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ORIGINAL RESEARCH

Models, structures, and the explanatory role of mathematics in empirical science

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Abstract

Are there genuine mathematical explanations of physical phenomena, and if so, how can mathematical theories, which are typically thought to concern abstract mathematical objects, explain contingent empirical matters? The answer, I argue, is in seeing an important range of mathematical explanations as *structural explanations*, where structural explanations explain a phenomenon by showing it to have been an inevitable consequence of the structural features instantiated in the physical system under consideration. Such explanations are best cast as deductive arguments which, by virtue of their form, establish that, *given* the mathematical structure instantiated in the physical system under consideration, the explanandum *had* to occur. Against the claims of platonists such as Alan Baker and Mark Colyvan, I argue that formulating mathematical explanations as structural explanations in this way shows that we can accept that mathematics can play an indispensable explanatory role in empirical science without committing to the existence of any abstract mathematical objects.

Keywords Mathematics · Models · Explanation · Structure · Indispensability

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This paper has had a long gestation period (partly due giving birth, moving jobs, and then giving birth again—to twins—in fairly quick succession), a consequence of which is that it is difficult to reconstruct a full list of all who have seen versions, to whom thanks are due. An early version was written during a period of research leave from the University of Liverpool, for which I remain grateful. I have presented versions of this paper at the Institute of Philosophy, Shanxi University, the University of Cambridge, the University of Leeds, the University of Manchester, the University of Nottingham, and the University of York, and I am grateful to audiences at those events for helpful discussions. The fast moving debate over mathematical explanation is such that the final version is substantially changed from the original, and in developing this version I have been particularly grateful to Juha Saatsi for an excellent set of written comments, as well as to two anonymous referees for this journal whose perceptive comments were extremely valuable.

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1 Introduction

Are there genuine mathematical explanations of physical phenomena, and if so, how can mathematical theories, which are typically thought to concern abstract mathematical objects, explain contingent empirical matters? Lange (2016), for example, argues that mathematical explanations of physical phenomena are a species of non-causal explanations that he calls *explanations by constraint*. But how can facts about spatiotemporally isolated mathematical objects can act as *constraints* on the physical world? The answer, I will argue, is in seeing an important range of mathematical explanations as *structural explanations*, where structural explanations explain a phenomenon by showing it to have been an inevitable consequence of the structural features instantiated in the physical system under consideration. Such explanations are best cast as deductive arguments which, by virtue of their form, establish that, *given* the mathematical structure instantiated in the physical system under consideration, the explanandum *had* to occur. The constraints placed on the world by the mathematical premises in these explanations are thus logical constraints: such explanations show that, given structural features of the physical system, their explananda were inevitable as a matter of logic.

Several questions arise out of this picture. First, does couching so-called mathematical explanations of physical phenomena as structural explanations establish that these are genuine *explanations*? A full answer to this question would require a full account of what it is to explain, and this is not something that I will pursue here (though I endorse much of what Lange (2016) has to say in defence of taking so-called 'explanations by constraint' as genuinely explanatory). My own view is there are features of these so-called 'explanations' that suggest that there is at least a case for including them as examples of genuine explanations. In particular, they supply important *modal* information about their explananda: they tell us why they had to occur given the structural features of the physical situation. They also offer opportunities for understanding provided by *unification*, through showing how apparently disparate phenomena are instances of a common structure.

Regardless, though, of whether what I call 'structural explanations' are genuine *explanations* or merely explanation-like (e.g. in providing some form of illumination/understanding of their target phenomena), what I am most keen to explore in this paper is a different question, that of whether the (explanatory- or explanation-like) theoretical role played by such structural 'explanations' offers support for mathematical platonism. Perhaps we might be moved to accept an account of explanation according to which all *genuine* explanations are causal. Nevertheless, as I will argue in Sect. 1, many so-called mathematical explanations of physical phenomena afford us at the very least important forms of *understanding* that are not available if focus on nominalistically-stated alternatives. So even if *supplying modal information* about an observed phenomenon, and *unifying disparate phenomena* turn out to be not enough to count as providing an explanation in a strict sense, these still remain important theoretical roles played by mathematics in science that go beyond what would be available if we confined ourselves to purely nominalistically-stated alternatives. And this raises the question of whether, if what I am calling structural 'explanations' succeed where purely non-mathematical descriptions fail in enhancing our understanding of the physical world in these kinds of ways, this amounts to an indispensable theoretical role that supports platonism. Attending to the nature of structural explanations shows that any attempt to argue from the indispensable theoretical role of structural explanations to mathematical platonism must fail, for structural explanations of physical phenomena do not require that our structure-characterising mathematical axioms are true of any *mathematical objects*, but only that they are true—or approximately true—when their non-logical terminology is interpreted to apply to systems of either actual, or idealized, *physical objects*. So admitting an indispensable theoretical role for mathematical-structural explanations does not support an inference to the existence of abstract mathematical objects.

The picture of mathematical explanations as structural explanations that I present here is sketched in Leng (2012) and Leng (2021), but it has not been developed in full detail in previously published work. This paper fills in the details of this sketch. In Sect. 1 I look at some examples of the alleged 'explanatory' role of mathematics in physical science, and agree with platonists such as Baker and Colyvan that there is important theoretical work done by mathematics in the examples they present that is not available if we focus solely on non-mathematical alternatives. I side with Baker and Colyvan there in saying that the theoretical role played by mathematics in these examples should be thought of as an 'explanatory' role, but even for those not convinced that this is genuine *explanation*, I argue that Baker and Colyvan have at the very least indicated an important theoretical role played by mathematics in physical science, and this raises the question of *how* mathematics is able to play this role, and in particular of whether the ability of mathematical theories to play this kind of role requires the existence of mathematical objects. The remainder of the paper considers the question of whether the existence of these kinds of mathematical explanations of physical phenomena supports the existence of mathematical objects. Section 2 characterises a class of mathematical explanations as structural explanations, arguing that they can be presented as deductively valid arguments whose premises include a mathematical theorem expressed modal structurally, together with empirical claims establishing that the conditions for the mathematical theorem are instantiated in the physical system under consideration. I suggest that these arguments should be thought of as genuinely explanatory by virtue of providing important modal information: they show that the phenomenon to be explained *had* to occur, given the structural features that are physically instantiated. Additionally, by identifying mathematical-structural features that necessitate the occurrence of the phenomenon to be explained, they offer opportunities for explanatory unification, showing apparently disparate phenomena to be consequences of the very same mathematical-structural features. I also show that these explanations, which can be understood in modal structural terms, involve no commitment to mathematical objects platonistically construed. In Sect. 3 I consider the application of this account to real cases where mathematical structure is instantiated not directly in physical systems, but only in an idealised model of a physical description (in what, following Bokulich, 2008 I will call 'structural model explanations'). I argue that the *explanatory* use of mathematics in these idealized model cases offers no further argument for realism than is already offered by the use of idealized models to represent physical phenomena. I also point to a helpful feature of the structural account as compared to mapping account of applications of mathematics: while it is true that in many cases the relation of mathematics to reality is of a map to a terrain, if the structural account is correct, mathematics does not explain simply by providing such a map, but by showing how mathematical-structural dependencies in mathematical models reflect mathematical-structural dependences in the physical world. I conclude, then, that viewing mathematical explanations of structural explanations provides an understanding of how mathematics can play a significant theoretical role in our understanding of physical phenomena that does not require us to adopt a platonist account of mathematical objects.

