's' 's not occurring free in φ) such that $\forall\xi(\xi \in s \leftrightarrow \varphi)$ holds. Therefore, by comprehension, there is a set r such that $\forall x(x \in r \leftrightarrow \neg(x \in x))$ holds, and so, by universal instantiation ($\forall\xi\varphi \vdash \varphi_{\tau/\xi}$), $r \in r \leftrightarrow \neg(r \in r)$ holds. However, suppose that $r \in r$ holds. Then, by comprehension and *modus ponens*, $\neg(r \in r)$ holds, and so it seems that, by adjunction, $r \in r \,\&\, \neg(r \in r)$ holds. Suppose next that $\neg(r \in r)$ holds. Then, by comprehension and *modus ponens*, $r \in r$ holds, and so it seems that, by adjunction, $r \in r \,\&\, \neg(r \in r)$ holds. Therefore, discharging the suppositions,

[4] Throughout, $\vdash$ expresses the relation of *logical consequence* (*i.e. following-from*) between certain *conclusions* and certain *premises*. Both premises and conclusions can be *multiple* (*i.e.* one, or none, or many), and, as will become clear in Sect. 3, in the context of substructural logics, the *collections* into which premise*s* and conclusion*s* are thus put together are not always *sets*, but are sometimes something *more fine-grained* such as *multisets* or *sequences* (since some substructural logics distinguish φ, φ from φ and some substructural logics distinguish ψ, φ from φ, ψ). Moreover, throughout, I assume the standard way of representing the logical properties of *being a logical truth* and of *being absurd* in terms of the logical relation of *logical consequence*, so that, letting $\oslash$ be the *empty collection* with the suitable fineness of grain, $\oslash \vdash \varphi$ represents that φ is a logical truth and $\varphi \vdash \oslash$ represents that φ is absurd (see Zardini, 2018, p. 269, fn 34; 2021d for some critical considerations concerning this kind of representation). Finally, throughout, I call principles of the form $\oslash \vdash \Gamma$ or $\Gamma \vdash \oslash$ '*laws*', principles of the form $\Gamma \vdash \Delta$ '*entailments*' (whereas I reserve '*implication*' and its relatives for the usual logical operation) and principles of the form 'If $\Gamma_0 \vdash \Delta_0$, $\Gamma_1 \vdash \Delta_1$, $\Gamma_2 \vdash \Delta_2 \ldots$ and $\Gamma_i \vdash \Delta_i$ hold, $\Gamma_{i+1} \vdash \Delta_{i+1}$ holds' '*metaentailments*' (see Zardini, 2021d for some terminological discussion).

reasoning by cases, $r \in r \vee \neg(r \in r)$—which, being the conclusion of the law of excluded middle, is a logical truth—entails $r \in r \,\&\, \neg(r \in r)$—which, being the premise of the law of noncontradiction, is absurd. The argument is apparently valid, the premises (comprehension and $r \in r \vee \neg(r \in r)$) are apparently true, but the conclusion ($r \in r \,\&\, \neg(r \in r)$) is apparently false (to the best of my knowledge, the first clear version of the paradox was given by Russell, 1903, pp. 523–528).

The Sorites paradox. One does not stop being *bald* by the addition of *one single hair*. Keeping fixed the other dimensions of comparison relevant for baldness and letting $B(i)$ express that a man with i hairs is bald, that entails the principle of *tolerance* according to which $B(i) \to B(i+1)$ holds. However, by tolerance, $B(1) \to B(2)$ holds, which, together with $B(1)$, by *modus ponens*, entails that $B(2)$ holds. Yet, by tolerance, $B(2) \to B(3)$ also holds, which, together with the previous *lemma* that $B(2)$ holds, by *modus ponens*, entails that $B(3)$ holds. With another $99,997$ structurally identical arguments, we reach the conclusion that $B(100,000)$ holds. It seems then that conclusion follows simply from tolerance and $B(1)$. The argument is apparently valid, the premises (tolerance and $B(1)$) are apparently true, but the conclusion ($B(100,000)$) is apparently false (to the best of my knowledge, the first clear version of the paradox was again given by Eubulides of Miletus).

The Preface paradox. One does not stop *knowing* something by the addition of *one single known conjunct*. Under suitable idealisations, that entails the principle of *collection* according to which, if one knows that P and that Q, one knows that [P and Q]. Moreover, let's suppose that Greg only has the usual, nonentailing kind of evidence about history, and let's take many, say 100,000, of his true epistemically best beliefs about it, where every belief is a belief in a proposition independent from the propositions that are the objects of the other beliefs (let these propositions be the propositions that H_1, that H_2, that H_3..., that $H_{100,000}$). Therefore, for every $1 \leq i \leq 100,000$, we should plausibly accept that Greg knows that H_i, although he plausibly does not know that [H_1 and H_2 and H_3 ... and $H_{100,000}$]. However, by collection, if Greg knows that H_1 and that H_2, he knows that [H_1 and H_2], and so, since, by adjunction, the antecedent holds, by *modus ponens* he knows that [H_1 and H_2]. Yet, by collection, it is also the case that, if Greg knows that [H_1 and H_2] and that H_3, he knows that [H_1 and H_2 and H_3], and so, since, by the previous lemma and adjunction, the antecedent holds, by *modus ponens* he knows that [H_1 and H_2 and H_3]. With another $99,997$ structurally identical arguments, we reach the conclusion that Greg knows that [H_1 and H_2 and H_3 ... and $H_{100,000}$]. It seems then that that conclusion follows simply from collection and, for every $1 \leq i \leq 100,000$, Greg's knowledge that H_i. The argument is apparently valid, the premises (collection and, for every $1 \leq i \leq 100,000$, Greg's knowledge that H_i) are apparently true, but the conclusion (Greg's knowledge that [H_1 and H_2 and H_3 ... and $H_{100,000}$]) is apparently false (to the best of my knowledge, the first clear version of the paradox was given by Makinson, 1965).

The Material-Implication paradox. '*If* Hellas Verona won the last *Serie A*, *then* they won exactly one *Serie A*' is *stronger* than '*Either* it is not the case that Hellas Verona won the last *Serie A* *or* they won exactly one *Serie A*' (for one thing, it is not the case that Hellas Verona won the last *Serie A*, and so, by the entailment of *addition* ($\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$), 'Either it is not the case that Hellas Verona won the last *Serie A* or they won exactly one *Serie A*' holds, but 'If Hellas Verona won the last *Serie A*, they won exactly one *Serie A*' does not). However, suppose that either it is not the case that Hellas Verona won the last *Serie A* or they won exactly one *Serie A* and suppose further that Hellas Verona won the last *Serie A*. Then, by the entailment of *disjunctive syllogism* ($\varphi, \neg\varphi \vee \psi \vdash \psi$), Hellas Verona won exactly one *Serie A*. Therefore, discharging the second supposition, by the metaentailment of *multipremise conditional proof* (if $\Gamma, \varphi \vdash \psi$ holds, $\Gamma \vdash \varphi \rightarrow \psi$ holds), 'Either it is not the case that Hellas Verona won the last *Serie A* or they won exactly one *Serie A*' entails 'If Hellas Verona won the last *Serie A*, they won exactly one *Serie A*'. The argument is apparently valid, the premise ('Either it is not the case that Hellas Verona won the last *Serie A* or they won exactly one *Serie A*') is apparently true, but the conclusion ('If Hellas Verona won the last *Serie A*, they won exactly one *Serie A*') is apparently false (to the best of my knowledge, the first clear version of the paradox was given by Faris, 1962, pp. 115–119).

There are many other paradoxes, but, as a matter of fact, these (possibly save for the Preface paradox) are the historically most salient ones *where the correctness of classical logic has most severely been put into question*.[5] It is natural to group paradoxes in general in *(natural) kinds*, and ask *whether* in particular some of the paradoxes in our list are *of the same kind*. But, to discuss better this question and surrounding issues, we first need to understand better *what it is* for two paradoxes to be of the same kind. The traditional definition of paradoxicality would seem right in appealing to a *subjective* element (signalled by 'apparently'), and etymology (Ancient Greek *para doxan*, beyond belief) concurs in revealing a perception of paradox as a moment of *disconnection between appearance and reality*. Paradoxicality consists in a certain general type of *mistake* (in the sense of an *appearance [of something] which fails to hold*).[6] It is then natural to assume that *two paradoxes are of the same kind iff they make the same kind of mistake*, where, the notion of mistake plausibly having an *aetiological* element, sameness of kind of mistake implies sameness of kind of the *cause* that brings about the mistake, where in turn, at least for the philosophical purposes of this introduction, the most relevant cause can be taken to be the cause *of the fact that is mistakenly represented* (rather than e.g. the cause *of the mistaken representation*).[7,8] Notice that this natural conception straightforwardly implies that

[5] Other historically salient challenges to classical logic (paradigmatically, the *intuitionistic* challenge originating with Brouwer, 1908) do not make use of paradoxes, but rely instead on other kinds of considerations.

[6] Typically, for the sake of simplicity, I talk in the singular about "a mistake" (or "the mistake"), although, as will become clear at the end of this section, on my view, in every paradox, a *fundamental mistake* actually gives rise to several *chains of mistakes* each determined by a specific presentation of the paradox, where all such chains of all paradoxes *culminate in a mistake of a certain general type that is what makes for paradoxicality*. On such a view, singular occurrences of 'mistake' and its relatives that are not related to talk of the "general type of mistake that is what makes for paradoxicality" are best understood as being about the fundamental mistake.

[7] However, it should be noted that, beyond the issue of kind individuation, some diagnoses of some paradoxes do importantly appeal as a *proximate cause* to something that still involves a *subjective* element (see fn 12 for a prime example).

[8] Throughout, by 'cause' I really mean 'main partial cause(s)'.

two paradoxes are of the same kind iff they have the same kind of solution:[9] for they are of the same kind iff they make the same kind of mistake, where the latter is the case iff, in them, the same kind of fact having the same kind of cause is mistakenly represented by the same kind of representation—but the solution to a paradox is exactly the *elimination* of its mistaken representation (by replacing it with one that corresponds to the relevant fact) accompanied by an *explanation* of why it is mistaken (by individuating the cause for why what obtains is the relevant fact rather than the one the mistaken representation would correspond to), where both elimination and explanation are of the same kind iff the mistake is, in the way just spelt out, of the same kind. Relatedly, notice that, on this natural conception, while it is of course possible to know that *two paradoxes are of the same kind* without knowing what kind their solution is, it is not possible to know *what kind a paradox is* without knowing what kind its solution is, since what kind a paradox is consists in what kind of mistake it makes (which determines what kind of elimination of its mistaken representation its solution includes), which in turn depends on what kind of cause brings about the mistake (which determines what kind of explanation of why the representation is mistaken its solution includes).

Observe then that it is reasonable to suppose that at least some of the paradoxes in our list are of the same kind. This is immensely plausible for the Liar paradox and Curry's paradox, as they both rely in the same way on the appearance of correlation *plus* selfreference (*i.e.* to make it appear that a proposition about truth is *equivalent with a logical function of itself*—in particular, with its own negation or with its own implication to a proposition respectively—in such a way that the logical interaction of the proposition with its logical function gives rise to an entailment that justifies the logical function)—it would then be immensely puzzling if, in the two cases, mistakes of two different kinds were linked with that same general type of appearance. Indeed, it is appealing to see a negation $\neg\varphi$ as a *special kind of implication* (implication from φ to the *absurdity constant* f) or, *vice versa*, see an implication $\varphi \rightarrow \psi$ as a *generalised kind of negation* (which "ψises" φ just as the negation $\neg\varphi$ "fises"—*i.e.* absurdises—φ).[10] It is then possible to give very natural versions of the Liar paradox and Curry's paradox that are totally analogous (see Zardini, 2015a for the details)—it would then be even more puzzling if, in the two cases, the mistakes occurred at different steps of the paradox, or if, even though occurring at the same step, they were nevertheless of two different kinds. It should be remarked though that this sameness of kind has recently been contested (by Priest, 1994; see Zardini, 2015a, p. 489, fn 43; Oms & Zardini, 2021, pp. 202–204 for some critical discussion).

[9] Plausible as it may be, the left-to-right implication is actually not always vindicated in the contemporary literature: for example, it becomes problematic from the point of view of the nowadays fashionable approach to paradox focusing on *cost-benefit analyses* (see Oms & Zardini, 2021, p. 209, fn 4 for some details). The right-to-left implication may come across as less plausible but it actually makes perfect sense for a proper *solution* to a paradox (as opposed to a mere *block* to it).

