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A Quasi-Variational-Hemivariational Inequality for Incompressible Navier-Stokes System with Bingham Fluid

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Abstract
In this paper we examine a class of elliptic quasi-variational inequalities, which involve a constraint set and a set-valued map. First, we establish the existence of a solution and the compactness of the solution set. The approach is based on results for an elliptic variational inequality and the Kakutani-Ky Fan fixed point theorem. Next, we prove an existence and compactness result for a quasi-variational-hemivariational inequality. The latter involves a locally Lipschitz continuous functional and a convex potential. Finally, we present an application to the stationary incompressible Navier-Stokes equation with mixed boundary conditions which model a generalized Newtonian fluid of Bingham type.

Keywords Bingham fluid · Variational–hemivariational inequality · Generalized subgradient · Slip boundary condition

Mathematics Subject Classification 35J87 · 49J40 · 76A05

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain. The paper is inspired by a class of stationary boundary value problems for the incompressible Navier-Stokes equation

$$-\operatorname{Div}\mathbb{S} + \operatorname{Div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega,$$

in which $\boldsymbol{u}$ denotes the fluid velocity vector, $\boldsymbol{f}$ stands for the external forces, p is the pressure, and a nonlinear constitutive relation between the extra stress tensor $\mathbb{S}$ and the symmetric part of the velocity gradient $\mathbb{D}$ models the Bingham fluid. The latter represents a

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viscoplastic material that behaves as a rigid body at low stresses (below a plasticity threshold) but flows as a viscous fluid at high stress. The aforementioned equation is supplemented by mixed boundary conditions, the homogeneous Dirichlet boundary condition on one part of the boundary, and a nonmonotone generalization of the slip condition of frictional type on another part. The slip condition is described by the Clarke subdifferential law of a locally Lipschitz continuous potential. Furthermore, the boundary value problem under consideration includes an implicit obstacle constraint set that can be related to the rate dissipation energy, turbulence of the flow, and additional constraints on the velocity of the fluid. To the best of the authors' knowledge, up to now, in the literature, there has not been any study on the above problem.

The rigid-viscoplastic models of Bingham fluid are of practical relevance in the industrial processes of fast material working. They commonly appear in industry to simulate the metal-forming problems as sheet or wire drawing with the flow through a die. For instance, such processes start with wire drawing, and then with drawing of tubes either free or with a floating plug. The extrusions of cylindrical bars at high temperatures, with floating glass as a lubricant, are also devised, see, for instance, [13, 29, 55] and the references therein. Our model is also relevant for a description of the flow in pipeline flows of yield-stress fluids such as concrete and cements. The nonmonotone slip law is justified by the nonsmoothness of the surface of a die or a pipe. Such complex flow patterns may lead to the slip weakening phenomena in which the tangential traction is a decreasing function of the tangential velocity.

The weak formulation of the Navier-Stokes equation with mixed boundary conditions leads to the following nonlinear abstract inequality. Find $u \in C$ such that $u \in K(u)$ and

$$\langle Au + B[u] - f, z - u\rangle + j^0(Mu, Mu; Mz - Mu) + \varphi(u, z) - \varphi(u, u) \geq 0 \ \text{ for all } z \in K(u), \tag{1.1}$$

where A, $B[\cdot]\colon V \to V^*$, $M\colon V \to X$, $K\colon C \to 2^C$, $\varphi\colon V \times V \to \mathbb{R}$, $f \in V^*$, V is a reflexive Banach space with the dual V^*, $\langle \cdot, \cdot \rangle$ is the duality bracket between V^* and V, X be a Hilbert space, C is a subset of V, and $j^0(w, v; z)$ denotes the generalized directional derivative of a locally Lipschitz continuous function $j(w, \cdot)$ at a point v in the direction z.

The problem (1.1) represents an elliptic quasi-variational inequality with implicit constraints. The existence of solution to (1.1) is demanding and, as far as we know, this abstract problem has not been studied up to now. In a particular case, if $B = 0$, $K(u) = K$, $C = K$, $\varphi(u, z) = \Phi(Nz)$, and N is a linear bounded operator, then (1.1) reduces to a variational-hemivariational inequality treated in [41]. In addition, if $j = 0$, then from (1.1) we arrive to an elliptic variational inequality of the second kind, see, e.g., [28]. When $B = 0$, $K(u) = K$ and $C = K$, then a simplified variant of problem (1.1) has been considered in [44]. If K is independent of the solution, we also refer to [44] for various other particular cases of (1.1) investigated in the literature. Existence results for elliptic quasi-variational inequalities can be found in many contributions, for instance [4, 7, 9, 25, 32, 33].

The existence and compactness of the set of weak solutions to (1.1), and the analysis of the stationary incompressible Navier-Stokes problem for a generalized Newtonian Bingham fluid represent two main traits of novelty of the work. We study the existence problem by a method that is different to the ones in [45, 46]. Our approach works without a relaxed monotonicity hypothesis on the subgradient extensively exploited recently in several papers [17, 44] and books [43, 57]. Moreover, the results on the incompressible Navier-Stokes problem are generalizations of [41] in the following three aspects: The convex potential depends on two variables, the set of unilateral constraints is allowed to depend on the solution, and the convection term is present in the equation. The main feature of the Navier-Stokes model is the presence of a nonlinearity of the form $k\partial j$ which leads to a weak formulation which is not a pure hemivariational inequality.

Various models in fluid mechanics have studied within the theory of variational-hemivariational inequalities: In [21, 42] for Navier-Stokes equations, in [18, 19] for non-Newtonian and generalized Newtonian fluids, in [17] for evolutionary Oseen problems. The models for the Bingham fluid have been considered in [2, 3, 5, 8, 12, 20, 47]. More details on variational and hemivariational inequalities could be found in the monographs [4, 6, 20, 35, 43, 50, 57] and the references therein.

The outline of the paper is as follows. After recalling prerequisites in Sect. 2, the main results on nonemptiness and compactness of the solution set to inequality (1.1) are stated and proved in Sect. 3. In Sect. 4 we discuss the Bingham model, its physical interpretation, and we provide its variational formulation. In Sect. 5 we demonstrate that the set of weak solutions in nonempty and compact. Finally, in Appendix we recall some properties of the convective term in the Navier-Stokes equation used in the paper.

2 Mathematical Preliminaries

In this section we set up notation and terminology, and recall some results that will be used in what follows, see [11, 15, 16, 43, 50].

Let $(X, \|\cdot\|_X)$ be a Banach space. Throughout the paper, X^* denotes the dual of X and $\langle\cdot,\cdot\rangle_{X^*\times X}$ is the duality brackets for the pair (X^*, X). A space X with the weak topology is denoted by X_w. Given a set $D \subset X$, we write $\|D\|_X = \sup\{\|x\|_X \mid x \in D\}$. For simplicity, when no confusion arises, we often skip the subscripts. The symbols $\rightharpoonup$ and $\to$ stand for the *weak convergence* and the *strong convergence*, respectively. For Banach spaces X, Y, we will use the notation $\mathcal{L}(X, Y)$ for the set of all linear continuous operators from X to Y. Given $A \in \mathcal{L}(X, Y)$, the *adjoint operator* $A^* \in \mathcal{L}(Y^*, X^*)$ is defined via $\langle A^*y^*, x\rangle := \langle y^*, Ax\rangle$ for every $y^* \in Y^*$ and $x \in X$. The notation $\|A\|$ stands for the *operator norm* in $\mathcal{L}(X, Y)$ defined by

$$\|A\| := \sup_{\|v\|_X\le 1} \|Av\|_Y = \sup_{v\neq 0} \frac{\|Av\|_Y}{\|v\|_X}.$$

An operator $A\colon X \to X^*$ is said to be *monotone*, if $\langle Au - Av, u - v\rangle \ge 0$ for all $u, v \in X$. It is called *maximal monotone*, if it is monotone, and $\langle Au - w, u - v\rangle \ge 0$ for any $u \in X$ entails $w = Av$. The operator A is said to be *bounded*, if it maps bounded sets to bounded sets. The operator A is called *pseudomonotone*, if it is bounded and $u_n \rightharpoonup u$ in X with $\limsup\langle Au_n, u_n - u\rangle \le 0$ implies $\langle Au, u - v\rangle \le \liminf\langle Au_n, u_n - v\rangle$ for all $v \in X$. If X is a reflexive Banach space, then $A\colon X \to X^*$ is pseudomonotone if and only if it is bounded and $u_n \rightharpoonup u$ in X with $\limsup\langle Au_n, u_n - u\rangle \le 0$ implies $\lim\langle Au_n, u_n - u\rangle = 0$ and $Au_n \rightharpoonup Au$ in X^*. It is known, see [43, Theorem 3.69] that if the operator A is bounded, hemicontinuous, and monotone, then it is pseudomonotone. The operator A is said to be *strongly monotone* if there is a constant $m_A > 0$ such that $\langle Au - Av, u - v\rangle \ge m_A\|u - v\|_X^2$ for all $u, v \in V$. Let $T\colon X \to 2^{X^*}$ be a set-valued map. The *domain* $D(T)$ and *graph* $\mathrm{Gr}(T)$ of T are defined, respectively, by $D(T) = \{x \in X \mid Tx \neq \emptyset\}$ and $\mathrm{Gr}(T) = \{(x, x^*) \in X \times X^* \mid x^* \in Tx\}$.

Let X be a Banach space and $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The mapping $\partial\varphi\colon X \to 2^{X^*}$ defined by

$$\partial\varphi(x) = \{x^* \in X^* \mid \langle x^*, v - x\rangle \le \varphi(v) - \varphi(x) \text{ for all } v \in X\}$$

is called the *(convex) subdifferential* of φ and an element $x^* \in \partial\varphi(x)$ is called a *subgradient* of φ at x. Let $h\colon X \to \mathbb{R}$ be a locally Lipschitz continuous function. The *generalized (Clarke) directional derivative* of h at the point $x \in X$ in the direction $v \in X$ is defined by

$$h^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

The *generalized subgradient* of h at x is a subset of the dual space X^* given by

$$\partial h(x) = \{\, \zeta \in X^* \mid h^0(x; v) \ge \langle \zeta, v \rangle \text{ for all } v \in X \,\}.$$

A locally Lipschitz continuous function h is said to be *regular* (in the sense of Clarke) at the point $x \in X$ if for all $v \in X$ the derivative $h'(x; v)$ exists and $h^0(x; v) = h'(x; v)$.

The convergence of sets in the sense of Mosco, see [16, 48], is recalled below.

Definition 2.1 Given a normed space Y, a sequence $\{C_n\}$ of closed and convex sets in Y, is said to converge to a closed and convex set $C \subset Y$ in the Mosco sense, denoted by $C_n \xrightarrow{M} C$ as $n \to \infty$, if the following two conditions are fulfilled.

(m_1) For any $z_n \in C_n$ with $z_n \rightharpoonup z$ in Y, up to a subsequence, we have $z \in C$.

(m_2) For any $z \in C$, there exists $z_n \in C_n$ with $z_n \to z$ in Y.

In addition to convex analysis, nonsmooth analysis and monotone operator theory, quasi-variational inequalities are often treated by a fixed point approach. The well-known Kakutani–Ky Fan fixed point theorem for a reflexive Banach space, see, e.g., [16, Corollary 1.7.42] reads as follows.

Theorem 2.2 *Let Y be a reflexive Banach space and $D \subseteq Y$ be a nonempty, bounded, closed and convex set. Let $\Lambda\colon D \to 2^D$ be a set-valued map with nonempty, closed and convex values such that its graph is sequentially closed in $Y_w \times Y_w$ topology. Then Λ has a fixed point.*

Finally, we recall an existence and uniqueness result for an elliptic variational inequality of the second kind with constraints. Let Y be a reflexive Banach space. Given an operator $G\colon Y \to Y^*$, a function $\Phi\colon Y \to \mathbb{R}$, and a set $E \subset Y$, we consider the following problem.

Problem 2.3 Find an element $u \in E$ such that

$$\langle Gu - g, v - u \rangle_{Y^* \times Y} + \Phi(v) - \Phi(u) \ge 0 \ \text{ for all } \ v \in E.$$

For this problem, we need the following hypotheses on the data.

Assumption 2.4 $\mathbf{H}(\mathbf{G})$: The operator $G\colon Y \to Y^*$ is such that

(i) it is pseudomonotone,

(ii) it is strongly monotone, i.e., there exists $m_G > 0$ such that $\langle Gv_1 - Gv_2, v_1 - v_2 \rangle \ge m_G \|v_1 - v_2\|_Y^2$ for all $v_1, v_2 \in Y$.

$\mathbf{H}(\Phi)$: The functional $\Phi\colon Y \to \mathbb{R}$ is convex and lower semicontinuous.

$\mathbf{H}(\mathbf{E})$: The set E is a nonempty, closed and convex subset of Y.

$\mathbf{H}(\mathbf{g})$: Let $g \in Y^*$.

Theorem 2.5 *Under Assumption 2.4, Problem 2.3 has a unique solution $u \in E$.*

Theorem 2.5 is a version of the classical theorem of Lions-Stampacchia and its several extensions, see, for instance, [4, 9, 10, 28, 35, 39] and the references therein. Theorem 2.5 represents also a particular case of a result proved recently in [44, Theorem 18] for variational-hemivariational inequalities, where the function Φ may depend additionally on the solution. Further, in [44], the authors required also that G is coercive. However, this assumption is redundant there, since if G is strongly monotone then G is coercive in the following sense $\langle Gv, v\rangle = \langle Gv - G0, v\rangle + \langle G0, v\rangle \geq m_G \|v\|_Y^2 + \|G0\|_{Y^*}\|v\|_Y$ for all $v \in Y$.

3 Multivalued Quasi-Variational Inequality

In this section we study a class of elliptic quasi-variational inequalities which involves a constraint set and a set-valued map. We establish the existence of a solution and provide a corollary for quasi-variational-hemivariational inequalities with constraints.

