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ORIGINAL RESEARCH

Screening off generalized: Reichenbach's legacy

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Abstract
Eells and Sober proved in 1983 that screening off is a sufficient condition for the transitivity of probabilistic causality, and in 2003 Shogenji noted that the same goes for probabilistic support. We start this paper by conjecturing that Hans Reichenbach may have been aware of this fact. Then we consider the work of Suppes and Roche, who demonstrated in 1986 and 2012 respectively that screening off can be generalized, while still being sufficient for transitivity. We point out an interesting difference between Reichenbach's screening off and the generalized version, which we illustrate with an example about haemophilia among the descendants of Queen Victoria. Finally, we embark on a further generalization: we develop a still weaker condition, one that can be made as weak as one wishes.

Keywords Probabilistic support · Transitivity · Screening off · Reichenbach · Sufficient conditions

1 Introduction

In their instructive entry on Hans Reichenbach in the *Stanford Encyclopedia of Philosophy*, Clark Glymour and Frederick Eberhardt note that "the fruits of some of [Reichenbach's] insights are only belatedly having their full impact"; in addition they observe that some of these insights have re-emerged in recent philosophy "without notice of the connection" to Reichenbach's work.[1] As an example of the former they mention contemporary ideas about causality that can be traced back to Reichenbach's

[1] Glymour and Eberhardt (2008/2016), p. 2, p. 56.

This article belong to the topical collection on All things Reichenbach, edited by Erik Curiel and Flavia Padovani.

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The Direction of Time.[2] They find an example of the latter in the work by Michael Strevens and Harman and Kulkarni:

> Michael Strevens' *Bigger Than Chaos* (Strevens 2003) reprises the views and arguments of Reichenbach's doctoral thesis without the Kantian gloss. Gil Harman and Sanjeev Kulkarni's *Reliable Reasoning* (2007) adopts a view of induction very close to Reichenbach's.[3]

In the present paper we argue that Reichenbach's thoughts about so-called screening off provide another example of Glymour and Eberhardt's conclusions. For first, the idea that screening off guarantees transitivity of probabilistic support reappeared in recent philosophy without mention of the connection to Reichenbach's work. And second, new findings illustrate that the fruits of this idea are only belatedly having their full impact.

Here is how we propose to proceed. In Sect. 2 we recall screening off as it was defined by Reichenbach. We bring to mind the proof of Eells and Sober (1983) that screening off is a sufficient condition for the transitivity of probabilistic causality, and we recall Tomoji Shogenji's argument that the same applies to probabilistic support (Shogenji 2003). Neither Eells and Sober nor Shogenji refer to Reichenbach, but we conjecture that Reichenbach may have been aware of the result, given that he provided its mathematical backing.

In Sect. 3 we turn to new findings. We introduce what we call 'generalized screening off', by which we mean the weaker sufficient condition for transitivity described by William Roche in (2012), preceded by Patrick Suppes in (1986). We explain that generalized screening off has an interesting property that Reichenbach's original screening off lacks: if p supports q and q supports r, then r might be more strongly supported by p than it is by q. In Sect. 4 we illustrate this possibility by means of a historical example concerning the transmission of haemophilia in the British royal family. In Sect. 5 we carry the generalization of Roche and Suppes further, developing a sufficient condition for transitivity that can be made arbitrarily weak. Somewhat surprisingly, as we show in Sect. 6, this new condition can occur in conjunction with the Simpson paradox, and we explain that the treatment of patients suffering from kidney stones, as described in Julious and Mullee (1994), exhibits this possibility.

2 Reichenbach's screening off

Let p, q and r be three propositions. We say that p supports q and that q supports r probabilistically, if the following inequalities hold:

$$P(q|p) > P(q) \quad \text{and} \quad P(r|q) > P(r)\,. \tag{1}$$

[2] Glymour and Eberhardt (2008/2016), p. 2.

[3] Ibid., p. 56.

For future use, we note that (1) can be written in two equivalent forms:

$$P(q \wedge p) > P(q)P(p) \quad \text{and} \quad P(r \wedge q) > P(r)P(q) \tag{2}$$

or alternatively

$$P(q|p) > P(q|\neg p) \quad \text{and} \quad P(r|q) > P(r|\neg q)\,, \tag{3}$$

on condition that $P(p)$ and $P(q)$ are regular probabilities, i.e. the values 0 and 1 are excluded. In general probabilistic support is not transitive. It does not follow from (1) that p supports r:

$$P(r|p) > P(r)\,, \tag{4}$$

or equivalently $P(r \wedge p) > P(r)P(p)$ or $P(r|p) > P(r|\neg p)$.

In his posthumous book *The Direction of Time* Hans Reichenbach coined the term 'screening off' to describe a particular kind of probabilistic relation.[4] We say that q screens off p from r if and only if

$$P(r|q \wedge p) = P(r|q) \quad \text{and} \quad P(r|\neg q \wedge p) = P(r|\neg q)\,. \tag{5}$$

Tomoji Shogenji showed in 2003 that under (5) probabilistic support *is* transitive: with screening off, (4) follows from (1). A similar argument had been given earlier by Ellery Eells and Elliott Sober, who proved that probabilistic causality is transitive under screening off, calling it a Markov condition.[5]

Neither Shogenji nor Eells and Sober refer to Reichenbach. However, it can be argued that the sufficiency of screening off for the transitivity of probabilistic support was already demonstrated by Reichenbach himself. On page 160 of *The Direction of Time* Reichenbach considers two events, we call them p and r, which have a common cause, q. He shows that, if q supports p and q supports r, and moreover q screens off p from r, then it follows that p supports r. On page 189 Reichenbach then considers, instead of a common cause, a linear chain in which p supports q and q supports r. If q screens off p from r, then the same mathematics that Reichenbach uses on page 160 shows again that p supports r. Here is Reichenbach's reasoning in more detail.

Let q be the common cause of both p and r, which implies that q supports p and q supports r. Reichenbach then shows that under the constraint (5) the following relation holds:

$$P(p \wedge r) - P(p)P(r) = P(q)[1 - P(q)][P(p|q) - P(p|\neg q)][P(r|q) - P(r|\neg q)]\,, \tag{6}$$

[4] Reichenbach (1956/1999), p. 189.

