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Set Valued Equilibrium Problems Without Linear Structure

Irene Benedetti[1] · **Anna Martellotti**[1]

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Abstract
In this paper we give several existence results for solutions of equilibrium problems in topological spaces without linear structure. To this end we introduce a new concept of convexity for maps and multivalued maps in spaces without linear structure. The discussion on convexity is enriched with some example useful to compare the new conditions with the existing one in literature. Finally, we apply the existence results obtained to a Nash equilibrium problem and to a maximization of a binary relation.

Keywords Equilibrium problem · Minimax inequalities · Set valued analysis · Convex-like maps

Mathematics Subject Classification 47J22 · 90C33 · 49J35 · 49J53 · 47N10

1 Introduction

Multivalued equilibrium problems arise in many areas of science and engineering, including economics, physics, and optimization. They are of particular interest due to their wide range of applications in decision-making problems where multiple competing criteria need to be balanced.

Let $\alpha \in \mathbb{R}$, X be a topological space, Y any set and let $G : X \times Y \multimap \mathbb{R}$ be a multivalued map (multimap for short), the problem under study is the following

$$\text{find} \quad y_0 \in Y \quad \text{such that} \quad G(x, y_0) \cap [\alpha, \infty) \neq \emptyset \quad \forall x \in X.$$

The equilibrium problems were initially introduced in a non multivalued setting by Fan in [20] as a generalization of the classical minimax equality, Browder in [12] considered this problem in the particular case of variational inequalities, then they have been extended in various directions by many authors. A popular research line focuses on generalization of convexity (concavity) of the function. It spreads in two different ways: in the first one as the

I. Benedetti
irene.benedetti@unipg.it

A. Martellotti
martellottianna@gmail.com

1 Department of Mathematics and Computer Science, University of Perugia, Via Vanvitelli, 1, Perugia, 06123, Italy

domain is embedded in a linear structure, the convexity of the map is surrogated by quasi-convexity; according to the definition of De Finetti ([14]) and then again largely extended and readapted, up to the recent definition of α-convexity proposed by Ruiz Galan in [36]. We refer to the book of Kassay and Radulescu ([28]) for a more complete discussion. The second line of investigation focuses on generalization of convexity (concavity) when the domains are embedded in topological spaces without linear structure. One significant starting point in this framework is the definition of convex-like functions introduced by Ky Fan ([19]): this concept was generalized by Granas and Liu ([24]) with finitely convex-like functions, and by Geraghty and Lin ([23]) with the definition of t-convex functions. Alternatively, Horvath ([26]) introduced a suitable substitute of the linear structure the so called H-spaces; his point of view was adopted thereafter by several authors; we mention, for instance, the works of Bardaro and Ceppitelli ([3, 4]). Minimax results without linear structure have been proved by many authors, such as Borwein and Zhuang ([10]), Bukhvalov and Martellotti ([13]), Martellotti and Salvadori ([32]).

Other authors exploits the generalization of linear spaces, called G-spaces due to Park, we refer to [33] for the definition of G-spaces and their relations with other extension of linear structure. For instance, Ding and Park ([15]) prove existence results for generalized vector equilibrium problems and Kalmoun ([27]) establish KKM-type results in G-spaces. Another contribution on this topic is given by Ding, Park and Jung ([16]), here constrained multiobjective games where studied in L-spaces, a generalization of G-spaces introduced in [6].

Lastly, there are the so-called asymmetric versions of the minimax relationships, where the specularity of the required properties is no longer requested: these results are particularly valuable for the study of inequalities systems, see for example Pomerol ([34]), Borwein and Zhuang ([10]), Ha ([25]), and Fan-Glicksberg-Hoffmann ([21]), or even for theorems in the field of optimal investment problems, such as the one in Pratelli ([35]).

In this paper we discuss asymmetric set-valued equilibrium problems in topological spaces without linear structure.

The multivalued problem has been widely investigated under multivalued convexity or quasi-convexity conditions, as in Alleche and Radulescu ([1, 2]), Benedetti and Martellotti ([7]), Krystali and Varga ([30]), Lin, Ansari and Wu ([31]). On the contrary, the study in the literature is far more limited for this type of problems when no linear structure is considered: we mention the above cited results [15, 16, 27], and in addition, ([29]), where the authors establish topologically-based full characterizations of the existence of solutions to optimization-related problems without linear structure, replacing it by a KKM-structure.

The paper is organized as follows. In Sect. 2, we introduce some notions of semicontinuity for maps and multimaps and we recall results useful later on: particularly an alternative version of the Hahn-Banach separation theorem, obtained in [22].

Section 3 is focused on convexity for single valued and multivalued maps in spaces without linear structure. Firstly, we give the definition of convex-like maps introduced by Fan in [19], then we recall its generalization proposed by Granas and Liu in [24]. Finally, we generalize this last definition introducing the concept of *finitely* (λ, μ)*-lower complete* (respectively *finitely* (λ, μ)*-upper complete)* maps, extending the definition also for multivalued maps. We also provide original and interesting examples useful to understand the relations among them.

In Sect. 4 we give several existence results for solutions of an equilibrium problem in topological spaces without linear structure, replacing the usual assumptions of convexity/concavity with the alternative forms of convexity/concavity introduced in Sect. 3. Firstly, we prove an existence result under the multivalued finitely (λ, μ)-inf completeness assumption in the homogeneous case ($\alpha = 0$). It is not possible to obtain a non homogeneous version of Theorem 4.1 without requiring some additional assumptions on the parameters λ, μ. This is due to the fact that unlike convex-likeness, finitely (λ, μ)-inf completeness is in general not preserved through translation unless $\alpha > 0$ and $\lambda + \mu \leq 1$ or $\alpha < 0$ and $\lambda + \mu \geq 1$. In order to obtain a general result for $\alpha \in \mathbb{R}$ we replace multivalued finitely (λ, μ)-inf completeness in the first variable, by the finite-inf convexlikeness in the first variable (see Theorem 4.2). This theorem can be seen as an asymmetric minimax theorem, for the set of assumptions holding for F and G are heavily asymmetric, we present a table with possible alternative forms of this result that can be easily deduced. Furthermore, we consider the non-homogeneous case for (λ, μ)-inf complete set valued maps. We prove an existence result for $\alpha > 0$ and $\lambda + \mu \leq 1$ (see Theorem 4.3), from it the specular case $\alpha < 0$, $\lambda + \mu > 1$ can be easily obtained. We will show in Example 4.2 that the theorem does not apply for $\alpha > 0$ and $\lambda + \mu > 1$.

In all the obtained results, we replace the assumption of compactness of the domain of the involved multimaps with the multivalued version of the so-called coercivity condition (see (H5) Theorem 4.1) introduced initially by Brezis, Nieremberg and Stampacchia in the milestone paper [11] and then frequently used in literature in the study of this type of problems, see for instance [1, 2, 7, 17, 30]. While, we assume a generalization of upper semicontinuity for multivalued maps (see Definition 2.3) introduced in [7]. This concept in a stronger form and in the single valued setting has been introduced by Tian in [38] and then has been used to obtain equilibrium existence results by many authors, see e.g. [8] and [37]. A similar type of semi-continuity for multivalued maps has been introduced in [18] to characterize equilibria of set valued maps. We refer to Sect. 2 for a detailed discussion on the relationships that exist among these various concepts of semi-continuity.

In Sect. 5, we compare the obtained results with the previous literature on the subject, in particular with the results obtained in [7] and [10].

Since equilibrium problems are a point of interest for many applications such as non-cooperative Game Theory, Fixed Point Theorems and Variational Inequalities and so on, several applications of our results are proposed in Sect. 6. Particularly, applying a single valued version of Theorem 4.2, we prove a result of existence of Nash equilibrium and an existence theorem for maximal elements of binary relations defined over a paracompact topological space.

Finally, in Appendix to complete the discussion on the several concepts of convexity without linear structure present in the literature, we provide an example showing that finitely convex-likeness defined in [24] is a genuine extension of Fan's Definition given in [19].

2 Preliminaries

This section recalls the fundamental notions used in the paper, namely semicontinuity for maps and multimaps, and we recall results useful later on, in particular an alternative version of the Hahn-Banach separation theorem obtained in [22].

In this section X denotes a topological space and Y any set.

Definition 2.1 Let $\alpha \in \mathbb{R}$, a map $f : X \times X \to \mathbb{R}$ is said to be *α-upper semicontinuous* (respectively *α-lower semicontinuous*) in the first variable if for every pair $(x_0, y_0) \in X \times X$ such that $f(x_0, y_0) < \alpha$ ($f(x_0, y_0) > \alpha$) there exists a neighbourhood U of x_0 such that $f(z, y_0) < \alpha$ ($f(z, y_0) > \alpha$) for every $z \in U$.

Tian in [38] proposed the definition of α-transfer semicontinuity for single valued maps as follows.

Definition 2.2 ([38], Definition 8) Let $\alpha \in \mathbb{R}$, a map $f : X \times X \to \mathbb{R}$ is said to be *α-transfer upper semicontinuous* (respectively *α-transfer lower semicontinuous*) in the first variable if for every pair $(x_0, y_0) \in X \times X$ such that $f(x_0, y_0) < \alpha$ $(f(x_0, y_0) > \alpha)$ there are a point $y' \in X$ and a neighbourhood U of x_0 such that $f(z, y') < \alpha$ $(f(z, y') > \alpha)$ for every $z \in U$.

This concept has been used by several authors (see for instance [8]). Scalzo in [37] defines f to be *positively quasi-transfer continuous* whenever the above requirement holds for every $\alpha > 0$. The same concept in the multivalued setting has been proposed in [7] Remark 5.3. Here we introduce a slight generalization.

Definition 2.3 Let $\alpha \in \mathbb{R}$, a multimap $F : X \times Y \multimap \mathbb{R}$ is said to be *α-transfer inclusion quasi continuous* in the first variable if for every pair (x_0, y_0) such that $F(x_0, y_0) \subset]\alpha, +\infty)$ there are a finite set $Y_0 \subset Y$ and a neighbourhood U of x_0 such that for each $z \in U$ there is $y' \in Y_0$ such that $F(z, y') \subset]\alpha, +\infty)$.