2 Why think that mathematics does genuine explanatory work?

Since Alan Baker's (2005) paper introducing the philosophy of mathematics world to the curious case of the periodical magicicada cicadas, much has been written on the alleged existence of mathematical explanations of physical phenomena. Typically, discussion has been divided along platonist/anti-platonist lines, with most platonists agreeing that there are such explanations, and most anti-platonists disagreeing (notable exceptions are Brown (2012) on the platonist side, and Leng (2012) on the anti-platonist side). For those who reject the claim that mathematics does genuine explanatory work in our scientific theories, a standard strategy has been to point to the *nominalistic content* of putative mathematical explanations of physical phenomena, holding that while these explanations may be characterised mathematically, all the genuine explanatory work in these explanations is carried by their nominalistic content, with mathematics being used as a convenient—and perhaps indispensable—way of indexing the explanatorily relevant physical facts. (Examples of strategies along these lines include Brown, 2012; Daly & Langford, 2009; Melia, 2000; Saatsi, 2011) In Leng (2012) I side with platonists including Baker and Colyvan (2011) in suggesting that if we focus on the nominalistic content of mathematical explanations of physical phenomena, we lose explanatory power.

Take for example Brown's account of the cicada case. Brown (2012, p. 10) uses the notions of *cycle factorizability* and *non-factorizability* to pick out nominalistically characterizable features of cicada life-cycles that he takes are ultimately responsible for the prime-length period phenomenon. Although there is a clear link between these notions and the mathematical notions of 'composite' and 'prime' as applied to numbers, Brown notes that nonetheless they are intelligible in non-mathematical terms (a cicada cycle is cycle factorizable if and only if it can be broken into repeated shorter cycles of equal duration without leaving any years out). A cycle is non-factorizable if and only if its associated number (of years) is prime, hence the relevance of talk of prime numbers in indexing the standard evolutionary explanation of cicada period length. But the real explanatory work, Brown contends, is done by the nominalistically kosher feature of cycle lengths that is indexed by prime numbers (non-factorizability).

It certainly seems right that it is cycle-non-factorizability (along with the relevant evolutionary facts that the explanation presupposes about periodic predators) that is responsible for the cicada's behaviour. In that sense, the non-factorizability of the 13 and 17 year cycles does explain why those cycles were chosen. But even though an adequate explanation can be afforded in terms of the nominalistically acceptable notion of cycle-factorizability, there is at least a sense in which, by refusing to appeal to the more general notion of prime number as it relates to non-factorizable cycles, this explanation remains impoverished. By framing the explanation in terms of prime numbers (with the non-factorizable cycles being those that are indexed by prime numbers) we can make use of our knowledge of prime and composite numbers in order to understand more about the possibilities for similar periodic behaviour. For example, the fundamental theorem of algebra, which tells us that composite numbers have a unique prime decomposition, can tell us that, of composite cycles, cycle lengths with fewer distinct prime factors would be preferable. So, for example, a 4-cycle would be preferable to a 6-cycle since it has only one prime factor (2) rather than two (2, 3), so while a 4 cycle would meet 2-cycle predators every time it occurred, it would only meet 3-cycle predators every fourth cycle (once every 12 years). On the other hand, a 6-cycle creature would meet 2-cycle and 3-cycle predators every time it occurred, making that a worse choice of cycle length in conditions where 2-cycle and 3-cycle predators occur. Such extrapolations concerning potential periodic behaviour come naturally when the explanation is framed in term of prime numbers, given our familiarity with their patterns, but are lost if we drop that framing and instead focus directly on the indexed property of cycle-factorizability. The mathematical framing thus offers easy access to a range of *modal information* concerning what would have happened had different cycle lengths been chosen, that is not present if we focus solely on cycle-factorizability. Along a similar vein, the well known 'Bridges of Königsberg' explanation using Euler's theorem not only shows why a certain kind of walk is impossible, but also provides information about what kinds of bridge/landmass configurations would be required to make possible a Eulerian walk.

Focussing on cycle-factorizability also prevents us from seeing connections with other phenomena that are naturally indexed with prime numbers, but which have nothing to do with cycle lengths. A teacher may come to realise that classes of 30 students are easier to work with than classes of 25, since in splitting into groups the latter can only be split evenly into 5 groups of 5 pupils, while the former has the option of 15 pairs, 6 groups of 5, 5 groups of 6, 3 groups of 10, or 2 of 15. Better choices of cycle lengths (for the purpose of avoiding predators) turn out to be worse choices of class sizes (for the purpose of allowing maximal opportunities for group work). Of course we *could* introduce a separate notion of collection-factorizability to apply to collections of discrete individuals, where a collection is factorizable if it can be broken up into a number of smaller collections of equal size without remainder. But there is obviously a common pattern here, and we are surely best placed to appreciate and understand that common pattern once we see the natural associations between collections of individuals, repeating cycles, and the prime and composite numbers that are used to index both. Along similar lines, Baker (2017) points to another example of a use of prime vs composite cycles as part of an explanation of a physical phenomena: an explanation of why fixed gear bikes where the numbers of cogs on front and back wheel are coprime see less wear from braking than bikes where the pairs are not coprime. I agree with Baker that couching all of these explanations in mathematical terms provides them with a *topic-generality* that adds a level of explanatoriness that goes beyond what is available if we focus on the nominalistic content of each explanation. While nominalistic versions of each explanation are available that succeed in showing that the nominalistically characterizable features of the particular systems in question sufficed to guarantee that the observed phenomenon would occur, the mathematical explanations serve to add another explanatory dimension, the *ability to unify* a range of what at first glance may seem like different phenomena. This additional dimension, I would like to suggest, is a structural one: the mathematical explanations show in each case that the explanandum occurred *as a consequence of structural features of the physical system* that can be characterised mathematically. As the same theorem involving the same mathematical structure is involved in each case, the topic generality of mathematical explanations allows us to see each of these disparate phenomena as a consequence of one and the same structural feature[1].

The work done by the mathematical framing in the typical examples of candidate mathematical explanations of physical phenomena, both in *providing modal information* about the explanandum and offering possibilities of *unification* of the phenomenon to be explained with apparently disparate phenomena supports our understanding of those phenomena in such a way that suggests to me at least that it is worthy of being called *explanatory*. In what follows, I will accept that examples such as the number theoretic explanation of cicada behaviour and the graph theoretic explanation of the impossibility of completing a Eulerian walk through Königsberg are genuine mathematical explanations of physical phenomena. I will offer an account of how these explanations work and argue that, if they do work in this way, our use of these explanations in empirical science does not commit us to mathematical platonism. Some readers may remain unconvinced, however, that the virtues I have pointed to of these so-called mathematical 'explanations' (i.e., providing modal information and unifying apparently disparate phenomena) suffice to show that these uses of mathematics are genuinely *explanatory*. But even if we agreed not to use the 'e' world to describe them, to the extent that we think that these theoretical virtues are virtues worth having, there remains a question of how mathematics can serve these functions (of providing modal information and theoretical unification), and whether using mathematics for these purposes presupposes platonism. For the reader who is not convinced my use of the 'e' word to talk about these examples, I hope the structural account I offer of how mathematics works to provide modal information and possibilities of unification will still be of interest in showing that if we wish to use mathematical theories for these purposes, doing so will not commit us to the existence of mathematical objects.

[1] Talk of shared structural features may suggest to some readers a Platonist interpretation in the form of the ante rem structuralism of Shapiro (1997). In Sect. 3.3 I show that this Platonist interpretation can be avoided by adopting a modal structuralist understanding of the notion of mathematical structure (following Hellman, 1989).