[5] Shogenji (2003), footnote 2; Eells and Sober (1983). In general, probabilistic causality and probabilistic support are formalized in the same way, and in this paper we will treat them as being on a par.

where he explicitly assumes that the unconditional probabilities are regular.[6]

Since q is the common cause of p and r, both $P(p|q) - P(p|\neg q)$ and $P(r|q) - P(r|\neg q)$ are positive. The unconditional probabilities $P(q)$ and $1 - P(q)$ are also positive, so the right side of (6) is positive. Therefore the left side is positive, which means that $P(p \wedge r) > P(p)P(r)$, or equivalently $P(r|p) > P(r|\neg p)$, i.e. p supports r. As we explain in detail in Appendix A, (6) is equivalent to

$$P(r|p) - P(r|\neg p) = [P(q|p) - P(q|\neg p)]\,[P(r|q) - P(r|\neg q)]\,, \qquad (7)$$

which implies that, under screening off, if p supports q, and q supports r, then p supports r.

On the relevant pages Reichenbach does not use the word ‘transitive’. Rather he speaks about the ‘relation of causally between’, by which he means that p probabilistically causes r through the intermediary q.[7] Yet it seems to us that his mathematical argument makes the transitivity of probabilistic support under screening off rather clear. We therefore conjecture that Reichenbach did realise that screening off is a sufficient condition for the transitivity of probabilistic causality, and thus of probabilistic support. If this conjecture is correct, it provides still another example of Glymour and Eberhardt’s observation that some of Reichenbach’s ideas “have reemerged in recent philosophy without notice of the connection”. The rest of this paper shows that the idea of screening off also illustrates their other observation, namely that “the fruits of some of [Reichenbach’s] insights are only belatedly having their full impact”.

3 Generalized screening off

An interesting generalization of screening off was offered by William Roche in (2012). Unbeknownst to Roche, this generalization had already been obtained by Patrick Suppes in (1986).[8] Instead of (5), which is Reichenbach’s condition of screening off, Roche and Suppes require only

$$P(r|q \wedge p) \geq P(r|q) \quad \text{and} \quad P(r|\neg q \wedge p) \geq P(r|\neg q)\,, \qquad (8)$$

where the equals signs have been replaced by inequalities. Although (8) is weaker than (5), it nevertheless entails the transitivity of probabilistic support.[9]

In order to explain how exactly (8) entails the transitivity of probabilistic support, we make use of a paper that Shogenji published in 2017.[10] There Shogenji explains

[6] Reichenbach’s actual formula is $w - ab = c(1-c)(u-r)(v-s)$, see Reichenbach (1956/1999), p. 160. On substituting the definitions that he gives of these variables, and adapting them to modern notation, the formula takes on the form (6).

[7] Reichenbach (1956/1999), pp. 188–190.

[8] Roche (2012); Suppes (1986). We thank Bill Roche for having drawn our attention to Suppes’ paper.

[9] The inequalities (8) generalize Reichenbach’s screening off condition in the context of probabilistic support, but there have been other generalizations in other contexts, notably in that of the common cause principle. See Hitchcock and Rédei (2020); Hofer-Szabó, Rédei and Szabó (2013); Mazzola (2019); Mazzola et al. (2020); Wronski (2014). Thanks to an anonymous reviewer for having made this point.

[10] Shogenji (2017).

what he calls "mediated confirmation", and identifies the conditions for transitivity of probabilistic support in various settings. In doing so he derives a very insightful identity. We will use a modified version of this identity to show not only how (8) guarantees transitivity, but also (in Sect. 5) how an even weaker constraint than (8) can do the job. In Appendix A we modify and prove Shogenji's identity in a way that is better tailored to our needs than is Shogenji's original expression. Our version deals directly with $P(r|p) - P(r|\neg p)$, rather than $P(r|p) - P(r)$, and is as follows:

$$P(r|p) - P(r|\neg p) = \kappa(p, r; q) + \kappa(p, r; \neg q) + \sigma(p, r; q)\,, \tag{9}$$

where

$$\begin{aligned}
\kappa(p, r; q) &= \big[P(r|q \wedge p) - P(r|q \wedge \neg p)\big]\frac{P(q|p)P(q|\neg p)}{P(q)} \\
\kappa(p, r; \neg q) &= \big[P(r|\neg q \wedge p) - P(r|\neg q \wedge \neg p)\big]\frac{P(\neg q|p)P(\neg q|\neg p)}{P(\neg q)} \\
\sigma(p, r; q) &= \big[P(r|q) - P(r|\neg q)\big]\big[P(q|p) - P(q|\neg p)\big]\,.
\end{aligned} \tag{10}$$

(9) is an identity, i.e. it is valid for any propositions p, q and r, whether or not there is screening off, and whether or not there is any probabilistic support.[11]

The three terms in (9) illustrate that screening off allows degrees, and they also show how exactly the support that p gives to r (via q) is helped or hindered. While $\kappa(p, r; q)$ is a measure of the degree to which r is screened from p by q, the term $\kappa(p, r; \neg q)$ measures the degree to which r is screened from p by $\neg q$. The last term, $\sigma(p, r; q)$, is the degree of support that p gives to r through q, which is helped or hampered by the positivity or the negativity of the kappa's.

If there is screening off *à la* Reichenbach, that is if q screens off p from r in the Reichenbachian way, then (5) holds, and therefore

$$\kappa(p, r; q) = 0 \quad \text{and} \quad \kappa(p, r; \neg q) = 0\,.$$

In this case (9) reduces to

$$P(r|p) - P(r|\neg p) = \sigma(p, r; q)\,. \tag{11}$$

From the definition of $\sigma(p, r; q)$ we see that it has to be positive if p supports q and q supports r. Thus the transitivity of support in the presence of screening off is a special case of the Shogenji identity.

However, (9) tells us more. For from (9) it can be shown that probabilistic support is also transitive under the generalized screening off of Roche and Suppes. The weakened constraint (8) is equivalent to

$$P(r|q \wedge p) \geq P(r|q \wedge \neg p) \quad \text{and} \quad P(r|\neg q \wedge p) \geq P(r|\neg q \wedge \neg p)\,, \tag{12}$$

[11] On condition that all the conditional probabilities are well defined.

which means that

$$\kappa(p, r; q) \geq 0 \quad \text{and} \quad \kappa(p, r; \neg q) \geq 0\,. \tag{13}$$

Under (1) $\sigma(p, r; q)$ is positive (and not simply non-negative), so we see from (13) that the sum of the three terms on the right of (9) is positive, and thus the left-hand side of (9) is positive too, which means that p supports r. This shows that the Roche-Suppes condition is indeed sufficient for probabilistic transitivity.