Recently, a similar concept, with $\alpha = 0$, has been used in [18] to characterize the equilibria of set valued maps. Namely, in [18] the following definition is introduced.

Definition 2.4 ([18], Definition 2.2) A multimap $F : X \times Y \multimap \mathbb{R} \cup \{+\infty\}$ is said to be $\overline{\mathbb{R}}_+$-*strongly-transfer-lower semicontinuous* in the first variable if for every pair (x_0, y_0) such that $F(x_0, y_0) \subset [0, +\infty]$ there are a point $y' \in Y$ and a neighbourhood U of x_0 such that $F(z, y') \subset [0, +\infty]$, for each $z \in U$.

Notice that Definition 2.3 is an obvious generalization of the upper semicontinuity for multivalued maps. Moreover, Tian's assumption Definition 2.2 clearly implies the α-transfer inclusion quasi-continuity for single valued maps (Definition 2.3), but the converse does not hold as shown by the following example.

Example 2.1 Let $g : \mathbb{R} \to \mathbb{R}$ be the Dirichlet map

$$g(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $f(x, y) = g(x - y)$. Then the map f is 0-transfer inclusion quasi continuous, but it is not 0-transfer lower semicontinuous in Tian's sense. In fact, for instance notice that $f(0, e) = 1$, but whatever y_1 we fix, and whatever neighbourhood U of 0 we fix, it will contain some interval $[-\varepsilon, \varepsilon]$ so that $U - y_1$ will contain both rational and irrational numbers; therefore f will assume the value $-1 < 0$ infinitely many times.

On the contrary, choosing $y_1 = 0$ and $y_2 \in \mathbb{R} \setminus \mathbb{Q}$ we certainly have for every $z \in [-\varepsilon, \varepsilon]$ that either $z \in \mathbb{Q}$ so that $z - y_2 \in \mathbb{R} \setminus \mathbb{Q}$ or $z \notin \mathbb{Q}$ and then $z - y_1 = z \in \mathbb{R} \setminus \mathbb{Q}$; in both cases $f = 1 > 0$.

On the other side, Definition 2.2 and Definition 2.3 are not comparable to the concept in Definition 2.4. Indeed, here is an example of a map satisfying Definition 2.2 without fulfilling Definition 2.4.

Example 2.2 Consider the map $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} -1 & \text{if } x < \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$$

It is evident that f is 0-transfer lower semicontinuous according to Definition 2.2, but for $x_0 = \frac{1}{2}$ no neighbourhood of x_0 at whatever level y can avoid values where $f = -1$.

Moreover, the map f in Example 2.1 satisfies Definition 2.3 but not Definition 2.4. Finally, the following is an example of a map that is $\overline{\mathbb{R}}_+$-strongly-transfer-lower semicontinuous (Definition 2.4), but not 0-transfer inclusion continuous (Definition 2.3), and therefore automatically not 0-lower semicontinuous (Definition 2.2).

Example 2.3 The map $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$$

is nonnegative for every $(x, y) \in [0, 1] \times [0, 1]$ and so it trivially satisfies Definition 2.4. On the other hand, $f(x_0, y_0) > 0$ if and only if $x_0 = y_0$. Therefore, given $x_0 = y_0 \in [0, 1]$, for any chosen neighbourhood U of x_0, there cannot exist a finite set $Y \subset [0, 1]$ such that for every $z \in U$, there exists $y \in Y$ such that $f(z, y) > 0$. In fact, $f(z, y) > 0$ if and only if $y = z$, so only for infinitely many y.

In order to prove one of the main results of this paper we will use the following result from [22] representing a new version of the Hanh-Banach Theorem.

Theorem 2.1 *Let $T \subset \mathbb{R}^l$ be a nonempty set such that*

(1) *For every $\mathbf{u} \in T$, $\max_{1 \leq i \leq l} u_i \geq 0$,*
(2) *there exists $\lambda, \mu > 0$ such that for every $\mathbf{u}, \mathbf{v} \in T$ and for every $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in \mathbb{R}^l$, $\varepsilon > 0$, there exists $\mathbf{w} \in T$ such that*

$$\mathbf{w} \ll \lambda\mathbf{u} + \mu\mathbf{v} + \boldsymbol{\varepsilon},$$

where $\ll$ is the lexicographic order in $\mathbb{R}^l$.

Then there exist $c_1, \dots c_l \geq 0$ with $\sum_{i=1}^{l} c_i = 1$ such that

$$\sum_{i=1}^{l} c_i u_i \geq 0 \quad \forall\, \mathbf{u} \in T.$$

3 Convexity for Multimaps Without Linear Structure

In this section we introduce some definitions of convexity for multimaps without linear structure that will be useful in the following section to prove multivalued minimax relationships.

In the whole section X and Y denote two arbitrary sets.

In [19] Ky Fan introduced the following condition extending the concept of convexity for functions defined on a set without linear structure.

Definition 3.1 A map $f : X \times Y \to \mathbb{R}$ is said to be *convex-like (concave-like) in* x if for every $x_1, x_2 \in X$ and for every $t \in [0, 1]$ there exists some $x \in X$ such that

$$f(x, y) \leq tf(x_1, y) + (1-t)f(x_2, y) \quad \forall y \in Y$$

$$(f(x, y) \geq tf(x_1, y) + (1-t)f(x_2, y) \quad \forall y \in Y).$$

In [24] Granas and Liu proposed the following weaker form of convex-likeness.

Definition 3.2 A map $f : X \times Y \multimap \mathbb{R}$ is said to be *finitely convex-like (finitely concave-like) in* x if for every $\{y_1, \dots, y_n\} \subset Y$, for every $x_1, x_2 \in X$ and for every $t \in [0, 1]$ there exists $x \in X$ such that

$$f(x, y_i) \leq tf(x_1, y_i) + (1-t)f(x_2, y_i), \quad i = 1, \dots, n$$

$$(f(x, y_i) \geq tf(x_1, y_i) + (1-t)f(x_2, y_i), \quad i = 1, \dots, n).$$

Finitely convex-likeness is a genuine extension of Fan's Definition as we shall show in Appendix with a suitable example.

We generalize the definition proposed by Granas and Liu ([24]) in the following way.

Definition 3.3 Given $\lambda, \mu \in]0, \infty[$, a map $f : X \times Y \to \mathbb{R}$, is said to be *finitely (λ, μ)-lower complete* (respectively *finitely (λ, μ)-upper complete) in* x if for every x_1, x_2, for every $\{y_1, \dots, y_n\} \subset Y$ and for every $\varepsilon > 0$, there exists $x \in X$ such that

$$f(x, y_i) \leq \lambda f(x_1, y_i) + \mu f(x_2, y_i) + \varepsilon, \quad i = 1, \dots, n$$

$$(f(x, y_i) \geq \lambda f(x_1, y_i) + \mu f(x_2, y_i) - \varepsilon, \quad i = 1, \dots, n.)$$

It is straightforward that a finitely convex-like function is (λ, μ)-lower complete for $\lambda \in [0, 1]$ and $\mu = 1 - \lambda$. However, this new concept strictly extends Granas and Liu's definition; let us give a suitable elegant example.

Example 3.1 Let $c \in \mathbb{Q}$ with $c < \frac{1}{2}$. The set $\mathcal{P}$ of all polynomials in c, with degree ≥ 1 and with coefficients in $\mathbb{N}^+$ is countable. Let $j : \mathbb{N}^+ \to \mathcal{P}$ be a bijection. Define the map $\varphi : [0, 1] \to \mathbb{R}$ as follows

$$\varphi(x) = \begin{cases} j(k) \text{ if } & x = \dfrac{1}{k}, k \in \mathbb{N}^+ \\ 0 & \text{elsewhere.} \end{cases}$$

Let now $Y = \{y_1, y_2\}$ and define $f : [0, 1] \times Y \to \mathbb{R}$ as

$$f(x, y) = \begin{cases} \varphi(x) & \text{if} \quad y = y_1 \\ -\varphi(x) & \text{if} \quad y = y_2. \end{cases}$$

Then, easily, f would be finitely convexlike if and only if for every $t \in [0, 1]$, and every $x_1, x_2 \in [0, 1]$ the equality

$$\varphi(x_0) = t\varphi(x_1) + (1-t)\varphi(x_2)$$

holds for some $x_0 \in [0, 1]$. Indeed, once x_1, x_2 and t are fixed, there should exist x_0 such that $f(x_0, y_i) \leq tf(x_1, y_i) + (1-t)f(x_2, y_i)$, $i = 1, 2$, i.e.

$$\varphi(x_0) \leq t\varphi(x_1) + (1-t)\varphi(x_2)$$

and

$$\varphi(x_0) \geq t\varphi(x_1) + (1-t)\varphi(x_2).$$

Then one is immediately convinced that f is not convexlike. Indeed, choose $t \notin \mathbb{Q}$ and $x_2 \notin \mathbb{Q}$, $x_1 = \dfrac{1}{k}$. It results

$$t\varphi(x_1) + (1-t)\varphi(x_2) = t \cdot j(k) \notin \mathbb{Q}$$

while φ has only rational values.

However f is (λ, μ)-lower complete with $\lambda = \mu = c$. In fact, analogously to the previous reasoning, it is enough to show that, for every $x_1, x_2 \in [0.1]$, there exists $x_0 \in [0, 1]$ such that

$$\varphi(x_0) = c\left[\varphi(x_1) + \varphi(x_2)\right].$$

Now $\varphi(x_1) + \varphi(x_2)$ is either 0, or a polynomial in $\mathcal{P}$ because for every polynomial $p \in \mathcal{P}$, $cp \in \mathcal{P}$ and $p_1, p_2 \in \mathcal{P}$ implies $p_1 + p_2 \in \mathcal{P}$. In the first occurrence one can take $x_0 = 0$, while in the second case, since $c\left[\varphi(x_1) + \varphi(x_2)\right] \in \mathcal{P}$, take $x_0 = \dfrac{1}{j^{-1}\left(c\left[\varphi(x_1) + \varphi(x_2)\right]\right)}$.