3 Mathematical explanations as structural explanations.

I propose that the mathematically couched explanations of cicada behaviour and of the impossibility of Eulerian walks through Königsberg, along with other examples that have been offered in the literature on mathematical explanation are examples of what I, following Bolulich (2008) (who herself follows Peter Railton (1980) & Hughes (1989)) will call *structural explanations*. Structural explanations explain by showing an empirical phenomenon to be a consequence of the mathematical structure of the empirical situation. According to Bokulich (2008, p. 149),

> a structural explanation is one in which the explanandum is explained by showing how the (typically mathematical) structure of the theory itself limits what sorts of objects, properties, states, or behaviors are admissible within the framework of that theory, and then showing that the explanandum is in fact a consequence of that structure.

But how does one show that an explanandum is a consequence of the mathematical structure of a theory? In answering this question, as my own interest is specifically in the status of mathematical hypotheses in structural explanations, rather than following Bokulich's discussion of this matter directly, I will focus more closely on how mathematical theories characterize structures that can be used in structural explanations, rather than discussing mathematically-structured empirical theories, which is where Bokulich's own attention lies.

In order to see what could be meant by empirical phenomena being consequences of the *mathematical structure* of an empirical set up, it will be helpful to consider an understanding of mathematical theories that is common to most forms of structuralism in the philosophy of mathematics. Consider a pure mathematical theory, presented axiomatically. These axioms will typically include logical terminology and some primitive terms. For example, in the (2nd order) Peano axioms for number theory, we have primitive terms 'zero (0)', 'number (N)' and 'successor (s)', where '0' is a singular term, 'Nx' a unary predicate, and 's(x)' a unary function. The axioms can be expressed as follows:

1. N(0) ('zero is a number').
2. $\forall x(Nx \supset Ns(x))$ ('The successor of every number is a number').
3. $(\forall x)(Nx \supset s(x) \neq 0)$ ('Zero is not the successor of any number').
4. $(\forall x)(\forall y)((Nx \,\&\, Ny) \supset (x \neq y \supset s(x) \neq s(y)))$. ('distinct numbers have distinct successors').
5. $(\forall F)(\forall x)((F0 \,\&\, (\forall x)(Nx \supset (Fx \supset Fs(x)))) \supset (\forall x)(Nx \supset Fx))$ ('If any property is such that it applies to 0 and, if it applies to a number it also applies to that number's successor, than that property applies to all numbers.').

We can abbreviate the conjunction of these axioms as PA⟨0, N, s⟩ (indicating the primitive terms).

A question arises of what we should make of the primitive terminology in such axiomatizations. There are two basic approaches on the table. An 'assertory' understanding of an axiom system sees its primitive terms as independently meaningful, and aiming to pick out some specific objects, predicates, and functions. The axioms are then attempts to assert basic truths about these independently meaningful primitives. On the other hand, an 'algebraic' understanding sees the primitive terminology as not having a meaning independently of the axiom system in which they occur, and (much like the 'unknowns' in a system of equations with various unknowns) as being given their meaning contextually by the axioms themselves[2]. Clearly an algebraic understanding is appropriate for *some* systems of axioms: the axioms for group theory, for example, can be thought of as defining what would have to be true of any collection, G, of objects with binary operator + and distinguished element 0 in order to count as a group. There is no specific intended interpretation of 'G', '+' and '0' about which the axioms aim to assert truths. What the leading versions of mathematical structuralism (such as Stewart Shapiro's ante rem structuralism (Shapiro, 1997) and Geoffrey Hellman's modal structuralism (Hellman, 1989)) have in common is that they assume an algebraic understanding of all axiomatic theories[3].

The correctness of mathematical structuralism as a picture of pure mathematics is not what is at issue here; my interest is only in the 'algebraic' approach to axiom systems assumed by structuralists. What is important about the algebraic understanding of mathematical theories in this context is the sense it allows us to make of the notion of mathematical structure, and in particular of the notions of a system of objects instantiating a mathematical structure, and of truths that are 'true in virtue of' that structure. For a particular system to instantiate an axiomatically characterized mathematical structure is simply for the axioms characterizing the structure to be true when their primitive constants, predicates, and function symbols are given an appropriate interpretation in the terms of that system. We can, for example, find particular mathematical systems instantiating axiomatically characterized mathematical structures: the natural number structure has an instantiation in the sets if we interpret 0 as ∅, s(x) as the function that takes a set A to its singleton {A}, and the predicate Nx as being true of a set A if and only if A is in the intersection of all sets containing ∅ and closed under the operation of taking successors (i.e., if and only if $A \in \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \ldots\}$. But we can also find 'concrete' systems instantiating some mathematical structures: the group axioms, for example, can be interpreted as truths about the simple system consisting of symmetric rotations of a square. Here, G is the collection of possible rotations: (id = keep as is; r_1 = rotate 90° clockwise, r_2 = rotate 180°, r_3 = rotate 270° clockwise). Of these, 0 is interpreted as the 'id' rotation, and the binary + operation is the result of performing two operations consecutively (so that, e.g., $r_1 + r_2 = r_3$). And, to take us back to Baker's cicada example, if we idealize somewhat to forget about the eventual demise of the earth, the series

[2] The labels 'algebraic' and 'assertory' are due to Geoffrey Hellman (2003), but the debate over how to understand axiomatic theories is older, going back at least to Frege and Hilbert, who corresponded over this matter (with Frege on the assertory and Hilbert on the algebraic side of the debate Frege, 1980).

[3] In Leng (2007) I argue that an algebraic approach to mathematical theories is also shared by other contemporary philosophical accounts of mathematics, including fictionalism and full-blooded platonism.

of earth-years starting from a given 0 in which cicadas appear and continuing without end can be viewed as an instantiation of the natural number axioms, with the function ‘s(x)’ being interpreted as ‘the year following year x’.

Consider now a system of objects and relations (mathematical or physical) that instantiates an axiomatically characterised structure. There will of course be a range of truths about that given system. For example, in our set theoretic system instantiating the Peano axioms, when supplemented with definitions of the individual numbers and the ‘less-than’ relation ‘<’), it will be true that the object it calls ‘2’ (ss0, i.e. in this case, $\{\{\varnothing\}\}$ will be a member of the object it calls ‘3’ (sss0, i.e., $\{\{\{\varnothing\}\}\}$), and it will also be true that $2<3$. But only the latter of these is, I claim, true *in virtue of* the axiomatically characterised structure provided by PA⟨0, N, s⟩. The axiomatic setting helps us to understand this difference. When we supplement the axioms with the appropriate definitions, ‘$2<3$’ is a logical consequence of PA⟨0, N, s⟩ (and thus true in *all* interpretations of these axioms), whereas ‘$2\in 3$’ is not a logical consequence of the structure-characterising axioms. In general, if structurally characterized axioms are true when interpreted as about a particular system, then we can say that a truth about that system is true *in virtue of* the mathematical structure characterized by those axioms when it is an interpretation of a claim that follows logically from those axioms.

We can now make sense of the notion of a structural explanation to which I wish to draw attention, in cases where the structure in question is characterized by mathematical axioms. Such a structural explanation explains by showing (a) that the system to be explained can be viewed as an instance of a mathematical structure, and (b) showing that the explanandum is true in virtue of that structure, i.e., that it is a consequence of the characterizing axioms and relevant definitions (when suitably interpreted). As such, we can think of the general form of a structural explanation (involving axiomatically characterised mathematical structure) as an explanatory argument as follows:

[Mathematical Premise, MP] Mathematical theorem, modal-structurally characterised (i.e., of the form, ‘necessarily, in any system satisfying <Axioms>, <Theorem>’)[4].

[Empirical Premises, EP] Empirical claims justifying the claim that <Axioms> are true when interpreted as about the physical system under consideration.

Therefore.

[Explanandum] <Theorem> is true of the system under consideration.

Of course, when it comes to real-life examples of mathematical explanations of physical phenomena, some work may be needed in order to discover this general

[4] The modal structural characterisation follows Hellman (1989).

form in the explanations as provided. We will see below how it might work in some particular cases.