There is an interesting difference between Reichenbach's screening off and Roche's generalized variant. Under the former, $P(r|p) \leq P(r|q)$. That is to say, although probabilistic support is transitive, it is in general subject to a reduction in strength. For under ordinary or Reichenbachian screening off, we can show from (11) that

$$P(r|p) - P(r) = \frac{P(q|p) - P(q)}{1 - P(q)} \Big(P(r|q) - P(r) \Big)\,.$$

Since $P(q|p) \leq 1$, it follows that $\frac{P(q|p)-P(q)}{1-P(q)} \leq 1$, the equality holding only if $P(q|p) = 1$. Therefore, given that q supports r, it is the case that $P(r|p) - P(r) \leq P(r|q) - P(r)$, so $P(r|p) \leq P(r|q)$. However this decrease in probabilistic support is no longer necessary under generalized screening off. There the direction may even be reversed, for it can happen that there is sufficient 'leakage' to ensure that $P(r|p) > P(r|q)$. In the next section we give a historical example of this phenomenon.

4 The royal disease

As is well known, sperm cells and unfertilized egg cells contain only one DNA strand, which includes the X chromosome for the latter and either X or Y for the former. It is only at fertilization that sex is determined, XX being the double DNA helix of females and XY that of males.[12]

A gene is a sequence of nucleic acids in the DNA that codes for a particular phenotypical property, such as eye colour. In some cases the gene has two versions or *alleles*: a dominant one, A, and a recessive allele, a.[13] Thus there are three possible combinations:

- AA, in which the dominant allele occurs in both DNA strands, the one from the mother and the other from the father;
- Aa, in which the dominant allele comes from one parent and the recessive allele from the other;
- aa, in which both parents contribute a recessive allele.

Most genes are not sex-linked — familiar examples are the genes that correspond to eye colour. However, some genes *are* sex-linked. A case in point is the gene associated with haemophilia (by which we mean haemophilia B).

[12] In the following we shall speak of an X strand to mean a strand of DNA that contains the X chromosome, and a Y strand to mean a strand of DNA that contains the Y chromosome.

[13] In practice many properties are controlled by more than one gene; but for simplicity we ignore this complication. The sickness of haemophilia that we will discuss at length *is* related to just one gene.

Haemophilia, also called ‘the royal disease’, is associated with a gene that occurs only in the X strand, but never in the Y strand. The dominant allele assures the normal coagulation of blood, which is important in the healing of wounds. The recessive allele, however, does not: no coagulating agent is coded by it. A female with two A alleles in her X strands, or one A and one a allele, will experience normal blood coagulation. Only if she has a in both strands will she be a haemophiliac, and this is uncommon, since the recessive allele is very rare. For a male the situation is more risky. If he has the recessive allele in his X strand, he will be a haemophiliac, for the Y strand has no corresponding gene, and so there is no possibility to compensate for the malfunctioning a allele.

In this case of the gene associated with blood coagulation, let us call an AA female *normal*, an Aa female a *carrier*, and an aa female a *haemophiliac*. Males come in only two variants: normal if the X strand contains A, or haemophiliac if the X strand contains a. A carrier mother and a normal father will have daughters who are not haemophiliacs; they will be either normal or carriers. However on average one half of the carriers’ sons will be haemophiliacs. On the other hand a normal mother and a haemophiliac father will have normal sons, but their daughters will all be carriers, so on average one half of the daughters’ sons will be haemophiliac, as we have seen. This is the origin of the adage ‘haemophilia skips a generation’: although the sons of haemophiliac fathers are healthy (assuming the mothers are normal), the daughters are all carriers, so *their* sons are at risk. Table 1 gives an overview of all the (Boolean) logical relations between parents and their children having haemophilia.

Table 1 Transmission of haemophilia

father	mother	sons	daughters
normal	normal	normal	normal
haemo.	normal	normal	carriers
normal	carrier	normal or haemo.	normal or carrier
haemo.	carrier	normal or haemo.	carrier or haemo.
normal	haemo.	haemo.	carriers
haemo.	haemo.	haemo.	haemo.

To illustrate the fact that probabilistic support can increase under generalized screening off, let us look at a famous historical case in which the illness manifested itself. It is well documented that Queen Victoria was a carrier of the recessive allele associated with haemophilia: she was of genetic type Aa.[14] Victoria’s eighth child and fourth son, Leopold, inherited the fatal allele a from his mother: he was a haemophiliac. He died from cerebral bleeding at the age of thirty, after a fall down some stairs in the south of France, but not before siring Princess Alice. Alice carried the ominous allele and transmitted it to her son, Rupert, who was a haemophiliac. On the first of April,

[14] It is not known how she acquired it. Perhaps it was a spontaneous mutation of A into a in the royal ovaries, or perhaps in the semen of her father, Prince Edward Duke of Kent. Her mother, Victoria Duchess of Kent, is thought not to have been a carrier; but of course she might have had an extramarital affair with a haemophiliac. Such scurrilous possibilities are however as improbable as they are improper, and we will not pursue the matter further.

1928, Rupert was driving from Paris to Lyon when he tried to overtake another vehicle and crashed into a tree. He died soon afterward of cerebral bleeding, as his grandfather Leopold had done before him.

Let L^h and R^h stand for 'Leopold is a haemophiliac' and 'Rupert is a haemophiliac', respectively; and A^n, A^c and A^h stand for 'Alice is normal', 'Alice is a carrier' and 'Alice is a haemophiliac', respectively. Then $L^h \wedge A^n$ and $A^n \wedge R^h$ are impossible; that is, it is excluded that Alice could be normal if her father were a haemophiliac, and it is also excluded that Rudolf could be a haemophiliac if his mother were normal.