3.1 Variational Inequality with a Set-Valued Map

Let V be a reflexive Banach space with the dual V^*. The norm in V and the duality brackets for the pair (V^*, V) are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let X be a Hilbert space with the norm $\|\cdot\|_X$ and the inner product $\langle\cdot,\cdot\rangle_X$. The dual space to X is identified with X. Given a set $C \subset V$, a function $\varphi\colon V \times V \to \mathbb{R}$, operators A, $B[\cdot]\colon V \to V^*$, $M\colon V \to X$, set-valued maps $K\colon C \to 2^C$, $F\colon X \to 2^X$, and $f \in V^*$, we consider the following problem.

Problem 3.1 Find $u \in C$ such that $u \in K(u)$ and there is $w \in F(Mu)$, with

$$\langle Au + B[u] - f, z - u\rangle + \langle w, Mz - Mu\rangle_X + \varphi(u,z) - \varphi(u,u) \geq 0 \ \text{ for all } \ z \in K(u).$$

We need the following hypotheses on the data.

Assumption 3.2 $\mathbf{H(A)}$: The operator $A\colon V \to V^*$ is such that

(i) A is pseudomonotone,
(ii) A is strongly monotone, i.e., $\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A\|v_1 - v_2\|^2$ for all $v_1, v_2 \in V$ with $m_A > 0$.

$\mathbf{H(B)}$: The operator $B[\cdot]\colon V \to V^*$ is such that

(i) $B[u] = B(u,u)$, $\langle B(v,u), z\rangle = b(v; u, z)$ for all v, u, $z \in V$, where $b\colon V^3 \to \mathbb{R}$ is a trilinear form,
(ii) $\langle B(v,u), u\rangle \geq 0$ for all $u, v \in V$,
(iii) $B(v_n, u_n) \to B(v,u)$ in V^* for all $v_n \rightharpoonup v$ in V, $u_n \rightharpoonup u$ in V.

$\mathbf{H(C)}$: The set $C \subset V$ is nonempty, closed and convex.
$\mathbf{H(F)}$: The set-valued map $F\colon X \to 2^X$ is such that

(i) F has nonempty, closed and convex values in X,
(ii) $\mathrm{Gr}(F)$ is closed in $X \times X_w$,
(iii) $\|F(z)\|_X \leq b_1 + b_2\|z\|_X$ for all $z \in X$ with $b_1, b_2 \geq 0$.

H(K): The set-valued map $K\colon C \to 2^C$ has closed and convex values, $0 \in K(v)$ for all $v \in C$, and it is weakly Mosco continuous, i.e., for any $\{v_n\} \subset V$ such that $v_n \rightharpoonup v$ in V, one has $K(v_n) \xrightarrow{M} K(v)$.

H(M): Let $M \in \mathcal{L}(V, X)$ be a compact operator.

H($\boldsymbol{\varphi}$): The function $\varphi\colon V \times V \to \mathbb{R}$ is such that

(i) $\varphi(v, \cdot)\colon V \to \mathbb{R}$ is convex and lower semicontinuous on V for all $v \in V$,

(ii) there exists $\alpha_\varphi > 0$ such that

$$\varphi(u_1, v_2) - \varphi(u_1, v_1) + \varphi(u_2, v_1) - \varphi(u_2, v_2) \le \alpha_\varphi \|u_1 - u_2\| \, \|v_1 - v_2\|$$

for all $u_1, u_2, v_1, v_2 \in V$,

(iii) there exists a continuous function $c_\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(u_1, v) - \varphi(u_2, v) \le c_\varphi(\|v\|)\|u_1 - u_2\| \ \text{ for all } \ v, u_1, u_2 \in V,$$

(iv) for all $\{v_n\}, \{u_n\}, \{z_n\} \subset V$ such that $u_n \rightharpoonup u$ in V, $v_n \to v$ in V, $z_n \rightharpoonup z$ in V for some $v, u, z \in V$, it holds:

$$\limsup \big(\varphi(u_n, v_n) - \varphi(z_n, v_n)\big) \le \varphi(u, v) - \varphi(z, v).$$

H(f): Let $f \in V^*$.

($\mathbf{H_0}$): The following *smallness condition* holds $b_2 \|M\|^2 + \alpha_\varphi < m_A$.

In what follows we comment on hypothesis $H(\varphi)$.

Remark 3.3 Consider a function φ which is independent of the first variable, that is, $\varphi(u, v) = \varphi(v)$. Then, under $H(\varphi)$(i), hypotheses $H(\varphi)$(ii) and (iv) can be dropped off. It is clear that in this case, condition (ii) is trivially satisfied with $\alpha_\varphi = 0$. The condition (iv) is a consequence of $H(\varphi)$(i). Indeed, for any $v_n \to v$ in V and $z_n \rightharpoonup z$ in V, we have $\varphi(v_n) \to \varphi(v)$ since φ is a continuous function, and the weak lower semicontinuity of φ implies $\varphi(z) \le \liminf \varphi(z_n)$. These convergences entail

$$\limsup \big(\varphi(v_n) - \varphi(z_n)\big) = \lim \varphi(v_n) + \limsup \left(-\varphi(z_n)\right) \le \varphi(v) - \varphi(z).$$

Note also that condition $H(\varphi)$(ii) was already used in [44, 57] and the references therein, conditions (ii) and (iii) together were used in [59, Theorem 10], while a version of (iv) was employed in [49, 56], respectively.

In the following we establish the existence of a solution to Problem 3.1. The proof is based on a fixed point argument and will be done in several steps.

We begin with the following intermediate problem. Let $(v, w) \in C \times X$ be fixed.

Problem 3.4 Find $u \in C$ such that $u \in K(v)$ and

$$\langle Au + B(v, u) - f + M^* w, z - u\rangle + \varphi(v, z) - \varphi(v, u) \ge 0 \ \text{ for all } \ z \in K(v). \tag{3.1}$$

We consider a map $p\colon C \times X \to C$ defined by $p(v, w) = u$, where $u \in C$ is the unique solution to Problem 3.4 corresponding to $(v, w) \in C \times X$.

Lemma 3.5 *Under Assumption 3.2, Problem 3.4 has a unique solution and the solution map p is completely continuous.*

Proof **Step 1.** We shall prove that Problem 3.4 has a unique solution $u \in C$. We use Theorem 2.5 with the following notation $Y := V$, $Gu := Au + B(v, u)$ for $u \in V$, $\Phi(z) = \varphi(v, z)$ for $z \in V$, $E := K(v)$ and $g := f - M^*w$. Then, we rewrite Problem 3.4 in the following equivalent form: Find $u \in E$ such that

$$\langle Gu - g, z - u\rangle + \Phi(z) - \Phi(u) \geq 0 \ \text{ for all } \ z \in E. \tag{3.2}$$

We shall verify the following properties of the data.

Since the operator $B(v, \cdot)\colon V \to V^*$ is linear for all $v \in V$, we obtain from $H(A)$(ii) and $H(B)$(ii) that

$$\langle Gu_1 - Gu_2, u_1 - u_2\rangle = \langle Au_1 - Au_2, u_1 - u_2\rangle + \langle B(v, u_1 - u_2), u_1 - u_2\rangle \geq m_A \|u_1 - u_2\|^2$$

for all u_1, $u_2 \in V$. Hence, G is strongly monotone. The operator $B(v, \cdot)\colon V \to V^*$ is bounded and completely continuous from $H(B)$(iii), so it is pseudomonotone. The operator G is bounded and pseudomonotone as a sum of two bounded and pseudomonotone operators. Hence G satisfies condition $H(G)$. The other conditions $H(\Phi)$, $H(E)$ and $H(g)$ follow from $H(\varphi)$, $H(K)$ and $H(f)$, respectively. Thus, we apply Theorem 2.5 to deduce that Problem 3.4 has a unique solution $u \in E = K(v) \subset C$.

Step 2. We establish the boundedness estimate for the solution of Problem 3.4. We use $H(K)$ and choose $z = 0 \in K(v)$ in (3.2) to get

$$\langle Au + B(v, u) - g, -u\rangle + \varphi(v, 0) - \varphi(v, u) \geq 0$$

and

$$\langle Au + B(v, u) - (A0 + B(v, 0)), u\rangle \leq \langle A0 + B(v, 0) - g, -u\rangle + \varphi(v, 0) - \varphi(v, u). \tag{3.3}$$

Let $\eta \in V$ be arbitrary. We use hypothesis $H(\varphi)$(ii), (iii), and the triangle inequality to obtain

$$\begin{aligned} \varphi(v, 0) - \varphi(v, u) &\leq \alpha_\varphi \|v - \eta\| \, \|u - 0\| + \varphi(\eta, 0) - \varphi(\eta, u) \\ &\leq (\alpha_\varphi \|v - \eta\| + c_\varphi(\|\eta\|))\|u - 0\|. \end{aligned} \tag{3.4}$$

From (3.3), due to the strong monotonicity of $A(\cdot) + B(v, \cdot)$, (3.4), and the property $B(v, 0) = 0$, we have

$$m_A \|u\|^2 \leq \Big(\|A0 - g\| + \alpha_\varphi \|v - \eta\| + c_\varphi(\|\eta\|)\Big)\|u\|.$$

We take into account that $g := f - M^*w$ to obtain

$$m_A \|u\| \leq C_0 + \alpha_\varphi \|v\| + \|M\| \|w\|_X, \tag{3.5}$$

where

$$C_0 := \|A0\|_{V^*} + \|f\|_{V^*} + \alpha_\varphi \|\eta\| + c_\varphi(\|\eta\|) > 0 \tag{3.6}$$

is independent of (v, w).

Step 3. We shall show that p is completely continuous, that is, continuous from $V_w \times X_w$ to V.

Let $\{v_n\} \subset C$, $\{w_n\} \subset X$, $v_n \rightharpoonup v$ in V, $w_n \rightharpoonup w$ in X, and $u_n := p(v_n, w_n) \in K(v_n)$. We prove that $u_n \to u$ in V and $u = p(v, w) \in K(v)$. This will be done in two steps. First, we show $u_n \rightharpoonup u$ in V, and next we prove the strong convergence.

• **Weak convergence**. Since $u_n \in C$ and $u_n \in K(v_n)$, we have

$$\langle Au_n + B(v_n, u_n) - g_n, z - u_n\rangle + \varphi(v_n, z) - \varphi(v_n, u_n) \geq 0 \ \text{ for all } \ z \in K(v_n) \tag{3.7}$$

with $g_n := f - M^* w_n$. Now, taking advantage of the estimate proved in Step 2, we get the uniform estimate for the sequence of solutions $\{u_n\}$ of the form

$$m_A \|u_n\| \leq C_0 + \alpha_\varphi \|v_n\| + \|M\| \|w_n\|_X, \tag{3.8}$$

where $C_0 := \|A0\|_{V^*} + \|f\|_{V^*} + \alpha_\varphi \|\eta\| + c_\varphi(\|\eta\|) > 0$ is independent of n. Since $\{v_n\}$ and $\{w_n\}$ are bounded in V and X, respectively, from (3.8), it follows that $\{u_n\}$ lies in a bounded set in V. By the reflexivity of V, there exist an element $u_0 \in V$ and a subsequence of $\{u_n\}$, still denoted as before, such that $u_n \rightharpoonup u_0$ in V. We use conditions $u_n \in K(v_n)$ and $v_n \rightharpoonup v$ in V, hypothesis $H(K)$, and from (m_1) of Definition 2.1, we deduce $u_0 \in K(v)$.

Next, let $z \in K(v)$. We employ condition (m_2) in the Mosco convergence twice for $z \in K(v)$ and $u_0 \in K(v)$ and we find two sequences $\{z_n\}$ and $\{\eta_n\}$ with

$$z_n, \eta_n \in K(v_n) \ \text{ such that } \ z_n \to z \ \text{ and } \ \eta_n \to u_0 \ \text{ in } \ V, \ \text{as } n \to \infty. \tag{3.9}$$

We choose $z = \eta_n \in K(v_n)$ in (3.7) to get

$$\langle Au_n + B(v_n, u_n), u_n - \eta_n\rangle \leq \langle g_n, u_n - \eta_n\rangle + \varphi(v_n, \eta_n) - \varphi(v_n, u_n). \tag{3.10}$$

It follows from hypothesis $H(\varphi)$(iv) that

$$\limsup \big(\varphi(v_n, \eta_n) - \varphi(v_n, u_n)\big) \leq \varphi(v, u_0) - \varphi(v, u_0) = 0. \tag{3.11}$$

Since M^* is compact, we have $g_n := f - M^* w_n \to f - M^* w =: g$ in V^*. By a direct calculation, we obtain

$$\begin{aligned}
\limsup\langle Au_n, u_n - u_0\rangle &= \limsup\langle Au_n + B(v_n, u_n), u_n - \eta_n\rangle \\
&\quad + \limsup\langle Au_n + B(v_n, u_n), \eta_n - u_0\rangle + \lim\langle B(v_n, u_n), u_0 - u_n\rangle \\
&\leq \limsup\Big(\langle g_n, u_n - \eta_n\rangle + \varphi(v_n, \eta_n) - \varphi(v_n, u_n)\Big) \\
&\quad + \limsup\langle Au_n + B(v_n, u_n), \eta_n - u_0\rangle + \lim\langle B(v_n, u_n), u_0 - u_n\rangle \leq 0.
\end{aligned}$$

Here, we have used (3.7) and the convergences $u_n - \eta_n \rightharpoonup 0$ in V, $\eta_n - u_0 \to 0$ in V, and $B(v_n, u_n) \to B(v, u_0)$ in V^*. Hence, we deduce $u_n \rightharpoonup u_0$ in V and $\limsup\langle Au_n, u_n - u_0\rangle \leq 0$, which by the pseudomonotonicity of A yields

$$\langle Au_0, u_0 - v\rangle \leq \liminf\langle Au_n, u_n - v\rangle \ \text{ for all } \ v \in V. \tag{3.12}$$

On the other hand, we take $z = z_n \in K(v_n)$ in (3.7) to get

$$\langle Au_n, u_n - z_n\rangle \leq \langle B(v_n, u_n) - g_n, z_n - u_n\rangle + \varphi(v_n, z_n) - \varphi(v_n, u_n). \tag{3.13}$$

Subsequently, we combine (3.12), (3.13) and use $H(B)$(iii), $H(\varphi)$(iv) to obtain

$$\langle Au_0, u_0 - z\rangle \leq \liminf\langle Au_n, u_n - z\rangle \leq \limsup\langle Au_n, u_n - z\rangle$$

$$
\begin{aligned}
&= \limsup\langle Au_n, u_n - z_n\rangle + \lim\langle Au_n, z_n - z\rangle = \limsup\langle Au_n, u_n - z_n\rangle \\
&\le \lim\langle B(v_n, u_n) - g_n, z_n - u_n\rangle + \limsup\big(\varphi(v_n, z_n) - \varphi(v_n, u_n)\big) \\
&\le \langle B(v, u_0) - g, z - u_0\rangle + \varphi(v, z) - \varphi(v, u_0),
\end{aligned}
$$

where $g_n \to g$ in V^*. The boundedness of the operator A has been used to deduce $\lim\langle Au_n, z_n - z\rangle = 0$. Hence

$$
\langle Au_0 + B(v, u_0) - g, u_0 - z\rangle \le \varphi(v, z) - \varphi(v, u_0).
$$

Since $z \in K(v)$ is arbitrary, we deduce that $u_0 \in K(v)$ is a solution to the limit problem corresponding to (3.7), i.e., $u_0 = p(v, w)$. The uniqueness of the limit element u_0 implies that the whole sequence $\{u_n\}$ convergences weakly to u_0 in V.