Before we move to examples, however, it is worth noting up front a feature of structural explanations thus characterised, that may ring alarm bells given the history of accounts of explanation in the philosophy of science. By couching the general form of structural explanations as *deductive*, *explanatory arguments* that invoke a necessitated conditional (in their modal-structurally characterised mathematical premise) this account makes structural explanations look suspiciously close to covering law explanations, in this case a deductive-nomological explanation where a mathematical ‘law’ takes the place of an empirical one, providing ‘nomic expectability’ to the conclusion of the argument as explanandum. Despite the well-trodden concerns about the covering law model as a general account of explanation, here I will embrace this similarity. I note some relevant differences: (1) the restriction of our ‘law’ to a modal-structurally characterised mathematical theorem avoids some difficulties concerning the kinds of generalisations that can be cited as ‘laws’ in such explanations; (2) the requirement that the empirical premises serve to justify the claim that the axioms applied in the mathematical premises apply to the physical system under consideration avoids some concerns about permitting arguments with irrelevant premises to count as explanations. However, one feature that my account does share with the D–N model is its tolerance for explanatory symmetries: given that (in the famous ‘flagpole’ case) it is equally a theorem that if the length of the shadow is x, the height of the flagpole is y *and* that, if the height of the flagpole is y, the length of the shadow is x, we can just as well use information about the length of the shadow along with structurally interpreted mathematical results to provide a structural explanation of the height of the flagpole as we can use information about the height of the flagpole to explain the length of the shadow. I do not have the space for a full discussion of this case here, so I will simply note that this is a bullet that I am willing to bite[5].

[5] The issue of explanatory symmetries is also raised as a problem for Lange’s ‘explanations by constraint’, which also look like they are best cast as explanatory arguments. In *Because without Cause*, Lange tries to avoid symmetries by arguing that ‘reversed’ versions of his explanations by constraint are ruled out because they appeal to features that are “not understood to be constitutive of the physical arrangement with which [the explanatory why question] is concerned” (Lange, 2016, p. 43). Craver and Povich (2017) find this account wanting (though see Lange, 2018 for a reply). I wish to embrace the potential for empirical symmetries in part because I think the kinds of features that contextual information might determine to be *constitutive* of a physical arrangement when we consider a why question might well be such as to allow perfectly acceptable reversals. Those who hold that true explanations cannot admit of symmetries might wish to resist taking ‘displaying the phenomenon to be nomically expectable’ to be a way of explaining things. My own view, though, is that ‘showing it to be nomically expectable’ should be considered a perfectly good way of explaining a phenomenon, and it is only a prejudice in favour of the causal that prevents us from accepting that perfectly good explanations may sometimes run in more than one direction.

3.1 Examples

A simple example of a structural explanation is provided if one considers a rather mundane puzzle about the difference between mattress flipping and tyre rotating, discussed by Brian Hayes (2005) in a popular article in *American Scientist* (2005). We are advised to flip/rotate double-sided mattresses periodically in order to ensure even wear. There are four possible ways of fitting a mattress into a standard rectangular bed, and by flipping/rotating mattresses periodically, our aim is to cycle through these four configurations so that over its lifetime the mattress gets equal use in each position. Similarly, it was once considered good practice to move the tyres on a car around, so as to even out wear on the tyres, and it is prudent, if one is doing this, to ensure that the tyres are moved around evenly so that no one tyre spends too long in the same position. There is (what Hayes calls) a simple 'golden rule' for moving tyres: a single operation that one can do each time one moves the tyres to ensure that, after enough applications, each tyre will have occupied each of the four positions it can take exactly once, so that wear is even. If one simply rotates the tyres around the car a single turn at each change (making an arbitrary choice at the start of whether to move clockwise or anticlockwise), one can be confident that, after applying this same operation repeatedly all tyres will have occupied all positions. This means that we do not need to remember how we positioned the tyres on previous occasions. If we simply resolve always to move a single turn clockwise (say), we will ensure even wear without having to keep track of previous positions.

If we think of the three main symmetric operations one can do to 'flip' a mattress (i.e., rotate 180° across each of the three orthogonal axes through its centre point), clearly no single one of these on its own would provide us with a golden rule for mattress turning: if we were always to rotate around its short vertical axis, for example, we would find ourselves always sleeping on the same side of the mattress, with the head and foot flipping each time. Hayes wondered, though, whether there was a single *combination* of these operations which, if one cycled through that combination at each 'flipping', would ensure that, over a period of time, the mattress would take all possible configurations, thus avoiding the problem of remembering how the mattress was configured on previous configurations before choosing which operation to take next time.

A quick internet search of mattress flipping advice suggests that no such solution has been found: the best advice available, Hayes tells us, seems to be to practice seasonal flipping, e.g. flipping across the short horizontal axis for one season and then across the long horizontal axis the next, to cycle through the four possible mattress positions over a year. Why is this? Why isn't there a single combination of moves, which if one repeats that same combination at each flipping would ensure that all positions of the mattress would be taken? And why does the mattress case differ from the superficially similar problem of tyre rotation? The answers can be seen once one realizes that the rotations of mattress form a 4-group. Since groups are closed, any combination of any number rotations is equivalent to a single rotation. And since (as we have already noted) no single rotation provides a golden rule, no combination of these will either. Why are things different in the tyre case? Because, though both are 4-groups, the tyre group is (an instance of) the cyclic group of order 4 (the group of 2-dimensional rotations of the square that we saw earlier), whereas the mattress rotations instantiate the Klein 4-group, which contains no operation which, when repeatedly applied to itself, cycle through all the four operations in the group.[6] So the fact that the two rotation sets are instances of two different groups explains Hayes's explanandum: why is there a golden rule for tyres but not for mattresses?

Couched in our general terms, we can present this explanation of a contrast as involving two separate structural explanatory arguments, one showing that there is single move which, repeated, will cycle through all of the possible mattress positions, and another showing that there is a single move which, repeated, will cycle through all of the possible tyre positions. If we bundle the definitions of the Klein 4-group and the cyclic group of order 4 into our 'structure characterising' axioms respectively, one argument will take the form:

> [MP] Necessarily, if ⟨0, a, b, ab⟩ is a Klein 4-group, then there is no element in ⟨0, a, b, ab⟩ whose repeated application will cycle through all the members of the group.
>
> [EP], when '0' is interpreted as no movement, 'a' as flipping across the vertical axis, 'b' as flipping across the short horizontal axis, and 'ab' as flipping across the long horizontal axis, these rotations of a mattress form a Klein 4-group.
>
> Therefore.
>
> [Explanandum]: there is no mattress rotation whose repeated application will cycle through all the possible positions of the mattress.

Similarly, the 'tyres' argument will use the empirical premise that, when '0' is interpreted as no movement, 'a' as moving all tyres one space clockwise, a^2 as moving all tyres two spaces, and a^3 as moving all tyres 3 spaces clockwise (or one space anti-clockwise), these tyre rotations form a cyclic group of order 4, to conclude that there is a golden rule for tyre rotation.

To move from this 'toy' example to some of the examples of mathematical explanations of physical phenomena mentioned already, the famous bridges of Königsberg explanation (as discussed by Pincock, 2007) is relatively easily put in this form, as follows:

6 The group tables are as follows:

Klein 4-group

0	a	b	ab
a	0	ab	b
b	ab	0	a
ab	b	a	0

Cyclic group of order 4

0	a	a^2	a^3
a	a^2	a^3	0
a^2	a^3	0	a
a^3	0	a	a^2

[MP] Necessarily, if ⟨Nodes, Edges⟩ is an instance of a connected graph, then if ⟨Nodes, Edges⟩ permits a Eulerian walk it must have either zero or two nodes with an odd number of edges.