[10] This is reflected by the fact that, at a deep level of analysis on which *negation* and *implication* correspond in the *object language* to the *metalanguage* properties of *absurdity* and *entailment* respectively (Zardini, 2021b), the principles governing negation and implication are essentially the same: [$\Gamma, \varphi \vdash \Delta$ (*i.e.* $\Gamma, \varphi \vdash \Delta, \mathsf{f}$) holds iff $\Gamma \vdash \Delta, \neg\varphi$ holds] and [$\Gamma, \varphi \vdash \Delta, \psi$ holds iff $\Gamma \vdash \Delta, \varphi \rightarrow \psi$ holds] (see Zardini, 2021d for a yet deeper level of analysis from which the current one arguably flows).

It is also plausible that the Liar paradox and Russell's paradox are of the same kind, since correlation and comprehension would seem supported by the same kind of idea (*i.e.* to connect a *mundane* state of affairs such as snow's being white with a *semantic* state of affairs such as the sentence 'Snow is white' 's being true or with a *set-theoretic* one such as the set of white objects containing snow, both in the *ascending* direction and in the *descending* direction) and since the Liar paradox and Russell's paradox both rely in the same way on the appearance of correlation and comprehension respectively *plus* selfreference (*i.e.* to make it appear that a proposition about truth and containment respectively is *equivalent with a logical function of itself*—in particular, with its own negation in both cases—in such a way that the logical interaction of the proposition with its logical function gives rise to an entailment that justifies the logical function)—it would then be puzzling if, in the two cases, mistakes of two different kinds were linked with that same general type of appearance. Indeed, the *Grelling-Nelson paradox* can be got from Russell's paradox by replacing *sets* with *predicates* and *containing* with *being-true-of* (see Grelling & Nelson, 1908 for the details). Presumably, the Liar paradox and the Grelling-Nelson paradox are of the same kind, but the Grelling-Nelson paradox and Russell's paradox are totally analogous—it would then be even more puzzling if, in the latter two cases, the mistakes occurred at different steps of the paradox, or if, even though occurring at the same step, they were nevertheless of two different kinds.[11] It should be remarked though that this sameness of kind has long been contested (since Peano, 1906, p. 157 and then influentially by Ramsey, 1925, pp. 352–354; see Priest, 2003, pp. 155–157 for some critical discussion).

Using '*semantic paradoxes*' as a label for the kind of paradox instantiated by the Liar paradox and Curry's paradox (and '*set-theoretic paradoxes*' as a label for the kind of paradox instantiated by Russell's paradox), it also pays to ask *what the cause of the mistake in the semantic paradoxes* (and in the set-theoretic paradoxes, if these are the same kind as the semantic paradoxes) *is*. A popular diagnosis is that it is some sort of *selfreference* (Poincaré, 1906; however, the essence of the idea goes at least as back as Richard, 1905).[12] In one direction, such a diagnosis faces the challenge that there are selfreferential expressions that, even if expressing semantic properties, would seem *unproblematic* (e.g. 'This sentence refers to itself'). In the other direction, such a diagnosis faces the challenge that there are semantic paradoxes that would seem

[11] Incidentally, notice that the Grelling-Nelson paradox is a stumbling block for attempts (more anecdotical than documented) at solving the semantic paradoxes *by denying the meaningfulness of a selfreferential entity of the sort of the Liar sentence merely qua selfreferential* (not that such attempts would otherwise be very plausible, as witnessed, among other things, by the fact that meaningless sentences are not true and by the meaningfulness of 'The sentence mentioned in fn 11 of the paper 'Substructural Approaches to Paradox: An Introduction to the Special Issue' is not true', see also Zardini, 2008c, pp. 561–567): there is nothing *selfreferential* in the predicate 'x is not true of x' (what there is is the *identity* of its arguments).

[12] This broad kind of diagnosis allows for several versions that *vary* with respect to *how ultimate a cause* the relevant sort of selfreference is. For example, Poincaré (1906) himself maintains that the relevant sort of selfreference is only *illusory*, and so constitutes simply a further, upstream mistake, whose *purely objective* cause (*i.e.* the cause of the fact that is mistakenly represented rather than also the cause of the mistaken representation) is, further upstream, the existence of a predicative hierarchy. For a contrasting example, Priest (2003) maintains instead that the relevant sort of selfreference is *real*, and so constitutes the purely objective cause of the semantic paradoxes upstream of any mistake involved in them. Analogous comments apply to the alternative broad kind of diagnosis discussed in the next paragraph.

not to involve any kind of selfreference, as in *Yablo's paradox* (see Yablo, 1985, p. 340 for the details).[13]

Both these challenges motivate a different, equally popular diagnosis, according to which the cause of the mistake in the semantic paradoxes is some sort of *ungroundedness* (Herzberger, 1970; however, the essence of the idea goes at least as back as Langford, 1937). To exemplify with the paradigmatic case of truth, a sentence is *grounded* iff its truth or falsity can ultimately be traced back to the obtaining or not of *nonsemantic facts* by applications of the principle that *the truth or falsity of 'P' depends on whether P*. It's easy to see that both the Liar sentence and the Yablo sentences generate a *nonwellfounded dependence chain* (*circular* in the former case and *infinitely descending* in the latter case) and are thus ungrounded. In one direction, such a diagnosis faces the challenge that there are ungrounded expressions that would seem *unproblematic* (e.g. 'Every sentence is such that, if it is true, it is true').[14] In the other direction, such a diagnosis faces the challenge that there are semantic paradoxes that would seem *not to involve any kind of ungroundedness*. Consider, for example, a Curry sentence κ' whose consequent is unproblematically true (say, 'EZ likes *pastissada*'). Assuming that the truth of the consequent suffices for the truth of an implication,[15] κ' *would seem grounded, yet it gives rise to a version of Curry's paradox*: while 'EZ likes *pastissada*' is unproblematically true, it should not be provable by Curry-style reasoning (Zardini, 2015a, pp. 469–485 makes essentially the same point against views that postulate *indeterminacy* (Bočvar, 1938) or *overdeterminacy* (Priest, 1979) as an essential ingredient—let alone cause—of a semantic paradox). It therefore remains very much an open question what the cause of the mistake in the semantic paradoxes is (see Zardini, 2015a, p. 492; 2019a for a proposal that has at least the merit of being able to meet the challenges presented in this and the last paragraph).

Moving on to the other paradoxes in our list, it is much less plausible that the Sorites paradox is of the same kind as the semantic paradoxes: the crucial appearance in the former case would seem one to the effect that *a big change comes about only through a series of small changes*, whereas the crucial appearance in the latter case would seem one to the effect that *a proposition is equivalent to a logical function of itself*—it would then be puzzling if, in the two cases, the same kind of mistake were linked with such two different general types of appearances (however, see e.g. Priest, 2010 for a recent argument in favour of sameness of kind and Oms & Zardini, 2021

[13] To the best of my knowledge, the first semantic paradox with an *infinitely descending chain* is actually given by Herzberger (1970, p. 150), who presents an *infinitely-descending-chain* version of the *Truth-Teller paradox* (whose *selfreferential* version—'This sentence is true'—to the best of my knowledge is also in modern times first given in print by Herzberger, 1970, p. 149, who presents it as already well-known at his time; such a version was in effect already discussed in medieval times, see the relevant passage of the tract *Insolubilia monacensia* in de Rijk, 1966, pp. 106–107).

[14] I should add that, both in the case of this challenge and in the case of the first challenge for the selfreference diagnosis, on my own diagnosis (Zardini, 2019a), the relevant sentences do carry the cause of the semantic paradoxes, but in a form that makes them unproblematic. Thus, in both cases, one way for the other diagnoses to meet the challenge would be to develop an account of selfreference or ungroundedness—*qua* causes of the semantic paradoxes—that similarly identifies a form of these that can be carried by unproblematic sentences.

[15] An assumption that we can force to hold by taking the implication to be *material* or, anyway, by switching from implication to *disjunction* of [negation of antecedent] and consequent.

for a criticism).[16] Similar considerations apply to the relation between the Preface paradox (where the crucial appearance would seem one to the effect that *ignorance comes about only through a series of additions of known conjuncts*) and the semantic paradoxes; it is however plausible that the former is of the same kind as the Sorites paradox (see my contribution to this volume for an argument in favour of sameness of kind). Moreover, it is not at all plausible that the Material-Implication paradox is of the same kind as the semantic paradoxes: the crucial appearance in the former case would seem one to the effect that *a disjunction generates an implicational link from the negation of one of its disjuncts to its other disjunct*—it would then be very puzzling if, in the two cases, the same kind of mistake were linked with such two different general types of appearances. Finally, it is not at all plausible that the Material-Implication paradox is of the same kind as the Sorites paradox—again, it would be very puzzling if, in the two cases, the same kind of mistake were linked with such two different general types of appearances.

Notice that already the less-than-paradigmatic κ'-version of Curry's paradox belies the traditional definition of paradoxicality, since that is a situation where apparently true premises apparently entail an apparently *true* conclusion (López de Sa & Zardini, 2007, p. 246). Indeed, *analogous points can be made for all the other paradoxes in our list*. For the Liar paradox, consider the Liar sentence 'This sentence is not true or EZ likes *pastissada*' and the fact that Liar-style reasoning licences the true conclusion that EZ likes *pastissada* (*mutatis mutandis* for Russell's paradox). For the Sorites paradox, consider a soritical series for baldness starting with Yul Brynner (in his fifties) and the fact that Sorites-style reasoning licences the true conclusion that Sean Connery (in his fifties) is bald (Oms & Zardini, 2019, p. 8, fn 14). For the Preface paradox, consider Greg's situation and the fact that Preface-style reasoning licences the true conclusion that Greg knows that [H_1 and H_2 and H_3 ... and H_{10}]. For the Material-Implication paradox, consider the circumstance that it is not the case that Hellas Verona won the last *Serie A* and the fact that [Material-Implication]-style reasoning licences the true conclusion that, if Hellas Verona won the last *Serie A*, they won at least one *Serie A*. The traditional definition breaks down across the board.[17]

[16] This is not deny that *some of the (philosophical, logical, natural-scientific etc.) materials employed* in the solution to the paradox in the former case might also be employed in the solution to the paradox in the latter case (for example, it might be that, in both cases, the solution employs the same nonclassical logic). Such sameness *of materials employed in the solution* remains indeed a possibility for those two cases, but one that in itself falls very much short of sameness *of kind of solution* or of sameness *of kind of paradox* in the sense explained in the fifth last paragraph. An analogous comment applies to all other pairs of cases considered in this paragraph.

[17] For good measure, a similar point can be made about *the premises' being apparently true*. For Curry's paradox, consider the Curry sentence 'If EZ does not like *pastissada*, then, if this sentence is true, χ' and the fact that Curry-style reasoning is triggered by the false premise that EZ does not like *pastissada*. For the Liar paradox, consider the Liar sentence 'This sentence is not true or EZ likes *pastissada*' and the fact that Liar-style reasoning is triggered by the false premise that EZ does not like *pastissada* (*mutatis mutandis* for Russell's paradox). For the Sorites paradox, consider a soritical series for baldness ending with Carlos Valderrama and the fact that Sorites-style reasoning is triggered by the false premise that Diego Maradona is bald. For the Preface paradox, consider Greg's situation and the fact that Preface-style reasoning is triggered by the false premise that Greg knows that [H_1 and H_2 and H_3 ... and $H_{99,991}$]. For the Material-Implication paradox, consider the circumstance that it is neither the case that Hellas Verona won the last *Serie A* nor is it the case that they have never won a *Serie A* and the fact that [Material-Implication]-style reasoning is triggered by the false premise that either Hellas Verona won the last *Serie A* or they have never won a

Focusing now on the paradigmatic example represented by Curry's paradox, one natural reaction to the point of the last paragraph is to say that what is paradoxical in Curry's paradox is not that apparently true premises apparently entail an apparently false conclusion, but that apparently *a priori (known, or knowable, or possibly justifiable etc.)* premises apparently entail a conclusion that is apparently *not a priori*. However, even setting aside the fact that e.g. someone going through a transcendental proof of the external world based on the existence of a certain kind of content is almost invariably not subject to paradox-generating appearances, a version of Curry's paradox with an *a priori* consequent such as e.g. Fermat's Last Theorem belies this *epistemological modification* of the traditional definition of paradoxicality (which also would seem unable to cope with the point of fn 17).[18] Another natural reaction to the

Footnote 17 continued
Serie A. The traditional definition of paradoxicality doubly breaks down across the board. Notice that a similar point can*not* be made about *the argument's being apparently valid*, which, contrary to the premises' being apparently true and the conclusion's being apparently false, would thus seem essential to paradoxicality. Notice also that such variations would seem more *extrinsic* to the Liar paradox and Russell's paradox (and to Curry's paradox in the case of apparently false premises, although, in that case, the extrinsicality is actually specious in that it can be eliminated by considering a version of Curry's paradox that employs as logical operation "*displication*" (φ's holding does not require ψ's holding) instead of *implication* (φ's holding requires ψ's holding), as explained in Zardini, 2015a, pp. 489–490), since, in those cases, the apparently true conclusion or the apparently false premise would seem merely to act as a relief valve or as a blasting cap respectively for the real workings of the paradox. Notice finally that the same point can be made without appeal to such variations by considering other paradoxes such as e.g. the *Church-Fitch paradox of knowability* (see Fitch, 1963, pp. 138–139 for the details and Salerno, 2009, pp. 34–37 for the origins) or the *argument for the entailment of ex contradictione quodlibet* ($\varphi \ \& \ \neg\varphi \vdash \psi$, see Lewis & Langford, 1959, pp. 250–251 for the details; to the best of my knowledge, the argument is due to the Parvipontanian school, in particular to Adam of Balsham or William of Soissons, see John of Salisbury's *Metalogicon*, book II, chapter 10), where what is paradoxical in the first place is simply that the premise (the principle that every truth is knowable and any contradiction respectively) apparently entails the conclusion (the principle that every truth is known and an arbitrary sentence respectively), since in neither case is the premise particularly plausible, let alone apparently true. Thanks to Ricardo Santos for originating this last point.