• **Strong convergence**. It remains to show the strong convergence of $\{u_n\}$ to u_0 in V. From condition (m_2) of the Mosco convergence for $u_0 \in K(v)$, we are able to find a sequence $\{\eta_n\} \subset K(v_n)$ such that $\eta_n \to u_0$ in V as $n \to \infty$. We choose η_n as a test function in (3.7) to get

$$
\langle Au_n + B(v_n, u_n) - g_n, \eta_n - u_n\rangle + \varphi(v_n, \eta_n) - \varphi(v_n, u_n) \ge 0 \;\text{ for all }\; n \in \mathbb{N}.
$$

This implies

$$
\langle Au_n + B(v_n, u_n), u_n - \eta_n\rangle \le \langle g_n, u_n - \eta_n\rangle + \varphi(v_n, \eta_n) - \varphi(v_n, u_n). \tag{3.14}
$$

Exploiting (3.14), $H(\varphi)$(iv) and the convergences $u_n - \eta_n \rightharpoonup 0$ in V, $\eta_n \to u_0$ in V, and $g_n \to g$ in V^*, we have

$$
\begin{aligned}
&\limsup\langle Au_n + B(v_n, u_n), u_n - u_0\rangle \qquad (3.15)\\
&\quad \le \limsup\langle Au_n + B(v_n, u_n), u_n - \eta_n\rangle \\
&\qquad + \limsup\langle Au_n + B(v_n, u_n), \eta_n - u_0\rangle \\
&\quad \le \limsup\langle g_n, u_n - \eta_n\rangle + \limsup(\varphi(v_n, \eta_n) - \varphi(v_n, u_n)) \\
&\qquad + \limsup\langle Au_n + B(v_n, u_n), \eta_n - u_0\rangle \le 0.
\end{aligned}
$$

We recall that $A(\cdot) + B(v_n, \cdot)$ is strongly monotone and $B(v_n, u_0) \to B(v, u_0)$ in V^*. Therefore, by (3.15), it follows

$$
\begin{aligned}
&m_A \limsup \|u_n - u_0\|^2 \\
&\quad \le \limsup\langle Au_n + B(v_n, u_n) - (Au_0 + B(v_n, u_0)), u_n - u_0\rangle \\
&\quad \le \limsup\langle Au_n + B(v_n, u_n), u_n - u_0\rangle - \liminf\langle Au_0 + B(v_n, u_0), u_n - u_0\rangle \le 0.
\end{aligned}
$$

As $v_n \rightharpoonup v$ in V we have by $H(B)$(iii) that $B(v_n, u_0) \to B(v, u_0)$ in V^*. This yields with $u_n \rightharpoonup u_0$ in V that $\langle Au_0 + B(v_n, u_0), u_n - u_0\rangle \to 0$. Therefore, we have $\|u_n - u_0\| \to 0$ as $n \to \infty$. This shows the complete continuity of the solution map p. □

The following is the main result of the paper.

Theorem 3.6 *Under Assumption 3.2, Problem 3.1 has a solution.*

Proof Let $v \in C \subset V$ and $w \in X$ be fixed. Recall that $p \colon C \times X \to C$ denotes the solution map for Problem 3.4.

Step 1. Consider the set-valued map $\Lambda \colon D \to 2^D$ defined by

$$\Lambda(v, w) := \big(p(v, w), F(Mp(v, w))\big) = (u, F(Mu)) \ \text{ for } \ (v, w) \in D, \tag{3.16}$$

where

$$D := \{ (v, w) \in C \times X \mid \|v\| \le r_1, \ \|w\|_X \le r_2 \} \tag{3.17}$$

with some $r_1, r_2 > 0$. We establish that Λ fulfills the conditions in Theorem 2.2. First, we show that for suitable constants $r_1, r_2 > 0$, the values of the map Λ lie in D. Let

$$r_1 := \frac{C_0 + b_1\|M\|}{m_A - b_2\|M\|^2 - \alpha_\varphi} \ \text{ and } \ r_2 := b_1 + b_2\,\|M\|\, r_1$$

with $C_0 := \|A0\|_{V^*} + \|f\|_{V^*} + \alpha_\varphi\|\eta\| + c_\varphi(\|\eta\|) > 0$, see (3.6). Note that by (H_0), r_1 is well defined. Suppose that $\|v\| \le r_1$ and $\|w\|_X \le r_2$. From (3.5) we have

$$\begin{aligned} m_A\|u\| &\le C_0 + \|M\|\|w\|_X + \alpha_\varphi\|v\| \le C_0 + \|M\|r_2 + \alpha_\varphi r_1 \\ &\le C_0 + b_1\|M\| + \frac{(b_2\|M\|^2 + \alpha_\varphi)(C_0 + b_1\|M\|)}{m_A - b_2\|M\|^2 - \alpha_\varphi} = m_A r_1. \end{aligned}$$

This entails $\|u\| \le r_1$. Further, we have

$$\|F(Mu)\|_X \le b_1 + b_2\|M\|\|u\| \le b_1 + b_2\|M\|r_1 = r_2.$$

Thus, we have positive constants r_1 and r_2 in the definition (3.17) of the set D such that $\Lambda(v, w) \subset D$ for all $(v, w) \in D$. Moreover, the values of Λ are nonempty, closed and convex sets, by the analogous properties of F.

Subsequently, we show that the graph of Λ is sequentially weakly closed in $D \times D$. Let $(v_n, w_n) \in D$, $(v_n, w_n) \rightharpoonup (v, w)$ in $V \times X$, $(\overline{v}_n, \overline{w}_n) \in \Lambda(v_n, w_n)$, and $(\overline{v}_n, \overline{w}_n) \rightharpoonup (\overline{v}, \overline{w})$ in $V \times X$. We show that $(\overline{v}, \overline{w}) \in \Lambda(v, w)$. By the definition of Λ, we have

$$\overline{v}_n = p(v_n, w_n) \ \text{ and } \ \overline{w}_n \in F(Mp(v_n, w_n)). \tag{3.18}$$

Using the complete continuity of the map p proved in Lemma 3.5 and the continuity of the operator M, we get $p(v_n, w_n) \to p(v, w)$ in V and $Mp(v_n, w_n) \to Mp(v, w)$ in X. Together with (3.18), this implies

$$\overline{v} = p(v, w) \ \text{ and } \ \overline{w} \in F(Mp(v, w)).$$

The latter is a consequence of the closedness of the graph of F in the $X \times X_w$ topology. Hence $(\overline{v}, \overline{w}) \in \big(p(v, w), F(Mp(v, w))\big) = \Lambda(v, w)$, which proves the closedness of the graph of Λ.

Step 2. We apply Theorem 2.2 with $Y = V \times X$ and the map Λ given by (3.16) to deduce that there exists a fixed point of Λ. This means that $v^* = u^*$ and $w^* \in F(Mu^*)$, where $u^* \in C$, $u^* \in K(u^*)$ and it satisfies

$$\langle Au^* + B(u^*, u^*) - f, z - u^* \rangle + \varphi(u^*, z) - \varphi(u^*, u^*) + \langle M^* w^*, z - u^* \rangle \ge 0$$

for all $z \in K(u^*)$ with $w^* \in F(Mu^*)$. Finally, we conclude that $u^* \in C$ is the solution to Problem 3.1. The proof of the theorem is complete. □

The complete continuity of the map p in Lemma 3.5 of the above theorem implies the following compactness result.

Proposition 3.7 *Under Assumption 3.2, the solution set of Problem 3.1 is compact in V.*

Proof We sketch only the main steps. Let $\{u_n\}$, $n \in \mathbb{N}$, be a sequence of solutions to Problem 3.1. Thus $u_n \in C$ satisfies $u_n \in K(u_n)$ and there exists $w_n \in X$, $w_n \in F(Mu_n)$ with

$$\langle Au_n + B[u_n] - f, z - u_n\rangle + \langle w_n, Mz - Mu_n\rangle_X + \varphi(u_n, z) - \varphi(u_n, u_n) \geq 0 \tag{3.19}$$

for all $z \in K(u_n)$. We use $H(K)$ and choose $z = 0 \in K(u_n)$ in (3.19) to get

$$(m_A - b_2\|M\|^2 - \alpha_\varphi)\|u_n\|^2 \leq \big(\|A0 + f\|_{V^*} + b_1\|M\| + \alpha_\varphi\|\eta\| + c_\varphi(\|\eta\|)\big)\ \|u_n\|$$

for any $\eta \in V$. It follows from (H_0) that $\{u_n\}$ is uniformly bounded in V, and by the reflexivity of V there exists $u_0 \in V$ such that $u_n \rightharpoonup u_0$ in V. Since C is weakly closed in V, it is clear that $u_0 \in C$. From condition (m_1) of the Mosco convergence, we easily have $u_0 \in K(u_0)$. By $H(M)$, it follows that $Mu_n \to Mu_0$ in X. Using $H(F)$(ii), (iii), and selecting a subsequence, we may suppose that $w_n \rightharpoonup w_0$ in X and obtain $w_0 \in F(Mu_0)$. Next, we follow Step 3 of Theorem 3.6 and deduce that u_0 satisfies the limit inequality

$$\langle Au_0 + B[u_0] - f, z - u_0\rangle + \langle w_0, Mz - Mu_0\rangle_X + \varphi(u_0, z) - \varphi(u_0, u_0) \geq 0$$

for all $z \in K(u_0)$. Finally, as in Lemma 3.5, we use $H(A)$(ii) and $H(B)$(iii) to show the strong convergence of u_n to u_0 in V. □

3.2 Quasi-Variational-Hemivariational Inequality

We shall demonstrate that Theorem 3.6 can be used in the study of a class of quasi-variational-hemivariational inequalities. In addition to the framework used for Problem 3.1, we suppose that we are given a function $j : X \times X \to \mathbb{R}$. We consider the following problem.

Problem 3.8 Find $u \in C$ such that $u \in K(u)$ and

$$\langle Au + B[u] - f, z - u\rangle + j^0(Mu, Mu; Mz - Mu) + \varphi(u, z) - \varphi(u, u) \geq 0 \ \text{ for all } z \in K(u).$$

We need an additional hypothesis on the data.

H(j): The functional $j : X \times X \to \mathbb{R}$ is such that

(i) $j(w, \cdot)$ is locally Lipschitz continuous for all $w \in X$,
(ii) $\|\partial j(w, v)\|_X \leq d_1 + d_2\|w\|_X + d_3\|v\|_X$ for all $w, v \in X$ with $d_1, d_2, d_3 \geq 0$,
(iii) $X \times X \times X \ni (w, v, z) \mapsto j^0(w, v; z) \in \mathbb{R}$ is upper semicontinuous.

The existence result for the quasi-variational-hemivariational inequality in Problem 3.8 is a consequence of Theorem 3.6.

Theorem 3.9 *If hypotheses $H(A)$, $H(B)$, $H(C)$, $H(j)$, $H(K)$, $H(M)$, $H(\varphi)$, $H(f)$, and*

$$(d_2 + d_3)\,\|M\|^2 + \alpha_\varphi < m_A \tag{3.20}$$

hold, then the solution set of Problem 3.8 is nonempty and compact in V.

Proof First we prove the existence of a solution. Let the set-valued map $F\colon X \to 2^X$ be defined by $F(z) := \partial j(z,z)$ for $z \in X$. Here and in what follows the notation $\partial j(w,v)$ is used to denote the generalized gradient of $j(w,\cdot)$ at the point $v \in X$ for fixed $w \in X$. We claim that F satisfies hypothesis $H(F)$. For all $w, v \in X$ the set $\partial j(w,v)$ is nonempty, weakly compact, and convex in X, see, e.g. [43, Proposition 3.23(iv)]. Hence, $H(F)$(i) holds.

To show that $\mathrm{Gr}(F)$ is closed in $X \times X_w$, let $z_n \in X$, $z_n \to z$ in X, $z_n^* \in X$, $z_n^* \in F(z_n)$, and $z_n^* \rightharpoonup z^*$ in X. Then, by the definition of the generalized gradient, we have

$$\langle z_n^*, \xi\rangle \le j^0(z_n, z_n; \xi) \ \text{ for all } \ \xi \in X.$$

Passing to the upper limit, we use $H(j)$(iii) to get

$$\limsup \langle z_n^*, \xi\rangle \le \limsup j^0(z_n, z_n; \xi) \le j^0(z, z; \xi) \ \text{ for all } \ \xi \in X,$$

which means $z^* \in \partial j(z,z) = F(z)$. Thus, $H(F)$(ii) is satisfied. It is also clear that from $H(j)$(ii), we obtain

$$\|F(z)\|_X \le \|\partial j(z,z)\|_X \le d_1 + (d_2 + d_3)\|z\|_X \ \text{ for all } \ z \in X.$$

Hence, the condition $H(F)$(iii) holds with $b_1 := d_1$ and $b_2 := d_2 + d_3$.