[EP] When ‘Nodes’ is interpreted as ‘landmasses’ and ‘edges’ is interpreted as ‘bridges’, Königsberg in 1735 is an instance of a connected graph with four nodes with an odd number of edges.

Therefore.

[Explanandum] Königsberg in 1735 does not permit a Eulerian walk.

Finally, as mentioned before, while Baker’s cicada example requires something of an idealization to fit straightforwardly this model (assuming that the sequence of years in which cicadas appear has no end)[7], having made that idealization we can we can sketch the explanation in rough terms as follows (following Baker in adding a biological premise to fill in the evolutionary constraints):

[MP] Necessarily, if PA⟨0, N, s⟩, arithmetic progressions of length n and m that have the same first member overlap minimally when n and m are coprime, and a number m is coprime with all numbers n <2m iff m is prime.

[EP] When ‘0’ is interpreted as some first year in which two broods of periodical magicicada cicadas appear together, ‘N’ as interpreted as the collection of years including and following that first year, and ‘s’ is interpreted as ‘the year after’, PA⟨0, N, s⟩ hold, and the sequence of years in which a magicicadas with period length m occur form an arithmetic progression of length m.
[Biological premise] It is advantageous for cicadas to choose periods which minimize overlap with periods of other periodical creatures.

Therefore:

[Explanandum] Prime number periods are advantageous for cicadas.

Three questions arise. First of all, do so-called structural explanations deserve to be viewed as genuine explanations? Second, does the use of structure-characterizing mathematical axioms in these explanations commit us to assigning an explanatory role to abstract mathematical objects (or indeed, abstract mathematical structures)? And finally, can all or even most purported examples of mathematical explanations of empirical phenomena be accounted for as structural explanations? My answer to these questions, in brief, are: yes; no; and not quite. I will consider the first two of these questions here, before turning to the third in Sect. 3.

[7] This idealization is inessential. It is made for the convenience of using a straightforward instantiation of the Peano axiom structure in the sequence of years in our explanatory argument. Since the theorem we are using will also apply also to apply to finite initial segments of the natural numbers, we could avoid the idealization and talk instead about theorems that hold in finite initial segments of n. I prefer to make the idealization for simplicity in formulating the explanatory argument, since, as I will argue in the next section, introducing idealizations into our mathematical explanations of physical phenomena will often be required anyway, and doing so incurs no additional platonistic debt.

3.2 Are structural 'explanations' genuine explanations?

Structural explanations certainly provide answers to the kind of 'why' questions that we ask in demanding explanations of phenomena. Why do the cicadas have the period length they have? Because the sequence of successive years is an instance of a natural number structure; the sequence of years in which the cicadas appear is an instance of an arithmetic progression within that structure; in any natural number structure, arithmetic progressions with prime differences between terms will overlap minimally with other progressions; and non-overlapping periods are advantageous. Why is there no golden rule for mattress flipping when there is one for tyre rotation? Because the mattress operations are an instance of the Klein 4-group; in any Klein 4-group, no one element can be repeatedly applied to itself to cycle through all four operations; and a golden rule would require there to be such a cycle. Perhaps this in itself is enough to present these purported explanations as genuinely explanatory. But for those looking for something more, note that as I have presented these explanations, the explanans in each case involves appeal to a general (structural) law (whose modal status as a logically necessary truth is supported by the fact that the consequent is derivable from the antecedent): "necessarily, in any natural number structure, arithmetic progressions with prime differences between terms will overlap minimally with other progressions"; "necessarily, in any Klein 4-group, no one element can be repeatedly applied to cycle through all four operations". By deriving observed phenomenon from premises that include a modal-structural law (a claim about what must be true in all structures of a given sort), I have already noted that these structural explanations share important features of DN-explanations—they explain by providing what Salmon (1989, p. 57) calls "nomic expectability—the expectability on the basis of lawful connections".

Another reason to think of structural explanations as genuinely explanatory is that they meet the criteria required for 'distinctively mathematical explanations' outlined in Marc Lange's recent defence of non-causal explanations. Lange (2016, pp. 5–6) holds that what he calls 'distinctively mathematical explanations' work.

> by showing how the fact to be explained could not have been otherwise—indeed, was inevitable to a stronger degree than could result from the action of causal powers.

The modal elements of structural explanations as I have characterised them help to show how these explanations establish that, given the structural features of instantiated in the target system, the explanandum of a structural explanation could not have been otherwise. Modality occurs in these explanations at two points. First is in the modal-structural mathematical premise. Here the necessity at work is logical necessity, and our justification for taking the MPs as true is the existence of a derivation of the consequent from the antecedent. The second modal element in these explanations is the fact that they are deductively valid arguments: the inevitability of the explanandum *given the premises* is established through showing that it is a consequence of those premises. While this second modal element justifies the claim that these explanations work by showing how the fact to be explained could not have been otherwise, it is the first modal element that meets Lange's criterion for these explanations to count as 'distinctively mathematical'. By showing that their explananda follow from logically necessary truths about what holds in *any* structure satisfying certain structure-characterising axioms, these explanations display by their form that the inevitability of the explanandum is stronger than causal.

Finally, to the extent that unification is a form of explanation, by displaying physical phenomena as consequences of the mathematical structure instantiated in a physical system, it is easy to see how structural explanations can serve to unify. Structural explanations involve a mathematical premise which is a modal-structurally characterised mathematical theorem, as well as an empirical premise containing information to establish that the physical system under consideration is such as to satisfy the antecedent of the modal-structural theorem in the mathematical premise. Structural explanations of this sort can unify apparently disparate phenomena when it can be shown that the structural explanations of those phenomena appeal to the very same mathematical result.

3.3 Ontological commitments of structural explanations

As mentioned above, in the recent debate over platonism and anti-platonism in the philosophy of mathematics, the existence of genuine mathematical explanations of physical phenomena has been held to support mathematical platonism. And my talk above of 'instantiation of a mathematical structure' in a physical system might suggest that structural explanations as I have characterised them are no less committed to a form of platonism, in the form of realism about the mathematical structures instantiated. However, the modal-structural characterisation of the mathematical premises of structural explanations shows that an inference from the existence of mathematical-structural explanations of physical phenomena to platonism is not warranted. In fact, structural explanations as I have characterised them (where a mathematical structure is shown to be instantiated in an empirical system, so that truths about that system can be displayed as holding *in virtue of* that structure) require no specifically mathematical ontology.

Although mathematical theories (such as, in the examples we have considered, number theory, group theory, and graph theory) are used in structural explanations of physical phenomena, we are not required to assume that the axioms of such theories are true of a realm of abstract mathematical objects. Rather, as indicated by the modal-structural formulation of the MPs in our examples, we may simply view the pure mathematical theory that is involved in the explanation as telling us what *would have to be* true, *were* there a system instantiating the structure characterized by the axioms, something that we can discover simply by inquiring into the consequences of the axioms of the theory. Having shown mathematically that any system exhibiting a given structure has a particular feature (e.g., that in any instance of the Klein 4-group, no element can be repeatedly applied to itself to cycle through all four elements of the group), we can transfer this information to the concrete instantiation we have found. Structural explanations of this sort may make essential use of mathematical theories to explain empirical phenomena, but such essential use does not require us to posit the existence of a special realm of mathematical objects about which these theories assert truths, only that such a theory is, when appropriately interpreted, true of the concrete system whose behaviour we are trying to explain.