We shall now propose an extension of the previous definitions to real valued multifunctions. To this aim we have to introduce the concepts of *inf-bounded, sup-bounded* and *inf-closed, sup-closed values*.

Definition 3.4 $F : X \multimap \mathbb{R}$ is said to be

1. *inf-bounded valued* if $F(x)$ is a lower bounded set for every $x \in X$;
2. *sup-bounded valued* if $F(x)$ is a upper bounded set for every $x \in X$;
3. *bounded valued* if $F(x)$ is a bounded set for every $x \in X$.

If F has inf-bounded values is possible to define the function $f : X \to \mathbb{R}$ as

$$f(x) = \inf F(x),$$

and if F has sup-bounded values is possible to define the function $g : X \to \mathbb{R}$ as

$$g(x) = \sup F(x).$$

Definition 3.5 An inf-bounded valued map $F : X \to \mathbb{R}$ is said to be *inf-closed valued* if $f(x) \in F(x)$ for every $x \in X$.

Analogously, a sup-bounded valued map $F : X \to \mathbb{R}$ is said to be *sup-closed valued* if $g(x) \in F(x)$ for every $x \in X$.

A multivalued map $F : X \times Y \multimap \mathbb{R}$ with inf-bounded values is said to be:

- *inf convex-like in x* if the map $(x, y) \mapsto f(x, y) = \inf F(x, y)$ is convex-like in $x \in X$;

- *inf finitely convex-like in* x if the map $(x, y) \mapsto f(x, y) = \inf F(x, y)$ is finitely convex-like in $x \in X$;
- *multivalued finitely* (λ, μ)*-inf complete in* x if the map $(x, y) \mapsto f(x, y) = \inf F(x, y)$ is finitely (λ, μ)-lower complete in $x \in X$.

Analogously, a multivalued map $F : X \times Y \multimap \mathbb{R}$ with sup-bounded values is said to be:

- *sup concave-like in* x if the map $(x, y) \mapsto g(x, y) = \sup F(x, y)$ is concave-like in $x \in X$;
- *sup finitely concave-like in* x if the map $(x, y) \mapsto g(x, y) = \sup F(x, y)$ is finitely concave-like in $x \in X$;
- *multivalued finitely* (λ, μ)*-sup complete in* x if the map $(x, y) \mapsto g(x, y) = \sup F(x, y)$ is finitely (λ, μ)-upper complete in $x \in X$.

The following result follows easily.

Lemma 3.1 *Given* $\lambda, \mu \in]0, \infty[$, *if* $F : X \times Y \multimap \mathbb{R}$ *is a multimap with inf-closed values, then the following two conditions are equivalent.*

1. F *is multivalued finitely* (λ, μ)*-inf complete*;
2. *for each* $x_1, x_2 \in X$, *for every* $\{y_1, \ldots, y_n\}$ *and for every* $\varepsilon > 0$ *there exists* $x \in X$ *such that for each* y_i, $i = 1, \ldots, n$ *and each choice* $\xi_j \in F(x_j, y_i)$ *one finds*

$$F(x, y_i) \cap (-\infty, \lambda\xi_1 + \mu\xi_2 + \varepsilon] \neq \emptyset, \quad i = 1, \ldots, n.$$

For all the above definitions, characterizations analogous to Lemma 3.1 can be stated.

4 Existence Results

In this section we give several existence results for solutions of an equilibrium problem in topological spaces without linear structure. To this end, we substitute the usual assumptions of convexity/concavity with the types of convexity/concavity introduced in the previous section.

First we give the following result in the homogeneous case $\alpha = 0$.

Theorem 4.1 *Let* X *be a topological space*, Y *any set and let* $F, G : X \times Y \multimap \mathbb{R}$ *multimaps with bounded values*, *with* G *sup-closed valued*, *such that*

(H0) $F(x, y) \subset G(x, y)$;
(H1) *for every* $x \in X$ *there exists* $y \in Y$ *such that* $F(x, y) \subset]0, \infty)$;
(H2) F *is* 0*-transfer inclusion quasi-continuous in the first variable*;
(H3) *there exists* $\lambda, \mu > 0$ *such that* F *is multivalued finitely* (λ, μ)*-inf complete in the first variable*;
(H4) G *is sup concave-like in the second variable*;
(H5) *there exist a compact set* $K \subset X$ *and a finite set* Y_0 *such that for every* $\bar{x} \notin K$ *there exists* $y \in Y_0$ *such that* $F(\bar{x}, y) \subset]0, \infty)$.

Then there exists $y_0 \in Y$ *such that*

$$G(x, y_0) \cap [0, \infty) \neq \emptyset \quad \forall x \in X.$$

Proof Let $f = \inf F$, $g = \sup G$.

For each $y \in Y$ set $C(y) = \{x \in X | F(x, y) \cap (-\infty, 0] \neq \emptyset\}$, $\Gamma(y) = C(y) \cap K^c$, $H(y) = C(y) \cap K$.

By **(H1)** $\bigcap_{y \in X} C(y) = \emptyset$. In fact, if we assume that some $\bar{x} \in C(y)$ for each $y \in Y$ then we should find that $F(\bar{x}, y) \cap (-\infty, 0] \neq \emptyset$ for every $y \in Y$ thus contradicting **(H1)**.

Set $A(y) = [C(y)]^c$; then $\{A(y), y \in Y\}$ is a cover of X.

Let $\mathcal{F}(Y)$ be the class of finite subsets of Y; for each $E \in \mathcal{F}(Y)$ define

$$B_E = \bigcup_{y \in E} A(y).$$

Then $\{B_E, E \in \mathcal{F}(Y)\}$ is an open cover of X. In fact, for each $y_0 \in Y$ fixed, and any $x_0 \in A(y_0)$ there is, by **(H2)**, a finite subset $E \subset Y$ and a neighbourhood U of x_0 such that $U \subset \bigcup_{y \in E} A(y)$; hence x_0 is an interior point of this union.

Then, by the compactness of the set K, there are finitely many $E_1, \ldots, E_k \in \mathcal{F}(Y)$ such that

$$K = \bigcup_{i=1}^{k} B_{E_i};$$

a fortiori

$$K = \bigcup_{i=1}^{k} \left(\bigcup_{y \in E_i} A(y) \right) = \bigcup_{i=1}^{n} A(y_i).$$

Then, it follows that $\bigcap_{i=1}^{n} H(y_i) = \emptyset$.

On the other side, by **(H5)** $\bigcap_{y \in Y_0} \Gamma(y) = \emptyset$, otherwise there should exist $\bar{x} \notin K$ such that $\bar{x} \in \Gamma(y)$ for each $y \in Y_0$, then $F(\bar{x}, y) \cap (-\infty, 0] \neq \emptyset$ for each $y \in Y_0$ thus contradicting **(H5)**.

Now, setting $\widetilde{Y} = Y_0 \cup E_1 \cup \ldots E_k$, obviously we find

$$\bigcap_{y \in \widetilde{Y}} C(y) = \left(\bigcap_{y \in \widetilde{Y}} \Gamma(y) \right) \cup \left(\bigcap_{y \in \widetilde{Y}} H(y) \right) \subset \left(\bigcap_{y \in Y_0} \Gamma(y) \right) \cup \left(\bigcap_{y \in E_1 \cup \cdots \cup E_k} H(y) \right) = \emptyset. \quad (1)$$

We set now $\widetilde{Y} = \{y_1, \ldots, y_n\}$ and

$$H_f = \{(\mathbf{z}, r) \in \mathbb{R}^n \times \mathbb{R} | \text{ there exists } x \in X \text{ s.t. } f(x, y_i) \leq z_i + r, \forall i = 1, \ldots, n\}.$$

We will apply Theorem 2.1 to the set $T = H_f$. Since $\bigcap_{i=1}^{n} C(y_i) = \emptyset$, we have that for every $(\mathbf{z}, r) \in H_f$, $\max\{\max_{1 \leq i \leq n} z_i, r\} \geq 0$. In fact, assume by contradiction the existence of $(\mathbf{z}, r) \in H_f$ such that $\max\{\max_{1 \leq i \leq n} z_i, r\} < 0$, then

$$\max_{1 \leq i \leq n} f(x, y_i) \leq \max_{1 \leq i \leq n} z_i + r \leq 2 \max\{\max_{1 \leq i \leq n} z_i, r\} < 0.$$

So, we have $f(x, y_i) < 0$, for every $i = 1, \ldots, n$ or equivalently $F(x, y_i) \cap (-\infty, 0] \neq \emptyset$, $i = 1, \ldots, n$, and that is absurd as it would mean that $x \in \bigcap_{i=1}^{n} C(y_i)$. Thus, we have assumption **(1)** of Theorem 2.1.

Assumption **(2)** is given by assumption **(H3)**. Indeed, for every pair $(\mathbf{z_1}, r_1), (\mathbf{z_2}, r_2) \in H_f$ let $x_1, x_2 \in X$ be such that

$$f(x_1, y_i) \leq z_{1,i} + r_1, \quad f(x_2, y_i) \leq z_{2,i} + r_2, \quad i = 1, \ldots, n.$$

By assumption **(H3)** there exists $x \in X$ such that

$$f(x, y_i) \leq \lambda f(x_1, y_i) + \mu f(x_2, y_i) + \varepsilon \leq \lambda z_{1,i} + \lambda r_1 + \mu z_{2,i} + \mu r_2 + \varepsilon,$$

so $\lambda(\mathbf{z_1}, r_1) + \mu(\mathbf{z_2}, r_2) + \boldsymbol{\varepsilon} \in H_f$, where $\boldsymbol{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in \mathbb{R}^{n+1}$.