We are now in a position to apply Theorem 3.6 and deduce that there exists $u \in C$ such that $u \in K(u)$ and there is $w \in \partial j(Mu, Mu)$, and

$$\langle Au + B[u] - f, z - u\rangle + \langle w, Mz - Mu\rangle_X + \varphi(u,z) - \varphi(u,u) \ge 0 \ \text{ for all } \ z \in K(u). \tag{3.21}$$

By the definition of the generalized gradient, we have $\langle w, \xi\rangle_X \le j^0(Mu, Mu; \xi)$ for all $\xi \in X$ and

$$\langle w, Mz - Mu\rangle_X \le j^0(Mu, Mu; Mz - Mu) \ \text{ for all } \ z \in K(u).$$

Using the latter in the inequality, we conclude that $u \in C$ solves Problem 3.8.

Next, we prove the compactness of the solution set of Problem 3.8. We exploit arguments in Step 3 of Theorem 3.6 and Proposition 3.7. For completeness we shortly provide the main steps of the proof.

• **Compactness in** V_w. Let $\{u_n\}$, $n \in \mathbb{N}$, be a sequence of solution to Problem 3.8, i.e., $u_n \in C$ with $u_n \in K(u_n)$ satisfies

$$\langle Au_n + B[u_n] - f, z - u_n\rangle + j^0(Mu_n, Mu_n; Mz - Mu_n) + \varphi(u_n, z) - \varphi(u_n, u_n) \ge 0 \tag{3.22}$$

for all $z \in K(u_n)$. Testing (3.22) with $z = 0 \in K(u_n)$, we use $H(A)$(ii), $H(B)$(ii), $H(j)$(ii), (3.4), and [43, Proposition 3.23(iii)] to obtain

$$(m_A - (d_2 + d_3)\|M\|^2 - \alpha_\varphi)\|u_n\|^2 \le \big(\|A0 - f\|_{V^*} + d_1\|M\| + \alpha_\varphi\|\eta\| + c_\varphi(\|\eta\|)\big)\ \|u_n\|$$

for any $\eta \in V$. From (3.20), we deduce that $\{u_n\}$ is uniformly bounded in V. So we may suppose that, at least for a subsequence, it holds $u_n \rightharpoonup u_0$ in V. By the weak closedness of C, we have $u_0 \in C$. It is clear from (m_1) of the Mosco convergence that $u_0 \in K(u_0)$. The hypothesis $H(M)$ gives $Mu_n \to Mu_0$ in X.

Let us fix $z \in K(u_0)$. We use (m_2) in the Mosco convergence for $z \in K(u_0)$ and $u_0 \in K(u_0)$ to find sequences $\{z_n\}$ and $\{\eta_n\}$ with

$$z_n, \eta_n \in K(u_n) \ \text{ such that } \ z_n \to z \ \text{ and } \ \eta_n \to u_0 \ \text{ in } \ V, \text{ as } n \to \infty. \tag{3.23}$$

We test (3.22) with $z=\eta_n$ to get

$$\langle Au_n+B[u_n],u_n-\eta_n\rangle \le \langle f,u_n-\eta_n\rangle + j^0(Mu_n,Mu_n;M\eta_n-Mu_n)+\varphi(u_n,\eta_n) - \varphi(u_n,u_n). \tag{3.24}$$

From $H(j)$(iii), we have

$$\limsup j^0(Mu_n,Mu_n;M\eta_n-Mu_n)\le j^0(Mu_0,Mu_0;Mu_0-Mu_0)=0. \tag{3.25}$$

As in (3.11), we obtain

$$\limsup \big(\varphi(u_n,\eta_n)-\varphi(u_n,u_n)\big)\le \varphi(u_0,u_0)-\varphi(u_0,u_0)=0. \tag{3.26}$$

We pass to the upper limit in (3.24), and use $H(B)$(iii), (3.25), (3.26) to deduce

$$\begin{aligned}\limsup\langle Au_n,u_n-u_0\rangle &= \limsup\langle Au_n+B[u_n],u_n-\eta_n\rangle\\ &\quad+\limsup\langle Au_n+B[u_n],\eta_n-u_0\rangle+\lim\langle B[u_n],u_0-u_n\rangle\le 0.\end{aligned}$$

We invoke $H(A)$(i) and have

$$\langle Au_0,u_0-v\rangle\le\liminf\langle Au_n,u_n-v\rangle \ \text{ for all } \ v\in V. \tag{3.27}$$

Next, let us choose $z=z_n$ in (3.22) and combine the latter with (3.27) to get

$$\begin{aligned}\langle Au_0,u_0-z\rangle &\le\liminf\langle Au_n,u_n-z\rangle\le\limsup\langle Au_n,u_n-z\rangle\\ &=\limsup\langle Au_n,u_n-z_n\rangle+\lim\langle Au_n,z_n-z\rangle\\ &=\limsup\langle Au_n,u_n-z_n\rangle\\ &\le\limsup\langle B[u_n]-f,z_n-u_n\rangle+\limsup j^0(Mu_n,Mu_n;Mz_n-Mu_n)\\ &+\limsup\big(\varphi(u_n,z_n)-\varphi(u_n,u_n)\big)\le\langle B[u_0]-f,z-u_0\rangle\\ &+j^0(Mu_0,Mu_0;Mz-Mu_0)+\varphi(u_0,z)-\varphi(u_0,u_0).\end{aligned}$$

Since $z\in K(u_0)$ is arbitrary, we conclude that $u\in C$ is a solution to Problem 3.8.

• **Compactness in** V. We show $\|u_n-u_0\|\to 0$. We apply condition (m_2) of the Mosco convergence for $u_0\in K(u_0)$ to find a sequence $\{\eta_n\}\subset K(u_n)$ such that $\eta_n\to u_0$ in V as $n\to\infty$. We choose $z=\eta_n$ as a test function in (3.22) and use it in the following inequality

$$\begin{aligned}&\limsup\langle Au_n+B[u_n],u_n-u_0\rangle\\ &\le\limsup\langle Au_n+B[u_n],u_n-\eta_n\rangle+\limsup\langle Au_n+B[u_n],\eta_n-u_0\rangle\\ &\le\limsup\langle f,u_n-\eta_n\rangle+\limsup j^0(Mu_n,Mu_n;M\eta_n-Mu_n)\\ &\quad+\limsup(\varphi(u_n,\eta_n)-\varphi(u_n,u_n))+\limsup\langle Au_n+B[u_n],\eta_n-u_0\rangle\le 0.\end{aligned}$$

The key point is to apply the strong monotonicity of $A(\cdot)+B[\cdot]$ with constant $m_A>0$. We have

$$m_A\limsup\|u_n-u_0\|^2\le\limsup\langle Au_n+B[u_n]-(Au_0+B[u_0]),u_n-u_0\rangle$$

$$\leq \limsup \langle Au_n + B[u_n], u_n - u_0 \rangle - \liminf \langle Au_0 + B[u_0], u_n - u_0 \rangle$$
$$\leq 0.$$

Hence we conclude that $u_n \to u_0$ in V. This completes the proof of the compactness of the solution set of Problem 3.8 in V. □

The following is a consequence of the first part of Theorem 3.9.

Remark 3.10 Let the solution set of Problem 3.1 with $F(z) = \partial j(z, z)$ for $z \in X$ be denoted by $\mathbb{S}_1$, and the solution set of Problem 3.8 be denoted by $\mathbb{S}_2$. Under the hypotheses of Theorem 3.9 it is clear that $\emptyset \neq \mathbb{S}_1 \subset \mathbb{S}_2$. For the converse implication, we still do not have a clear answer, and it remains an interesting question of further research. We are able to show the converse implication in the following particular cases.

Case (i). Let $j := 0$. Then $F(\cdot)$ reduces to $\{0\}$ and Problem 3.8 with $j = 0$ is equivalent to: Find $u \in C$ such that $u \in K(u)$ and there is $w \in F(Mu) = \{0\}$, with

$$\langle Au + B[u] - f, z - u \rangle + \varphi(u, z) - \varphi(u, u) \geq 0 \;\text{ for all }\; z \in K(u).$$

Hence we get Problem 3.1.

Case (ii). Let $\varphi := 0$ and $K(\cdot) := V$. Let $u \in C$ be a solution of Problem 3.8. This means $u \in K(u)$ and

$$\langle Au + B[u] - f, z - u \rangle + j^0(Mu, Mu; Mz - Mu) \geq 0 \;\text{ for all }\; z \in V.$$

We set $z := v + u$ for $v \in V$ to obtain

$$\langle Au + B[u] - f, v \rangle + j^0(Mu, Mu; Mv) \geq 0 \;\text{ for all }\; v \in V.$$

Equivalently, we have $j^0(Mu, Mu; Mv) \geq \langle f - Au - B[u], v \rangle$ for all $v \in V$. Thus $f - Au - B[u] \in F(Mu)$. So there is $w \in F(Mu)$ such that $f - Au - B[u] = w$, and therefore $u \in C$ is a solution to Problem 3.1.

We conclude this section with comments on the functional j and a particular case of Problem 3.8.

Remark 3.11 (1) Note that under $H(j)$(i) and (ii), condition $H(j)$(iii) is equivalent to say that the generalized gradient operator $\partial j \colon X \times X \to 2^X$ has a closed graph in the $X \times X \times X_w$ topology. If j is independent of the first argument, condition $H(j)$(iii) is automatically satisfied, by [43, Proposition 3.23(ii)].

(2) In hypothesis $H(j)$ we do not require the so-called *relaxed monotonicity condition* of the generalized gradient, extensively used in the literature for hemivariational inequalities, see, for instance, [17, 43, 44, 57]. The relaxed monotonicity hypothesis on the locally Lipschitz continuous function $j \colon X \to \mathbb{R}$ is stated as follows, see [57, p. 124]: There is a constant $\alpha_j \geq 0$ such that

$$\langle v_1^* - v_2^*, v_1 - v_2 \rangle \geq -\alpha_j \, \|v_1 - v_2\|_X^2 \;\text{ for all }\; v_i^* \in \partial j(v_i),\; v_i \in X,\; i = 1, 2$$

or equivalently

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \, \|v_1 - v_2\|_X^2 \;\text{ for all }\; v_i \in X,\; i = 1, 2.$$

We have removed the relaxed monotonicity condition in $H(j)$, which permits to cover a wider class of boundary conditions involving nonmonotone laws described by nonconvex and non-differentiable functions.

(3) Further, if, in addition to $H(j)$, the function $j(w, \cdot)$ is supposed to be convex, then Problem 3.8 reduces to the following elliptic quasi-variational inequality of the second kind: Find $u \in C$ such that $u \in K(u)$ and

$$\langle Au + B[u] - f, z - u\rangle + \psi(u, z) - \psi(u, u) \geq 0 \ \text{ for all } \ z \in K(u),$$

where $\psi(u, z) = j(Mz, Mu) + \varphi(u, z)$.

4 An Incompressible Navier-Stokes Equation for Bingham Fluid

In this section we analyse a mathematical model for the incompressible Navier-Stokes equation with mixed boundary conditions which naturally leads to an elliptic quasi-variational-hemivariational inequality with an implicit constraint set. We state the physical model, discuss its ingredients, formulate hypotheses, and provide the variational formulation.

4.1 Problem Statement

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with Lipschitz boundary Γ. The boundary is partitioned into two disjoint and measurable sets Γ_0 and Γ_1 such that their $(d–1)$-dimensional Hausdorff measure is denoted by $|\Gamma_0| > 0$ and $|\Gamma_1| > 0$, respectively. The classical formulation of the steady-state flow problem is the following.

Problem 4.1 Find a flow velocity $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, an extra stress tensor $\mathbb{S}\colon \mathbb{M}^d \to \mathbb{M}^d$, and a pressure $p\colon \Omega \to \mathbb{R}$ such that $\boldsymbol{u} \in U(\boldsymbol{u})$ and

$$-\operatorname{Div}\mathbb{S} + \operatorname{Div}(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = \boldsymbol{f} \qquad \text{in } \ \Omega, \tag{4.1}$$

$$\begin{cases} \mathbb{S} = \mathbb{T}(\mathbb{D}\boldsymbol{u}) + g\,\dfrac{\mathbb{D}\boldsymbol{u}}{\|\mathbb{D}\boldsymbol{u}\|} & \text{if } \ \mathbb{D}\boldsymbol{u} \neq \boldsymbol{0} \\ \|\mathbb{S}\| \leq g & \text{if } \ \mathbb{D}\boldsymbol{u} = \boldsymbol{0} \end{cases} \qquad \text{in } \ \Omega, \tag{4.2}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \ \Omega, \tag{4.3}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \ \Gamma_0, \tag{4.4}$$

$$\begin{cases} u_\nu = 0 \\ -\boldsymbol{\tau}_\tau(\boldsymbol{u}) \in k(\boldsymbol{u}_\tau)\partial j_\tau(\boldsymbol{u}_\tau) \end{cases} \qquad \text{on } \ \Gamma_1. \tag{4.5}$$

The objects in Problem 4.1 are subsequently introduced. Let

$$\begin{aligned} \widetilde{V} &:= \{\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega}; \mathbb{R}^d) \mid \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} = 0 \text{ on } \Gamma_0,\ v_\nu = 0 \text{ on } \Gamma_1\}, \\ V &:= \text{ closure of } \widetilde{V} \text{ in } H^1(\Omega; \mathbb{R}^d), \qquad H = L^2(\Omega; \mathbb{R}^d), \end{aligned} \tag{4.6}$$

and $U\colon V \to 2^V$ be a set-valued map defined by

$$U(\boldsymbol{u}) := \{\boldsymbol{v} \in V \mid r(\boldsymbol{v}) \leq m(\boldsymbol{u})\} \ \text{ for } \ \boldsymbol{u} \in V. \tag{4.7}$$

On the space V we consider two norms, the standard one $\|\boldsymbol{v}\| = \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)}$, and the norm given by $\|\boldsymbol{v}\|_V = \|\mathbb{D}\,\boldsymbol{v}\|_{L^2(\Omega;\mathbb{M}^d)}$ for $\boldsymbol{v} \in V$. By Korn's inequality, see, e.g., [18, Theorem 8], these norms are equivalent on V. It is well known that the trace operator denoted by

$$\gamma\colon V \subset H^1(\Omega;\mathbb{R}^d) \to L^2(\Gamma;\mathbb{R}^d) \tag{4.8}$$

is linear, continuous and compact, see [43, Theorem 2.21]. For simplicity, instead of $\gamma\boldsymbol{v}$, we often write $\boldsymbol{v}$. We denote its norm in the space $\mathcal{L}(V, L^2(\Gamma;\mathbb{R}^d))$ by $\|\gamma\|$.