[18] A broadly related modification insists instead on the premises' apparently *concerning a certain subject matter* and the conclusion's apparently *not concerning it* ("You can't prove stuff about EZ's tastes simply by reasoning about truth!"; I hasten to add that frankly I don't know whether there is an account of subject matters that would make this modification minimally plausible). However, even setting aside the fact that e.g. someone going through a teleological proof of the existence of God based on the existence of a certain flower is almost invariably not subject to paradox-generating appearances, a version of Curry's paradox with a truth-theoretic consequent such as e.g. 'If this sentence is true, then, if its antecedent is true, its consequent is true' belies this *topical modification* of the traditional definition of paradoxicality (which at least would seem able to cope with the point of fn 17). Notice that both the epistemological and the topical modifications—or, for that matter, the unmodified traditional definition—might try to address the *undergeneration* problem by merely requiring that there be *an instance of the form* of the offending argument (not necessarily the very same instance constituted by that argument) that exhibits the postulated features. Without insisting too much on the fact that, unsurprisingly, that move would make one swing towards an *overgeneration* problem (e.g. the argument resulting by simply substituting 'hirsute' and its relatives for 'bald' and its relatives in our version of the Sorites paradox is not at all paradoxical), I should record that, once one goes in for *abstracting to the form of the argument*, it seems to me that the most natural way of doing so is to go in for the next reaction to be discussed in the main text. Be that as it may, such a move would not seem to go to the heart of the undergeneration problem, as *it questionably makes the paradoxicality of the offending argument depend on features exhibited by other arguments* (to appreciate how questionable that is, consider the fact that, *even if our language were in principle so limited as to allow only the formulation of arguments lacking the postulated features*, the offending argument would seem paradoxical all the same, *cf* Zardini, 2021d).

point of the last paragraph is to say that what is paradoxical in Curry's paradox is not that apparently true premises apparently entail an apparently false conclusion, but that *everything (if anything) in the relevant range of propositions* (possibly together with apparently true auxiliary premises) apparently entails *everything (if anything) in the relevant range of propositions*.[19] Now, what is implicit behind this move is presumably that it is bad to entail everything, in turn presumably because not everything holds. If so, on this move, *the paradoxicality of Curry's paradox depends on the fact that not everything holds*. However, the remarkable fact is that, even under the assumption of *trivialism* (the claim that everything holds, see Priest, 2000), Curry's paradox is still paradoxical (another remarkable fact that can be used to the same effect is the one mentioned in the parenthetical observation at the end of fn 18).[20] Notice also that a shortcoming common to both this latter reaction and the modification of the former reaction discussed in fn 18—when one considers how they can be generalised beyond the case of Curry's paradox—is that of failing to characterise correctly—of all things!—an *archparadigmatic* case of paradox such as the Liar paradox: while it is true that the paradox can be extended by applying *ex contradictione quodlibet* to conclude everything, paradox is hit as soon as contradiction is (Zardini, 2021d).

A better reaction to the point of the second last paragraph is to say that what is paradoxical in Curry's paradox is that, *apparently, even if the conclusion failed to hold,*[21] *all the elements of the putative proof of the conclusion would still be available*. In other (more precise) words, *apparently, even if the conclusion failed to hold, the premises would*[22] *still be true and the argument would still be valid*. Since it is known full well that one instance of the argument in question (fn 3) is the one featuring the conclusion and the premises in question, it is presumably known full well that that counterfactual

19 The proviso '(if anything)' is there to cover the cases where the argument actually involves no nonauxiliary premises (as in our original version of Curry's paradox, which only involves the auxiliary premises of correlation and of the existence of a sentence such as κ) or no conclusions (as in a version of Curry's paradox with displication, see fn 17). The restriction 'in the relevant range of propositions' only becomes relevant when one considers how the reaction can be generalised beyond the case of Curry's paradox to characterise correctly also e.g. the Sorites paradox.

20 In general, one should distinguish between what, *within* a situation, a representational object is and what, *with respect to* a situation, a representational object is. For example, consider the opposite of trivialism: *voidism* (the claim that nothing holds). Although it is not the case that, within the void situation, the proposition that nothing holds is true, with respect to the void situation that proposition is true. Given this distinction, the point in the main text is to be understood not as the trivial point that, within the trivial situation, Curry's paradox is paradoxical (for, within the trivial situation, everything holds!), but as the substantial point that, with respect to the trivial situation, Curry's paradox is paradoxical (while, with respect to the trivial situation, Curry's paradox fails to have many other properties, e.g. it fails to involve negation). Thanks to Diogo Santos for pressing me on this point.

21 I say 'failed to hold' rather than 'were false' because, especially in the context of paradox, one might be open to the idea that *some sentences are both true and false* (Priest, 1979), in which case the appearance that, even if the conclusion were false, all the elements of the putative proof of the conclusion would still be available need not be a mistake (since then the conclusion could still also be true in virtue of the proof!).

22 The proposed account can be extended to cover paradoxes with premises that are not apparently true (fn 17) by replacing this '*would*' with '*could*'; throughout, for ease of exposition, I stick however to the somewhat simpler version with 'would'—all I say applies *mutatis mutandis* to the version with 'could'. The proposed account can also be extended to cover conceptions of logical consequence that include *requirements on premises and conclusions other than downwards truth preservation* (such as e.g. *upwards falsity preservation*)—the details of such extensions are however better left for other occasions.

implication fails to hold; yet, it does appear to hold, wherein lies the paradox.[23] For example, if the conclusion is 'EZ likes *pastissada*', what is paradoxical in the resulting version of Curry's paradox is that, apparently, even if 'EZ likes *pastissada*' failed to hold, all the elements (correlation, the existence of the relevant Curry sentence, *modus ponens*, unipremise conditional proof *etc.*) of the putative proof of 'EZ likes *pastissada*' would still be available.

It's important to realise that the proposed account relies on an *understanding of counterfactual implication that does very substantial work*. For we must be able to conceive of a situation where, apparently, the conclusion fails to hold, and so a situation where, apparently, also all unproblematic ways of getting to the conclusion involve either a premise or a logical principle that fails to hold (which, in the cases where the conclusion is a logical truth, makes the situation an apparently [impossible and indeed counterlogical] one). Moreover, we must be able to discriminate such unproblematic ways from the problematic one constituted by the putative proof, in such a way that we're able so to conceive of the situation in question that, apparently, every premise and logical principle involved in the putative proof holds in the situation (which, given that one instance of the argument in question is the one featuring the conclusion in question, again makes the situation an apparently [impossible and indeed counterlogical] one).[24] Therefore, the proposed account is *correct* (if it is) because our ability to judge the crucial counterfactuals *tracks* our implicit grasp of the

[23] While the *argument* in question appears to be valid (fn 28), given the appearance of that counterfactual implication presumably the *argument instance* in question does not appear to be valid. (I'm here assuming that the notion of validity is also applicable to argument instances in a way that is not derivative on its applying to arguments (Zardini, 2021a).) In view of this gap between the appearance of a universal generalisation and the appearance of one of its instances, one might consider a more streamlined account according to which what is paradoxical in Curry's paradox is that, *apparently, [the argument instance is invalid even though the argument is valid]* (or variations thereof, like e.g. the one according to which what is paradoxical in Curry's paradox is that *[it is not the case that, apparently, the argument instance is valid] even though, [apparently, the argument is valid]*). However, such a simple appearance involving the invalidity of the argument instance arguably falls short of capturing the full force of paradoxicality, since the sheer invalidity of an argument instance is quite compatible with the truth of its premises favouring or even forcing the holding of its conclusion (for example, the truth of 'This is a sample of H_2O' forces the holding of 'This is a sample of water', although the argument instance from the former to the latter is invalid). One might also consider a less dramatic account according to which what is paradoxical in Curry's paradox is that, *[apparently, even if the conclusion failed to hold, the premises would still be true] even though, [apparently, the argument is valid]*. However, such distinct appearances isolating the lack of connection between premises and conclusion from the validity of the argument arguably also fall short of capturing the full force of paradoxicality, since they are typically present just as well in nonparadoxical cases of candidate counterexamples to classically valid arguments (for example, [apparently, even if 'Sue believes that Phosphorus is Hesperus' failed to hold, 'Sue believes that Phosphorus is Phosphorus' and 'Phosphorus is Hesperus' would still be true] even though, [apparently, the entailment of *indiscernibility of identicals* ($\varphi(\ldots\tau_0\ldots), \tau_0 = \tau_1 \vdash \varphi(\ldots\tau_1\ldots)$) holds]).

[24] The application of the proposed account to certain *recherché* cases also requires paying due heed to the *hyperintensionality* that *terms* such as 'the conclusion' and 'the premises' exhibit within such counterfactual implication. For example, given suitable expressive resources, correlation could be formulated as a single sentence, and we could then consider a Curry sentence κ'' whose consequent is correlation itself. *κ'' gives rise to a version of Curry's paradox just as much as κ' does, yet it is not the case that, apparently, even if correlation failed to hold, correlation (and the other premise) would still be true (and the argument would still be valid)*. However, even in this version of Curry's paradox, even though *correlation is both the conclusion of the argument and the premise of the argument*, it is still the case that, apparently, even if *the conclusion of the argument* failed to hold, *the premise of the argument* would still be true (in the sense that, apparently, even if *correlation as the conclusion of the argument* failed to hold, *correlation as the premise*

distinction between unproblematic and problematic ways of getting to a conclusion;[25] the proposed account is also *illuminating* (if it is) because our ability to judge the crucial counterfactuals at the same time (virtuously circularly) *articulates* our implicit grasp of the distinction between unproblematic and problematic ways of getting to a conclusion.

It is appealing to extend the proposed account from the *particular* case of Curry's paradox to paradoxicality in *general*: a paradox is a situation where, apparently, even if the conclusion failed to hold, all the elements of the putative proof of the conclusion would still be available.[26] The proposed account of paradoxicality then fits nicely with the idea that paradoxicality consists in a certain general type of mistake, by providing a specification of exactly what that general type is: *we're mistakenly led to judge that, even if the conclusion failed to hold, all the elements of the putative proof of the conclusion would still be available*. Fixing on a specific presentation of a paradox, this general type of mistake might be determined by the fact that, *while we correctly judge that all the elements of the putative proof of the conclusion are available, we're mistakenly led to judge that, even if the conclusion failed to hold, they would still be so*, in which case we will after all have to come to terms with the fact that the premises really do entail the conclusion (so that—recalling from fn 22 that it is inessential that the elements of the putative proof include true premises—we *must* reject[27] one of the premises or accept the conclusion); alternatively (and more relevantly in the case of substructural approaches to paradox), the general type of mistake might be determined by the fact that *we're mistakenly led to judge that all the elements of the putative proof of the conclusion are available in the first place*,[28] in which case—recalling again from fn 22 that it is inessential that the elements of the putative proof include true premises—we did after all rightly sense that the premises really do not entail the conclusion (so that we *can* accept the premises and reject the conclusion, and accept that, even if the conclusion failed to hold, all the *valid* elements in the vicinity of the

Footnote 24 continued
of the argument would still be true). It is the appearance of this counterfactual implication that accounts for the problematicity in this case of the idea that correlation as a premise of the argument forces itself to hold as the conclusion of the argument. Thanks to Giuseppe Spolaore for prompting this point.