By Theorem 2.1 there exist $c_0, \ldots, c_n \geq 0$ with $\sum_{i=0}^{n} c_i = 1$ such that

$$\sum_{i=1}^{n} c_i z_i + c_0 r \geq 0 \quad \forall (\mathbf{z}, r) \in H_f.$$

Since for every $r \in \mathbb{R}$ we have that $(f(x, y_1) + r, \ldots, f(x, y_n) + r, -r) \in H_f$, there exist $(c_0, c_1, \ldots, c_n) \in \mathbb{R}^{n+1}$, with $c_i \geq 0$ for every i, $\sum_{i=0}^{n} c_i = 1$ such that

$$\sum_{i=1}^{n} c_i f(x, y_i) + \sum_{i=1}^{n} c_i r - c_0 r \geq 0,$$

so

$$\sum_{i=1}^{n} c_i f(x, y_i) + \left(\sum_{i=1}^{n} c_i - c_0 \right) r \geq 0 \quad \forall r \in \mathbb{R}. \tag{2}$$

Hence $\sum_{i=1}^{n} c_i - c_0 = 0$, otherwise we could choose r such that (2) is not satisfied. So

$$1 = \sum_{i=1}^{n} c_i + c_0 = 2c_0,$$

then $c_0 = \dfrac{1}{2} > 0$. Hence

$$\sum_{i=1}^{n} \frac{c_i}{c_0} = 1$$

and since $(f(x, y_1), \ldots, f(x, y_n), 0) \in H_f$ we get by (2)

$$\sum_{i=1}^{n} \frac{c_i}{c_0} f(x, y_i) > \sum_{i=1}^{n} c_i f(x, y_i) \geq 0.$$

Since $f \le g$ this implies

$$\sum_{i=1}^{n} \frac{c_i}{c_0} g(x, y_i) \ge 0 \quad \forall x \in X.$$

Since g is concave-like in y, because of assumption **(H4)**, we obtain that for some $y_0 \in Y$,

$$g(x, y_0) \ge 0, \quad \forall x \in X.$$

Finally, as G has sup-closed values this implies the claimed result $G(x, y_0) \cap [0, \infty) \neq \emptyset$ for every $x \in X$. $\square$

It is not possible to obtain a non homogeneous version of Theorem 4.1 without requiring some additional assumptions on the parameters λ, μ. This is due to the fact that unlike convex-likeness, finitely (λ, μ)-inf completeness is in general not preserved through translation unless $\alpha > 0$ and $\lambda + \mu \le 1$ or $\alpha < 0$ and $\lambda + \mu \ge 1$.

The following example illustrates this pathology in the case $\alpha > 0$.

Example 4.1 Let X be an arbitrary set and $Y = \{y\}$, consider $\varphi : X \to [0, +\infty)$. We have that for every $x_1, x_2 \in X$ and for every $\varepsilon > 0$

$$\varphi(x_j) \le \varphi(x_1) + \varphi(x_2) + \varepsilon, \quad j = 1, 2.$$

Define $f : X \times Y \to \mathbb{R}$ as $f(x, y) = \varphi(x)$, it immediately holds that f is finitely $(1, 1)$-inf complete.

Choose $\alpha > 0$ such that

$$i = \inf_{x \in X} f(x, y) - \alpha < 0,$$

we show that $g = f - \alpha$ is not finitely (λ, μ)-inf complete for every choice of λ, μ with $\lambda + \mu > 1$, i.e. for every $\lambda, \mu > 0$ with $\lambda + \mu > 1$ there exist x_1, x_2 and $\varepsilon > 0$ such that

$$\lambda g(x_1, y) + \mu g(x_2, y) + \varepsilon < i.$$

Indeed, since $i < 0$ implies $(\lambda + \mu)i < i$, we can fix $\varepsilon > 0$ such that

$$(\lambda + \mu)i + (\lambda + \mu + 1)\varepsilon < i$$

and choose $x_1, x_2 \in X$ such that

$$g(x_j, y) < i + \varepsilon, \quad j = 1, 2.$$

Then

$$\begin{aligned} \lambda g(x_1, y) + \mu g(x_2, y) + \varepsilon &\le \lambda i + \lambda\varepsilon + \mu i + \mu\varepsilon + \varepsilon \\ &= (\lambda + \mu)i + (\lambda + \mu + 1)\varepsilon < i. \end{aligned}$$

This is not the case, for convex-like multivalued maps. Thus, we replace multivalued finitely (λ, μ)-inf completeness in the first variable by the finite-inf convexlikeness in the first variable (see assumption (H3)) getting the following general version of the result for $\alpha \in \mathbb{R}$.

Theorem 4.2 *Let X be a topological space, Y any set, $\alpha \in \mathbb{R}$ and let $F, G : X \times Y \multimap \mathbb{R}$ be two multimaps with bounded values, with G sup-closed valued, and suppose that they satisfy assumptions* (**H0**), (**H1**), (**H2**), (**H4**), (**H5**) *of Theorem 4.1 with the half line* $]0, +\infty)$ *replaced by* $]\alpha, +\infty)$ *and*

(**H3**)$'$ $x \multimap F(x, y)$ *is finitely inf convex-like in* x.

Then there exists $y_0 \in Y$ such that

$$G(x, y_0) \cap [\alpha, +\infty) \neq \emptyset \quad \forall x \in X.$$

Proof Let $f = \inf F$, $g = \sup G$ and $\widetilde{Y} = \{y_1, \ldots, y_n\}$ as in the proof of Theorem 4.1. The proof is identical to the one of Theorem 4.1 until $\bigcap_{i=1}^n C(y_i) = \emptyset$ is proven. We set now

$$E_f = \{(\mathbf{z}, r) \in \mathbb{R}^n \times \mathbb{R} |\text{ there exists } x \in X \text{ s.t. } f(x, y_i) < z_i + r, \forall i = 1, \ldots, n\}.$$

Since f is finitely convex-like in the first variable, the set E_f is convex. We shall now prove that $(\mathbf{0}, \alpha) \notin E_f$.

By contradiction, if $(\mathbf{0}, \alpha) \in E_f$ then for some $\bar{x} \in X$ we should have that $f(\bar{x}, y_i) < \alpha$ for each $i = 1, \ldots, n$ and therefore $F(\bar{x}, y_i) \cap (-\infty, \alpha[\neq \emptyset$ for each $i = 1, \ldots, n$. But then $\bar{x} \in \displaystyle\bigcap_{i=1}^n C(y_i)$ which contradicts the choice of $y_1, \ldots, y_n$.

Since E_f is open and convex, by the Hahn-Banach separation Theorem, we can now separate $\overline{E_f}$ and $(\mathbf{0}, \alpha)$, in the large sense, namely there exists $(\mathbf{v}, \bar{r}) \neq \mathbf{0}$ such that

$$\mathbf{v} \cdot \mathbf{z} + \bar{r} \cdot r \geq \bar{r} \cdot \alpha \tag{3}$$

for $(\mathbf{z}, r) \in \overline{E_f}$. Observe that the set

$$H_f = \{(\mathbf{z}, r) \in \mathbb{R}^n \times \mathbb{R} |\text{ there exists } x \in X \text{ s.t. } f(x, y_i) \leq z_i + r, \forall i = 1, \ldots, n\}$$

is contained in $\overline{E_f}$.

In fact, if $(\mathbf{z}, r) \in H_f$ then $\left(\mathbf{z} + \frac{1}{n}\mathbf{u}, r + \frac{1}{n}\right) \in E_f$, with $\mathbf{u} = (1, \ldots, 1)$, so

$$\lim_n \left(\mathbf{z} + \frac{1}{n}\mathbf{u}, r + \frac{1}{n}\right) = (\mathbf{z}, r),$$

implying $(\mathbf{z}, r) \in \overline{E_f}$. Hence $(\mathbf{v}, \bar{r})$ separates H_f from $(\mathbf{0}, \alpha)$ in the large sense.

Now, following exactly the same argument as in [10] one proves that each component of $\mathbf{v} = (v_1, \ldots, v_n)$ is greater or equal than zero and that $\bar{r} > 0$.

In fact, again by the Hahn-Banach separation Theorem, one can find $(v_1, v_2, \ldots, v_n, \bar{r}) \neq 0$ with $v_i \geq 0$, $\bar{r} \geq 0$ because $E_f + \mathbb{R}_+^{n+1} \subset E_f$. Indeed if $y = (\mathbf{z}, r) \in E_f$ and $y' = (\mathbf{z}', r')$ with $z_i' \geq z_i$ for each $i = 1, \ldots, n$ and $r' \geq r$, we have

$$f(x, y_i) \leq z_i + r \leq z_i' + r,'$$

so $y' \in E_f$. As $(0, 1 + \max_{1 \leq i \leq n} f(x, y_i)) \in E_f$, for $\mathbf{z} = \mathbf{0}$ in (3) one obtains $\bar{r} > 0$.

Since for every $r \in \mathbb{R}$ and every $x \in X$ the point $(f(x, y_1) + r, \ldots, f(x, y_n) + r, -r) \in E_f$ then

$$\sum_{i=1}^n v_i f(x, y_i) + \sum_{i=1}^n v_i r - \bar{r} \cdot r \geq \bar{r} \cdot \alpha;$$

dividing by $\bar{r}$ one reaches

$$\sum_{i=1}^{n} \frac{v_i}{\bar{r}} f(x, y_i) + \left(\sum_{i=1}^{n} \frac{v_i}{\bar{r}} - 1 \right) r \geq \alpha$$

that holds for each $x \in X$ and each $r \in \mathbb{R}$; thus the term between brackets is 0. We then obtain

$$\sum_{i=1}^{n} \frac{v_i}{\bar{r}} f(x, y_i) \geq \alpha \tag{4}$$

for every $x \in X$. Since immediately $f \leq g$ this in turn implies

$$\sum_{i=1}^{n} \frac{v_i}{\bar{r}} g(x, y_i) \geq \alpha. \tag{5}$$

Since g is concave-like in the second variable, because of assumption **(H4)** we obtain that for some $y_0 \in Y$,

$$g(x, y_0) \geq \alpha, \quad \forall x \in X.$$

Finally, as G has sup-closed values, this implies the claimed result

$$G(x, y_0) \cap [\alpha, +\infty) \neq \emptyset$$

for every $x \in X$. □

Remark 4.1 Theorem 4.2 holds also if the intervals appearing in **(H1)**, **(H2)** and **(H5)** are closed, provided the interval defining $C(y)$ is the open halfline $(-\infty, \alpha[$.