There are, of course, modal commitments incurred in viewing these explanations as involving claims about what would have to be true, were any system to instantiate the axioms. As I have said, the necessity at hand here is logical necessity (where, 'necessarily, if ⟨Axioms⟩ then ⟨Theorem⟩' holds if and only if ⟨Theorem⟩ is a logical consequence of ⟨Axioms⟩). So the nominalist who wishes to adopt this account of explanation and hold that the structural laws involved in these explanations are literally true will have to commit to the truth of some logical necessities (or, equivalently, to the truth of some claims about what follows from our axioms). But these modal commitments are no more than are already incurred in the leading fictionalist accounts of mathematics. Hartry Field, for example, is clear in *Science without Numbers* (1980) and papers following (including Field (1984), Field (1989) about the requirement to include primitive modal operators in his fictionalist account of mathematics, endorsing the considerations in favour of primitive modality outlined by Georg Kreisel (1967). Likewise I defend the use of such operators in my own version of 'easy road' nominalism in Leng (2007) arguing that attempts to reduce modal claims to truths about set theoretic models fail since it is modal facts themselves that determine whether a set theoretic reduction is adequate. There will of course be those who will worry that the nominalist appeal to modality is problematic—perhaps because we often make use of the mathematical machinery of model theory to discover modal truths, or because they think that modal truths *just are* truths about set theoretic models. However, to the extent that these worries about the modal commitments of fictionalism arise, they arise already independently of the issue of the use of modal truths (about what follows from structure-characterizing axioms) in structural explanations. So this modal element of the mathematical explanations we have considered raises no new problem for mathematical fictionalists.

4 From structural explanations to structural model explanations

Are most, or even many, mathematical explanations of physical phenomena best understood as structural explanations, explaining by showing that their target system is an instance of a mathematical structure? I have argued that the cicada explanation can be understood as a structural explanation, where the axioms of number theory are interpreted as truths about the system of years consisting of some initial years in which cicadas appear, with the 'successor' relation being the 'the following year' relation. And I have presented a very simple example of a structural explanation involving group theory and a relevant difference between features of the cyclic group of order 4 and the Klein 4-group. In the first of these examples we had to introduce an element of idealization to allow the axioms to be interpreted as truths: we had to assume that the sequence of Earth years continued without end. (This idealization though false, was innocuous enough given that it was only behaviour at a finite initial segment that was needed for the explanation.) The second example required no idealization, but was admittedly rather simple, as is the explanation in the example given of the Königsberg bridges. It is difficult to find many serious examples of genuine structural explanations of this sort in empirical science (though group theory is a powerful tool in chemistry when applied to symmetries of molecules). While for some finite mathematical structures we can find physical instantiations, these structures are often so simple that any empirical phenomena we might try to explain with reference to the structure will be independently obvious already (as, arguably, is the case with the mattress-flipping explanation). And even for finite mathematical structures, these may be more clearly instantiated in idealized models of empirical phenomena rather than directly. Furthermore, most mathematical structures are not finite, and so may not have a physical instantiation (or are at best approximately instantiated as in the case of modelling years using the natural numbers or, as, for example, when we consider localized physical space to be an instantiation of Euclidean geometry). While our scientific theories are mathematical to the core, where complex mathematical structures, and sophisticated, genuinely informative explanations, are involved in these theories it is generally the case that much work needs to be done to fit the mathematics to physical reality. Simple instantiation of a structure is rare. More often a process of modelling must occur in order to bring the phenomena into contact with mathematical theory. Thus, most interesting structural explanations in mathematics will take the form of what, again following Bokulich (2008, p. 147), I will call *structural model explanations*, explanations where a mathematical structure is instantiated not in a physical system but in an idealized model of that system.

4.1 Mathematical explanations as structural model explanations

The most basic form of a structural model explanation explains by hypothesizing a model instantiating the structure of a given mathematical theory, showing that some facts about that model are true in virtue of that structure, and then relating that model to some empirical phenomenon to be explained by means of the model. The 'modelling' relation is as ever a complex one: it may involve resemblance, approximation, or perhaps more formal mappings (isomorphisms, homomorphisms), and will very often involve viewing the phenomenon modelled as related not to the whole structure in the model, but to some smaller substructure embedded in that model. The modelling relation may be described formally by means of a partial structures approach, as developed, e.g., by Bueno, French and Ladyman (see, e.g., French (2000), Bueno et al.(2002)), though we may find that the ultimate tie of model to modelled will be looser than can be formally characterized by such a theory. Indeed, it is likely that the formal framework of partial structures may only apply after a degree of modelling and idealization has already occurred, so as to exhibit relations between various models of increasing abstraction, rather than model and reality itself, which may be tied by fundamentally informal links (a possibility acknowledged by defenders of the partial structures view). Pincock (2012) provides a book-length study of the use of mathematical models to represent physical phenomena. Rather than attempt a full discussion of this complex issue how idealized models represent, though, let us simply assume that where models are aptly chosen, we can draw inferences about the real systems they represent. In such cases, we may ask, how can we transfer explanatory considerations from model to system modelled?

Given an axiomatically characterized mathematical theory, then, we can imagine that that theory is instantiated in some model system. Suppose that this model system has a subsystem that is held to be 'apt'—to be appropriately related by some informal 'representation' relation to an empirical system we wish to investigate (often, this subsystem will be the model system itself). The relation between the subsystem of our mathematical model and the empirical system under investigation may of course be mediated by further models—e.g., we may first need to abstract an idealized system from the empirical system we wish to consider, and then show that this idealized system bears appropriate structural relations to the subsystem of our original mathematical theory. But without going into the complexities of how such a relation may be established (and therefore of how it is the subsystem in the model is held to aptly represent the system modelled), let us simply assume for now that the subsystem of our model that instantiates a mathematical structure is indeed held to be true (enough) to the empirical system under investigation. Suppose now that we show some properties to be true in this subsystem of our mathematical model simply in virtue of the structure it instantiates. And suppose that these properties of the subsystem are seen to correspond (via our loose modelling relation) to empirical properties observed in the empirical system under investigation. Then an explanation of these empirical properties may be that they hold of the empirical system in virtue of its mathematical structure: given that the empirical system is well modelled by a subsystem of the larger mathematical structure, the empirical phenomena observed were to be expected, as (interpreted) consequences of the axioms characterizing that structure.

An example will help us to understand the processes at work in this loose sketch. Given that there is wind at all, at any point in time there will be at least one point on the earth's surface with no wind. Why is this? The 'Hairy Ball Theorem' from topology provides an explanation. In order to give this explanation we must start by preparing the empirical phenomena (wind patterns) for mathematical description. This involves some idealization. First, we forget the fact that the layer of air in the atmosphere above the earth's surface has depth, and that wind movements are different at different depths. Instead we think of a single layer of air on the earth's surface, with no depth. We then need to think about the direction and strength of wind at various points on the earth's surface. Over a large scale, we can measure prevailing wind direction at a point by a weather vane centred on that point—this gives us a horizontal direction of the wind as a 2-dimensional tangent to the earth's surface at that point (e.g., as coming from a North North Easterly direction). We can also measure its speed/strength at that point by means of a spinning anemometer (measuring the number of rotations of spinning cups in a given time period). To each point where measuring instruments are located, our measurements therefore allow us to associate a direction and a magnitude: or, in mathematical terms, a vector (measured using an appropriate measurement scale), where the direction of each vector is always at a tangent to the earth's surface. Extrapolating this to the small scale, we can think of wind direction as being defined at each point on the earth's surface by a tangent vector. Furthermore, we can assume that changes to the direction and magnitude of these vectors as we move across the earth's surface are continuous. Thus we can think of the essential features of the wind as being represented by a continuous function corresponding points on the earth's surface to vectors.