The probability that Rupert is a haemophiliac, given that his mother Alice is a carrier, is the chance that she passes on the recessive allele, a, to her son, and that is one half. This probability is not changed by adding the condition that Alice's father, Leopold, is a haemophiliac or is not a haemophiliac. Thus $P(R^h|A^c \wedge L^h) = P(R^h|A^c \wedge \neg L^h) = \frac{1}{2}$, and this entails $\kappa(L^h, R^h; A^c) = 0$, as we immediately see from (10). On the other hand $\neg A^c \wedge L^h = A^h \wedge L^h$, since it is impossible that Alice could be normal, given that her father was a haemophiliac. Therefore $P(R^h|\neg A^c \wedge L^h) = P(R^h|A^h \wedge L^h)$, and under the condition that his mother is a haemophiliac, it is certain that Rupert will inherit the fatal allele, so $P(R^h|\neg A^c \wedge L^h) = 1$. However, $\neg A^c \wedge \neg L^h = A^n \wedge L^n$, since if Leopold is not haemophiliac, he is normal, and then his daughter cannot be a haemophiliac. So $P(R^h|\neg A^c \wedge \neg L^h) = P(R^h|A^n \wedge L^n) = 0$, for it is impossible for a normal mother to bear a haemophiliac son. Thus $P(R^h|\neg A^c \wedge L^h) - P(R^h|\neg A^c \wedge \neg L^h) = 1$, which entails $\kappa(L^h, R^h; \neg A^c) > 0$, as we again see from (10). Thus A^c does not screen off R^h from L^h in the ordinary sense, but it does so in the generalized sense of Sect. 3. The possibility arises therefore that Leopold's haemophilia makes Rupert's haemophilia more likely than Alice's being a carrier does. Such is indeed the case, as we show in detail in Appendix B.[15]

Detailed calculations are actually not needed in order to understand why Rupert's haemophilia is made more likely by Leopold being a haemophiliac than by his mother being a carrier. For if we know that Alice is a carrier, then there is an even chance of Rupert having haemophilia; but if the only thing we know is that Leopold has haemophilia, then Alice may be either a carrier or a haemophiliac, and in the latter case Rupert would necessarily be a haemophiliac. This means that Rupert's being a haemophiliac is made more likely by Leopold's being a haemophiliac than it is by Alice's being a carrier: $P(R^h|L^h) > P(R^h|A^c)$.[16]

[15] This is distinct from the question as to which of the two data—Leopold's or Alice's genetic condition—gives more support to the thesis that Rupert is a haemophiliac. The problem with this question is that there are many different measures of probabilistic support, and they are not all ordinally equivalent to one another. By simply asking which conditional probability, $P(R^h|L^h)$ or $P(R^h|A^c)$, is the greater, we avoid this source of ambiguity.

[16] At this juncture, the reader might have asked herself why we do not use the language of Directed Acyclic Graphs to make the point. After all, DAGs were designed to handle problems involving probabilistic dependencies. Since at the heart of a DAG is the Markov condition, of which screening off *à la* Reichenbach is a special case, it is clear that the linear graph $L^h \longrightarrow A^c \longrightarrow R^h$, in which A^c screens off R^h from L^h in the generalized manner of Roche (and not in that of Reichenbach), is *not* a DAG. Nevertheless, by treating Alice as a three-variable node (n, c, h), rather than as a two-variable one $(c, \neg c)$, we *do* obtain a common or garden DAG. So why not use them? The answer is that the DAGs do not tell us the interesting fact that $P(R^h|L^h) > P(R^h|A^c)$ without a detailed calculation that is in fact much the same as the one given in Appendix B. So although the language of DAGs could have been used, it does not shed new light on this particular problem.

5 A further generalization

The Suppes-Roche condition (8) is weaker, and therefore more general, than Reichenbach's condition (5). Can it be relaxed even further without losing the transitivity? That is, can it be further generalized while maintaining the positivity of $P(r|p) - P(r|\neg p)$? Indeed it can.[17] For we know that $\sigma(p, r; q)$ is positive, since we have assumed that p supports q and q supports r. Moreover the right-hand side of (9) could still be positive even if either $\kappa(p, r; q)$ or $\kappa(p, r; \neg q)$ were negative. It would be enough if the other were positive and sufficiently great to ensure that

$$\kappa(p, r; q) + \kappa(p, r; \neg q) \geq 0\,. \tag{14}$$

This is a weaker condition than (13), for the latter requires that $\kappa(p, r; q)$ and $\kappa(p, r; \neg q)$ be both non-negative, but (14) only requires that the *sum* of the two be non-negative. From the Shogenji identity (9) we see immediately that (14) is sufficient to guarantee the transitivity of probabilistic support.

Is (14) the weakest possible sufficient condition? One would like to find the weakest possible sufficient condition, say C, such that, if p supports q and q supports r, then C implies that p supports r:

$$\Big\{\big[P(q|p) > P(q|\neg p)\big]\&\big[P(r|q) > P(r|\neg q)\big]\Big\} \longrightarrow \Big\{C \longrightarrow \big[P(r|p) > P(r|\neg p)\big]\Big\}\,. \tag{15}$$

We could try for C the inequality

$$\kappa(p, r; q) + \kappa(p, r; \neg q) + \sigma(p, r; q) > 0\,,$$

but that would be trivial. True, the Shogenji identity (9) teaches us that this inequality implies $P(r|p) > P(r|\neg p)$. However, in order to do so, it does not need the antecedent, $\big\{\big[P\big(q|p\big) > P(q|\neg p)\big]\&\big[P(r|q) > P(r|\neg q)\big]\big\}$. A nontrivial condition that guarantees the transitivity of probabilistic support must *require* the antecedent condition, namely that p supports q, and that q supports r, in order to be worthy of the name. The equivalent

$$\kappa(p, r; q) + \kappa(p, r; \neg q) > -\sigma(p, r; q)\,,$$

is of course also unacceptable, but this form gives us an inkling of how to avoid the triviality: we could replace C by a set of conditions in the following way.

Let ε be a real number in the open unit interval, i.e. $0 < \varepsilon < 1$. Instead of the trivial C, consider

$$\kappa(p, r; q) + \kappa(p, r; \neg q) \geq -(1 - \varepsilon)\,\sigma(p, r; q)\,, \tag{16}$$

[17] See Atkinson and Peijnenburg (2021).

which we shall call condition C_ε. Since ε can take any value between 0 and 1, we should regard C_ε as a continuous set of conditions, one for each value of ε. If we decrease the value of ε more probabilities are allowed. That is, if $\varepsilon_1 < \varepsilon_2$, then C_{ε_2} is a proper subset of C_{ε_1}, where we have used the same symbol C_ε to refer to the constraint (16) as well as to the set of probabilities subject to the constraint.