Observe that in Theorem 4.2 we borrowed the idea of [7] to weaken the usual compactness assumption appearing in minimax relationships.

Theorem 4.2 is an asymmetric minimax Theorem, for the set of assumptions holding for F and G are heavily asymmetric. Acting on the proof one can easily deduce alternative forms of Theorem 4.2, for instance according to Table 1.

Table 1 Alternative versions of Theorem 4.2

	(H1)	(H2)	(H3)	(H4)	(H5)
Theorem 4.2	F	F	F	G	F
Alternative 1	G	G	F	F or G	G
Alternative 2	G	G	G	F	G

The first line is Theorem 4.2. To prove the Alternative 1 represented in the second line, one defines the $C(y)$ via G instead of F and again gets a finite family having empty intersection; then one acts on E_f, that is again convex; now simply noticing that $\inf G \leq f$ it is easy to deduce that again $(\mathbf{0}, \alpha) \notin E_f$ and hence the separation argument follows. Finally one uses either (5) or (directly (4)) depending upon whether assumption **(H4)** holds for G or F.

Analogous reasonings would apply to Alternative 2.

We turn now to the non-homogeneous case for (λ, μ)-inf complete set valued maps. It is clear that for $\alpha > 0$ and $\lambda + \mu \leq 1$, if f is finitely lower (λ, μ)-complete with respect to x, so is $f - \alpha$. Therefore we have the following non homogeneous version of Theorem 4.1.

Theorem 4.3 *Let X be a topological space, Y any set, let $\alpha > 0$ and let $F, G : X \times Y \multimap \mathbb{R}$ multimaps with bounded values, with G sup-closed valued and suppose that they satisfy assumptions* (**H0**), (**H1**), (**H2**), (**H4**), (**H5**) *of Theorem 4.1 with the half line $]0, +\infty)$ replaced by $]\alpha, +\infty)$ and*

(**H3**)$''$ *there exists $\lambda, \mu > 0$ with $\lambda + \mu \leq 1$ such that F is multivalued finitely (λ, μ)-inf complete in the first variable.*

Then there exists $y_0 \in Y$ such that

$$G(x, y_0) \cap [\alpha, \infty) \neq \emptyset \quad \forall x \in X.$$

The specular case $\alpha < 0$, $\lambda + \mu > 1$ can be equivalently treated.

On the other side, the following example shows that the result does not hold when $\alpha > 0$ and $\lambda + \mu > 1$, even in the single valued case.

Example 4.2 Let $X = Y = [0, 1]$ and consider $f : X \times Y \to \mathbb{R}$ defined as

$$f(x, y) = (1 - y)\sqrt{x} + y\frac{\sqrt{1 - x^2}}{2}.$$

Since f is continuous in x and X is compact, assumptions **(H2)** and **(H5)** for $F \equiv f$ are trivially satisfied. For $\alpha = \sqrt{\sqrt{5} - 2}$, assumption **(H1)** is easily satisfied: it is enough to choose $y = 1$ for $x \leq \alpha^2$ and $y = 0$ for $x > \alpha^2$.

The function f is finitely (λ, μ)-inf complete in x for $\lambda = \mu = 1$, since $f(x, y) \geq 0$ for every (x, y).

We need to verify assumption **(H4)**, i.e. for every pair $y_1, y_2 \in Y$ and for every $t \in [0, 1]$ there exists $y_0 \in Y$ such that

$$f(x, y_0) \geq t f(x, y_1) + (1 - t) f(x, y_2), \quad \forall x \in X.$$

Define $f_1(x) = \sqrt{x}$ and $f_2(x) = \dfrac{\sqrt{1 - x^2}}{2}$, one gets

$$\begin{aligned} t f(x, y_1) + (1 - t) f(x, y_2) &= t(1 - y_1) f_1(x) + t y_1 f_2(x) + (1 - t)(1 - y_2) f_1(x) \\ &\quad + (1 - t) y_2 f_2(x) = [t(1 - y_1) + (1 - t)(1 - y_2)]\, f_1(x) \\ &\quad + [t y_1 + (1 - t) y_2]\, f_2(x) \\ &= [1 - t y_1 - (1 - t) y_2]\, f_1(x) + [t y_1 + (1 - t) y_2]\, f_2(x). \end{aligned}$$

For $y_t = t y_1 + (1 - t) y_2 \in [0, 1]$ one has

$$t f(x, y_1) + (1 - t) f(x, y_2) = (1 - y_t) f_1(x) + y_t f_2(x) = f(x, y_t), \quad \forall x \in X,$$

so one has to choose $y_0 = y_t$.

Nevertheless, the thesis does not hold, i.e. there are no $y_0 \in [0, 1]$ such that $f(x, y_0) \geq \alpha$, for every $x \in X$.

Indeed, assume by contradiction that there exists such y_0, we would have

$$f(0, y_0) \geq \alpha \quad \text{and} \quad f(1, y_0) \geq \alpha,$$

or, equivalently,

$$\frac{y_0}{2} \geq \alpha \quad \text{and} \quad 1 - y_0 \geq \alpha,$$

then

$$2\alpha \leq y_0 \leq 1 - \alpha,$$

so $\alpha \leq \dfrac{1}{3}$. But $\alpha \cong 0,4795 > 0,\overline{3}$, a contradiction.

5 Comparison with Previous Results

Theorem 4.1 in the case of one multifunction can be compared with [7] Theorem 5.3, where diagonally multivalued quasi-convexity is assumed in place of multivalued finitely (λ, ν)-inf completeness. In this section X is a convex subset of a Hausdorff topological vector space.

Definition 5.1 ([7], Definition 4.5) A multimap $F : X \times X \multimap \mathbb{R}$ is said to be *diagonal multivalued transfer quasi-convex* (*diagonal multivalued transfer quasi-concave*) in the first variable if for every finite set $\{x_1, \dots, x_n\} \subset X$ there exists $\{y_1, \dots, y_n\} \subset X$ such that for every $Y \subset \{y_1, \dots, y_n\}$, say $Y = \{y_{j1}, \dots, y_{js}\}$, for every $y \in \mathrm{co}Y$, and for every $t_{j\ell} \in F(x_{j\ell}, y)$, there exists $t \in F(y, y)$ such that

$$t \leq \max_{1 \leq \ell \leq s} t_{j\ell} \left(t \geq \min_{1 \leq \ell \leq s} t_{j\ell} \right).$$

The previous definition can be charactherized in terms of the single valued map $f : X \times X \to \mathbb{R}$, defined as $f(x, y) = \inf F(x, y)$, $(x, y) \in X \times X$.

Definition 5.2 ([5], Definition 2) A map $f : X \times X \multimap \mathbb{R}$ is said to be *diagonal transfer quasi-convex* (*diagonal transfer quasi-concave*) in the first variable if for every finite set $\{x_1, \dots, x_n\} \subset X$ there exists $\{y_1, \dots, y_n\} \subset X$ such that for every $Y \subset \{y_1, \dots, y_n\}$, say $Y = \{y_{j1}, \dots, y_{js}\}$, and for every $y \in \mathrm{co}Y$, it follows

$$f(y, y) \leq \max_{1 \leq \ell \leq s} f(x_{j\ell}, y) \left(f(y, y) \geq \min_{1 \leq \ell \leq s} f(x_{j\ell}, y) \right).$$

Proposition 5.1 *Let $F : X \times X \multimap \mathbb{R}$ have bounded and closed values; then F is diagonal multivalued transfer quasi-convex if and only if the map $f : X \times X \to \mathbb{R}$, $f(x, y) = \inf F(x, y)$, $(x, y) \in X \times X$ is diagonal transfer quasi-convex.*

For the reader's convenience, we remind the statement of [7] Theorem 5.3 in the same inequality direction as in Theorem 4.1.

Theorem 5.1 *Let $F : X \times X \multimap \mathbb{R}$ be a multimap with bounded and sup-closed values, satisfying*

(H1) *for each* $x \in X$, $F(x,x) \subset [0,+\infty)$;
(H2) *for every pair* (x_0, y_0) *such that* $F(x_0, y_0) \subset (-\infty, 0[$, *there exists a neighbourhood* U *of* y_0 *such that, for each* $z \in U$, $F(x_0, z) \subset (-\infty, 0[$;
(H3) $x \multimap F(x,y)$ *is diagonally transfer multivalued quasi-convex in* x;
(H4) *there exist a relatively compact subset* $K \subset X$ *and a finite set* $X_0 \subset X$ *such that for each* $\bar{y} \notin K$ *there is some* $\bar{x} \in X_0$ *and a neighbourhood* U *of* $\bar{y}$ *with* $F(\bar{x}, z) \subset (-\infty, 0[$ *for each* $z \in U$.

Then there exists $y_0 \in X$ *such that*

$$F(x, y_0) \cap [0, +\infty) \neq \emptyset \quad \forall x \in X.$$

We shall provide an example of a multivalued map to which one can apply Theorem 4.1 but not Theorem 5.1.