This 'prepared description' (to use Nancy Cartwright's terminology (Cartwright, 1983, p. 15)) of wind behaviour, achieved by a number of idealizations as well as by applying a mathematical measurement scale[8], enables us to see the phenomenon of wind movement in a wider mathematical perspective. In particular, we have described the wind as a tangential vector field on the surface of the Earth. If we now take that surface to be topologically equivalent to a sphere (ignoring inconvenient tunnels) then we can apply a theorem of topology to the wind system we have described. According to the 'Hairy Ball Theorem' of topology, "there does not exist an everywhere nonzero tangent vector field on the 2-sphere $\mathbf{S}^2$" (Weisstein, web resource). So given that there is any wind at all on the earth's surface (so that the vector field in our model is not everywhere zero), there must be some point on the earth's surface where the value of the tangent vector representing the wind speed and direction is zero (i.e., there is no wind at that point). What happens in the area around a point with zero wind? Well, wind cannot flow in or out of that point as we are hypothesizing that wind speed and direction is zero there, so while there may still be zero wind at adjacent points, when the wind in the vicinity of this zero point is non-zero once more it must initially spiral around the no-wind area, as in a cyclone. To envisage what is going on we may imagine the hairy ball of the theorem's title. What the theorem tells us is that one cannot continuously comb a hairy ball flat—the best we can do is to create a cowlick with a hair sticking straight up in the middle and adjacent hairs circling around. (The flat hairs represent non-zero tangent vectors; the hair that remains sticking up would of course no longer be a tangent vector, so in order for the combing to remain a continuous tangential vector field it must be zeroed.)

We have, therefore, a mathematical explanation of an empirical phenomenon (the inevitable occurrence of certain wind patterns)[9]. The explanation is a model explanation: we do not apply the mathematics directly to the empirical phenomenon, but first prepare the phenomenon for mathematical description through a process of

[8] Field (1980) explains how the use of mathematics in such measurements can be dispensed with. Without actually following Field in dispensing with this use of mathematics, Field's machinery should convince us that our initial use of real numbers in measurement are merely a means of quantitatively representing qualitative differences between wind strength and direction at various points: there is some nominalistic content to these measurements, even though they are mathematically indexed. Beyond this measurement step, though, I wish to suggest that the subsequent use of a mathematical theory to model wind behaviour so-measured is an essential explanatory use whose value does not solely reside in its representational content.

[9] In fact, as Alan Baker (2005) has pointed out, examples such as these are somewhat tenuous, since the phenomenon to be explained is not one that has been independently noticed or even verified: it is more a prediction of the mathematics than a previously noted puzzle crying out for explanation. Nevertheless, since I am presuming for the purposes of this paper that there are some genuine mathematical explanations of empirical phenomena, rather than trying to establish the existence of genuine examples, I have chosen to stick with this example for its relative simplicity. It provides, at least, an explanation in the sense that, had the phenomenon been noted prior to the mathematical prediction, it would have explained that phenomenon.

idealization and abstraction. Our prepared description of wind on the earth's surface as a tangent vector field enables us to apply the resources of topology to this description and so to derive a conclusion about the properties of this vector field. And returning to the original phenomenon modelled, we are able to give a physical interpretation of this conclusion, stating what this should means we should expect about actual wind behaviour. This explains actual wind behaviour structurally, to the extent that it is shown to be a consequence of the mathematical structure of the physical system, so the explanation is a structural model explanation.

Must a structural model explanation to be true in order to explain? In Bokulich's discussion of such explanations, she focuses on structural explanations where the model is a classical system and the phenomena to be explained are quantum phenomena, so that the mathematical structure appealed to in the structural explanation, using the mathematical of classical mechanics rather than quantum mechanics, is not straightforwardly 'true of' the system to be modelled. But whether, in these proposed structural model explanations, the models are close enough in structural terms to the phenomena modelled to be genuinely explanatory is something that need not immediately concern us here. In all model explanations, the issue of how close a model must relate to the phenomenon modelled (and therefore of how 'true' the model is to the phenomenon it models—or, we may say, how 'apt' the model is) in order to be explanatory is a complex and contentious matter, but this is not the sense of truth that matters for the purposes of the metaphysical question of what status should be given to models in model explanations. What we need to ask is, in cases where we do think that the model in a structural model explanation is *close enough* to the system modelled in relevant respects to be genuinely explanatory, must we accept the existence of the model system itself in order to accept this explanation as genuine? What is the metaphysical status of the idealized models appealed to in structural model explanations, and in particular, does commitment to the existence of mathematical explanations as structural model explanations incur an undesirable commitment to abstracta?

There is a strong tradition in discussions of models in the philosophy of science of thinking of models as merely imaginary objects. For example, Peter Godrey-Smith (2009, p. 102) characterises "model-based" science as.

> a style of theoretical work in which an imaginary system is introduced and investigated—an imaginary population, ecology, neural network, stock market, or society. The behavior of the imaginary system is explored, and this is used as the basis for an understanding of more complex real-world systems.

This suggests a picture of model building as analogous with storytelling: although we appear to speak as if the objects in our models really exist, we are actually just telling a story that fleshes out the supposition that there are such things, without commitment to the truth of that supposition. (Similarly, with theorizing in the context of a pure mathematical theory, we can view theorists as working out the consequences of the supposition that the axioms of that theory are true, without any commitment to the actual truth of those axioms.) The literal truth of statements uttered in describing the models used in model explanations may not be required for those explanations to be explanatory if all we are doing by uttering those statements is elaborating on what *would* be the case *were* there objects of the sort described.

I do not have the space here develop a nominalist understanding of ideal models in empirical science. In Leng (2010) I endorse an account of models as representations that builds on Kendall Walton's (1990) account of representation as "prop oriented make-believe", an account that has been developed further in the modelling literature in various ways e.g. by Frigg (2010), Toon (2010) and Salis (2019). To the extent that a fictionalist account of models in science can be defended, the idealised models in 'structural model explanations' do not need to exist in order to be utilised in explaining physical phenomenon.

What I would like to suggest, then, is that whether we really *believe* that there are abstract systems of ideal objects instantiating an axiomatically characterized mathematical structure and appropriately related to a system of physical objects, or merely *pretend* that there are such things, makes no difference to our ability to exploit the 'framing effect' of a structural model explanation in enabling us to see empirical systems as essentially mathematically structured. In mathematical theorizing we discover what would be true of any system instantiating the axioms (including in any subsystem of such a system), and in describing a model (whether real or imagined) that allows us to see a physical system as structurally related to a (real or imagined) subsystem of a system instantiating a mathematical structure, we are able to conclude that certain empirical interpretations of our mathematical results ought to be true of the physical system under discussion simply by virtue of its mathematical structure. The framing effect of seeing wind patterns on the earth's surface as modelled by a tangent vector field no more requires the existence of the mathematical system one imagines than does the framing effect of seeing those same wind patterns as modelled by a hairy ball that one is trying to comb flat requires the real existence of said ball. In either case the appropriateness of the imagined model allows us to frame facts about the system modelled as holding in virtue of its sharing the structure of the model system. And in either case, the explanatory work done by the model is that it shows us that those facts were to be expected given the structure of the situation modelled.