Let $\mathbb{M}^d$ denote the class of symmetric $d \times d$ matrices,

$$\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\top}) \tag{4.9}$$

represent the symmetric part of the velocity gradient, the *total stress* tensor is defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = -p\,\mathbb{I} + \mathbb{S}(\mathbb{D}\boldsymbol{u}) \ \text{ in } \ \Omega,$$

where $\mathbb{S}$ is the *extra stress tensor* and $\mathbb{I}$ is the $d \times d$ identity matrix. Denoting by $\boldsymbol{\nu}$ the unit *outward normal* on Γ, the *traction vector* is given by $\boldsymbol{\tau}(\boldsymbol{u}, p) = \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{\nu}$ on the boundary. The normal and tangential components of the velocity and of the traction on the boundary are denoted by $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, and by $\tau_\nu(\boldsymbol{u}, p) = \boldsymbol{\tau}(\boldsymbol{u}, p)\cdot\boldsymbol{\nu}$, $\boldsymbol{\tau}_\tau(\boldsymbol{u}) = \boldsymbol{\tau}(\boldsymbol{u}, p) - \tau_\nu(\boldsymbol{u}, p)\boldsymbol{\nu}$, respectively. We have

$$\mathbb{S}_\nu(\boldsymbol{u}) = \tau_\nu(\boldsymbol{u}, p) + p, \quad \mathbb{S}_\tau(\boldsymbol{u}) = \boldsymbol{\tau}_\tau(\boldsymbol{u}) \ \text{ on } \ \Gamma. \tag{4.10}$$

Also, Div and div are the divergence operators for tensor and vector valued functions, respectively, i.e., $\mathrm{Div}(\mathbb{S})_i := \sum_{j=1}^d \frac{\partial}{\partial x_j}\mathbb{S}_{ij}$ and $\mathrm{div}(\boldsymbol{u}) := \sum_{i=1}^d \frac{\partial}{\partial x_i}u_i$. Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d$, their tensor product is the second-order tensor $\boldsymbol{z}$ defined by $\boldsymbol{z} = \boldsymbol{v}\otimes\boldsymbol{w} = (v_i w_j)_{1\le i,j\le d}$. For simplicity, we often do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega\cup\Gamma$. The inner products and norms on $\mathbb{R}^d$ and $\mathbb{M}^d$ are denoted by the standard notation, and very often the subscripts are omitted.

We need the following hypotheses on the data of Problem 4.1.

Assumption 4.2 $\mathbf{H}(\mathbb{T})$: The function $\mathbb{T}\colon \Omega\times\mathbb{M}^d \to \mathbb{M}^d$ is such that

(i) $\mathbb{T}(\cdot, \mathbb{E})$ is measurable on Ω for all $\mathbb{E} \in \mathbb{M}^d$,
(ii) $\mathbb{T}(\boldsymbol{x}, \cdot)$ is continuous on $\mathbb{M}^d$ for a.e. $\boldsymbol{x} \in \Omega$, and $\mathbb{T}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0}$ for a.e. $\boldsymbol{x} \in \Omega$,
(iii) $\|\mathbb{T}(\boldsymbol{x}, \mathbb{E})\| \le a_1(x) + a_2\,\|\mathbb{E}\|$ for all $\mathbb{E} \in \mathbb{M}^d$, a.e. $\boldsymbol{x} \in \Omega$ with $a_1 \in L^2(\Omega)$, $a_1, a_2 > 0$,
(iv) $(\mathbb{T}(\boldsymbol{x}, \mathbb{E}_1) - \mathbb{T}(\boldsymbol{x}, \mathbb{E}_2)) : (\mathbb{E}_1 - \mathbb{E}_2) \ge \alpha\,\|\mathbb{E}_1 - \mathbb{E}_2\|^2$ for all $\mathbb{E}_1, \mathbb{E}_2 \in \mathbb{M}^d$, a.e. $\boldsymbol{x} \in \Omega$ with $\alpha > 0$.

$\mathbf{H}(\mathbf{f}, \mathbf{g})$: Let $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$, $g \in L^2(\Omega)$, $g \ge 0$.
$\mathbf{H}(\mathbf{j}_\tau)$: The function $j_\tau\colon \Gamma_1\times\mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j_\tau(\cdot, \boldsymbol{\xi})$ is measurable on Γ_1 for all $\boldsymbol{\xi} \in \mathbb{R}^d$,
(ii) $j_\tau(\boldsymbol{x}, \cdot)$ is locally Lipschitz continuous for a.e. $\boldsymbol{x} \in \Gamma_1$,
(iii) $\|\partial j_\tau(\boldsymbol{x}, \boldsymbol{\xi})\| \le b_1(x) + b_2\,\|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_1$ with $b_1 \in L^2(\Gamma_1)$, $b_1, b_2 \ge 0$,
(iv) $j_\tau(\boldsymbol{x}, \cdot)$ or $-j_\tau(\boldsymbol{x}, \cdot)$ is regular for a.e. $\boldsymbol{x} \in \Gamma_1$ (see Sect. 2).

$\mathbf{H}(\mathbf{k})$: The function $k\colon \Gamma_1\times\mathbb{R}^d \to \mathbb{R}$ is such that

(i) $k(\cdot, \boldsymbol{\xi})$ is measurable on Γ_1 for all $\boldsymbol{\xi} \in \mathbb{R}^d$,

(ii) $k(\boldsymbol{x}, \cdot)$ is continuous on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_1$,
(iii) $0 < k_0 \leq k(\boldsymbol{x}, \boldsymbol{\xi}) \leq k_1$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_1$ for some $k_0, k_1 \in \mathbb{R}$.

$\mathbf{H}(\mathbf{r}, \mathbf{m})$: The functions $r\colon V \to \mathbb{R}$ and $m\colon H \to \mathbb{R}$ are such that

(i) r is positively homogeneous, convex and lower semicontinuous,
(ii) m is continuous, $m_0 := \inf_{v \in H} m(v) > 0$ and $r(0) \leq m_0$.

4.2 Physical Interpretation

The stationary Navier-Stokes system of equations (4.1) represents the conservation law, $\boldsymbol{f}$ is the volume density of given forces, and $\mathrm{Div}(\boldsymbol{u} \otimes \boldsymbol{u})$ represents the convective (inertia) term. The Bingham constitutive law (4.2) is a nonlinear relation between the extra stress tensor $\mathbb{S}$ and the symmetric part of the velocity gradient $\mathbb{D}\boldsymbol{u}$ in which g stands for the plasticity threshold (yield stress). It states that the norm of the extra stress is limited by a maximal value g called the yield limit. If the strict inequality is satisfied at low stress, there are no deformations and the fluid behave as a rigid body. If equality holds at high stress, then the body initiates to behave as a fluid. The examples of the constitutive function $\mathbb{T}\colon \Omega \times \mathbb{M}^d \to \mathbb{M}^d$ in (4.2) can be found in, e.g., [2, 5, 12, 40]. The mathematical models that involve the function $\mathbb{T}$ have the form

$$\mathbb{T}(\boldsymbol{x}, \mathbb{E}) = \mu(\|\mathbb{E}\|)\mathbb{E} \quad \text{for} \quad \mathbb{E} \in \mathbb{M}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \tag{4.11}$$

where $\mu\colon [0, \infty) \to \mathbb{R}$ is a given viscosity function, are called generalized Newtonian fluids. If $\mu(r) = \mu_0$ for $r \geq 0$ with $\mu_0 > 0$ a given viscosity constant, then (4.11) reduces to $\mathbb{T}(\boldsymbol{x}, \mathbb{E}) = \mu_0 \mathbb{E}$. This is the linear law for the usual Newtonian fluid, and it clearly satisfies $H(\mathbb{T})$. The constitutive law (4.2) reduces to the Bingham model of Newtonian fluid (if $\mu(r) = \mu_0$ for $r \geq 0$) and to the Navier-Stokes system (when $g = 0$). Examples of Bingham fluids include cosmetics and personal care products, water suspensions of clay, concrete, cements, volcanic lava and magmas, etc. The Bingham visco-plactic flows appear in drilling engineering and industrial models including heavy oils in reservoirs, water within clay soils, drilling mud, ceramic pastes, sewage sludges and processes of fast material working, see, for instance, [13, 29, 55], and [14] for a review on numerical simulations.

The solenoidal (divergence free) condition (4.3) states that the fluid is incompressible. The homogeneous Dirichlet boundary condition (4.4) means that the fluid adheres to the wall, see [30]. The first condition in (4.5) is called the no leak (impermeability) boundary condition and the second one is the multivalued nonmonotone slip condition. Here we give some examples.

(a) The classical Navier slip condition of the form $\boldsymbol{\tau}_\tau(\boldsymbol{u}) = -\kappa\, \boldsymbol{u}_\tau$ on Γ_1 states that the tangential velocity is proportional to the shear stress. It is a prototype of the condition (4.5) and was introduced in [51]. It corresponds to (4.5) with the quadratic potential $j_\tau(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2}\kappa \|\boldsymbol{\xi}\|^2$ for $\boldsymbol{x} \in \Gamma_1$, $\boldsymbol{\xi} \in \mathbb{R}^d$, and an adhesive constant $\kappa > 0$. It is clear that in this law a slip is instantaneous whenever $\boldsymbol{\tau}_\tau \neq \mathbf{0}$, see [37].

(b) The following model is based on the Tresca friction law, see, e.g., [43], and was proposed in [22]:

$$u_\nu = 0, \quad \|\boldsymbol{\tau}_\tau\|_{\mathbb{R}^d} \leq g_1, \quad \boldsymbol{\tau}_\tau(\boldsymbol{u}) \cdot \boldsymbol{u}_\tau + g_1 \|\boldsymbol{u}_\tau\|_{\mathbb{R}^d} = 0 \text{ on } \Gamma_1. \tag{4.12}$$

Here, the modulus of friction g_1 is assumed to be a continuous and strictly positive function. The nonlinear slip boundary conditions of frictional type in (4.12) can be equivalently rewritten as

$$u_\nu = 0, \quad -\boldsymbol{\tau}_\tau \in g_1\,\partial \|\boldsymbol{u}_\tau\|_{\mathbb{R}^d} \ \text{ on } \ \Gamma_1. \tag{4.13}$$

This entails that (4.12) is a particular case of the boundary condition (4.5) with $k(\boldsymbol{x},\boldsymbol{\xi}) = g_1(\boldsymbol{x})$ and $j_\tau(\boldsymbol{x},\boldsymbol{\xi}) = \|\boldsymbol{\xi}\|_{\mathbb{R}^d}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_1$.

(c) The following different example of boundary condition (4.5) has been treated in [36]:

$$\begin{cases} \|\boldsymbol{\tau}_\tau(\boldsymbol{u})\|_{\mathbb{R}^d} \le \alpha + \beta(\|\boldsymbol{u}_\tau - \boldsymbol{w}_\tau\|_{\mathbb{R}^d}) \\ \boldsymbol{\tau}_\tau(\boldsymbol{u}) \cdot (\boldsymbol{u}_\tau - \boldsymbol{w}_\tau) = -(\alpha + \beta(\|\boldsymbol{u}_\tau - \boldsymbol{w}_\tau\|_{\mathbb{R}^d}))\|\boldsymbol{u}_\tau - \boldsymbol{w}_\tau\|_{\mathbb{R}^d} \end{cases} \text{on } \ \Gamma_1, \tag{4.14}$$

where $\alpha\colon \Gamma_1 \to (0,\infty)$ and $\beta\colon \Gamma_1 \times [0,\infty) \to [0,\infty)$ are prescribed functions such that for a.e. $\boldsymbol{x} \in \Gamma_1$, $\beta(\boldsymbol{x}, r) = 0$ if and only if $r = 0$, while $\boldsymbol{w}_\tau$ denotes the tangential velocity of the wall surface at Γ_1. The condition (4.14) is motivated by a generalization of three slip boundary conditions. They are: The Navier slip condition in [51], the nonlinear Navier-type slip conditions used to model the wall slip of non-Newtonian fluids in [36], and the threshold slip condition of *"friction type"* studied by Fujita et al. [22–24, 53, 54].

Note that the nonlinear Navier-Fujita slip condition (4.14) is again a particular case of condition (4.5) in Problem 4.1 with functions $k(\boldsymbol{x},\boldsymbol{\xi}) = \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x}, \|\boldsymbol{\xi}\|_{\mathbb{R}^d})$ and $j_\tau(\boldsymbol{x},\boldsymbol{\xi}) = \|\boldsymbol{\xi}\|_{\mathbb{R}^d}$ for a.e. $\boldsymbol{x} \in \Gamma_1$, all $\boldsymbol{\xi} \in \mathbb{R}^d$. The function j_τ satisfies hypothesis $H(j_\tau)$ below with $b_1(x) = 1$ and $b_2 = 0$. The slip boundary conditions of frictional type have been studied widely for various fluid models, see, e.g., [23, 30, 31, 34, 38, 60].

(d) The condition (4.5) is much more general than (4.14) since it involves nonmonotone relations (graphs) described by nonconvex superpotentials j_τ, for concrete examples, see [41, Example (60)], and the references therein. This type of the slip condition may appear when the part Γ_1 of the boundary is rough. It may result in a law in which the tangential traction is a decreasing function of the tangential velocity. Choosing various nonconvex locally Lipschitz continuous functions j_τ, we obtain nonsmooth and nonmonotone extensions of the slip boundary conditions discussed shortly above.