[25] And so the proposed account—when generalised as in the next paragraph—cannot be regarded as *reducing paradoxicality to something graspable in terms of independently understood concepts*, in contrast to all the other proposals considered in this section. In view of the *poor track record of reductive definitions of central philosophical notions*, that is arguably a virtue rather than a vice.

[26] The putative proof is an *argument*, and, in the framework of this introduction (fn 4), arguments can be *multiple-conclusioned*. Two clarifications are then in order concerning the interaction of the proposed account of paradoxicality with multiple-conclusionedness. Firstly, while, for ease of exposition, I stick in this discussion to arguments with *at most one* conclusion, the generalisation to arguments with *many* conclusions is straightforward. Secondly, arguments with *no conclusions* and arguments with *no premises* represent collections of *jointly exclusive* or *jointly exhaustive* sentences respectively: the proposed account should then be so understood that, in the former case, it is vacuously the case that the conclusion fails to hold and, in the latter case, it is vacuously the case that the premises are true.

[27] Throughout, by 'reject' and its relatives, I simply mean *considered nonacceptance* of the relevant sentence (so that rejecting a sentence does not imply accepting its negation).

[28] Notice that that judgement is implicit in the judgement that, even if the conclusion failed to hold, all the elements of the putative proof of the conclusion would *still* be available.

elements of the invalid proof would still be available).[29] (Further, in either case, those still *general types* of mistake will in turn be determined by *more specific types* that are those that constitute the different *kinds* of paradox that have been discussed in this section.) The proposed account also fits nicely with the gloss on paradoxicality that (with Dan López de Sa) I gave in earlier works (López de Sa & Zardini, 2007, p. 246; 2011, pp. 472–473) to the effect that, in a paradox, despite the apparent validity of the argument, *the premises apparently do not rationally support the conclusion*, since the appearance of that *epistemological* fact is plausibly explained by the appearance of the *metaphysical* fact that, even if the conclusion failed to hold, the premises would still be true and the argument would still be valid.[30]

[29] Satisfying a desideratum foreshadowed in fn 17, it is thus central to the proposed account of paradoxicality that *paradox involves argument structure* (fns 4, 26). Paradoxicality consists in *a certain general type of appearance of an incongruous logical fact*. This is reflected in the circumstance that, as has just been explained in the main text, fixing on a specific presentation of a paradox, the *main division of solutions to the paradox* consists in *whether they accept that the problematic logical fact obtains* (*i.e. accept that the argument is valid*). (Compare with the traditional definition of paradoxicality, according to which, fixing on a specific presentation of a paradox (something that the definition does not typically consider, but that we can surely add), the main division of solutions to the paradox consists in *what they reject*: one of the premises, the validity of the argument or the conclusion. By contrast, on the proposed account, fixing on a specific presentation of a paradox, the division of solutions to the paradox *in terms of acceptance/rejection of premises/conclusions* is *derivative* on their main division in terms of whether they accept that the argument is valid, and consists in *whether they forbid or allow that all the premises are accepted while the conclusion is rejected*.) However, I hasten to add that the distinction between being a premise or conclusion and being a logical principle is arguably *not robust across different presentations of the same paradox*: often enough, unless one goes in for some implausibly restrictive line to the effect that "only first-order logic [or whatever, EZ] is logic", what are premises or conclusions in one presentation of a paradox are naturally replaced by close relatives that are logical principles in another presentation of the same paradox. For an example of the former kind of case, correlation as an apparently true premise in our presentation of the Liar paradox is naturally replaced by the apparently valid entailments $\varphi \vdash T(\ulcorner\varphi\urcorner)$ and $T(\ulcorner\varphi\urcorner) \vdash \varphi$ (close relatives of correlation which would seem on equal footing with it) in another presentation of that paradox; for an example of the latter kind of case, the apparently false conclusion $T(\ulcorner\lambda\urcorner)$ & $\neg T(\ulcorner\lambda\urcorner)$ in our presentation of the Liar paradox is naturally replaced by the apparently valid entailment φ & $\neg\varphi \vdash \mathsf{f}$ (a close relative of the claim that φ & $\neg\varphi$ does not hold which would seem on equal footing with it) in another presentation of that paradox. And, of course, *vice versa*, what are logical principles in one presentation of a paradox are naturally replaced by close relatives that are premises or conclusions in another presentation of the same paradox (as, given their symmetry, the examples just given also show). What *is* robust across different presentations of the same paradox is the *fundamental mistake* involved in the paradox (fn 6), and so its *solution*, both of which might concern a premise or conclusion in one presentation of the paradox and a logical principle in another presentation of the paradox (and, of course, different views differ as to what that mistake and that solution are). Every paradox realises itself in a *variety of appearances of an incongruous logical fact*. Thanks to Ricardo Santos for urging me to be more explicit about this material.

[30] It is well-known that *sensitivity* of a reason for a proposition (if the proposition failed hold, the reason would not be available) is not always a necessary condition on the reason's rationally supporting the proposition. For example, I know that my father ate bread yesterday based on the reason that, while I haven't seen him yesterday, I know that he has always very consistently been eating bread with his meals all his life. Yet, even if my father did not eat bread yesterday, that reason would still be available. However, this kind of example crucially concerns a *nondeductive* reason for a proposition, whereas the reasons at stake in a paradox are typically *deductive*. And it is much more plausible that deductive reasons must be sensitive in order rationally to support a proposition (witness the palpable futility of proving with a certain alleged proof method that that proof method does not prove everything). (Notice that the proposed account of paradoxicality is formulated in terms of a counterfactual implication of the form $\varphi > \psi$, whereas insensitivity is formulated in terms of a negation of a counterfactual implication of the form $\neg(\varphi > \neg\psi)$; throughout, I plausibly assume that the instances in question of the former form imply the corresponding instances of the latter form.) Having noted all that, there would indeed seem to be paradoxes where the

3 Substructurality

A logic is *structural* iff it *includes all the structural principles of classical logic, substructural* otherwise. In turn, a principle is *structural* iff it *does not concern particular object-language expressions*.[31] For example, adjunction is not structural in that it concerns the particular object-language expression &, whereas the metaentailment of *contraction* (see below) is structural in that it only concerns the metalanguage expression ','. There are *infinitely many* structural principles of classical logic (for example, for every i, consider the principle that there is an entailment with exactly i premises), but the *most salient ones* (in general but also for this introduction and this volume) are the entailment of *reflexivity*:

(I) $\varphi \vdash \varphi$,

the metaentailment of *monotonicity*:

(K) If $\Gamma_0 \vdash \Delta$ holds, $\Gamma_1, \Gamma_0 \vdash \Delta$ holds, and, if $\Gamma \vdash \Delta_0$ holds, $\Gamma \vdash \Delta_1, \Delta_0$ holds,

the metaentailment of *transitivity*:

(S) If $\Gamma_0 \vdash \Delta_0, \varphi$ holds and $\Gamma_1, \varphi \vdash \Delta_1$ holds, $\Gamma_1, \Gamma_0 \vdash \Delta_0, \Delta_1$ holds,

the metaentailment of *contraction*:

(W) If $\Gamma, \varphi, \varphi \vdash \Delta$ holds, $\Gamma, \varphi \vdash \Delta$ holds, and, if $\Gamma \vdash \Delta, \varphi, \varphi$ holds, $\Gamma \vdash \Delta, \varphi$ holds

and the metaentailment of *commutativity*:

Footnote 30 continued
reasons at stake are nondeductive. A straightforward example can be produced along the lines of the paradox of sufficiency of Sect. 4, by replacing *sufficiency* with *favouring* and *logical consequence* with *defeasible consequence*. Letting F and $\Vdash$ express favouring and defeasible consequence respectively, just like $\varphi \vdash T(\ulcorner\varphi\urcorner)$ and $T(\ulcorner\varphi\urcorner) \vdash \varphi$ are characteristic of the notion of truth, the principle that, if $\varphi \Vdash \psi$ holds, $F(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$ holds and the defeasible entailment $\varphi, F(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Vdash \psi$ would seem characteristic of the notion of favouring, yet they give rise to a variation of Curry's paradox with a sentence κ''' identical with $F(\ulcorner\kappa'''\urcorner, \ulcorner\chi\urcorner)$ as intermediate and not unquestionably absurd conclusion based on deductive reasons, where in turn, by the defeasible entailment just mentioned, κ''' is a nondeductive reason for the final and unquestionably absurd conclusion χ. The proposed account can be extended to cover this specific type of paradox by replacing 'failed to hold' with 'failed to hold and easily so'; letting the *ease sensitivity* of a reason for a proposition consist in its being the case that, if the proposition failed to hold and easily so, the reason would not be available, it is much more plausible that even nondeductive reasons must be ease sensitive. Thanks to Domingos Faria and Mauricio Suárez for helping to bring out these ideas.

[31] To the best of my knowledge, the term 'substructural' has been proposed by the late Kosta Došen at the Workshop *Logics without Structural Rules* at the Eberhard Karls University of Tübingen in 1990 (Wansing, 1996, p. 115; see also Došen, 1993, which appears in the outcome of the workshop Došen & Schröder-Heister, 1993). For what it's worth, as Došen (1993, p. 1) explicates, 'sub' is supposed to indicate both "less" (in the sense of *subtracting from the structural principles of classical logic*) and "under" (in the sense of *affecting the structure at the foundations of classical logic*). (I also wouldn't be terribly surprised if Marxian 'superstructure' provided some inspiration.) As for 'structure'-talk in relation to the target principles, that comes straight from Gentzen (1934, p. 191) (*Struktur-Schlussfiguren*), who however, by taking the entailment of reflexivity (see below in the main text) not to be one of them, would seem to be operating with a stricter understanding of what a structural principle is (*i.e.* a *transformation [from some entailments to an entailment] that does not concern particular object-language expressions* (*i.e.* a structural (in our sense) metaentailment (in our sense))).

(C) If $\Gamma_0, \varphi, \psi, \Gamma_1 \vdash \Delta$ holds, $\Gamma_0, \psi, \varphi, \Gamma_1 \vdash \Delta$ holds, and, if $\Gamma \vdash \Delta_0, \varphi, \psi, \Delta_1$ holds, $\Gamma \vdash \Delta_0, \psi, \varphi, \Delta_1$ holds[32]

(see Zardini, 2018, pp. 242–247 for a brief overview, for each of these principles, of the main philosophical reasons for denying the principle, whether or not those are related to the paradoxes in our list). The first three principles can be summed up by saying that logical consequence corresponds to a *Tarski-Scott closure operation* (Tarski, 1930; Scott, 1974), and, under extremely minimal assumptions, the last two by saying that premises and conclusions can be represented as forming *sets*, so that, restricting to the principles in our list, a logic is in effect *structural* iff it *corresponds to a Tarski-Scott closure operation with sets of premises and conclusions*, *substructural* otherwise.