Now add $\sigma(p, r; q)$ to both sides of (16):

$$\kappa(p, r; q) + \kappa(p, r; \neg q) + \sigma(p, r; q) \geq \varepsilon\, \sigma(p, r; q)\,, \tag{17}$$

which is equivalent to $P(r|p) - P(r|\neg p) \geq \varepsilon\sigma(p, r; q)$, because of the Shogenji identity. The right-hand side of this inequality could be negative, for although ε is positive, $\sigma(p, r; q)$ could be negative or zero. Therefore C_ε by itself does not entail the consequent, $P(r|p) > P(r|\neg p)$. However the antecedent (namely that p supports q and q supports r) implies that $\sigma(p, r; q)$ is positive, so C_ε and the antecedent together do imply the consequent. That is

$$\Big\{\big[P(q|p) > P(q|\neg p)\big]\&\big[P(r|q) > P(r|\neg q)\big]\Big\} \longrightarrow \Big\{C_\varepsilon \longrightarrow \big[P(r|p) > P(r|\neg p)\big]\Big\}.$$

C_ε is a nontrivial sufficient condition for the transitivity of probabilistic support. Unlike the trivial condition C, it does require the positivity of $\sigma(p, r; q)$, which is implied by the antecedent (1).

Since $\varepsilon < 1$, (16) is weaker than (14). Indeed, ε labels a continuum of conditions of the form (17), one for each value of ε greater than, but not equal to zero. There is no nontrivial weakest condition of the form C_ε, because for any such condition there always exists a weaker condition, for example $C_{\varepsilon/2}$. One can make the condition as weak as one likes.

Let us summarize. We have distinguished three versions of screening off, each of which is a sufficient condition for the transitivity of probabilistic support:

- the original one, introduced by Reichenbach (1956/1999),
- a generalized, weaker version of Roche (2012) and Suppes 1983,
- a further generalization, which encompasses a continuum, as we explained in the present section.

As we have seen, all three can be understood with the help of (9), which reveals that the support which p gives to r is equal to the sum of three components, namely $\kappa(p, r; q)$, $\kappa(p, r; \neg q)$, and $\sigma(p, r; q)$. Each of these components contributes in its own way to the strength with which probabilistic support is transmitted from p to r through an intermediate proposition q. When $\kappa(p, r; q)$ and $\kappa(p, r; \neg q)$ are both *zero*, Reichenbach's standard screening off holds sway. Then it falls to $\sigma(p, r; q)$ alone to ensure that the support that p affords r is positive, albeit not as great as the support that q imparts to r. If however the sum of $\kappa(p, r; q)$ and $\kappa(p, r; \neg q)$ is *positive*, then the generalized screening off *à la* Roche and Suppes transpires. Now the support is enhanced above the contribution of $\sigma(p, r; q)$ alone, and it can even exceed the support that q gives to r. We illustrated this possibility with the story about the descendants of Queen Victoria. Finally, the domain in which the sum of $\kappa(p, r; q)$ and $\kappa(p, r; \neg q)$ is *negative* includes our further generalization in which there is a continuum of very weak conditions. In this case the support that p affords r is less than the positive contribution of $\sigma(p,r;q)$, so it is much smaller than the support that q gives to r.[18]

Table 2 displays the various situations and summarizes the interplay of the three terms on the right-hand side of (9). The numbers in the leftmost column refer to the equations or inequalities in which the case in question is defined. A dash '–' indicates that the relevant quantity may be positive, zero or negative.

Table 2 Domains of $\kappa(p,r;q)$ and $\kappa(p,r;\neg q)$

	$\kappa(p,r;q)$	$\kappa(p,r;\neg q)$	$\kappa(p,r;q)+\kappa(p,r;\neg q)$
Reichenbach's screening off (5)	0	0	0
Suppes-Roche generalization (13)	≥ 0	≥ 0	≥ 0
Intermediate form (14)	–	–	≥ 0
Our further generalization (16)	–	–	$\geq -(1-\varepsilon)\,\sigma(p,r;q)$

In the next section we take a closer look at our further generalization (16), in which the sum of the two kappa's can be negative. We spell out an intriguing connection of this generalization with another effect, that of Simpson, and we illustrate the connection with a medical example.

6 Digging deeper

In the present section, we will dig deeper into the third kind of sceening off. We explain that it has a remarkable property: in contrast to the other two, the third kind of screening off can occur simultaneously with the Simpson paradox. We will illustrate this possibility with a classical study about the removal of kidney stones.[19]

The Simpson effect, as we prefer to call it, was a surprise when it was first discovered, but has by now become common knowledge.[20] It is defined by the following conditions: p and r are positively correlated unconditionally, but negatively correlated when conditioned on q and on $\neg q$:

$$\begin{aligned} P(r|p) &> P(r|\neg p) \\ P(r|q\wedge p) &< P(r|q\wedge\neg p) \\ P(r|\neg q\wedge p) &< P(r|\neg q\wedge\neg p)\,. \end{aligned} \tag{18}$$

The second and third inequalities of (18) imply that both $\kappa(p,r;q)$ and $\kappa(p,r;\neg q)$ are negative (see (9)). The first inequality, on the other hand, which states that p confirms r, means that $\kappa(p,r;q)+\kappa(p,r;\neg q)+\sigma(p,r;q)$ is positive, whence it

18 Thanks to Miklós Rédei and Robert Rynasiewicz for pressing us to spell this out explicitly.

19 Julious and Mullee (1994); Atkinson and Peijnenburg (2021).

20 Yule (1903); Simpson (1951); Sprenger and Weinberger (2021).

follows that $\sigma(p, r; q)$ must also be positive. This can be so if one of the following is true:

(a) p confirms q and q confirms r
(b) p disconfirms q and q disconfirms r.[21]

If (a), and p confirms r — the first inequality of (18) — then there is transitivity of probabilistic support. And in view of the second and third inequalities of (18), we see that we have to do with transitivity in the domain where (14) is violated, that is, where $\kappa(p, r; q) + \kappa(p, r; \neg q) < 0$.