We comment on the implicit constraint set in (4.7). A natural choice for the function $r\colon V \to \mathbb{R}$ can be the rate dissipation energy (or drag) function $r(\boldsymbol{v}) = \frac{\nu_0}{2}\int_\Omega \|\mathbb{D}\boldsymbol{v}\|^2\,dx$ which measures the drug due to viscosity, $\nu_0 > 0$ is the viscosity coefficient. Another possibility is the vorticity function $r(\boldsymbol{v}) = \int_\Omega \|\mathrm{rot}\,\boldsymbol{v}\|^2\,dx$ which measures the turbulence in the flow through the L^2-norm in space of the vorticity. We can also select the velocity tracking function $r(\boldsymbol{v}) = \int_\Omega \|\boldsymbol{v} - \boldsymbol{v}_0\|^2\,dx$, see [1], [27, p.192], [58, Sect. 7.4.3] and the references therein. A standard option for the function $m\colon H = L^2(\Omega;\mathbb{R}^d) \to \mathbb{R}$ can be $m(\boldsymbol{v}) = a + \int_\Omega \|\boldsymbol{v}(x)\|\varrho(x)\,dx$ with $\varrho \in H$, $\varrho \ge 0$, and $a > 0$.

4.3 Variational Formulation

We now derive the variational formulation of Problem 4.1. Let $\boldsymbol{u}$, $\mathbb{S}$ and p be sufficiently smooth functions which satisfy (4.1)–(4.5). Let $\boldsymbol{v} \in U(\boldsymbol{u}) \subset V$. We multiply the equation (4.1) by $\boldsymbol{v} - \boldsymbol{u}$ and integrate over Ω to obtain

$$\int_\Omega (-\operatorname{Div}\mathbb{S}) \cdot (\boldsymbol{v} - \boldsymbol{u})\,dx + \int_\Omega \operatorname{Div}(\boldsymbol{u} \otimes \boldsymbol{u}) \cdot (\boldsymbol{v} - \boldsymbol{u})\,dx + \int_\Omega \nabla p \cdot (\boldsymbol{v} - \boldsymbol{u})\,dx$$
$$= \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u})\,dx. \tag{4.15}$$

We denote by I_1, I_2 and I_3 the corresponding terms on the left hand side of (4.15). We apply the following Green formulas

$$\int_\Omega (u\,\mathrm{div}\,\boldsymbol{v} + \nabla u \cdot \boldsymbol{v})\,dx = \int_\Gamma u\,(\boldsymbol{v}\cdot\boldsymbol{\nu})\,d\Gamma \ \text{ for } \ u \in H^1(\Omega),\ \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d),$$

$$\int_\Omega (\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}) + \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v})\,dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma \ \text{ for } \ \boldsymbol{v} \in H^1(\Omega),\ \boldsymbol{\sigma} \in H^1(\Omega;\mathbb{S}^d),$$

see [43, Theorems 2.24 and 2.25]. We obtain

$$I_1 = \int_\Omega \mathbb{S} : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx - \int_{\partial\Omega} (\mathbb{S}\,\boldsymbol{\nu})\cdot(\boldsymbol{v}-\boldsymbol{u})\,d\Gamma,$$

$$I_2 = -\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + \int_{\partial\Omega} (\boldsymbol{u}\otimes\boldsymbol{u})\nu\cdot(\boldsymbol{v}-\boldsymbol{u})\,d\Gamma,$$

$$I_3 = -\int_\Omega \mathrm{div}(\boldsymbol{v}-\boldsymbol{u})\,p\,dx + \int_{\Gamma_0\cup\Gamma_1} (v_\nu - u_\nu)\,p\,d\Gamma.$$

For I_1 and I_3 we use the properties $\mathrm{div}\,\boldsymbol{u} = \mathrm{div}\,\boldsymbol{v} = 0$ in Ω, $\boldsymbol{u} = \boldsymbol{v} = 0$ on Γ_0, and $u_\nu = v_\nu = 0$ on Γ_1. It is clear that $I_3 = 0$. From the decomposition formula, see [43, (6.33)] and (4.10), we get

$$\int_{\partial\Omega} (\mathbb{S}\,\boldsymbol{\nu})\cdot(\boldsymbol{v}-\boldsymbol{u})\,d\Gamma = \int_{\Gamma_1} \boldsymbol{\tau}_\tau(\boldsymbol{u})\cdot(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\,d\Gamma.$$

So, we have

$$I_1 + I_3 = \int_\Omega \mathbb{S} : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx - \int_{\Gamma_1} \boldsymbol{\tau}_\tau(\boldsymbol{u})\cdot(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\,d\Gamma. \tag{4.16}$$

Subsequently, we consider I_2. We employ the relation $(\boldsymbol{u}\otimes\boldsymbol{u})\nu\cdot(\boldsymbol{v}-\boldsymbol{u}) = (\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u}))\,u_\nu$ on $\partial\Omega$ to have

$$I_2 = -\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + \int_{\partial\Omega} (\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u}))\,u_\nu\,d\Gamma.$$

Let $b\colon H^1(\Omega;\mathbb{R}^d)^3 \to \mathbb{R}$ be the trilinear form defined by

$$b(\boldsymbol{v};\boldsymbol{u},\boldsymbol{z}) = \sum_{i,j=1}^d \int_\Omega v_j\,\frac{\partial u_i}{\partial x_j}\,z_i\,dx \quad \text{for} \quad \boldsymbol{v},\boldsymbol{u},\boldsymbol{z} \in H^1(\Omega;\mathbb{R}^d). \tag{4.17}$$

On the other hand, Property (vi) in Appendix and $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω imply

$$\begin{aligned} b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) &= -\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx - \int_\Omega (\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u}))\,\mathrm{div}\,\boldsymbol{u}\,dx \\ &\quad + \int_{\partial\Omega} (\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u}))\,u_\nu\,d\Gamma \\ &= -\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + \int_{\partial\Omega} (\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u}))\,u_\nu\,d\Gamma. \end{aligned}$$

Thus, we deduce $I_2 = b(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u})$. Hence, by the relation $I_1 + I_2 + I_3 = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,dx$, we have

$$\int_\Omega \mathbb{S} : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - \int_{\Gamma_1} \boldsymbol{\tau}_\tau(\boldsymbol{u})\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau)\,d\Gamma = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,dx. \quad (4.18)$$

Let $\Omega_+ := \{\boldsymbol{x}\in\Omega \mid \|\mathbb{D}\boldsymbol{u}(\boldsymbol{x})\| > 0\}$ and $\Omega_0 := \{\boldsymbol{x}\in\Omega \mid \|\mathbb{D}\boldsymbol{u}(\boldsymbol{x})\| = 0\}$. We use condition (4.2) and the Cauchy-Schwarz inequality to get

$$\begin{aligned}\int_{\Omega_+} \mathbb{S} : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx &= \int_{\Omega_+}\left(\mathbb{T}(\mathbb{D}\boldsymbol{u}) + g\,\frac{\mathbb{D}\boldsymbol{u}}{\|\mathbb{D}\boldsymbol{u}\|}\right) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx \\ &= \int_{\Omega_+} \mathbb{T}(\mathbb{D}\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + \int_{\Omega_+} g\,\frac{\mathbb{D}\boldsymbol{u}}{\|\mathbb{D}\boldsymbol{u}\|} : \mathbb{D}\boldsymbol{v}\,dx - \int_{\Omega_+} g\,\|\mathbb{D}\boldsymbol{u}\|\,dx \\ &\le \int_\Omega \mathbb{T}(\mathbb{D}\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + \int_{\Omega_+} g\,\|\mathbb{D}\boldsymbol{v}\|\,dx - \int_\Omega g\,\|\mathbb{D}\boldsymbol{u}\|\,dx.\end{aligned} \quad (4.19)$$

On the other hand, by the condition $\|\mathbb{S}\| \le g$ in Ω_0, we have

$$\int_{\Omega_0} \mathbb{S} : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx = \int_{\Omega_0} \mathbb{S} : \mathbb{D}\boldsymbol{v}\,dx \le \int_{\Omega_0} \|\mathbb{S}\|\,\|\mathbb{D}\boldsymbol{v}\|\,dx \le \int_{\Omega_0} g\,\|\mathbb{D}\boldsymbol{v}\|\,dx. \quad (4.20)$$

Adding the inequalities (4.19) and (4.20), we deduce

$$\int_\Omega \mathbb{S} : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx \le \int_\Omega \mathbb{T}(\mathbb{D}\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + \int_\Omega g\,(\|\mathbb{D}\boldsymbol{v}\| - \|\mathbb{D}\boldsymbol{u}\|)\,dx. \quad (4.21)$$

From the condition (4.5) and the definition of the subgradient, it follows that

$$-\boldsymbol{\tau}_\tau(\boldsymbol{u})\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau) \le k(\boldsymbol{u}_\tau)\,j_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau-\boldsymbol{u}_\tau) \ \text{ on } \ \Gamma_1. \quad (4.22)$$

We use the inequalities (4.21) and (4.22) in (4.18), and arrive at the following variational formulation of Problem 4.1.

Problem 4.3 Find a flow velocity $\boldsymbol{u} \in U(\boldsymbol{u})$ such that

$$\begin{aligned}&\int_\Omega \mathbb{T}(\mathbb{D}\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v}-\boldsymbol{u})\,dx + b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + \int_\Omega g\,(\|\mathbb{D}\boldsymbol{v}\| - \|\mathbb{D}\boldsymbol{u}\|)\,dx \\ &+ \int_{\Gamma_1} k(\boldsymbol{u}_\tau)\,j_\tau^0(\boldsymbol{u}_\tau;\boldsymbol{v}_\tau-\boldsymbol{u}_\tau)\,d\Gamma \ge \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u})\,dx \ \text{ for all } \ \boldsymbol{v}\in U(\boldsymbol{u}).\end{aligned}$$

Problem 4.3 is called a nonlinear *elliptic quasi-variational-hemivariational inequality* with implicit constraints. Note that in Problem 4.3 the pressure and the extra stress have been eliminated. We conjecture that any smooth solution to Problem 4.3 is a solution to Problem 4.1. The recovery of the associated pressure and the extra stress, and interpretation of slip boundary condition in (4.5) from the weak formulation is an interesting open problem.

5 Solvability of the Bingham Model

We shall prove the following result on the solution set of Problem 4.3.

Theorem 5.1 *Under Assumption 4.2 and the following smallness condition*

$$\sqrt{2}\, b_2 k_1 \, \|\gamma\|^2 < \alpha, \tag{5.1}$$

the set of solutions to Problem 4.3 is nonempty and compact in V.

Proof Let $X = L^2(\Gamma_1; \mathbb{R}^d)$ and $C = V$. We introduce the following operators and functions defined by

$$A\colon V \to V^*, \ \ \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \mathbb{T}(\mathbb{D}\boldsymbol{u}) : \mathbb{D}\boldsymbol{v}\, dx, \ \ \boldsymbol{u}, \boldsymbol{v} \in V, \tag{5.2}$$

$$B\colon V \to V^*, \ \ \langle B[\boldsymbol{u}], \boldsymbol{v}\rangle = -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \mathbb{D}\boldsymbol{v}\, dx, \ \ \boldsymbol{u}, \boldsymbol{v} \in V, \tag{5.3}$$

$$J\colon X \times X \to \mathbb{R}, \ \ J(\boldsymbol{w}, \boldsymbol{u}) = \int_{\Gamma_1} k(\boldsymbol{w})\, j_\tau(\boldsymbol{u})\, d\Gamma, \ \ \boldsymbol{w}, \boldsymbol{u} \in X, \tag{5.4}$$

$$\varphi\colon V \to \mathbb{R}, \ \ \varphi(\boldsymbol{v}) = \int_\Omega g\, \|\mathbb{D}\boldsymbol{v}\|\, dx, \ \ \boldsymbol{v} \in V, \tag{5.5}$$

$$\boldsymbol{f}_1 \in V^*, \ \ \langle \boldsymbol{f}_1, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \ \ \boldsymbol{v} \in V, \tag{5.6}$$

$$M\colon V \to X, \ \ M\boldsymbol{v} = \boldsymbol{v}_\tau, \ \ \boldsymbol{v} \in V. \tag{5.7}$$

We use the notation (5.2)–(5.7) and consider the following auxiliary quasi-variational-hemivariational inequality: Find $\boldsymbol{u} \in V$ such that $\boldsymbol{u} \in U(\boldsymbol{u})$ and

$$\langle A\boldsymbol{u} + B[\boldsymbol{u}] - \boldsymbol{f}_1, \boldsymbol{v} - \boldsymbol{u}\rangle + J^0(M\boldsymbol{u}, M\boldsymbol{u}; M\boldsymbol{v} - M\boldsymbol{u}) + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}) \geq 0 \ \text{ for all } \ \boldsymbol{v} \in U(\boldsymbol{u}). \tag{5.8}$$

We shall prove, by applying Theorem 3.9, that the problem (5.8) has a solution. To this end we verify the hypotheses of Theorem 3.9 in several steps.