I should stress that I'm adopting a literal—and so in a certain respect rather strict—understanding of what it takes for one of the principles in our list not to hold in a logic. In particular, let the *implicational analogue* of a metaentailment be the entailment where the premises and conclusion are got by replacing [the relevant entailment $\varphi_0, \varphi_1, \varphi_2 \ldots, \varphi_i \vdash \psi_0, \psi_1, \psi_2 \ldots, \psi_j$ in the metaentailment] with $\varphi_0 \rightarrow (\varphi_1 \rightarrow (\varphi_2 \ldots \rightarrow (\varphi_i \rightarrow (\neg\psi_0 \rightarrow (\neg\psi_1 \rightarrow (\neg\psi_2 \ldots \rightarrow \psi_j))) \ldots)$ (so that, for example, the implicational analogue of the metaentailment from $\varphi \vdash \psi$ to $\chi, \varphi \vdash \psi$ is the entailment $\varphi \rightarrow \psi \vdash \chi \rightarrow (\varphi \rightarrow \psi)$). Then, quite a few logics star an implication for which, for some principles in our list, the implicational analogues of those principles do not hold while their consequence relation is so defined that the principles themselves do (literally) hold (for example, already in the *3-valued Łukasiewicz logic* (Łukasiewicz, 1920), $\varphi \rightarrow (\varphi \rightarrow \psi) \vdash \varphi \rightarrow \psi$ does not hold while its consequence relation is so defined that (literally), if $\varphi, \varphi \vdash \psi$ holds, $\varphi \vdash \psi$ holds). By my count, then, that is simply *not* a substructural logic (although one could of course use its materials to define a substructural logic in a natural way, for example by setting $\varphi_0, \varphi_1, \varphi_2 \ldots, \varphi_i \vdash \psi_0, \psi_1, \psi_2 \ldots, \psi_j$ to hold in the new substructural logic iff $\varphi_0 \rightarrow (\varphi_1 \rightarrow (\varphi_2 \ldots \rightarrow (\varphi_i \rightarrow (\neg\psi_0 \rightarrow (\neg\psi_1 \rightarrow (\neg\psi_2 \ldots \rightarrow \psi_j))) \ldots)$ is a logical truth in the old structural one). In itself, that is of course a terminological point, but I'm making it because I want to focus on those logics that say something peculiar not simply about the rather lofty topic of *(typically embedded) implications* (we already had e.g. *conditional logics* (e.g. Nute, 1984) for that), but also about the much more down-to-earth topics of the *combination of premises*, *the combination of conclusions* and *logical consequence*.[33]

[32] If we envisage failure of (C), we should reformulate many other (structural and nonstructural) principles *to recover their intended (order-insensitive) force*. For example, if (C) fails, to recover the intended force of (K) (which allows for addition of premises or conclusions *anywhere*), we should reformulate it as "If $\Gamma_2, \Gamma_0 \vdash \Delta$ holds, $\Gamma_2, \Gamma_1, \Gamma_0 \vdash \Delta$ holds, and, if $\Gamma \vdash \Delta_2, \Delta_0$ holds, $\Gamma \vdash \Delta_2, \Delta_1, \Delta_0$ holds".

[33] Substructural *logics* have been around for a while: indeed, both *Aristotelian* and *Stoic* logic would seem substructural in that, for one example, they would seem to deny that some valid entailments have no premises and, for another example, would seem to deny some principle in our list (e.g. (I)). Moving on to modern times, we owe to Gentzen (1934) (influenced by the works of Hertz and Hilbert, see e.g. Hertz, 1923; Hilbert, 1928 respectively) *the individuation of something like the principles in our list* (see the comment in fn 31 concerning his divergent classification of (I)), although, given his adamant insistence that $\varphi_0, \varphi_1, \varphi_2 \ldots, \varphi_i \vdash \psi_0, \psi_1, \psi_2 \ldots, \psi_j$ "contentwise means exactly the same as" $\varphi_0 \,\&\, \varphi_1 \,\&\, \varphi_2 \ldots \,\&\, \varphi_i \rightarrow \psi_0 \vee \psi_1 \vee \psi_2 \ldots \vee \psi_j$ (p. 180; that of course does not cover the cases where either collection is empty, for which Gentzen is then forced to make special stipulations), it is

An early indication of the *superficiality* of the notion of a substructural logic is given by how carefully phrased the characterisation given in the second last paragraph must be. As a first stab at defining substructurality, we could say that a principle is structural iff it does not concern particular *logical operations*, but that would be too relaxed, since e.g. identity is not an operation (*i.e.* a *function whose codomain is identical with its domain*) but the law of *indiscernibility of identicals*, which only concerns identity, is not structural.[34] As a reaction to that, we could say that a principle is structural iff it does not concern particular *logical notions*, but that would still be too relaxed, since e.g. the properties of being a bachelor and of being unmarried are not logical but the entailment from 'x is a bachelor' to 'x is unmarried', which only concerns the properties of being a bachelor and of being unmarried, is not structural. As a reaction to that, we could say that a principle is structural iff it does not concern particular *notions*, but that would now be too strict, since e.g. premise combination is a particular notion but (K), which concerns premise combination, is structural. As a reaction to that, we could say that a principle is structural iff it does not concern particular *notions expressed by object-language expressions*, but that would still be too strict, since e.g. the object language may contain an expression expressing premise combination but (K), which concerns premise combination, is structural. As a reaction to that, we could say (and have said) that a principle is structural iff it does not concern particular *object-language expressions*, but even that is not too strict only if ',' is not an object-language expression, with the result that *there is a philosophically interesting distinction between structural and nonstructural principles only insofar as there is a philosophically interesting sense in which ',' is not an "object-language expression"*. And that is in turn questionable on at least two counts. Firstly, ',' is certainly not part of the usual informal metalanguage employed in theorising about a logic—rather, it is part and parcel of the *symbolism that is being theorised about*, both at the semantic and at the proof-theoretic level, which makes it in a very natural sense an "object-

Footnote 33 continued
actually unlikely that he himself understood the principles as being about logical consequence and the combination of premises and of conclusions. Be that as it may, on the strict understanding of what it takes to be a substructural logic spelt out in this paragraph, to the best of my knowledge, we arguably have to wait until Lewy (1958) for the first *really substructural logic* (which denies (S), see also the discussions in Geach, 1958; Smiley, 1959, with Bolzano, 1837—at least according to George, 1983; 1986—as a (not so modern) precursor)—and it is actually one lying outside the mainstream traditions mentioned in Sect. 5! (Two clarifications about the main claim in the last sentence. Firstly, Gentzen (1934) offers a substructural *presentation* of intuitionistic logic (one that restricts the number of occurrences of conclusions to at most 1). However, one must distinguish *a mere presentation of a logic* (a particular semantics, or a particular deductive system, or a list of logical truths *etc.*) from *the logic itself* (the *constitutive* principles (*i.e.* the principles entering into the definition) of a general theory of logical properties such as e.g. logical consequence), and intuitionistic logic is arguably multiple-conclusioned (for example, the entailment of *abjunction* ($\varphi \vee \psi \vdash \varphi, \psi$) would seem to make perfect sense also from an intuitionistic perspective, see Dragalin, 1979 for a multiple-conclusion presentation of intuitionistic logic). Secondly, Lambek (1958) offers a substructural *system* that calculates well-formedness (one where (K), (W) and (C) do not hold). However, one must distinguish *any old system* (for the purposes of this introduction, anything that has the same format as some logic) from *a logic proper* (which, as I've mentioned above, is something that constitutes a theory of logical properties), and the Lambek calculus was manifestly not offered as a logic (although it could be, see Došen, 1993, pp. 17–20).)

[34] For this reason, the usual label of 'operational principles' for nonstructural principles would not seem felicitous.

language expression". True, it does not come up in the definition of a well-formed formula, but that hardly marks a watershed of philosophical interest. Secondly, there are logics where ',' is an "object-language expression" even in the pedantic sense of coming up in the definition of a well-formed formula (von Kutschera, 1968), which then presumably makes the question whether a principle is structural *logic-relative* (and makes even, say, adjunction structural relative to some weirdly formulated logic where, analogously to the role played by ',' in a standardly formulated logic, & (alongside ',') is used to build up entailments but does not come up in the definition of a well-formed formula), thereby robbing it of much of its philosophical interest.

Even setting aside all these niceties, it is doubtful that anything like the notion of a structural principle and the subsequent notion of a substructural logic can mark a logically or philosophically interesting distinction. To take two paradigmatic examples, *closure under uniform substitution* (if $\Gamma_0 \vdash \Delta_0$ holds and $\Gamma_1 \vdash \Delta_1$ results from it by a uniform substitution, $\Gamma_1 \vdash \Delta_1$ holds) and *compactness* (if $\Gamma_0 \vdash \Delta_0$ holds, there are finite subcollections Γ_1 and Δ_1 of Γ_0 and Δ_0 respectively such that $\Gamma_1 \vdash \Delta_1$ holds) are presumably "structural principles" on any reasonable understanding of that notion. Moreover, they are usually supposed to be principles "of" "classical logic" and, in our context, the grounds for rejecting either part of that supposition are flimsy. Taking them in reverse order, could "classical logic" be taken to be something like *standard second-order classical logic* (which is not compact) rather than simply something like *first-order classical logic*? Hardly so: on a natural understanding of what it is to be a principle, very few logics—which do not include many logics paradigmatically considered to be structural, such as e.g. propositional classical logic—have all the structural principles of something like standard second-order classical logic (for example, think of the principle that, if a subset s of the consequence relation is recursively enumerable, for some Γ and Δ it is the case that $\Gamma \vdash \Delta$ holds and does not belong to s). Further, could a principle "of" a logic be taken to be something like a principle *constitutive* (fn 33) *of* the logic (where presentations of classical logic usually do not feature closure under uniform substitution and compactness among its constitutive principles) rather than simply something like a principle *holding in* the logic? Hardly so: on many natural presentations of classical logic, very few principles—which do not include many principles paradigmatically considered to be structural, such as e.g. (W)—are constitutive of it (for example, think of a presentation of classical logic in terms of sets of premises and conclusions). It is therefore very hard to see how, in our context, closure under uniform substitution and compactness could fail to count as "structural principles of classical logic", with the consequence that every logic that is either not closed under uniform substitution or not compact will count as substructural. That lumps together, say, *Carnap-style modal logic* (Carnap, 1946), *infinitary logic* (Henkin, 1955) and *linear logic* (Girard, 1987) as substructural logics. What kind of logical or philosophical insight can one expect to gain by thinking about such an unsavoury congeries of logics? It would seem more sensible to focus on a more unified, proper subset of structural principles (as is in effect done in research on substructural logics), like e.g. those principles concerning *standard algebraic properties of premise combination and conclusion combination*.[35]

It is tempting to think that *structural principles are located at a level different from and indeed prior to that of nonstructural principles* (and to speculate that *the advantages of substructural approaches to paradox that will be presented in Sect. 4 flow from that putative fact*), and, in fact, structural principles are commonly so thought of (and even occasionally so speculated about, see e.g. Ripley, 2015a, p. 310). For example, it is common to think that *premise combination*—the level at which a structural principle such as e.g. (K) operates—is different from and indeed prior to *conjunction*, in the sense that the logical properties of premise combination *are constituted independently of* those of conjunction and indeed *help to constitute* those of conjunction. However, a little reflection suffices to show how problematic the common thought is for virtually all the logics of interest for this introduction, and that, quite to the contrary, in these logics, *it is conjunction that is prior to premise combination* (in fact, *premise combination consists in a certain kind of conjunction*).[36]

In spite of its usual representation as set (or multiset, or sequence, or whatnot, see fn 4) formation, premise combination as usually understood[37] is much more than that, as e.g. φ, ψ has the force of representing φ and ψ *as holding together* (by contrast, possibly apart from irrelevant and obvious containment facts, $\{\varphi, \psi\}$ neither represents anything as holding nor does it represent any things as doing something together). But *that is exactly the kind of combination performed by the logical operation of conjunction*, for ' 'φ' holds and 'ψ' holds' has exactly the force of representing φ and ψ as holding together. The combination of premises φ and ψ consists in the conjunction that φ holds and ψ holds.[38] It might be tempting to reply to this point by going *expressivist* and say that to accept φ, ψ is simply to accept φ and to accept ψ. However, since φ, ψ precisely represents φ and ψ as holding together, to accept φ, ψ must be equivalent with accepting that φ and ψ hold together, given which the expressivist temptation falls afoul of the fact that one may justifiedly accept φ and justifiedly accept ψ while justifiedly not accepting that φ and ψ hold together (for example, one

[35] Such focus would not directly include (I) or (S). This tendency would be reinforced by the remainder of this section, where it'll turn out that, *as far as substructural approaches to paradox are concerned, a direct role of (I) and (S) is indeed marginal*.

[36] To emphasise, the issue is about (premise combination and) a *nonlinguistic* logical *operation* of conjunction, not a *linguistic* logical *operator* of conjunction. Obviously, the language of a logic might lack a logical operator of conjunction and the logic still exhibit premise combination; even in such a case however the question remains whether there is nevertheless a logical operation of conjunction—intelligible given what the logic overall is even if not *de facto* included in usual presentations of the logic (fn 33)—that is prior to premise combination.

[37] This qualification is presupposed throughout this introduction, with the only exception of the second next paragraph, where it is temporarily suspended.