Example 5.1 Let g be the map in Example 2.1, $X = [0,1]$, $f : X \times X \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} 0 & \text{if } x = 0, y \neq 0 \\ g(x-y) & \text{if } x \neq 0, y \neq 0, x \neq y, \\ 1 & \text{if } x \neq 0, y = 0 \\ 2 & \text{if } x = y \end{cases}$$

and $F : X \times X \multimap \mathbb{R}$ defined as $F(x,y) = [f(x,y), 3]$. Since $F(x,x) \subset]0,+\infty)$, assumption **(H1)** in Theorem 4.1 and in Theorem 5.1 is satisfied. As for assumption **(H2)** we have the following situation: **(H2)** in Theorem 5.1 is automatically satisfied since $F(x,y) \subset [0,+\infty)$ for every $(x,y) \in X \times X$; as for Theorem 4.1 $F(x_0,y_0) \subset]0,+\infty)$ if $f(x_0,y_0) = 1$ or $f(x_0,y_0) = 2$. This in turn happens in the following three cases:

(1) $x_0 \neq 0$ and $y_0 = 0$
(2) $x_0 \neq 0$, $y_0 \neq 0$ and $x_0 - y_0 \in \mathbb{R} \setminus \mathbb{Q}$
(3) $x_0 = y_0$

Choose $Y_0 = \{0\}$ for all the above cases. So, for every $z \in [0,1]$ we have

$$f(z,0) = \begin{cases} 1 \text{ if } z \neq 0 \\ 2 \text{ if } z = 0 \end{cases}$$

Hence, either $F(z,0) = [1,3] \subset]0,+\infty)$ for every $z \in]0,1]$, or $F(0,0) = [2,3] \subset]0,+\infty)$. Hence in the whole $[0,1]$ one has $F(z,0) \subset]0,+\infty)$, i.e. F is 0-transfer inclusion quasi continuous in the first variable, thus, assumption **(H2)** in Theorem 4.1 is satisfied as well. X being compact, assumption **(H4)** in Theorem 5.1 and assumption **(H5)** in Theorem 4.1 are trivially satisfied. It remains to discuss the convexity assumptions. Since $\sup F(x,y) = 3$ for every $(x,y) \in X \times X$, **(H4)** in Theorem 4.1 is immediate. Let us prove that **(H3)** in Theorem 4.1 is satisfied. To this aim we prove that f is $(2,2)$-lower complete, i.e. for any choice of x', x'', there exists $x \in X$ such that

$$f(x,y) \leq 2f(x',y) + 2f(x'',y) \quad \forall y \in X. \tag{6}$$

Let $x = 0$. We have two cases:

(I) for $y \neq 0$, $f(0,y) = 0$ and so (6) is automatically verified;

(II) for $y=0$, $f(0,0)=2$, $f(x',0)=f(x'',0)=1$ if $x'\neq 0$ and $x''\neq 0$, $f(x',0)=f(x'',0)=2$ if $x'=x''=0$, so

$$2f(x',0)+2f(x'',0)\geq 4>f(0,0)$$

and (6) is satisfied also in this case.

On the other side, **(H3)** in Theorem 5.1 fails to be true, namely there exists $\{x_1,\ldots,x_n\}\subset X$ such that for each $\{y_1,\ldots,y_n\}\subset X$ with $n>1$ there exist $J\subset\{1,\ldots,n\}$ and $y\in \mathrm{co}\{y_i, i\in J\}$ with

$$f(y,y)>\max_{j\in J} f(x_j,y). \tag{7}$$

Indeed, let $\{x_1,x_2\}\subset X\cap\mathbb{Q}$, with $x_1\neq x_2$. Let $\{y_1,y_2\}\subset X$,

(a) assume $y_1\neq y_2$, take $J=\{1,2\}$, so that $\mathrm{co}\{y_1,y_2\}\cap\mathbb{Q}\neq\emptyset$. Then there exists $y\in \mathrm{co}\{y_1,y_2\}\cap\mathbb{Q}$, with $y\neq x_1$ and $y\neq x_2$, where $f(y,y)=2$ and $f(x_i,y)=g(x_i-y)=0$ for $i=1,2$.
(b) suppose $y_1=y_2$, so $\mathrm{co}\{y_1,y_2\}=\{y\}$ with $y=y_1=y_2$. Thus, $f(y,y)=2$. For $y\neq x_1$ and $y\neq x_2$, we have $f(x_i,y)\leq 1$ and (7) is satisfied. For $y=x_1$ we consider $J=\{2\}$ obtaining

$$\max_{j\in J} f(x_j,y)=f(x_2,x_1)\leq 1.$$

For $y=x_2$ we consider $J=\{1\}$ getting

$$\max_{j\in J} f(x_j,y)=f(x_1,x_2)\leq 1;$$

thus, (7) follows also in this case. So applying Proposition 5.1 we get that F it is not multivalued transfer quasi-convex.

In order to deduce minimax equalities of the classical form

$$\sup_{y\in Y}\inf_{x\in X} f(x,y)=\inf_{x\in X}\sup_{y\in Y} f(x,y)$$

one usually needs to make use of a translation of the function f. Since, finitely (λ,μ)-lower completeness, as already observed, is not inherited by translation, the suitable equilibrium result to consider is the one contained in Theorem 4.2.

So, Theorem 4.2 specializes in the following form for single valued maps.

Theorem 5.2 *Let X be a topological space, Y any set, $\alpha\in\mathbb{R}$ and let $\varphi,\gamma: X\times Y\to\mathbb{R}$ satisfy*

(H0) $\varphi(x,y)\leq\gamma(x,y)$ *for every* $(x,y)\in X\times Y$;
(H1) *for each $x\in X$ there exists $y\in Y$ such that $\varphi(x,y)>\alpha$;*
(H2) *for every pair (x_0,y_0) such that $\varphi(x_0,y_0)>\alpha$ there are a finite set $Y_0\subset Y$ and a neighbourhood U of x_0 such that for each $z\in U$ there is $y'\in Y_0$ such that $\varphi(z,y')>\alpha$;*
(H3) $x\mapsto\varphi(x,y)$ *is finitely convex-like in x;*
(H4) $y\mapsto\gamma(x,y)$ *is concave-like in y;*

(H5) *there exist a compact set $K \subset X$ and a finite set $Y_0 \subset Y$ such that for every $\bar{x} \notin K$ there exists $y \in Y_0$ such that $\varphi(\bar{x}, y) > \alpha$.*

Then there exists $y_0 \in Y$ such that

$$\gamma(x, y_0) \geq \alpha \quad \forall x \in X.$$

Proof Considering two multimaps $F : X \times Y \multimap \mathbb{R}$ and $G : X \times Y \multimap \mathbb{R}$ defined respectively as

$$F(x, y) = \{\varphi(x, y)\}, \quad G(x, y) = [\varphi(x, y), \gamma(x, y)],$$

the result trivially follows from Theorem 4.2. □

Remark 5.1 In the case of one single valued map Theorem 5.2 is a strict generalization of the minimax result in [10]. In particular, we weaken the assumptions of convex likeness, of lower semicontinuity in the first variable, and of compactness of the domain, requiring finite convex likeness and conditions (**H2**) and (**H5**) instead.

6 Applications

Since to prove the next result we will use a single valued version of Theorem 4.2 with x and y swapped, we state here this result for the reader's convenience.

Theorem 6.1 *Let X be a topological space, $\alpha \in \mathbb{R}$ and let $\varphi : X \times X \to \mathbb{R}$ satisfy*

- **(H1)** *for each $x \in X$ $\varphi(x, x) > \alpha$;*
- **(H2)** *for every pair (x_0, y_0) such that $\varphi(x_0, y_0) > \alpha$ there are a finite set $X_0 \subset X$ and a neighbourhood U of y_0 such that for each $z \in U$ there is $x' \in X_0$ such that $\varphi(x', z) > \alpha$;*
- **(H3)** *$y \mapsto \varphi(x, y)$ is finitely convex-like in y;*
- **(H4)** *$x \mapsto \varphi(x, y)$ is concave-like in x;*
- **(H5)** *there exists a compact set $K \subset X$ and a finite set $X_0 \subset X$ such that for every $\bar{y} \notin K$ there exists $x \in X_0$ with $\varphi(x, \bar{y}) > \alpha$.*

Then there exists $x_0 \in X$ such that

$$\varphi(x_0, y) \geq \alpha \quad \forall y \in X.$$

6.1 Nash Equilibria

A noncooperative game is defined by a finite set of players $I = 1, \ldots, n$; each player has a set of strategies X_i, which is a nonempty subset of a topological space E_i, and $u_i : X \to \mathbb{R}$ is the payoff function of player i, where $X = X_1 \times \cdots \times X_n$. So we can define a noncooperative game in the normal form as

$$G = (X_i, u_i)_{i \in I}.$$

When the player i chooses a strategy $x_i \in X_i$, the situation of the game is described by the vector $x = (x_1, \ldots, x_n) \in X$. For each player $i \in I$ denote by $X_{-i} = \prod_{j \neq i} X_j$ the cartesian product of the sets of strategies of players $j \neq i$ and $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Note that $x = (x_i, x_{-i})$, $i = 1, \ldots, n$.

Definition 6.1 (Nash equilibrium of a game) A strategy profile $x^* \in X$ is a pure strategy *Nash equilibrium* of a game G if

$$u_i(y_i, x^*_{-i}) \leq u_i(x^*) \quad \forall y_i \in X_i.$$

Definition 6.2 The function $\Phi : X \times X \to \mathbb{R}$ is said to be the aggregator function defined at each $(x, y) \in X \times X$ by

$$\Phi(x, y) = \sum_{i=1}^{n} u_i(x_1, \dots, y_i, \dots, x_n) = \sum_{i=1}^{n} u_i(y_i, x_{-i}).$$

Moreover we define $f : X \times X \to \mathbb{R}$ as

$$f(x, y) = \Phi(x, x) - \Phi(x, y) = \sum_{i=1}^{n} (u_i(x) - u_i(y_i, x_{-i})), \quad x, y \in X \times X. \tag{8}$$

Theorem 6.2 *Let X be a topological space, K a compact subset of X and let $f : X \times X \to \mathbb{R}$ satisfy*

- **(N1)** *for every pair (x_0, y_0) such that $f(x_0, y_0) > 0$ there are a finite set $X_0 \subset X$ and a neighbourhood U of y_0 such that for each $z \in U$ there is $x' \in X_0$ such that $f(x', z) > 0$;*
- **(N2)** *$y \mapsto f(x, y)$ is finitely convex-like in y;*
- **(N3)** *$x \mapsto f(x, y)$ is concave-like in x;*
- **(N4)** *there exists a finite set $X_0 \subset X$ such that for each $\bar{y} \notin K$ there is some $x \in X_0$ with $f(x, \bar{y}) > 0$.*

Then the game G has a Nash equilibrium.

Proof. To get the claimed result we will apply Theorem 6.1. By the definition of $f : X \times X \to \mathbb{R}$ we have that

$$f(x, x) \geq 0 \quad \forall x \in X. \tag{9}$$

By Remark 4.1 we can substitute condition **(H1)** in Theorem 6.1 by (9). All the other assumptions are trivially satisfied and so exists $x^* \in X$ such that

$$f(x^*, y) \geq 0 \quad \forall y \in X,$$

i.e. the game G has a Nash equilibrium.