What about case (b)? At first sight we do not seem to have a case of transitivity, but appearances are deceptive. For we can make use of an observation of Reichenbach:

> If $[P(p|q) < P(p|\neg q)]$ and also $[P(r|q) < P(r|\neg q)]$, we find, once more, $[P(p \wedge r) > P(p)P(r)]$; in this case, q and $\neg q$ have merely changed places.[22]

Following Reichenbach's lead, we see that p indeed confirms $\neg q$, and $\neg q$ confirms r.[23] So (b) can be interpreted after all as a case of the transitivity of probabilistic support, on condition that we regard $\neg q$ rather than q as the mediator between p and r. By interchanging q and $\neg q$, we find that (b) mirrors (a).

For a given value of ε, the constraint C_ε is expressed by (17); but the Simpson set — the set of probabilities that satisfy (18) — also includes probabilities such that $\kappa(p, r; q) + \kappa(p, r; \neg q) + \sigma(p, r; q) < \varepsilon\, \sigma(p, r; q)$, on condition of course that the first inequality of (18) holds. The conclusion is that the Simpson set is not a subset of the set of probabilities that satisfy the condition C_ε: there are probabilities belonging to the Simpson set that violate (17). On the other hand, the set of probabilities that satisfy the condition C_ε is not a subset of the Simpson set either, for (17) is consistent with

$$P(r|q \wedge p) < P(r|q \wedge \neg p) \quad \text{and} \quad P(r|\neg q \wedge p) > P(r|\neg q \wedge \neg p)\,,$$

or with

$$P(r|q \wedge p) > P(r|q \wedge \neg p) \quad \text{and} \quad P(r|\neg q \wedge p) < P(r|\neg q \wedge \neg p)\,,$$

but the Simpson inequalities (18) are not. These options imply that one of $\kappa(p, r; q)$ and $\kappa(p, r; \neg q)$ is positive, while the other is negative, this being consistent with the possibility that $\kappa(p, r; q) + \kappa(p, r; \neg q)$ be negative. So the set defined by C_ε has an overlap with the Simpson set, but is not a subset of it.

---

21 Here p disconfirms q means that $P(q|p) < P(q|\neg p)$.

22 Reichenbach (1956/1999), p. 160. We have substituted our notation for Reichenbach's, see also the paragraph in Sect. 2 in which Eq. (6) occurs.

23 Proof:

$$\{P(q|p) < P(q|\neg p)\} \longrightarrow \{P(\neg q|p) = 1 - P(q|p) > 1 - P(q|\neg p) = P(\neg q|\neg p)\}$$
$$\{P(r|q) < P(r|\neg q)\} \longrightarrow \{P(r|\neg q) > P(r|q)\}$$

.

The fact that the two sets overlap is somewhat surprising. At first sight, co-existence of transitivity with the Simpson effect seems counterintuitive, for the two results seem radically opposed to one another. Transitivity means that a property is transmitted from one object to another via an intermediate object: if p confirms q and q confirms r, then under transitivity p confirms r. The Simpson effect, on the other hand, appears to *obstruct* transmission: although p confirms r, it fails to do so when conditioned on q or $\neg q$.

A good illustration of this overlap between the Simpson set and the set defined by the constraint C_ε is the classic paper on the removal of kidney stones by Julious and Mullee (1994). They drew attention to a study that had been made by Charig and coworkers of the success rates of two kinds of operations to remove kidney stones (renal calculi): open surgery versus percutaneous nephrolithotromy (the penetration of the skin and kidney by a tube, through which the stone is removed).[24]

Julious and Mullee concentrated on 700 operations that were performed on patients with kidney stones, one half by percutaneous nephrolithotomy and the other by open surgery. An operation was deemed successful if no stones greater than 2 mm in diameter were present in the operated kidney three months after the operation; and success rates were compared for stones that were smaller or larger than 2 cm in diameter.

For each of these 700 operations, define the following propositions:

r : the operation was successful
p : percutaneous nephrolithotomy was performed
$\neg p$: open surgery was performed
q : the stone that was removed was less than 2 cm in diameter
$\neg q$: the stone that was removed was at least 2 cm in diameter

Since the number of percutaneous nephrolithotomies was equal to the number of open surgeries (namely 350), $P(p) = 0.5$. The numbers given by Charig et al. correspond to the following conditional probabilities (relative frequencies):[25]

$$\begin{aligned} P(r|p) &= 0.83 \quad P(r|\neg p) = 0.78 \\ P(r|q \wedge p) &= 0.87 \quad P(r|q \wedge \neg p) = 0.93 \\ P(r|\neg q \wedge p)) &= 0.69 \quad P(r|\neg q \wedge \neg p) = 0.73 \end{aligned} \tag{19}$$

From these probabilities we calculate

$$P(r|p) - P(r|\neg p) = 0.05\,,$$

[24] Charig et al. (1986).

[25] Julious and Mullee incorrectly give the percentage of successes for percutaneous nephrolithotomies with stones of diameter less than 2 cm as 83%, whereas according to Charig et al. it should be 87%. This is presumably a copying error, for with 83% the probability distribution would be inconsistent whereas with 87% the distribution is consistent and there is indeed a Simpson effect.

so p supports r, i.e. percutaneous nephrolithotomy improves the chance of success. On the other hand,

$$
\begin{aligned}
P(r|q \wedge p) - P(r|q \wedge \neg p) &= -0.06 \\
P(r|\neg q \wedge p) - P(r|\neg q \wedge \neg p) &= -0.04\,,
\end{aligned}
\tag{20}
$$

So percutaneous nephrolithotomy *decreases* the chance of success for stones of less than 2 cm diameter, and *also* for stones at least as large as 2 cm. This is an example of the Simpson paradox, which was the burden of the paper of Julious and Mullee.

However we can also calculate

$$
\begin{aligned}
P(q|p) - P(q|\neg p) &= 0.053 \\
P(r|q) - P(r|\neg q) &= 0.164\,,
\end{aligned}
$$

so p supports q, and q supports r, and these numbers yield $\sigma(p, r; q) = 0.087$. On condition that

$$
\varepsilon \le \frac{P(r|p) - P(r|\neg p)}{\sigma(p, r; q)} = \frac{0.05}{0.087} = 0.57\,,
\tag{21}
$$

this also belongs to the set defined by the condition C_ε. Thus the kidney stone data illustrate the overlap between the Simpson set and the set defined by condition C_ε, for any positive ε that satisfies (21).