[38] Relatedly, it would seem that sometimes one can *entertain* (accept, deny, reject *etc.*) φ, ψ *in itself, independently of any possible inference that may be drawn from it*. The assumption that premise combination consists in a certain kind of conjunction has an easy job at accounting for the entertainability fact: that fact obtains *precisely because* φ, ψ *has (no more than) the force of representing* φ *and* ψ *as holding together, and that is exactly the kind of combination performed by the logical operation of conjunction*. Such entertainability would however be a *mystery* if conjunction were not prior to premise combination—for, in that case, how could it be that the entertainment of independently constituted premise combination floats so freely of the use of premises?

may justifiedly accept that Benfica will win the next Portuguese *Liga* and justifiedly accept that Atlético will win the next Spanish *Liga* while justifiedly not accepting that Benfica will win the next Portuguese *Liga* together with Atlético's winning the next Spanish *Liga*). Moreover, most of our thought about premise combination *is in the context of thought about entailment and does not involve acceptance of* φ, ψ—the expressivist temptation is therefore subject to a particularly acute version of the *Frege-Geach problem* (Geach, 1965).[39]

Notice that the conjunction that the premise combination φ, ψ consists in is not 'φ and ψ' but ''φ' holds and 'ψ' holds'. This is crucial for the entailment $\varphi, \psi \vdash \varphi \,\&\, \psi$ to be correctly rendered as an entailment connecting φ's holding and ψ's holding with $\varphi \,\&\, \psi$'s holding instead of being incorrectly rendered as an entailment connecting (letting φ mean that P and ψ mean that Q) its being the case that P and Q with its being the case that P and Q or as an entailment connecting 'φ and ψ' (synonymous with $\varphi \,\&\, \psi$) holding with $\varphi \,\&\, \psi$'s holding (Zardini, 2018, pp. 257–258).[40] Relatedly, notice also that this treatment does not assume that, when one *competently infers* from φ *and* ψ, one must grasp what φ, ψ represents. Presumably, one can competently infer without grasping any sort of *metalinguistic* concept, or even without grasping any sort of *conjunctive* concept. In such cases, one displays a *sensitivity* to the state of affairs represented by φ, ψ without being able to grasp that state of affairs, and we should long ago have learnt to accommodate for this kind of sensitivity on account of the fact that one (think of higher-level animals and small children) can competently infer from φ *to* ψ without grasping any sort of *metalinguistic* concept (such as those of sentence and of entailment), or even without grasping any sort of *implicational* concept (such as the one expressed in 'If φ, then ψ').

The master argument in favour [of the claim that premise combination consists in a certain kind of conjunction] advanced in the second last paragraph is reinforced by three auxiliary arguments. A first auxiliary argument takes its lead from the fact that there is really nothing in the most general notion of premise combination that requires its usual understanding. The point is perhaps most direct in the dual case of *conclusion combination*. How should one understand the combined conclusions φ, ψ? At such a level of abstraction, there is simply no (right) answer to this (wrong) question: φ, ψ could represent φ and ψ *as holding alternatively* (as is usually understood) or *as holding together* (as could equally naturally be understood) or goodness knows. There are therefore different kinds of conclusion combinations, and the most natural way of grounding their difference involves the corresponding logical operations: on such explanation, the first option is tantamount to what is expressed by ''φ' holds or 'ψ' holds'—thereby making conclusion combination a certain kind of disjunction—while

[39] One could take another tack and put forth the view that φ, ψ is sort of *syncategorematic* in that it does not really represent or mean anything outside of a $\vdash$-context like e.g. $\varphi, \psi \vdash \chi$, in which case the whole compound means that it is logically necessary that, if φ holds and ψ holds, χ holds (or whatever your favourite gloss on $\vdash$-contexts is). That is not a particularly plausible view (fn 38), and, more importantly, on any plausible gloss on $\vdash$-contexts I know of, it does nothing to avoid use of *conjunction* (or of its relative: *universal quantification*) for combining premises.

[40] To take the conjunction that the premise combination φ, ψ consists in to be the *metalinguistic* ''φ' holds and 'ψ' holds' rather than something *object-language* (like e.g. 'It is the case that φ and it is the case that ψ') is indeed more in line with the arguable fact that *logical consequence is primarily a relation among linguistic entities* (Zardini, 2021c), but is not crucial for the purposes of this introduction.

(as we've already seen in the second last paragraph) the second option is tantamount to what is expressed by ' 'φ' holds and 'ψ' holds'—thereby making conclusion combination a certain kind of conjunction. The same point can then be made in the case of premise combination (see Zardini, 2021d for a style of presentation of a logic where both premise combination and conclusion combination can go both in conjunctive mode and in disjunctive mode, with arbitrary embeddings of one mode into the other one).[41,42]

A second auxiliary argument takes its lead from the fact that, in many nonclassical and substructural logics just as well as in classical logic, *conjunction is fully intersubstitutable with premise combination*, in the sense that $\Gamma_0, \varphi, \psi, \Gamma_1 \vdash \Delta$ holds iff $\Gamma_0, \varphi \,\&\, \psi, \Gamma_1 \vdash \Delta$ holds (see Zardini, 2021d for a style of presentation of a logic that builds in that principle). Such full intersubstitutability would be a *mystery* if conjunction were not prior to premise combination—for, in that case, how could it be that one of the fundamental logical operations so perfectly matches independently constituted premise combination? One possible explanation would be given by the assumption (made e.g. by Beall & Ripley, 2018, p. 751) that *conjunction expresses in the object language premise combination*. However, it is not clear how such an assumption could be correct. Firstly, why should one of the fundamental logical operations be there to express premise combination—that would seem to make that logical operation capriciously *redundant* and make elementary logic weirdly *reflexive*. Secondly, it would not seem that a straightforward *object-language* sentence such as e.g. 'Snow is white and grass is green' represents what 'Snow is white', 'Grass is green' does: for one thing, the latter—whatever it exactly represents—is arguably *about the sentences* 'Snow is white' and 'Grass is green' (fn 40), whereas 'Snow is white and grass is green' is definitely not. Thirdly, the point made in the second last paragraph applies *also* to the assumption in question: if conjunction expressed premise combination, the entailment $\varphi \,\&\, \psi \vdash \varphi \,\&\, \psi$ would correspond to the entailment $\varphi, \psi \vdash \varphi \,\&\, \psi$, which it does not. Fourthly, a relative of the point made in fn 38 applies to the assumption in question: if conjunction expressed premise combination, it would be a *mystery* how a conjunction could be entertained independently of any possible inference that may be drawn from it—but that is most definitely *not* a mystery. Contrary to the formidable explanatory challenges thus faced by the assumption that conjunction is not prior to premise combination, the opposite assumption that premise combination consists in a certain kind of conjunction has an easy job at accounting for the full-intersubstitutability fact:

[41] Relatedly, it is routine to introduce the difference between premise combination and conclusion combination by giving some gloss to the effect that premises are combined "as in a conjunction" while conclusions are combined "as in a disjunction". The extreme *commonality* and *spontaneity* of such a gloss is some evidence that it not only *serves a pedagogical purpose* but that it also *tracks the objective fact* that premise combination and conclusion combination as usually understood consist in a certain kind of conjunction and disjunction respectively (while alas missing the objective fact emphasised in this paragraph that premise combination and conclusion combination need not be understood as usual).

[42] A more sophisticated version of this argument can be run in terms of the difference between *distinct kinds of conjunction-like* (or *of disjunction-like*) combination (a more *interactive* one and a more *selective* one, see Minc, 1972) instead of the difference between a *conjunctive kind* of combination and a *disjunctive one*.

that fact obtains *precisely because premise combination consists in a certain kind of conjunction* (and because ' 'φ' holds' is fully intersubstitutable with φ)[43].

A third auxiliary argument takes its lead from the fact that many substructural logics enjoy *a very natural model-theoretic semantics*, and *in such semantics premise combination is just defined in terms of conjunction*, along the lines of something to the effect that $\varphi, \psi \vdash \chi$ holds iff every model where $\varphi \;\&\; \psi$ has a designated value is a model where χ has a designated value. Given the *naturalness* of the semantics, that is strong evidence that premise combination consists in a certain kind of conjunction.[44]

There are therefore strong reasons for thinking that *premise combination consists in a certain kind of conjunction*, and analogous reasons are available for thinking that *conclusion combination consists in a certain kind of disjunction* and that *entailment consists in a certain kind of implication*. I emphasise that such reasons rely on assumptions about the target logic that, while almost always unquestionable for virtually all the logics of interest for this introduction (and for many structural logics including classical logic),[45] might not be such for other (substructural or structural) logics, and the following conclusions about "logics" should accordingly be understood as implicitly so qualified. One can then understand the fact that certain structural principles hold or do not hold in a logic *as the result of the fact that the corresponding principles for conjunction, disjunction or implication* (specifically, the particular conjunction that underlies premise combination, the particular disjunction that underlies conclusion combination and the particular implication that underlies entailment) *hold or do not hold in the logic*. For one example, (K) holds in classical logic but not in nonmonotonic logics because $\varphi \;\&\; \psi$ entails φ in classical logic but not in nonmonotonic logics (in the latter case, understanding & as expressing the particular conjunction that underlies premise combination in nonmonotonic logics). For another example, (S) holds in classical logic but not in nontransitive logics because $\varphi \to \psi$ and $\psi \to \chi$ entail $\varphi \to \chi$ in classical logic but not in nontransitive logics (in the latter case, understanding $\to$ as expressing the particular implication that underlies entailment in nontransitive logics)[46]. For yet another example, (W) holds in classical logic but not in noncontractive

[43] The parenthetical part of the explanation only needs *a dedicated truth predicate for [the original language that does not contain that predicate]* (although it may contain other truth predicates), and so it is not subject to the complications induced by the semantic paradoxes (Zardini, 2018, p. 268, fn 30).

[44] A broadly related argument is offered in Zardini (2018, p. 273, fn 50), however under stronger assumptions about logical consequence than those that are being made in this introduction.

[45] The only exception is given by the assumption made by the third auxiliary argument, which is certainly at least questionable for some nonreflexive logics and for some nontransitive logics (as well as for classical logic).

[46] Analogously to the other examples, this might not be the implication that is most prominent in certain *presentations* of a certain nontransitive logic (fn 33). The logic **K3LP** of Cobreros *et al.* (2012) provides an illustration of this point. **K3LP** works with 3 linearly ordered values $1, 1/2, 0$ and, in the framework of Zardini (2008a; 2008b, pp. 93–174), assumes that 1 is the only designated value as well as that $\mathsf{tol}(1) = \{1, 1/2\}$ (where tol is a tolerance function that interacts with an implication function impl so that, for every value v_0, v_1, $\mathsf{impl}(v_0, v_1)$ is designated iff $v_1 \in \mathsf{tol}(v_0)$), and one natural possibility for the further characterisation of tol is that $\mathsf{tol}(1/2) = \{1, 1/2, 0\}$. Further setting $\mathsf{impl}(1, 0) = 0$, the resulting impl is the target implication, for which the entailment in the main text does not hold (where entailment is such that, for every model, if every premise has a designated value, some conclusion has a value v_0 such that, for some designated value v_1, $v_0 \in \mathsf{tol}(v_1)$), although, for several reasons, in the presentation of Cobreros *et al.* (2012), the most prominent implication

logics because φ entails $\varphi \mathbin{\&} \varphi$ in classical logic but not in noncontractive logics (in the latter case, understanding & as expressing the particular conjunction that underlies premise combination in noncontractive logics).

Assuming that this is right, it implies the need for a *reconceptualisation of substructural logics*, not as logics that fundamentally deny some structural principle of classical logic, but as logics that *fundamentally deny some principle of a certain specific kind that conjunction, disjunction or implication obey in classical logic*—that is, the kind of principles that determine that classical logic has the structural principles it has. That arguably does make substructural logics less categorically different from structural nonclassical logics than is commonly assumed: they all *fundamentally deny some principle of the logical operations*. The real difference is in that they centre on logical operations *other than negation*.