6.2 Maximization of Binary Relation

A *binary relation U* on a set K is a multimap, $U : K \multimap K$, from K into itself with possibly empty values. We can write $y \in U(x)$ to mean that y stands in the relation U to x. A *maximal element* of the binary relation U is a point x such that no point y satisfies $y \in U(x)$, i.e. $U(x) = \emptyset$. Thus, denoting with $K_0 = \{x : U(x) \neq \emptyset\}$, the set of maximal elements of U is equal to $K \setminus K_0$. (see e.g. [9]).

In [7] the authors proved an existence theorem of maximal elements for a convex set K. In the sequel we shall extend the existence result to binary relations U defined on a paracompact topological space K which is not assumed to be neither metric nor convex.

Theorem 6.3 *Let K be a paracompact topological space and $U : K \multimap K$ be a multimap satisfying*

(a) $x \notin U(x), \forall x \in K$;
(b) *for every* $x', x'' \in K_0$, *there exists* $\bar{x} \in K_0$ *such that* $U(\bar{x}) \subset U(x') \cap U(x'')$;
(c) *for every* $x \in K$ *there exists* $x' \in K$ *such that* $x \in \left[U^c(x')\right]^o$;
(d) *there exist a relatively compact set* $A \subset K$ *and a finite set* X_0 *such that for every* $\bar{y} \notin A$ *there exist* $x \in X_0$ *such that* $\bar{y} \notin U(x)$.

Then $K \setminus K_0 \neq \emptyset$.

Proof We shall apply Theorem 6.1 with $\alpha = -\varepsilon$ with $0 < \varepsilon < 1$ to $\varphi(x, y) = -\chi_{U(x)}(y)$. By **(a)** $\varphi(x, x) = 0 > -\varepsilon$.

Assumption **(c)** ensures that **(H2)** is fulfilled. Indeed, from **(c)** for every $y \in K$ there exists $x \in K$ such that $y \in [U^c(x)]^o$; therefore

$$\mathcal{A} = \{\left[U^c(y)\right]^o,\ y \in K\}$$

is an open cover of K. Being K paracompact by assumption, $\mathcal{A}$ has a locally finite open refinement $\mathcal{B}$, i.e. for every $y \in K$ there exists a neighbourhood of y, $W(y)$, that intersects only finitely many sets in the cover $\mathcal{B}$, say $B_1, \ldots, B_k$. From the fact that $\mathcal{B}$ is a refinement of the cover $\mathcal{A}$ we have that for every B_i, $i = 1, \ldots, k$ there exists $x_i \in K$ such that $B_i \subset [U^c(x_i)]^o$. Now,

$$W(y) = \bigcup_{j=1}^{k} (W(y) \cap B_j) \subset \bigcup_{j=1}^{k} \left[U^c(x_j)\right]^o \subseteq \bigcup_{j=1}^{k} U^c(x_j).$$

We have proven that for every $y \in K$ there are a neighbourhood, $W(y)$, of y and a finite set $\{x_1, \ldots, x_k\} \subset K$ such that for every $z \in W(y)$, $z \notin U(x_i)$ for some $i = 1, \ldots, k$, i.e. $\varphi(x_i, z) = 0 > -\varepsilon$.

Requirement **(H5)** follows directly from **(d)**.

Let us prove **(H3)**. For $\{x_1, \ldots, x_n\} \subset K$ fixed, for each $y', y'' \in K$ and $t \in [0, 1]$ we have

$$t\varphi(x_i, y') + (1 - t)\varphi(x_i, y'') = \begin{cases} 0 & \text{if } y', y'' \notin U(x_i), \\ -t & \text{if } y' \in U(x_i),\ y'' \notin U(x_i), \\ t - 1 & \text{if } y' \notin U(x_i),\ y'' \in U(x_i), \\ -1 & \text{if } y', y'' \in U(x_i). \end{cases}$$

If $\{x_1, \ldots, x_n\} \cap K_0 \neq \emptyset$, note that from **(b)** it follows that $\bigcap_{i=1}^{n} U(x_i) \neq \emptyset$. Pick $\overline{y} \in \bigcap_{i : x_i \in K_0} U(x_i)$, then

$$\varphi(x_i, \overline{y}) = -1 \leq t\varphi(x_i, y') + (1 - t)\varphi(x_i, y''), \quad \text{for } x_i \in K_0$$

and

$$\varphi(x_i, \overline{y}) = 0 = t\varphi(x_i, y') + (1 - t)\varphi(x_i, y''), \quad \text{for } x_i \notin K_0.$$

Otherwise, when $\{x_1, \ldots, x_n\} \cap K_0 = \emptyset$, $U(x_i) = \emptyset$ for every $i = 1, \ldots, n$, so that

$$\varphi(x_i, \overline{y}) = 0 = t\varphi(x_i, y') + (1-t)\varphi(x_i, y'')$$

for every $\overline{y} \in K$.

It only remains to prove assumption **(H4)**, i.e. for each $x', x'' \in K$ and $t \in [0, 1]$ there exists $\overline{x} \in K$ such that

$$\varphi(\overline{x}, y) \geq t\varphi(x', y) + (1-t)\varphi(x'', y), \quad \forall\, y \in K.$$

We shall consider two cases

(i) $x' \notin K_0$ or $x'' \notin K_0$
(ii) $x' \in K_0$ and $x'' \in K_0$

In case (i) let for instance $x' \notin K_0$, i.e. $U(x') = \emptyset$, let $\overline{x} = x'$. So $\varphi(\overline{x}, y) = 0$ for every $y \in K$.

In case (ii) choose $\overline{x}$ according with assumption **(b)**. For $y \in K$ only two cases occur, either $y \in U(\overline{x})$, or $y \notin U(\overline{x})$. If $y \in U(\overline{x})$, then $y \in U(x') \cap U(x'')$, hence

$$-1 = \varphi(\overline{x}, y) = t\varphi(x', y) + (1-t)\varphi(x'', y).$$

When $y \notin U(\overline{x})$, then

$$0 = \varphi(\overline{x}, y) \geq t\varphi(x', y) + (1-t)\varphi(x'', y).$$

whichever values $\varphi(x', y)$, $\varphi(x'', y)$ are.

Therefore, from Theorem 6.1, $x_0 \in K$ exists such that $\varphi(x_0, y) \geq -\varepsilon$ for every $y \in K$, that is $\chi_{U(x_0)}(y) = 0$ for every $y \in K$, since $0 < \varepsilon < 1$, i.e. $U(x_0) = \emptyset$. □

Remark 6.1 We consider the particular case of a binary relation U induced by a utility function $u : K \to \mathbb{R}$, i.e.

$$U(x) = \{y \in K : \ u(y) > u(x)\}.$$

1. If $u : K \to \mathbb{R}$ is upper semicontinuous, assumption **(c)** is satisfied.
2. Observe that in this situation assumption **(b)** is automatically satisfied and the following alternative holds:

$$x \in U(y) \quad \Longrightarrow y \notin U(x).$$

   Under this implication condition **(c)** and **(d)** in Theorem 6.2 of [7] respectively imply **(c)** and **(d)** in Theorem 6.3. Hence for this kind of binary relation Theorem 6.3 extends the above mentioned result.

Appendix

As anticipated in Sect. 3, we show now an example of a map that is finitely convex-like, but fails to be convex-like. In order to provide the example we need the following lemmata.

Lemma 7.1 *Given $\gamma \in (\mathbb{R}\setminus\mathbb{Q}) \cap [0, 1]$, there exists $\beta \in \mathbb{R}\setminus\mathbb{Q}$ such that $\gamma\beta \notin \mathbb{Q}$ and $\gamma\beta^2 \notin \mathbb{Q}$.*

Proof Let γ be in $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$ be fixed. By contradiction, we assume that for every $\beta \in \mathbb{R} \setminus \mathbb{Q}$ either $\gamma\beta \in \mathbb{Q}$ or $\gamma\beta^2 \in \mathbb{Q}$. Surely both $\gamma\beta$ and $\gamma\beta^2$ can not be both rationals, because then we would have $\beta = \dfrac{\gamma\beta^2}{\gamma\beta} \in \mathbb{Q}$, while we are assuming $\beta \in \mathbb{R}\setminus\mathbb{Q}$. Hence, for every $\beta \in \mathbb{R}\setminus\mathbb{Q}$ only one between $\gamma\beta$ and $\gamma\beta^2$ has to be rational. Thus $\gamma\beta^2 - \gamma\beta \notin \mathbb{Q}$, so every equation

$$\gamma\beta^2 - \gamma\beta - \lambda = 0$$

with $\lambda \in \mathbb{Q}^+$ has no solution $\beta \notin \mathbb{Q}$. Equivalently

$$\gamma \left(\beta^2 - \beta - \frac{\lambda}{\gamma} \right) = 0$$

has no irrational solutions. Now

$$\beta_1, \beta_2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\lambda}{\gamma}},$$

so it necessarily has to be $\sqrt{\dfrac{1}{4} + \dfrac{\lambda}{\gamma}} \in \mathbb{Q}$, for every $\lambda \in \mathbb{Q}$, which is clearly false. Indeed it would imply that $\dfrac{1}{4} + \dfrac{\lambda}{\gamma} \in \mathbb{Q}$ and so $\dfrac{\lambda}{\gamma} \in \mathbb{Q}$ for every $\lambda \in \mathbb{Q}$. Then we would have $\dfrac{\gamma}{\lambda} \in \mathbb{Q}$, hence $\gamma = \lambda \dfrac{\gamma}{\lambda} \in \mathbb{Q}$, which is absurd as we assume $\gamma \notin \mathbb{Q}$. $\square$

Lemma 7.2 *Given k irrational numbers $z_1, \ldots, z_k \in [0, 1]$ there exists an irrational z_0 such that $z_i z_0 \notin \mathbb{Q}$, $i = 1, \ldots, k$.*

Proof We prove first that the claim is true for $k = 2$.