- We verify $H(A)$. We use $H(\mathbb{T})$(iii) and Hölder's inequality to obtain

$$\begin{aligned}\textstyle\int_\Omega \mathbb{T}(\mathbb{D}\boldsymbol{u}) : \mathbb{D}\boldsymbol{v}\, dx &\leq \Big(\int_\Omega (2a_1^2(\boldsymbol{x}) + 2a_2^2\|\mathbb{D}\boldsymbol{u}\|^2)\, dx\Big)^{1/2} \Big(\int_\Omega \|\mathbb{D}\boldsymbol{v}\|^2\, dx\Big)^{1/2}\\ &\leq \Big(2\|a_1\|^2_{L^2(\Omega)} + 2a_2^2\|\boldsymbol{u}\|_V^2\Big)^{1/2} \|\boldsymbol{v}\|_V \leq \sqrt{2}\left(\|a_1\|_{L^2(\Omega)} + a_2\|\boldsymbol{u}\|_V\right)\|\boldsymbol{v}\|_V\end{aligned}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$ which implies $\|A\boldsymbol{u}\|_{V^*} \leq \sqrt{2}\left(\|a_1\|_{L^2(\Omega)} + a_2\|\boldsymbol{u}\|_V\right)$. Thus, A is a bounded operator. From hypothesis $H(\mathbb{T})$(iv), it follows

$$\begin{aligned}\langle A\boldsymbol{v}_1 - A\boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle &= \int_\Omega (\mathbb{T}(\mathbb{D}\boldsymbol{v}_1) - \mathbb{T}(\mathbb{D}\boldsymbol{v}_2)) : \mathbb{D}(\boldsymbol{v}_1 - \boldsymbol{v}_2)\, dx\\ &\geq \alpha \int_\Omega \|\mathbb{D}(\boldsymbol{v}_1 - \boldsymbol{v}_2)\|^2\, dx = \alpha\, \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^2 \ \text{ for all } \ \boldsymbol{v}_1, \boldsymbol{v}_2 \in V,\end{aligned}$$

and A is a strongly monotone operator with constant $m_A = \alpha$. By employing [16, Theorem 1.5.2] (Krasnoselskii's theorem for Nemytskii operators) together with $H(\mathbb{T})$ we deduce that A is continuous from V to V^*. Since the operator A is bounded, monotone and hemicontinuous (being continuous), by [43, Theorem 3.69(i)], we conclude that A is pseudomonotone, i.e., $H(A)$ holds.

• We verify $H(B)$ for the operator $B[\cdot]\colon V \to V^*$ defined by (5.3). The proof is given in Properties (i), (iii) and (v) of the appendix.

• We verify $H(j)$ for the functional J given in (5.4). We use hypothesis $H(j_\tau)$(i)–(iii), $H(k)$, and [15, Theorem 5.6.39], to deduce that $J(\boldsymbol{w}, \cdot)$ is Lipschitz continuous on every bounded set for all $\boldsymbol{w} \in X$, which implies condition $H(j)$(i). Based on hypothesis $H(j_\tau)$(iv) and [43, Theorem 3.47(v), (vii)], we have

$$\partial J(\boldsymbol{w}, \boldsymbol{u}) = \int_{\Gamma_1} k(\boldsymbol{w}) \partial j_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))) \, d\Gamma, \tag{5.9}$$

$$J^0(\boldsymbol{w}, \boldsymbol{v}; \boldsymbol{z}) = \int_{\Gamma_1} k(\boldsymbol{w}(\boldsymbol{x})) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}); \boldsymbol{z}(\boldsymbol{x})) \, d\Gamma \tag{5.10}$$

for all $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z} \in X$. Next, let $\boldsymbol{w}, \boldsymbol{u} \in X$ and $\boldsymbol{u}^* \in X^*$, $\boldsymbol{u}^* \in \partial J(\boldsymbol{w}, \boldsymbol{u})$. Hence, $\boldsymbol{u}^*(x) \in k(\boldsymbol{w}(\boldsymbol{x})) \partial j_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_1$. By the growth condition $H(j_\tau)$(iii) and $H(k)$(iii), we have

$$\|\boldsymbol{u}^*(\boldsymbol{x})\|^2 \le 2 k_1^2 \, (b_1^2(\boldsymbol{x}) + b_2^2 \, \|\boldsymbol{u}(\boldsymbol{x})\|^2) \ \text{ for a.e. } \boldsymbol{x} \in \Gamma_1.$$

Integrating the last inequality on Γ_1, we obtain $\|\boldsymbol{u}^*\|_{X^*} \le d_1 + d_3 \|\boldsymbol{u}\|_X$, where $d_1 = 2^{1/2} k_1 \|b_1\|_{L^2(\Gamma_1)}$ and $d_3 = 2^{1/2} b_2 k_1$. We infer that the hypothesis $H(j)$(ii) is satisfied with constants $d_2 = 0$, and d_1, d_3 as described.

Subsequently, we will verify the upper semicontinuity property $H(j)$(iii). Let $\boldsymbol{w}_n \to \boldsymbol{w}$ in X, $\boldsymbol{v}_n \to \boldsymbol{v}$ in X, and $\boldsymbol{z}_n \to \boldsymbol{z}$ in X, where $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{z} \in X$. From [43, Theorem 2.39], by passing to a subsequence if necessary, we may suppose

$$\boldsymbol{w}_n(\boldsymbol{x}) \to \boldsymbol{w}(\boldsymbol{x}), \ \ \boldsymbol{v}_n(\boldsymbol{x}) \to \boldsymbol{v}(\boldsymbol{x}), \ \ \boldsymbol{z}_n(\boldsymbol{x}) \to \boldsymbol{z}(\boldsymbol{x}) \ \text{ in } \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Gamma_1$$

and $\|\boldsymbol{w}_n(\boldsymbol{x})\|_{\mathbb{R}^d} \le w_0(\boldsymbol{x})$, $\|\boldsymbol{v}_n(\boldsymbol{x})\|_{\mathbb{R}^d} \le v_0(\boldsymbol{x})$, $\|\boldsymbol{z}_n(\boldsymbol{x})\|_{\mathbb{R}^d} \le z_0(\boldsymbol{x})$ a.e. on Γ_1 with w_0, v_0, $z_0 \in L^2(\Gamma_1)$. We use the continuity of $k(\boldsymbol{x}, \cdot)$ for a.e. $\boldsymbol{x} \in \Gamma_1$ and the upper semicontinuity of $j_\tau^0(\boldsymbol{x}, \cdot; \cdot)$ for a.e. $\boldsymbol{x} \in \Gamma_1$, see [43, Proposition 3.23(ii)], to obtain

$$\begin{aligned}
\limsup k(\boldsymbol{w}_n(\boldsymbol{x})) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}_n(\boldsymbol{x}); \boldsymbol{z}_n(\boldsymbol{x})) &\le \limsup (k(\boldsymbol{w}_n(\boldsymbol{x})) - k(\boldsymbol{w}(\boldsymbol{x}))) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}_n(\boldsymbol{x}); \boldsymbol{z}_n(\boldsymbol{x})) \\
&\quad + \limsup k(\boldsymbol{w}(\boldsymbol{x})) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}_n(\boldsymbol{x}); \boldsymbol{z}_n(\boldsymbol{x})) \\
&\le k(\boldsymbol{w}(\boldsymbol{x})) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}); \boldsymbol{z}(\boldsymbol{x}))
\end{aligned}$$

for a.e. $\boldsymbol{x} \in \Gamma_1$. We apply Fatou's lemma to get

$$\begin{aligned}
\limsup J^0(\boldsymbol{w}_n, \boldsymbol{v}_n; \boldsymbol{z}_n) &= \limsup \int_{\Gamma_1} k(\boldsymbol{w}_n(\boldsymbol{x})) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}_n(\boldsymbol{x}); \boldsymbol{z}_n(\boldsymbol{x})) \, d\Gamma \\
&\le \int_{\Gamma_1} \limsup k(\boldsymbol{w}_n(\boldsymbol{x})) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}_n(\boldsymbol{x}); \boldsymbol{z}_n(\boldsymbol{x})) \, d\Gamma \\
&\le \int_{\Gamma_1} k(\boldsymbol{w}(\boldsymbol{x})) \, j_\tau^0(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}); \boldsymbol{z}(\boldsymbol{x})) \, d\Gamma = J^0(\boldsymbol{w}, \boldsymbol{v}; \boldsymbol{z}),
\end{aligned} \tag{5.11}$$

where the last equality is a consequence of (5.10). This proves the upper semicontinuity in $H(j)$(iii) and concludes the proof of condition $H(j)$.

• We verify $H(K)$ for the set-valued map $U\colon V \to 2^V$ defined by (4.7). By hypothesis $H(r,m)$(ii), it is clear that $\mathbf{0}_V \in K(\boldsymbol{v})$ for all $\boldsymbol{v} \in V$. Let $\boldsymbol{v} \in V$ and $\{\boldsymbol{u}_n\} \subset K(\boldsymbol{v})$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ as $n \to \infty$ with $\boldsymbol{u} \in V$. By the lower semicontinuity of r, we have $r(\boldsymbol{u}) \le \liminf r(\boldsymbol{u}_n) \le m(\boldsymbol{v})$. Thus, the set $K(\boldsymbol{v})$ is closed for all $\boldsymbol{v} \in V$. For any $\boldsymbol{v} \in V$, let $\boldsymbol{u}, \boldsymbol{w} \in K(\boldsymbol{v})$ and $\lambda \in (0,1)$ be arbitrary. The convexity of r implies

$$r(\lambda \boldsymbol{u} + (1-\lambda)\boldsymbol{w}) \le \lambda r(\boldsymbol{u}) + (1-\lambda) r(\boldsymbol{w}) \le \lambda m(\boldsymbol{v}) + (1-\lambda) m(\boldsymbol{v}) = m(\boldsymbol{v}),$$

and so $\lambda \boldsymbol{u} + (1-\lambda)\boldsymbol{w} \in K(\boldsymbol{v})$. Hence, $K(\boldsymbol{v})$ is a convex set. We deduce that the set-valued map $K\colon V \to 2^V$ has nonempty, closed, and convex values.

Let $\{\boldsymbol{v}_n\} \subset V$ be such that $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in V as $n \to \infty$ for some $\boldsymbol{v} \in V$. We shall verify that $K(v_n) \xrightarrow{M} K(v)$ by checking conditions (m_1) and (m_2) of Definition 2.1. For the proof of (m_1), let $\boldsymbol{u} \in K(\boldsymbol{v})$ be arbitrary and set $\boldsymbol{u}_n = \frac{m(\boldsymbol{v}_n)}{m(\boldsymbol{v})}\boldsymbol{u}$. Then, by using the positive homogeneity of r and the condition $m_0 > 0$, it follows

$$r(\boldsymbol{u}_n) = \frac{m(\boldsymbol{v}_n)}{m(\boldsymbol{v})} r(\boldsymbol{u}) \le m(\boldsymbol{v}_n),$$

which implies $\boldsymbol{u}_n \in K(\boldsymbol{v}_n)$ for every $n \in \mathbb{N}$. By the compactness of the embedding of V into H, and continuity of m, we have

$$\lim \|\boldsymbol{u}_n - \boldsymbol{u}\| = \lim \left\| \frac{m(\boldsymbol{v}_n)}{m(\boldsymbol{v})}\boldsymbol{u} - \boldsymbol{u} \right\| = \lim \frac{|m(\boldsymbol{v}_n) - m(\boldsymbol{v})|}{m(\boldsymbol{v})} \|\boldsymbol{u}\| = 0,$$

which entails $\boldsymbol{u}_n \to \boldsymbol{u}$ in V as $n \to \infty$. Hence, condition (m_1) follows. To prove condition (m_2), let $\{\boldsymbol{u}_n\} \subset V$ be such that $\boldsymbol{u}_n \in K(\boldsymbol{v}_n)$ and $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in V as $n \to \infty$ for some $\boldsymbol{u} \in V$. We use the continuity of m, the weak lower semicontinuity of r, and compact embedding $V \hookrightarrow H$ again, to obtain

$$r(\boldsymbol{u}) \le \liminf r(\boldsymbol{u}_n) \le \liminf m(\boldsymbol{v}_n) = m(\boldsymbol{v}).$$

Thus, $\boldsymbol{u} \in K(\boldsymbol{v})$, which implies (m_2). Hence, the condition $H(K)$ is verified.

• From the trace theorem, see, e.g., [43, Theorem 2.21], it is known that the operator M defined by (5.7) is bounded, linear and compact, and therefore, $H(M)$ holds.

• We verify $H(\varphi)$ for the function φ defined by (5.5). In view of Remark 3.3, it suffices to check $H(\varphi)$(i) and (iii). By Hölder's inequality, we have

$$\begin{aligned} \varphi(\boldsymbol{v}_1) - \varphi(\boldsymbol{v}_2) &= \int_\Omega g\,(\|\mathbb{D}\boldsymbol{v}_1\| - \|\mathbb{D}\boldsymbol{v}_2\|)\,dx \\ &\le \int_\Omega g\,\|\mathbb{D}(\boldsymbol{v}_1 - \boldsymbol{v}_2)\|\,dx \le \|g\|_{L^2(\Omega)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \ \text{ for all } \ \boldsymbol{v}_1, \boldsymbol{v}_2 \in V. \end{aligned}$$

The Lipschitz continuity of φ implies its lower semicontinuity. The convexity of φ is obvious. Hence $H(\varphi)$(i) and (iii) also hold.

• It follows from $H(f,g)$ that the functional f_1 defined by (5.6) satisfies $H(f)$.

• Finally, we use $m_A = \alpha$, $d_2 = 0$, $d_3 = \sqrt{2} b_2 k_1$, $M = \gamma$ and $\alpha_\varphi = 0$ (see Remark 3.3), to infer that the smallness condition (3.20) is a consequence of (5.1).

Having verified all hypotheses of Theorem 3.9, we deduce from it that the auxiliary inequality problem (5.8) has a solution. We observe that Problems 4.3 and (5.8) are in fact equivalent. The equivalence follows easily from (5.3), the equality $\langle B[\boldsymbol{u}], \boldsymbol{v}-\boldsymbol{u}\rangle = b(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u})$ for $\boldsymbol{u}, \boldsymbol{v} \in V$ (see (A.5) in Appendix), and (5.10). Finally, the compactness of the solution set is a consequence of Theorem 3.9. This completes the proof. □

We note that for some classical friction laws, the smallness condition can be removed.

Remark 5.2 The smallness condition (5.1) in Theorem 5.1 becomes trivial, for instance, if $b_2 = 0$, which holds for, e.g., $j_\tau(\boldsymbol{x}, \boldsymbol{\xi}) = \kappa(\boldsymbol{x}) \|\boldsymbol{\xi}\|$ with the slip threshold $k \in L^\infty(\Gamma_1)$.

We conclude with two simple examples which illustrate the weak Mosco convergence in the hypothesis $H(K)$ and hypothesis $H(j_\tau)$.