Indeed, they typically *do not centre on implication either* and thus centre on logical operations (*i.e.* conjunction and disjunction) all of whose argument places are *upwards monotonic* (where, given an $i$ary logical operation $\circ$, for every $j \leq i$, its jth argument place (as occupied by φ_j in $\circ(\varphi_0, \varphi_1, \varphi_2 \ldots, \varphi_j \ldots, \varphi_i)$) is *upwards monotonic* iff, if ψ is at least as strong as χ, $\circ(\varphi_0, \varphi_1, \varphi_2 \ldots, \psi \ldots, \varphi_i)$ entails $\circ(\varphi_0, \varphi_1, \varphi_2 \ldots, \chi \ldots, \varphi_i)$).[47] That is clear for nonmonotonic, noncontractive and noncommutative logics. The situation is more nuanced for nontransitive logics. Under the assumption that *the entailment-underlying implication of a nontransitive logic is reducible in the usual fashion to disjunction and negation*,[48] (S) in its basic form without side premises and side conclusions ultimately boils down to the entailment from $\neg\varphi \vee \psi$ and $\neg\psi \vee \chi$ to $\neg\varphi \vee \chi$ (see e.g. Weir, 2015 for a logic where (S) does not hold but its basic form does and Zardini, 2021b for its critical discussion). In turn, given those premises, a suitable version of the principle of *selection of conjunction over disjunction* (licensing the entailment $\varphi_0 \mathbin{\&} (\varphi_1 \vee \varphi_2) \vdash (\varphi_0 \mathbin{\&} \varphi_1) \vee \varphi_2$[49] also in the scope of a disjunction) yields $((\neg\varphi \vee \psi) \mathbin{\&} \neg\psi) \vee \chi$, given which selection [of conjunction over disjunction] as applied to the first disjunct yields $\neg\varphi \vee (\psi \mathbin{\&} \neg\psi) \vee \chi$. If the second disjunct can be ruled out by a suitable version of the principle of noncontradiction, that yields $\neg\varphi \vee \chi$, thereby verifying (S) in its basic form. Focusing on nontransitive logics that satisfy the reducibility assumption concerning their entailment-underlying implication, this analysis makes it clear that (S) in its basic form is a more complex principle than other ones in our list, involving as it does both a principle concerning *the interaction between conjunction and disjunction* (such as the suitable version of selec-

Footnote 46 continued
is another one (such that, for every v_0, v_1, the implication function on $\langle v_0, v_1\rangle$ corresponds to $\mathsf{max}(\mathsf{neg}(v_0), v_1)$, where neg is a negation function such that, for every v, $\mathsf{neg}(v) = 1 - v$), for which the entailment in the main text does hold.

[47] Naturally, substructural logics that are thus centred would be congenial to the tendency that has emerged at the end of the ninth last paragraph.

[48] That is typically equivalent with the assumption that *the entailment-underlying implication of a nontransitive logic is reducible in the usual fashion to conjunction and negation*. For logics where the latter assumption holds but the former one doesn't, an argument similar to the one to follow in the main text nevertheless still holds.

[49] That entailment is of course a relative of the entailment of *distribution of conjunction over disjunction* ($\varphi_0 \mathbin{\&} (\varphi_1 \vee \varphi_2) \vdash (\varphi_0 \mathbin{\&} \varphi_1) \vee (\varphi_0 \mathbin{\&} \varphi_2)$), but is *weaker than it e.g. in a noncontractive logic* (Zardini, 2019a, pp. 181–182).

tion of conjunction over disjunction)—contrary to the way in which other structural principles like (K), (W) and (C) only involve principles concerning *the internal properties of conjunction* and *the internal properties of disjunction*—and even a principle concerning *negation* (such as the suitable version of the principle of noncontradiction). Relatedly, the analysis makes it clear that *there are at least two very different ways in which a logic might be nontransitive*: by denying the suitable version of selection of conjunction over disjunction or by denying the suitable version of the principle of noncontradiction, where only the former way conforms to the reconceptualisation of substructural logics as logics that fundamentally deny some of the principles that conjunction, disjunction and implication obey in classical logic and that determine that classical logic has the structural principles it has (that it does so conform is also confirmed in a particularly revealing way by the presentation of classical logic mentioned at the end of the fifth last paragraph).

4 Substructural approaches to paradox

There are indeed approaches to paradox that use a substructural logic:[50] the development and discussion of such approaches is the general topic of this volume. Indeed, not only are substructural approaches to paradox as *prima facie* viable as any—they enjoy several noteworthy advantages over more traditional *structural nonclassical* ones. Firstly, substructural approaches to paradox often *revise classical logic without revising the fundamental principles governing logical operations*. For the purposes of this introduction (see Zardini, 2021b; 2021d for deeper levels of analysis), and restricting throughout to the sentential level, these can be taken to be the *pairs of principles determining how weak and how strong a sentence with that operation as main operation is* (Zardini, 2019b, pp. 172–173, 179): for negation, the law of excluded middle and the law of noncontradiction; for conjunction, adjunction and the entailment of *simplification* ($\varphi \,\&\, \psi \vdash \varphi$ and $\varphi \,\&\, \psi \vdash \psi$); for disjunction, addition and abjunction; for implication, unipremise conditional proof and *modus ponens*. But why is it a good thing to maintain the fundamental principles governing logical operations? Well, for example, the law of excluded middle and the law of noncontradiction would seem correct at least for an understanding of $\neg\varphi$ as *covering every way in which* φ *fails*, but structural nonclassical approaches to the semantic and set-theoretic paradoxes (Bočvar, 1938; Asenjo, 1966), as well as to the Sorites paradox (Tye, 1990; Ripley, 2005), typically do deny either of those principles, thereby implausibly committing

[50] Substructural *approaches to paradox* have emerged significantly more recently than substructural *logics* (fn 33). On the strict understanding of what it takes to be a substructural logic spelt out in Sect. 3, to the best of my knowledge, the first fairly systematic and really substructural approach to the Liar paradox and Curry's paradox is the noncontractive approach of Zardini (2011) (Stepanov, 2007 contains an earlier partial treatment in the same direction); the first fairly systematic and really substructural approach to Russell's paradox is the noncontractive approach of Grišin (1974); the first fairly systematic and really substructural approach to the Sorites paradox is the nontransitive approach of Zardini (2008a; 2008b) (Weir, 1998, pp. 792–794; Béziau, 2006 contain earlier brief considerations in the same direction); the first fairly systematic and really substructural approach to the Preface paradox is the nontransitive approach of my contribution to this volume; the first fairly systematic and really substructural approach to the Material-Implication paradox is the nonmonotonic approach of Read (1988).

themselves to the incoherence of the notion of failure. One might then wonder what the point is of vindicating the notion of truth at the cost of jettisoning the related notion of failure (Zardini, 2011, p. 514; 2014b, pp. 193–196). Similarly, unipremise conditional proof and *modus ponens* would seem correct at least for an understanding of $\varphi \rightarrow \psi$ as *covering every way in which* φ *suffices for* ψ, but structural nonclassical approaches to the semantic and set-theoretic paradoxes typically do deny either of those principles (Priest, 2006; Goodship, 1996), thereby implausibly committing themselves to the incoherence of the notion of sufficiency. One might then wonder what the point is of vindicating the notion of truth at the cost of jettisoning the related notion of sufficiency (Zardini, 2011, p. 517).

Secondly, substructural approaches to paradox often *provide a unified solution to the paradoxes of a certain kind*. For one example, as I've argued in Sect. 2, the Liar paradox and Curry's paradox are of the same kind. Yet, the former stars negation whereas the latter stars implication, and structural correlation-friendly nonclassical approaches to the semantic paradoxes block the Liar paradox by denying either the law of excluded middle or the law of noncontradiction, but block Curry's paradox by denying either unipremise conditional proof or *modus ponens*. Pending an unlikely account explaining how these two denials flow from a common source, such approaches do not provide a unified solution to the semantic paradoxes (Zardini, 2015a). For another example, it is immensely plausible that *the tolerance version of the Sorites paradox* (the one presented in Sect. 2) is of the same kind as *the lack-of-sharp-boundaries version of the Sorites paradox* (which can be got from the one presented in Sect. 2 by replacing tolerance with the principle of *lack of sharp boundaries* according to which $\neg(B(i) \,\&\, \neg B(i+1))$ holds and *modus (ponendo) ponens* with the entailment of *modus ponendo tollens* ($\varphi, \neg(\varphi \,\&\, \psi) \vdash \neg\psi$) together with the entailment of *double-negation elimination* ($\neg\neg\varphi \vdash \varphi$), see Oms & Zardini, 2019, pp. 6–7 for the details). Yet, the former stars implication whereas the latter stars negation and conjunction, and, for the implication and conjunction that arguably most adequately capture the spirit of tolerance and lack of sharp boundaries respectively, structural tolerance-friendly nonclassical approaches to the Sorites paradox (Goguen, 1969; Ripley, 2005) either [block the tolerance version of the Sorites paradox by accepting tolerance while denying *modus ponens*, but block the lack-of-sharp-boundaries version of the Sorites paradox by rejecting lack of sharp boundaries by means of disputing the idea that one between $B(i)$ and $\neg B(i)$ holds] or [block the tolerance version of the Sorites paradox by rejecting tolerance by means of disputing the idea that $B(i+1)$ follows from (a sentence that holds together with) $B(i)$, but block the lack-of-sharp-boundaries version of the Sorites paradox by accepting lack of sharp boundaries while denying *modus ponendo tollens*]. In either case, pending an unlikely account explaining how the two moves in question flow from a common source, such approaches do not provide a unified solution to the Sorites paradox (Zardini, 2019b, p. 170, fn 5, p. 179, fn 23; Oms & Zardini, 2021, pp. 212–213, fn 16).

Relatedly, and by way of transitioning to the third point, some other times the problem for structural nonclassical approaches to paradox is not that *there are two different logical operations* at play—rather, *there are none*, only *a notion of a kind such that nonclassical approaches typically vindicate its characterising principles*. For one example (suggested by some of the considerations in the second last paragraph), just like $\varphi \vdash T(\ulcorner\varphi\urcorner)$ and $T(\ulcorner\varphi\urcorner) \vdash \varphi$ are characteristic of the notion of truth, the laws $\oslash \vdash \varphi, F'(\ulcorner\varphi\urcorner)$ and $\varphi, F'(\ulcorner\varphi\urcorner) \vdash \oslash$ would seem characteristic of the notion of *failure*, yet they give rise to a variation of the Liar paradox with no logical operation at play. For another example (also suggested by some of the considerations in the second last paragraph), just like $\varphi \vdash T(\ulcorner\varphi\urcorner)$ and $T(\ulcorner\varphi\urcorner) \vdash \varphi$ are characteristic of the notion of truth, the principle that, if $\varphi \vdash \psi$ holds, $S(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$ holds and the entailment $\varphi, S(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \vdash \psi$ would seem characteristic of the notion of *sufficiency*,[51] yet they give rise to a variation of Curry's paradox with no logical operation at play.[52,53] For yet another example, just like $\varphi \vdash T(\ulcorner\varphi\urcorner)$ and $T(\ulcorner\varphi\urcorner) \vdash \varphi$ are characteristic of the notion of truth, the *material* entailment $B(i) \vdash B(i+1)$ (a close relative of tolerance which would seem on equal footing with it) would seem characteristic of the *vagueness* of the notion of *baldness* (see Zardini, 2008b, pp. 27–28, 175–176; 2015b, pp. 221–222; 2019b, p. 169, fn 4 for more details on material validity), yet it gives rise to a variation of the Sorites paradox with no logical operation at play (Zardini, 2019b, p. 176).[54]

Thirdly, substructural approaches to paradox often *afford the only way to uphold certain compelling principles concerning the original notions with their intended force*. For one example, even more compelling than the *convergence* version of correlation we've been working with is its *divergence* version according to which both $\neg(\varphi \mathbin{\&} \neg T(\ulcorner\varphi\urcorner))$ and $\neg(\neg\varphi \mathbin{\&} T(\ulcorner\varphi\urcorner))$ hold, and that is naturally understood as having the force of making $\varphi \mathbin{\&} \neg T(\ulcorner\varphi\urcorner)$ and $\neg\varphi \mathbin{\&} T(\ulcorner\varphi\urcorner)$ absurd—but virtually no structural approach can uphold the divergence version of correlation with such a force (Heck, 2012; Zardini, 2013a). For another example, tolerance is naturally understood as having the force of making $B(i+1)$ follow from $B(i) \to B(i+1)$ and $B(i)$—but

[51] In the recent literature (e.g. Beall & Murzi, 2013), a similar point has been made with the notion of *validity* (see Zardini, 2014a, p. 357, fn 13 for some brief indications concerning the history of the *paradoxes of logical properties*), but that is actually subject to several problems (Zardini, 2013b; 2014a), which are however overcome by shifting to a notion such as sufficiency.

[52] Curry's paradox is thus the motor behind a few main themes of this introduction: it both provides the initial motivation for questioning popular diagnoses of the semantic paradoxes as well as the traditional definition of paradoxicality (Sect. 2) and also affords a cluster of paradigmatic arguments for the superiority of substructural approaches to the semantic paradoxes (this section).