7 Conclusion

We started this paper by recalling two claims by Glymour and Eberhardt: first, that some of Reichenbach's ideas re-emerged in recent philosophy without notice of the connection, and second, that more than once the fruits of his ideas only belatedly attained their full impact. In this paper we have argued that Reichenbach's thoughts on screening off illustrate both of their claims.

In *The Direction of Time* of 1956, Reichenbach introduced the term 'screening off', although the general idea behind it can already be found in his earlier work, notably his *Wahrscheinlichkeitslehre* (Reichenbach 1935). Tomoji Shogenji proved in 2003 that screening off is a sufficient condition for the transitivity of probabilistic support, and a similar proof concerning probabilistic causality had been given by Ellery Eells and Elliott Sober in (1983).

We conjectured in Sect. 2 of this paper that Reichenbach in 1956 may have been aware of this result. In Sect. 3 we discussed the work of William Roche and Patrick Suppes, who succeeded in weakening the condition of screening off while preserving the transitivity of probabilistic support. This generalized screening off, as we have called it, permits a possibility that is excluded by Reichenbach's approach: if p supports q and q supports r, then it may happen that $P(r|p)$ is greater than $P(r|q)$. In order to show that this possibility is not purely theoretical, but may occur in real life, we gave in Sect. 4 an example based on the transmission of haemophilia among the descendants of Queen Victoria. We then weakened generalized screening off further in Sect. 5, and developed a sufficient condition for the transitivity of probabilistic support which can be made as weak as one likes. In Sect. 6 we explained that this further generalization allows for the possibility of a Simpson effect, which we illustrated by means of a study of kidney stone treatment in Julious and Mullee (1994).

We conclude by presenting Table 3, which is Table 2 augmented by the Simpson effect.

Table 3 Domains of $\kappa(p, r; q)$, $\kappa(p, r; \neg q)$ and the Simpson effect

	$\kappa(p, r; q)$	$\kappa(p, r; \neg q)$	$\kappa(p, r; q) + \kappa(p, r; \neg q)$
Reichenbach's Screening off (5)	0	0	0
Suppes-Roche generalization (13)	≥ 0	≥ 0	≥ 0
Intermediate form (14)	–	–	≥ 0
Our further generalization (16)	–	–	$\geq -(1-\varepsilon)\,\sigma(p, r; q)$
Simpson effect (18)	<0	<0	$> -\sigma(p, r; q)$

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A Shogenji's Identity Modified

Lemma The following identities hold:

$$\frac{P(r \wedge p|q) - P(r|q)P(p|q)}{P(p)P(\neg p)}\,P(q) = \big[P(r|q \wedge p) - P(r|q \wedge \neg p)\big]\,\frac{P(q|p)P(q|\neg p)}{P(q)} \tag{22}$$

$$\frac{P(r \wedge p) - P(r)P(p)}{P(p)P(\neg p)} = P(r|p) - P(r|\neg p)\,, \tag{23}$$

$$P(p)P(\neg p)[P(q|p) - P(q|\neg p)] = P(q)P(\neg q)[P(p|q) - P(p|\neg q)]\,. \tag{24}$$

Proof

$$
\begin{aligned}
P(r|q\wedge p) - P(r|q\wedge\neg p) &= \frac{P(r\wedge q\wedge p)P(q\wedge\neg p) - P(r\wedge q\wedge\neg p)P(q\wedge p)}{P(q\wedge p)P(q\wedge\neg p)}\\
&= \frac{P(r\wedge q\wedge p)[P(q) - P(q\wedge p)] - [P(r\wedge q) - P(r\wedge q\wedge p)]P(q\wedge p)}{P(q\wedge p)P(q\wedge\neg p)}\\
&= \frac{P(r\wedge q\wedge p)P(q) - P(r\wedge q)P(q\wedge p)}{P(q\wedge p)P(q\wedge\neg p)}\\
&= \frac{P(r\wedge p|q) - P(r|q)P(p|q)}{P(q\wedge p)P(q\wedge\neg p)}P^2(q)
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{P(r\wedge p|q) - P(r|q)P(p|q)}{P(p)P(\neg p)}P(q) &= \left[\frac{P(r|q\wedge p) - P(r|q\wedge\neg p)}{P(q)}\right]\frac{P(q\wedge p)P(q\wedge\neg p)}{P(p)P(\neg p)}\\
&= \left[P(r|q\wedge p) - P(r|q\wedge\neg p)\right]\frac{P(q|p)P(q|\neg p)}{P(q)}\,,
\end{aligned}
$$

which is (22).

In the special case that q is the tautology, this reduces to

$$
\frac{P(r\wedge p) - P(r)P(p)}{P(p)P(\neg p)} = P(r|p) - P(r|\neg p)\,,
$$

which is (23); and from this

$$
\begin{aligned}
&P(p)P(\neg p)[P(r|p) - P(r|\neg p)]\\
&= P(r\wedge p) - P(r)P(p) = P(r)P(\neg r)[P(p|r) - P(p|\neg r)]\,,
\end{aligned}
$$

where we have used the symmetry between r and p. This is an identity, and so we may replace r by q to obtain (24).

Theorem The following identity holds:

$$
\begin{aligned}
P(r|p) - P(r|\neg p) = &\left[P(r|q\wedge p) - P(r|q\wedge\neg p))\right]\frac{P(q|p)P(q|\neg p)}{P(q)}\\
&+\left[P(r|\neg q\wedge p) - P(r|\neg q\wedge\neg p))\right]\frac{P(\neg q|p)P(\neg q|\neg p)}{P(\neg q)}\\
&+\left[P(r|q) - P(r|\neg q)\right]\left[P(q|p) - P(q|\neg p)\right]. \qquad (25)
\end{aligned}
$$

This identity is equivalent to Shogenji's, but it is in a form more convenient for our purposes.