4.2 Explanatory models, or explanatory structures?

The introduction of idealized models that represent actual physical phenomenon into our account of explanation may introduce a new wrinkle, however, as it has sometimes been suggested that showing that phenomenon to be explained holds in a structurally similar model of a target system cannot suffice to explain that phenomenon. For example, James R. Brown argues that it is *because* mathematics generally finds application via enabling us to form tractable models of physical phenomena that mathematics cannot play a genuine explanatory role:

> Mathematics hooks on to the world by providing representations in the form of structurally similar models. The fact that it works this way means that it cannot explain physical facts, except in some derivative sense that is far removed from

the doctrines of explanation employed in indispensability arguments. (Brown, 2012, p. 8)

In Brown's view, the role that structural models have in our theories is to provide representations of physical systems. These may aid understanding, by being easier to work with than the systems themselves (and allowing us to use familiar descriptive tools). But providing this kind of aide to understanding is a *derivative sense* of explaining, and does not amount to mathematics playing a genuine explanatory role. Despite Brown's own platonism (which he holds on independent grounds), Brown's view of the role of mathematics in explanations is thus in line with those nominalists who hold that all the genuine explanatory work in so-called mathematical 'explanations' of empirical phenomena resides in the nominalistic content that the mathematics helps us to grasp.

By separating out the structural elements of structural model explanations from the model elements, we can see where Brown and others go wrong in this regard. Brown is absolutely right that, to the extent that the role played by mathematics is to provide tractable *models* of empirical phenomena, the ease of understanding that results from having such tractable models is only 'explanatory' in a derivative sense—the models may make the ultimately nominalistic features of the systems easier for us to grasp, but they are not playing an explanatory role in showing us why the phenomena we observe *had* to be true. However, focus on mathematics as providing models diverts attention from the structural features of these models, which is where their explanatory work resides. What makes a structural explanation explanatory is not just that it displays some ultimately nonmathematical content to be true, but rather that it displays that content to be true *in virtue of* the mathematical structure of the empirical system under investigation: it shows it to be a consequence of mathematical axioms that are true under an empirical interpretation. And what makes a structural *model* explanation explanatory is again not (just) that it displays some nonmathematical content to be true, but that it shows it to be true in virtue of the mathematical structure of the situation to be explained, by relating that situation in an appropriate manner to a model that instantiates (or is a subsystem of a system that instantiates) a given mathematical structure. Structural explanations of either sort show why the observed phenomenon had to happen or was to be expected given the mathematical structure of the empirical system under study. The nominalistic content of these explanations, on the other hand, do no such thing: the insight the mathematical structure provides is lost if we simply focus on the content of the true descriptive claim that the empirical situation is such as to make the models used in these explanations appropriate.

The added explanatory work done by representing empirical phenomena as essentially mathematically structured can make sense of a complaint that Otávio Bueno and Mark Colyvan have expressed about the limitations of 'mapping accounts' of the application of mathematics (such as that of Pincock (2004) and suggested in Leng (2005)). Mapping accounts try to explain the applicability of mathematics by noting that our mathematical theories and the physical world to which they apply are related by structural similarity relations, much like a map is related to a city. It is not surprising that we can learn things about a city from studying its map, given the structural similarity relation holding between the map and the city, and it is also not surprising that it is helpful for us in discussing the spatial arrangements of objects in a city in terms of the map rather than the city itself—it provides a useful simplification that can make navigation problems tractable and allow us to ignore the mass of irrelevant detail. But, Bueno and Colyvan (2011) note, it would be odd to think that the map of the city could by itself *explain* facts about the city (unless, perhaps, we discover that the map was the blueprint from which the city was built). More needs to be said in mapping accounts of applications to show how the mathematical theories we claim to be structurally related to the physical world can *explain* features of that world.

The difference between a mathematical theory and a road map in explanatory uses of mathematics is that, while both are models (and hence both are structurally similar to the reality they represent), only the mathematical theory involves a *structural model*, in the sense of representing the physical world as (approximating) an instantiation of (a substructure of) some wider mathematical structure. Rather than simply mirroring the structure of an empirical system (as in any model), a *structural model* represents that system as an instance of a mathematical structure. As such, it enables us to explain features of that system as holding *in virtue of* its mathematical structure, whenever they can be shown to be empirical interpretations of mathematical statements that are derivable from the structure's characterizing axioms. It is not mere similarity that matters in the use of mathematical models to explain, but rather, the realization that this similarity means that the inferential structure of our mathematical theories carries over to the empirical situation modelled, so that truths about the empirical situation can be seen as holding in virtue of its mathematical structure.

5 Conclusion

I have, in this paper, agreed with platonists including Baker and Colyvan that mathematics sometimes plays an indispensable explanatory role in empirical science, with mathematical hypotheses sometimes doing genuine explanatory (or at least, explanation-like) work that is not exhausted by the nominalistic content that those hypotheses enable us to represent. Nevertheless, I have argued, the involvement of mathematical hypotheses in these explanations does not support platonism. Mathematical hypotheses can play this kind of explanatory (or explanation-like) role even if there are no abstract mathematical objects, since the role mathematics plays in such explanations is of showing physical phenomena to be true in virtue of the mathematical structure instantiated (or approximately instantiated) in the physical system under study, rather than by appealing to abstract mathematical objects per se. In structural explanations, we have examples of distinctively mathematical explanations which show their explananda to hold by virtue of logical necessity given the mathematical structure instantiated in the physical system. When structure-characterising axioms are interpreted so as to be true of a particular physical system, we can generate mathematical explanations of physical phenomena that do not appeal to any abstract mathematical objects, but instead only require modal truths about what follows logically from our mathematical assumptions, together with the recognition that the assumptions of our mathematical theories are true when interpreted as about the physical system under examination.

Given, however, the amount of idealization that is generally required in order to apply mathematics to physical systems, most mathematical explanations of empirical phenomena will involve intermediate idealized models, rather than the direct instantiation of mathematical structures in physical systems, where what is directly explained by these structural explanations is features of an idealized model that instantiates our mathematical axioms. The presence of these models as intermediaries may raise concerns that such explanations are committed to abstract mathematical objects, or at the very least, to abstract idealizations of physical objects whose status is arguably as questionable as the abstract mathematical objects that mathematical fictionalists try to avoid. I have suggested that when idealized models are introduced into explanations, mathematical theories are used to provide structural explanations of the features of these models, which can then be used to explain features of the physical systems they model to the extent that the models provide apt representations. And I have proposed an understanding of idealized models in science that views them as a form of 'make-believe' or pretense. In such cases, when we speak of similarities between the physical system and the model, we are indirectly asserting that the physical system is the way *it* would have to be to make the pretense appropriate. So we can speak *as if* there are objects as imagined in our idealized theoretical models in order to represent how things are taken to be with the physical systems with which we are ultimately concerned. The models in structural model explanations need not, then, present any new worry the nominalist who takes it that fictions can be used to represent without the objects of fictions existing.

What this paper has not, of course, established is that *all* explanatory or explanation-like uses of mathematical hypotheses occur in the context of structural model explanations. I have not argued (and indeed I do not hold) that all explanation is structural explanation, and it is at least conceivable that examples could be found where mathematics plays a genuine explanatory role but where the explanation given is not structural. However, the dual role of mathematics identified in this discussion—as providing amenable, abstracted *models* of reality that are easy to work with, and as identifying features of those models that hold in virtue of *structure-characterizing mathematical axioms*, seems to me at least to get at some of the key elements of what it is so special about mathematics that makes it such a useful tool in both describing and explaining empirical phenomena. If there are other features of mathematics in application that have been overlooked, and that can be exploited to show mathematics to be explanatory in other ways, I would certainly be keen to hear of them. But what I hope to have shown in this paper is that there is a gap between showing that mathematics can play an indispensable explanatory (or explanation-like) role and showing that the existence of mathematical objects (or the truth of our mathematical theories) is required for mathematics to play such a role. We should not automatically infer, of the best explanation we have of a phenomena, that it is *true*, but only that it does indeed *explain*. The question then needs to be asked, "How does it explain?", and it is in the details of answering this question that we may hope to uncover the metaphysical commitments of our taking the explanation to be explanatory.

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