[53] We could also consider *principles concerning the original notion of truth and involving negation or implication*. For one example, the laws $\varphi, \neg T(\ulcorner\varphi\urcorner) \vdash \oslash$ (essentially: no sentence both says what holds and is nevertheless not true) and $\oslash \vdash \varphi, \neg T\ulcorner\varphi\urcorner$ (essentially: a sentence either says what holds or is at least not true) are at least as compelling as correlation, yet they give rise to a variation of the Liar paradox with no principle governing logical operations (not even principles governing negation!) at play. For another example, the entailment $\varphi, T(\ulcorner\varphi\urcorner) \to T(\ulcorner\psi\urcorner) \vdash \psi$ (essentially: a sentence and truth preservation from it to a sentence entail the latter sentence) and the principle that, if $\varphi \vdash \psi$ holds, $T(\ulcorner\varphi\urcorner) \to T(\ulcorner\psi\urcorner)$ holds (essentially: validity requires truth preservation) are at least as compelling as correlation, yet (taking a Curry sentence whose consequent ψ is fully intersubstitutable with $T(\ulcorner\psi\urcorner)$) they give rise to a variation of Curry's paradox with no principle governing logical operations (not even principles governing implication!) at play.

[54] Similarly to fn 53, we could also consider *principles concerning the notion of baldness and involving negation*: for example, $B(i), \neg B(i+1) \vdash \oslash$ (essentially: no number is a sharp boundary for B) and $\oslash \vdash \neg B(i), B(i+1)$ (essentially: either a number is already a negative case of B or the next number is still a positive case of B) are at least as compelling as tolerance, yet they give rise to a variation of the Sorites paradox with no principle governing logical operations (not even principles governing negation!) at play.

virtually no structural approach can uphold all the instances of tolerance with such a force (Zardini, 2019b, p. 170).

As *per* the argumentation of Sect. 3, typically centring on conjunction and disjunction, substructural logics typically centre on logical operations whose arguments are all upwards monotonic—that is, in effect, logical operations of *positive composition*. In the framework of Sect. 2, approaches to a paradox that use any such logic thus individuate the mistakenly represented fact of the paradox in a *peculiar behaviour of positive composition*. While such a take on a paradox might initially come across as rather surprising and unlikely given the feeling of familiarity and obviousness that positive composition emanates as opposed to other kinds of logical operations, importantly, even with substructural logics so reconceptualised, substructural approaches to paradox retain all the advantages expounded in this section, which can then be understood as evidence for the idea that the paradoxes in our list are indeed rooted in mistakes that we're led to make *when (explicitly or implicitly) operating with conjunction and disjunction in the course of a paradoxical reasoning*. Therefore, substructural approaches to paradox represent a powerful trend in contemporary philosophy of logic, which typically adopts a stimulatingly new attitude towards the paradox-monger: rather than fixing on his flamboyant nots, they try to unmask his trick by going after his nonchalant ands and ors.

5 Volume contents

The foregoing elucidations and expositions hopefully afford an interesting vantage point from which to appreciate the rich variety of developments and discussions of substructural approaches to paradox offered by the papers of this volume. More in detail, Ross Brady's and Edwin Mares' papers consider a type of approach to the semantic and set-theoretic paradoxes in the tradition of *relevant logics* (Orlov, 1928) that denies the implicational analogues of (K), (W) and (C). On the basis of general considerations concerning definitions, Brady pleads for a logic where implication represents a notion of meaning containment (under a specific understanding of what such containment amounts to) for which paradox-driving principles fail even though they hold for logical consequence and in which a disjunction is provable only if either disjunct is; he further takes note of the fact that several *prima facie* intelligible notions such as e.g. the one of failing cannot be added to the logic on pain of paradox. Mares extends the information-theoretic interpretation developed in Mares (2004) for stronger relevant logics to weaker relevant logics friendly to correlation and comprehension (the basic idea being that the implicational analogues of (K), (W) and (C) fail because implication represents a notion of information application that is sensitive to those structural features), submits that *prima facie* intelligible notions are not admissible on such an interpretation because they do not correspond to "positive" information conditions and demonstrates that, in the logics he considers, comprehension must be restricted to properties that correspond to such conditions. In the same tradition, Pilar Terrés' paper considers an approach to the Material-Implication paradox and related paradoxes that denies (K). Terrés distinguishes between a minimal, classical notion of logical consequence and of its accompanying operations (which looks for truth preservation) and an enriched, relevant one (which looks for the reasons for accepting a sentence), with either being selectable depending on the features of context: she argues that, while the Material-Implication paradox and related paradoxes are sound arguments in classical logic, they usually have unsound readings (involving the occurrence of an intensional operator) in a relevant logic.

Neil Tennant's and Peter Schröder-Heister and Luca Tranchini's papers consider a type of approach to the semantic and set-theoretic paradoxes in the tradition of *normal proofs* (Prawitz, 1965) that denies (less crucially) (K) and (more crucially) (S). Replying to an overgeneralisation objection levelled by Schröder-Heister & Tranchini (2017) that relies on the addition of a certain intuitive reduction procedure to normalisation, Tennant refines his proof-theoretic criterion of paradoxicality (according to which a paradoxical derivation is one whose normalisation does not terminate) by adopting natural-deduction generalised elimination rules (Schröder-Heister, 1981) and shows that, on the refined criterion, derivations of Russell's paradox that do not write comprehension into the rules of the system do not count as paradoxical, whereas derivations of the Liar paradox that write correlation into the rules of the system do count as paradoxical (since then there are normal proofs of the Liar sentence and of its negation, but no normal proof of absurdity). Schröder-Heister and Tranchini counterreply to Tennant by pointing out that generalised elimination rules call for adding yet a further intuitive reduction procedure to normalisation, one that however reinstates the overgeneralisation objection; they suggest that the general problem be at least partially tackled instead by imposing strict conditions on admissible reduction procedures in normalisation, requiring preservation of identity of derivations (under a specific understanding of what such identity amounts to).

Elia Zardini's paper considers an approach to the Sorites paradox in the tradition of *tolerant logics* (Zardini, 2008a; 2008b, pp. 93–174) that denies (S). Zardini observes that vagueness is also crucial in the situation of the Preface paradox and related epistemic and implicational paradoxes and that the Sorites paradox on the one hand and those paradoxes on the other hand are totally analogous, concluding that they are all of the same kind and proceeding to apply his favoured solution to the Sorites paradox also to those other paradoxes. In the same tradition, Pablo Cobreros, Paul Égré, Dave Ripley and Robert van Rooij's paper considers a type of approach to the Sorites paradox that denies either (K) or (S) or both. Drawing on Cobreros *et al.* (2015), Cobreros, Égré, Ripley and van Rooij add to the basic framework of tolerant logics a further interpretation of sentences driven by speakers' intuitions of assertability: they use that interpretation to define a nondeductive logic (which denies (K)) that allows arbitrary iterations of tolerant reasoning about similar objects (at least as long as false conclusions are not reached) and where tolerance itself is invalid, alongside a deductive logic (which denies (S)) that disallows such iterations but where tolerance itself cannot be used as a premise, alongside a nondeductive logic (which denies both (K) and (S)) that allows such iterations but where tolerance itself can be used as a premise (at least as long as it is not applied to contradictory cases such as borderline ones). Also in the same tradition, Eduardo Barrio, Lucas Rosenblatt and Diego Tajer's paper considers an approach to the semantic paradoxes (due to Cobreros *et al.,* 2013) that denies (S). Barrio, Rosenblatt and Tajer explore the addition to the object language of a predicate expressing the notion of validity: they uncover the incoherence in the combination of circumstances that, while (S) fails for the logic they consider, since the validity version of a Curry sentence is absurd in that logic, then, according to that logic, the paradoxical instances of (S) for that sentence hold for the notion of validity expressed in the object language (and that indeed, employing plausible stronger principles for validity in the style of Zardini, 2014a, according to that logic, every instance of (S) holds for the notion of validity expressed in the object language).

Zach Weber's and Petr Cintula and Francesco Paoli's papers consider a type of approach to the semantic and set-theoretic paradoxes in the tradition of *BCK-logics* (Tarski, 1936) that denies (W). Weber explores the addition to the object language of a predicate expressing the notion of provability (*i.e.* validity with no premises): assuming the same approach to the ensuing provability version of Curry's paradox, he highlights how principles that one way or another force (W) to hold for sentences about provability (even for those that are not provable in the logic) are then not admissible. Replying to charges brought up by Ripley (2015a, pp. 322–325; 2015b), Cintula and Paoli first remark that there are several versions of (S) that, by not building in contraction, hold on a noncontractive approach but not on a nontransitive one and then, by employing a conjunctive mode of conclusion combination, formulate a notion of a multiset-based consequent relation and its corresponding notion of a multiset-based closure operation that are compatible with failure of (W).

Lionel Shapiro's paper considers an approach to the semantic paradoxes in the tradition of *dialetheism* (Priest, 1979 and then especially the version of Goodship, 1996 taken up by Beall, 2015) but develops a related substructural logic that denies (S), (W) and (C). Shapiro proves that there is a mapping between what is valid in his substructural logic and what is valid in the logic that would seem to be used by Beall (2015)'s approach and contends on this basis that there is no fact of the matter as to which of the two logics Beall (2015)'s approach uses (and, more generally, that there is no fact of the matter as to whether an approach to a paradox uses a substructural logic). Ole Hjortland's paper considers several types of substructural approaches to the semantic paradoxes that deny either (I) or (S) or (W). Focusing on the question of the extent to which all such approaches deviate from classical logic, Hjortland notes that every such approach cannot accept principles of classical logic that build in the structural principle it denies and, conversely, that certain types of structural approaches have a syntactic presentation (of the kind also used in Shapiro's paper) under which what is denied is a structure-related feature that classical logic enjoys under that presentation (*i.e.* the absence in a sequent of an intermediate position between premises and conclusions). Julien Murzi and Lorenzo Rossi's paper considers an approach to the semantic paradoxes in the tradition of *groundedness* (Herzberger, 1970) that denies (I). Replying to critical points raised by Zardini (2013b, pp. 636–638) and Field (2017), Murzi and Rossi defend the good standing of a notion of validity typically understood as obeying something like the analogues of the principles for sufficiency mentioned in Sect. 4, doing so mainly by providing a fixed-point construction for a [material-implication]-like predicate where (I) as well as the relevant analogue of the second principle for sufficiency mentioned in Sect. 4 fails.

6 Volume acknowledgements

This volume originates from the highly successful and productive workshop *Substructural Approaches to Paradox*, which I organised in 2013 at the University of Barcelona in the framework of my Marie Skłodowska-Curie Intraeuropean Research Fellowship 301493 *A Noncontractive Theory of Naive Semantic Properties: Logical Developments and Metaphysical Foundations* and in which the vast majority of the volume's contributors took part. To the best of my knowledge, that was the first conference ever dedicated to the topic of substructural approaches to paradox—which has in the meanwhile proven to be one of the most fertile research areas in philosophy of logic in the last decade or so—and this is now the first volume ever dedicated to it.[55] I would like to thank all the contributors for enthusiastically accepting my invitation and for delivering such challenging pieces—it has been a great pleasure to engage in a philosophical dialogue with each of them. I would also like to thank all the anonymous referees for *Synthese* who helped with their careful and constructive comments to improve the papers even further. Special thanks go to the editors of *Synthese* for accepting my volume proposal, and in particular to Gila Sher, Catarina Dutilh Novaes and Otávio Bueno for their great guidance, great support and great patience at different stages of this project. Finally, thanks to Laura and Miguel for refreshingly not giving a damn about the topic: life doesn't (wholly) consist in substructural approaches to paradox.

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[55] At different stages, work on this volume was supported by the Marie Skłodowska-Curie Intraeuropean Research Fellowship 301493 *A Noncontractive Theory of Naive Semantic Properties: Logical Developments and Metaphysical Foundations*, by the FCT Research Fellowship IF/01202/2013 *Tolerance and Instability: The Substructure of Cognitions, Transitions and Collections* and by the Ramón y Cajal Research Fellowship RYC-2017-22883. Additionally, support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged. I've also benefited from the Project CONSOLIDER-INGENIO 2010 CSD2009-00056 of the Spanish Ministry of Science and Innovation *Philosophy of Perspectival Thoughts and Facts*, from the Project FFI2011-25626 of the Spanish Ministry of Science and Innovation *Reference, Self-Reference and Empirical Data*, from the Project FFI2012-35026 of the Spanish Ministry of Economy and Competition *The Makings of Truth: Nature, Extent, and Applications of Truthmaking*, from the Project FFI2015-70707-P of the Spanish Ministry of Economy, Industry and Competitiveness *Localism and Globalism in Logic and Semantics*, from the FCT Project PTDC/FER-FIL/28442/2017 *Companion to Analytic Philosophy 2*, from the Project 2019PIDPID-107667GB-I00 of the Spanish Ministry of Science and Innovation *Worlds and Truth Values: Challenges to Formal Semantics* and from the Project PID2019-105746GB-I00 of the Spanish Ministry of Science and Innovation *Linguistic Relativity and Experimental Philosophy*.

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