Proof From the rule of total probability,

$$\begin{aligned} P(p)P(r) &= \big[P(p|q)P(q) + P(p|\neg q)P(\neg q)\big]\big[P(r|q)P(q) + P(r|\neg q)P(\neg q)\big] \\ &= P(p|q)P(r|q)P^2(q) + P(p|\neg q)P(r|\neg q)P^2(\neg q) \\ &\quad + \big[P(p|q)P(r|\neg q) + P(p|\neg q)P(r|q)\big]P(q)P(\neg q) \\ &= P(p|q)P(r|q)P^2(q) + P(p|\neg q)P(r|\neg q)P^2(\neg q) \\ &\quad - \big[P(p|q) - P(p|\neg q)\big]\big[P(r|q) - P(r|\neg q)\big]P(q)P(\neg q) \\ &\quad + [P(p|q)P(r|q) + P(p|\neg q)P(r|\neg q)]P(q)P(\neg q) \\ &= P(p|q)P(r|q)P(q) + P(p|\neg q)P(r|\neg q)P(\neg q) \\ &\quad - \big[P(p|q) - P(p|\neg q))\big]\big[P(r|q) - P(r|\neg q)\big]P(q)P(\neg q) \\ &= P(p|q)P(r|q)P(q) + P(p|\neg q)P(r|\neg q)P(\neg q) \\ &\quad - \big[P(q|p) - P(q|\neg p\big]\big[P(r|q) - P(r|\neg q)\big]P(p)P(\neg p)\,. \end{aligned} \tag{26}$$

See Eq. (24) in the Lemma for a justification of the last line. Moreover

$$P(p \wedge r) = P(p \wedge r|q)P(q) + P(p \wedge r|\neg q)P(\neg q)\,, \tag{27}$$

so from (26) and (27) it follows that

$$\begin{aligned} P(p \wedge r) - P(p)P(r) &= \big[P(p \wedge r|q) - P(p|q)P(r|q)\big]P(q) \\ &\quad + \big[P(p \wedge r|\neg q) - P(p|\neg q)P(r|\neg q)\big]P(\neg q) \\ &\quad + \big[P(q|p) - P(q|\neg p\big]\big[P(r|q) - P(r|\neg q)\big]P(p)P(\neg p). \end{aligned}$$

Division throughout by $P(p)P(\neg p)$ and use of Eqs. (22) – twice – and (23) in the Lemma yield (25) immediately.

B Haemophilia

Let n, c, h stand for ‘normal’, ‘carrier’ and ‘haemophiliac’, respectively; and consider the following propositions[26]:

1. L^u: Prince Leopold of Albany is u, where $u = n, h$
2. H^v: Leopold’s wife Helena is v, where $v = n, c, h$
3. A^x: Leopold and Helena’s daughter Alice is x, where $x = n, c, h$
4. R^y: Alexander and Alice’s son Rupert is y, where $y = n, h$

We do not need to take note of the genetic type of Alice’s husband, Alexander of Teck, since we know he contributes a Y strand to his son, Rupert, which is of no relevance to the question whether Rupert is normal or haemophiliac. That is a consequence of whether he inherits a dominant or a recessive allele from his mother. The chance of

[26] We would like to thank a reviewer for suggesting a simplification of our original proof.

inheriting a recessive allele is one half:

$$P(R^h|A^c) = \frac{1}{2} . \tag{28}$$

Let p be the proportion of the alleles of type A in the population, so $q = 1 - p$ is the proportion of the alleles of type a. Since haemophilia is very rare, $q << p$; certainly $q < \frac{1}{2} < p$, which is all we need to know. The unconditional prior probabilities for Leopold and Rupert are

$$P(L^n) = P(R^n) = p \; ; \quad P(L^h) = P(R^h) = q \, ,$$

since only the X strand of their DNA is relevant. For Helena and Alice,

$$P(H^n) = P(A^n) = p^2 \; ; \quad P(H^c){=}P(A^c) = 2pq \; ; \quad P(H^h) = P(A^h) = q^2 \, ,$$

since both of their X strands are relevant. We suppose Leopold's and Helena's genetic types to be independent of one another, so $P(H^v|L^u) = P(H^v)$, where v can be n, c or h, whilst u can be n or h.[27]

Alice receives one recessive allele from her father, Leopold, and in the absence of information about Helena's genetic constitution, the probability that she also transmits a to her daughter, Alice, thereby making her daughter haemophiliac, is

$$P(A^h|L^h) = P(H^h) + \frac{1}{2}P(H^c) = q^2 + \frac{1}{2} \times 2pq = (q + p)q = q \, . \tag{29}$$

In fact we could have inferred this result immediately by working at the level of the genes, for the probability that the allele which Helena transmits to Alice is recessive is the proportion of the a allele in the genetic pool, namely q.

The conditional probability of R^h, given L^h, can be written

$$P(R^h|L^h) = P(R^h|A^h \wedge L^h)P(A^h|L^h) + P(R^h|\neg A^h \wedge L^h)P(\neg A^h|L^h) \, ,$$

an instance of the rule of total probability in conditional form. We know that $P(R^h|A^h \wedge L^h) = P(R^h|A^h)$, since the probability that Rudolf is a haemophiliac is a function only of his mother's genetic constitution. Alice cannot be normal, since her father is a haemophiliac, and therefore

$$\neg A^h \wedge L^h = (A^n \vee A^c) \wedge L^h = (A^n \wedge L^h) \vee (A^c \wedge L^h) = A^c \wedge L^h \, .$$

[27] Strictly speaking, the supposition of genetic independence is false: Leopold and Helena were both great-great-grandchildren of Frederick, Prince of Wales. But if we restrict ourselves to the recessive haemophilia allele, then there is indeed independence.

So $P(R^h|\neg A^h \wedge L^h) = P(R^h|A^c \wedge L^h) = P(R^h|A^c)$. Finally,

$$
\begin{aligned}
P(R^h|L^h) &= P(R^h|A^h)P(A^h|L^h) + P(R^h|A^c)\big(1 - P(A^h|L^h)\big) \\
&= 1 \times q + \frac{1}{2} \times (1-q) = \frac{1}{2}(1+q) > \frac{1}{2} = P(R^h|A^c),
\end{aligned}
$$

where we have used (28) and (29). So it is more likely that Rupert would be a haemophiliac if his maternal grandfather were a haemophiliac than if his mother were a carrier.

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