Example 5.3 Let the set-valued map $K \colon V \to 2^V$ be given by $K(v) = K_0 + G(v)$ for $v \in V$, where $K_0 \subset V$ is a nonempty, closed and convex set and $G \colon V \to V$ is a compact map. Then for any $\{v_n\} \subset V$ such that $v_n \rightharpoonup v$ in V we have $K(v_n) \xrightarrow{M} K(v)$ as $n \to \infty$.

Example 5.4 The following one dimensional example represents a nonconvex function which satisfies hypothesis $H(j_\tau)$. Let $\lambda > 0$ and $j_\lambda \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$j_\lambda(r) = \begin{cases} \sqrt{|r|^2 + \lambda^2} - \lambda & \text{if } |r| \le 1, \\ \left(\dfrac{1}{\sqrt{1+\lambda^2}} - 1\right)|r| + \ln|r| + \sqrt{1+\lambda^2} - \lambda - \dfrac{1}{\sqrt{1+\lambda^2}} + 1 & \text{if } |r| > 1 \end{cases}$$

for $r \in \mathbb{R}$. The derivative of j_λ is given by

$$j_\lambda'(r) = \begin{cases} \dfrac{r}{\sqrt{|r|^2+\lambda^2}} & \text{if } |r| \le 1, \\ \dfrac{1}{r} + \dfrac{1}{\sqrt{1+\lambda^2}} - 1 & \text{if } r > 1, \\ \dfrac{1}{r} - \dfrac{1}{\sqrt{1+\lambda^2}} + 1 & \text{if } r < -1 \end{cases}$$

for $r \in \mathbb{R}$. It can be observed that j_λ' is a continuous function, so $j_\lambda \in C^1(\mathbb{R})$ and $|\partial j_\lambda(r)| = |j_\lambda'(r)| \le 1$ for $r \in \mathbb{R}$. Hence, j_λ is Lipschitz continuous and regular. Moreover, the choice $r_1 = 3$, $r_2 = 0$, $\alpha = \frac{1}{3}$, and $\lambda = 1$ leads to the inequality $j_\lambda(\alpha r_1 + (1-\alpha) r_2) > \alpha j_\lambda(r_1) + (1-\alpha) j_\lambda(r_2)$. This shows that j_λ is nonconvex.

6 Conclusion and Outlook

In this paper we have studied a class of elliptic quasi-variational-hemivariational inequalities. The main results are on the existence of a solution and the compactness of the solution set. The analysis of the abstract inequality is used in the investigation of a mathematical model of steady-state incompressible Navier-Stokes problem for the flow of a Bingham fluid with mixed boundary conditions including nonmonotone friction. The weak formulation of the Bingham model comprises new ingredients: A convex potential of two variables, the convection term, and the set of unilateral constraints depending on the solution.

We have used the arguments based on results of the classical Lions-Stampacchia theory for variational inequalities, and a fixed point theorem for set-valued maps. First we have studied a quasi-variational inequality with a set-valued map. Then we have examined quasi-variational-hemivariational inequality. The relaxed monotonicity condition, often used in the literature, for the Clarke subgradient is avoided in our approach.

Some open problems related to this paper can be studied in the future.

(i) Study the existence of a solution of Problem 4.3 if $|\Gamma_0| = 0$.

(ii) Examine when Problem 3.1 with $F(z) = \partial j(z, z)$ and Problem 3.8 are equivalent.

(iii) Based on the weak formulation in Problem 4.3, recover the pressure and the extra stress such that we obtain the solution to the classical formulation in Problem 4.1.

(iv) By the discussion in this paper, we have established the existence and compactness results, while we did not get in touch with any regularity result. This seems to be an interesting direction of research.

(v) Study the non-stationary Navier-Stokes equations under the leak boundary condition of frictional type and implicit constraints leading to evolutionary quasi-variational-hemivariational inequalities.

(vi) The optimal control and inverse problems for evolutionary Bingham models can be another recent topic of studies.

Appendix

In this section we collect the properties of the nonlinear convective (inertia) term in the Navier-Stokes equation. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. From the Rellich-Kondrachov theorem, see [52, Sect. 2.6.1, Theorem 6.1 and Corollary 6.2], we have the following embeddings

$$\begin{cases} \text{if } d = 2, & \text{then } H^1(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact for all } q \in [1, \infty), \\ \text{if } d = 3, & \text{then } H^1(\Omega) \hookrightarrow L^q(\Omega) \text{ is continuous for all } q \in [1, 6], \\ & \text{and compact for all } q \in [1, 6). \end{cases} \tag{A.1}$$

Let V be the space defined by (4.6) and $b\colon H^1(\Omega; \mathbb{R}^d)^3 \to \mathbb{R}$ be the form defined by (4.17).

(i) The trilinear form b is well defined and continuous on $H^1(\Omega; \mathbb{R}^d)^3$ for $d = 2$ or 3, that is

$$|b(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{z})| \le c\, \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^d)} \|\boldsymbol{u}\|_{H^1(\Omega;\mathbb{R}^d)} \|\boldsymbol{z}\|_{H^1(\Omega;\mathbb{R}^d)}$$

for $\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{z} \in H^1(\Omega; \mathbb{R}^d)$ with $c > 0$, see [26, Lemma 2.1, p.284].

(ii) For $\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{z} \in H^1(\Omega; \mathbb{R}^d)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $v_\nu = 0$ on $\partial\Omega$, we have

$$b(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{z}) + b(\boldsymbol{v}; \boldsymbol{z}, \boldsymbol{u}) = 0 \quad \text{(the antisymmetry property)}, \tag{A.2}$$

$$b(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{u}) = 0. \tag{A.3}$$

For the proof, by the linearity of the form b, we observe that (A.2) and (A.3) are equivalent. We show (A.3) for smooth functions, then we conclude by density. Let $\boldsymbol{u} \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$ and $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$. By Green's formula, we have

$$b(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{u}) = \sum_{i,j=1}^{d} \int_\Omega v_j \frac{\partial u_i}{\partial x_j} u_i \, dx$$

$$= \frac{1}{2}\sum_{i,j=1}^{d} v_j \frac{\partial (u_i^2)}{\partial x_j}\, dx = -\frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}) \sum_{i=1}^{d} u_i^2\, dx + \int_{\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{\nu}) \sum_{i=1}^{d} u_i^2\, d\Gamma.$$

If $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $v_\nu = 0$ on $\partial\Omega$, the latter implies $b(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{u}) = 0$. The condition (A.3) follows by the density of $C^\infty(\overline{\Omega}; \mathbb{R}^d)$ into $H^1(\Omega; \mathbb{R}^d)$.

(iii) The operator $B\colon H^1(\Omega; \mathbb{R}^d)^2 \to \mathbb{R}$ defined by $\langle B(\boldsymbol{v}, \boldsymbol{u}), \boldsymbol{z}\rangle = b(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{z})$ for $\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{z} \in H^1(\Omega; \mathbb{R}^d)$ satisfies

$$\langle B(\boldsymbol{v}, \boldsymbol{u}), \boldsymbol{u}\rangle = 0$$

for $\boldsymbol{v}$, $\boldsymbol{u} \in H^1(\Omega; \mathbb{R}^d)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $v_\nu = 0$ on $\partial\Omega$. This property follows from (A.3).

(iv) If for $d = 2$ or 3, then for any $\boldsymbol{v}_0 \in H^1(\Omega; \mathbb{R}^d)$ fixed, the operator $B(\boldsymbol{v}_0, \cdot)\colon V \to V^*$ is clearly linear and bounded with

$$\|B(\boldsymbol{v}_0, \boldsymbol{u})\|_{V^*} \le c\, \|\boldsymbol{v}_0\|_{H^1(\Omega;\mathbb{R}^d)} \|\boldsymbol{u}\|_{H^1(\Omega;\mathbb{R}^d)} \text{ with some } c > 0.$$

(v) If $d \le 3$, then for $\boldsymbol{v}^n \rightharpoonup \boldsymbol{v}$ and $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ both in V, we have $B(\boldsymbol{v}^n, \boldsymbol{u}^n) \to B(\boldsymbol{v}, \boldsymbol{u})$ in V^*. Let $\boldsymbol{z} \in V$. From (A.2), by Hölder's inequality, we have

$$\Big| \sum_{i,j=1}^{d} \int_\Omega v_j \frac{\partial (u_i^n - u_i)}{\partial x_j} z_i\, dx \Big| = |b(\boldsymbol{v}, \boldsymbol{u}^n - \boldsymbol{u}, \boldsymbol{z})| \overset{(A.2)}{=} |-b(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{u}^n - \boldsymbol{u})|$$

$$= \Big| \sum_{i,j=1}^{d} \int_\Omega v_j \frac{\partial z_i}{\partial x_j} (u_i^n - u_i)\, dx \Big| \le \sum_{i,j=1}^{d} \|v_j\|_{L^4(\Omega)} \Big\| \frac{\partial z_i}{\partial x_j} \Big\|_{L^2(\Omega)} \|u_i^n - u_i\|_{L^4(\Omega)}.$$

Hence

$$\begin{aligned} |\langle B(\boldsymbol{v}^n, \boldsymbol{u}^n) - B(\boldsymbol{v}, \boldsymbol{u}), \boldsymbol{z}\rangle| &= \Big| \sum_{i,j=1}^{d} \int_\Omega \Big(v_j^n \frac{\partial u_i^n}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j} \Big) z_i\, dx \Big| \\ &= \Big| \sum_{i,j=1}^{d} \int_\Omega \Big\{ (v_j^n - v_j) \frac{\partial u_i^n}{\partial x_j} z_i + v_j \Big(\frac{\partial u_i^n}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \Big) z_i \Big\}\, dx \Big| \\ &\le \sum_{i,j=1}^{d} \Big(\|v_j^n - v_j\|_{L^4(\Omega)} \Big\| \frac{\partial u_i^n}{\partial x_j} \Big\|_{L^2(\Omega)} \|z_i\|_{L^4(\Omega)} \Big) \\ &\quad + \sum_{i,j=1}^{d} \|v_j\|_{L^4(\Omega)} \Big\| \frac{\partial z_i}{\partial x_j} \Big\|_{L^2(\Omega)} \|u_i^n - u_i\|_{L^4(\Omega)}. \end{aligned} \tag{A.4}$$

We use the compact embedding $H^1(\Omega; \mathbb{R}^d) \subset L^4(\Omega; \mathbb{R}^d)$ to get $\boldsymbol{v}^n \to \boldsymbol{v}$, $\boldsymbol{u}^n \to \boldsymbol{u}$ both in $L^4(\Omega; \mathbb{R}^d)$. Therefore, by the uniform boundedness of $\{\partial u_i^n / \partial x_j\}$ in $L^2(\Omega)$ for i, $j = 1, \ldots, d$, from (A.4), we have

$$\|B(\boldsymbol{v}^n, \boldsymbol{u}^n) - B(\boldsymbol{v}, \boldsymbol{u})\|_{V^*} = \sup_{\|\boldsymbol{z}\|_V \le 1} |\langle B(\boldsymbol{v}^n, \boldsymbol{u}^n) - B(\boldsymbol{v}, \boldsymbol{u}), \boldsymbol{z}\rangle| \overset{(A.4)}{\to} 0,$$

which completes the proof.

(vi) The application of the Green formula shows that

$$\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \mathbb{D}\boldsymbol{v}\,dx = -b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) - \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{v})\,\mathrm{div}\,\boldsymbol{u}\,dx + \int_{\partial\Omega} (\boldsymbol{u}\cdot\boldsymbol{v})\,u_\nu\,d\Gamma \tag{A.5}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$, where the symmetric part of the velocity gradient $\mathbb{D}\boldsymbol{v}$ is given by (4.9). In fact, for $\boldsymbol{u}, \boldsymbol{v} \in C^\infty(\overline{\Omega};\mathbb{R}^d)$, we have

$$\begin{aligned}
\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \mathbb{D}\boldsymbol{v}\,dx &= \frac{1}{2}\sum_{i,j=1}^{d}\int_\Omega \left(u_i\,u_j\,\frac{\partial v_i}{\partial x_j} + u_i\,u_j\,\frac{\partial v_j}{\partial x_i}\right) dx \\
&= \frac{1}{2}\sum_{i,j=1}^{d}\Big(-\int_\Omega v_i\,\frac{\partial(u_i u_j)}{\partial x_j}\,dx + \int_{\partial\Omega} u_i\,u_j\,v_i\,\nu_j\,d\Gamma \\
&\quad - \int_\Omega v_j\,\frac{\partial(u_i u_j)}{\partial x_i}\,dx + \int_{\partial\Omega} u_i\,u_j\,v_j\,\nu_i\,d\Gamma\Big) \\
&= -\frac{1}{2}\sum_{i,j=1}^{d}\int_\Omega \left(v_i\,\frac{\partial u_i}{\partial x_j}\,u_j + v_j\,\frac{\partial u_j}{\partial x_i}\,u_i + u_i\,v_i\,\frac{\partial u_j}{\partial x_j} + u_j\,v_j\,\frac{\partial u_i}{\partial x_i}\right) dx \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d}\int_{\partial\Omega} \Big\{ (u_i\,v_i)(u_j\,\nu_j) + (u_j\,v_j)(u_i\,\nu_i)\Big\}\,d\Gamma \\
&= -\frac{1}{2}\Big(b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + \int_\Omega ((\boldsymbol{u}\cdot\boldsymbol{v})\,\mathrm{div}\,\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{v})\,\mathrm{div}\,\boldsymbol{u})\,dx\Big) \\
&\quad + \frac{1}{2}\int_{\partial\Omega} \Big\{ (\boldsymbol{u}\cdot\boldsymbol{v})u_\nu + (\boldsymbol{u}\cdot\boldsymbol{v})u_\nu \Big\}\,d\Gamma \\
&= -b(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) - \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{v})\,\mathrm{div}\,\boldsymbol{u}\,dx + \int_{\partial\Omega} (\boldsymbol{u}\cdot\boldsymbol{v})\,u_\nu\,d\Gamma.
\end{aligned}$$

We can make the conclusion from the density of $C^\infty(\overline{\Omega};\mathbb{R}^d)$ into $H^1(\Omega;\mathbb{R}^d)$.

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