Case A. If $z_1 z_2 \in \mathbb{R}\setminus\mathbb{Q}$, we have that, since $z_i \notin \mathbb{Q}$, then $\dfrac{1}{z_i} \notin \mathbb{Q}$. Even $\dfrac{1}{z_1 z_2} \notin \mathbb{Q}$ so, for $z_0 = \dfrac{1}{z_1 z_2}$ we find $z_1 z_0 = \dfrac{1}{z_2} \notin \mathbb{Q}$ and $z_2 z_0 = \dfrac{1}{z_1} \notin \mathbb{Q}$. In order to have $z_0 \in [0, 1]$ it is enough to choose n such that $\dfrac{1}{n z_1 z_2} \le 1$.

Case B. $z_1 z_2 \in \mathbb{Q}$ and $z_1^2 \in \mathbb{Q}$; we choose $\beta \in (\mathbb{R}\setminus\mathbb{Q}) \cap [0, 1]$ such that $\beta z_1 \in \mathbb{Q}$. Then, if $z_0 = \beta z_1$, since $z_1^2 \in \mathbb{Q}$ and $\beta \in \mathbb{R}\setminus\mathbb{Q}$, we obtain that $z_1 z_0 = \beta z_1^2 \notin \mathbb{Q}$ and $z_2 z_0 = \beta(z_1 z_2) \notin \mathbb{Q}$.

Case C. $z_1 z_2 \in \mathbb{Q}$ and $z_1^2 \notin \mathbb{Q}$. Applying Lemma 7.1 with $\gamma = z_1^2$ we can obtain β such that $\beta z_1^2 \notin \mathbb{Q}$ and $\beta^2 z_1^2 \notin \mathbb{Q}$, so $\beta z_1 \notin \mathbb{Q}$. Now, if we pick $z_0 = \beta z_1$ we have $z_1 z_0 = \beta z_1^2 \notin \mathbb{Q}$ and $z_2 z_0 = \beta(z_1 z_2) \notin \mathbb{Q}$.

Let us prove now that the claim is true for $k > 2$.

We consider $J = \{i \in \{1, \ldots, k\}$ such that $z_i^2 \in \mathbb{Q}\}$.

Case A. Assume we have $J = \{1, \ldots, k\}$, so all the square numbers are rational. Then $z_i = \sqrt{\dfrac{p_i}{q_i}}$ with $p_i, q_i \in \mathbb{N}$, $i = 1, \ldots, k$. Let s be a prime number greater of all the p_i and the q_i's. $z_0 = \sqrt{s} \notin \mathbb{Q}$ then $z_0 z_1 = \sqrt{\dfrac{s p_i}{q_i}} \notin \mathbb{Q}$, because otherwise we would have that $\dfrac{s p_i}{q_i} = \dfrac{\sigma^2}{\tau^2}$ with $\sigma, \tau \in \mathbb{N}$, where the ratio is already reduced to the lowest terms. Hence $s p_i \tau^2 = q_i \sigma^2$.

In this equality only σ^2 could contain the factor s, with an even exponent as it is a square number, i.e. $\sigma^2 = \alpha s^{2n}$ with $\alpha \in \mathbb{N}$. Then $\frac{sp_i}{q_i} = \frac{\alpha s^{2n}}{\tau^2}$, hence $\frac{p_i}{q_i} = \frac{\alpha s^{2n-1}}{\tau^2}$, i.e. $\tau^2 p_i = \alpha q_i s^{2n-1}$. This time it is τ^2 that can contain the factor s, with an even exponent again, so $\tau^2 = s^{2k}$. So we proved that the ratio $\frac{\sigma^2}{\tau^2}$ are not reduced to the lowest terms.

Case B. Assume $J = \emptyset$, so all the square numbers are irrational. Thus, we have two subcases. Fix an index, let k be it, then consider all the products $z_i z_k$, $i = 1, \ldots, k-1$.

Subcase I. If $z_i z_k \notin \mathbb{Q}$ for every $i = 1, \ldots, k-1$ it is enough to consider $z_0 = z_k$.

Subcase II. If there are $z_i z_k \in \mathbb{Q}$, necessarily the number of products $z_i^2 z_k^2 \notin \mathbb{Q}$ with $i \neq k$ is smaller than $k-1$. By induction, choose $\beta \notin \mathbb{Q}$ such that $\beta z_i^2 z_k^2 \notin \mathbb{Q}$ for those irrational products, with $\beta z_k^2 \notin \mathbb{Q}$. Obviously for the rational products we have $\beta z_i^2 z_k^2 \notin \mathbb{Q}$. Then consider $z_0 = \sqrt{\beta} z_k$: it has to be $z_0 \notin \mathbb{Q}$ otherwise $z_0^2 = \beta z_k^2 \in \mathbb{Q}$. Similarly for $z_0 z_i = \sqrt{\beta} z_k z_i$, as its square is irrational too.

Case C. $J \subsetneq \{1, \ldots, k\}$, but $J \neq \emptyset$. In this case $z_i^2 \notin \mathbb{Q}$ just for a number of indices smaller than k. Then, by induction, there exists $\beta \in \mathbb{R} \backslash \mathbb{Q}$ such that $\beta z_i^2 \notin \mathbb{Q}$ with $i \notin J$ and, a fortiori, $\beta z_i^2 \notin \mathbb{Q}$, $i = 1, \ldots, k$. So, $z_0 = \sqrt{\beta} \notin \mathbb{Q}$ and $z_0 z_i \notin \mathbb{Q}$ as if $\sqrt{\beta} z_i \in \mathbb{Q}$ it would imply $\beta z_i^2 \in \mathbb{Q}$, a contradiction. □

Example 7.1 Let $X = Y = [0, 1]$ and consider $f : X \times Y \to \mathbb{R}$ where

$$f(x, y) = \begin{cases} 1 & \text{for } xy \in \mathbb{Q} \\ -1 & \text{for } xy \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

For $\{y_1, \ldots y_n\}$ fixed, we have three possible cases.

First case. $y_i \in \mathbb{Q}$, $\forall i = 1, \ldots, n$. In this case

$$f(x, y_i) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ \\ -1 & \text{for } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}, i = 1, \ldots, n$$

Therefore, whatever x', x'' it is enough to pick $x_0 \in \mathbb{R} \backslash \mathbb{Q}$ to obtain

$$f(x_0, y_i) = -1 \leq t f(x', y_i) + (1-t) f(x'', y_i) = \begin{cases} 1 \\ 2t - 1 \\ 1 - 2t \\ -1 \end{cases}$$

for every $t \in [0, 1]$.

Second case. $y_i \in \mathbb{R} \backslash \mathbb{Q}$, $\forall i = 1, \ldots, n$.

Similarly to the first case, since

$$t f(x', y_i) + (1-t) f(x'', y_i) = \begin{cases} 1 \\ 2t - 1 \\ 1 - 2t \\ -1 \end{cases}$$

it is enough to pick x_0 such that $f(x_0, y_i) = -1$, so we consider $x_0 \in \mathbb{Q}$.

Third case. $\{y_1, \ldots, y_n\} \cap \mathbb{Q} \neq \emptyset$ and $\{y_1, \ldots, y_n\} \cap \mathbb{R} \backslash \mathbb{Q} \neq \emptyset$.

Since for $x', x'' \in \mathbb{R}\backslash\mathbb{Q}$ we obtain $tf(x', y_i) + (1-t)f(x'', y_i) = -1$ whenever $y_i \in \mathbb{Q}$, also in this case we need $f(x_0, y_i) = -1$ for every $i = 1, \ldots, n$. For $y_i \in \mathbb{Q}$, we have necessarily to choose $x_0 \in \mathbb{R}\backslash\mathbb{Q}$, for $y_i \notin \mathbb{Q}$, we can choose $x_0 \in \mathbb{R}\backslash\mathbb{Q}$ according to Lemma 2.2.

On the contrary, let us prove that f is not convex-like. By contradiction, assume that for every $x_1, x_2 \in [0, 1]$ and for every $t \in [0, 1]$ there exists $x_0 \in [0, 1]$ such that

$$f(x_0, y) \leq tf(x_1, y) + (1-t)f(x_2, y) \quad \forall y \in Y.$$

Fixed $x_1 \in \mathbb{Q}$, $x_2 \notin \mathbb{Q}$ and $t = \dfrac{1}{2}$ we have

$$tf(x_1, y) + (1-t)f(x_2, y) \geq -1 \quad \forall y \in Y.$$

Hence there should exist x_0 such that $f(x_0, y) = -1$ for every $y \in [0, 1]$. That would imply that $x_0 y \notin \mathbb{Q}$ for every $y \in [0, 1]$, and this is impossible, since for $x_0 \in \mathbb{Q}$ and $y \in \mathbb{Q}$ we have $f(x_0, y) = 1$ and for $x_0 \in \mathbb{R}\backslash\mathbb{Q}$ there exists $\lambda \in \mathbb{Q}$ such that $y = \dfrac{\lambda}{x_0} \in [0, 1]$. Then $x_0 y = \lambda \in \mathbb{Q}$, so $f(x_0, y) = 1$, a contradiction.

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References

1. Alleche, B., Radulescu, V.D.: Further on set-valued equilibrium problems and applications to Browder variational inclusions. J. Optim. Theory Appl. **175**, 39–58 (2017)
2. Alleche, B., Radulescu, V.D.: Further on set-valued equilibrium problems in the pseudo-monotone case and applications to Browder variational inclusions. Optim. Lett. **12**, 1789–1810 (2018)

3. Bardaro, C., Ceppitelli, R.: Fixed point theorems and vector-valued minimax theorems. J. Math. Anal. Appl. **146**, 369–373 (1990)
4. Bardaro, C., Ceppitelli, R.: Some generalizations in H-modular spaces of Fan's best approximation theorem. J. Aust. Math. Soc. **56**, 291–302 (1994)
5. Baye, M.R., Tian, G., Zhou, J.: Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. Rev. Econ. Stud. **60**, 935–948 (1993)
6. Ben-El-Mechaiekh, H.: Abstract convexity and fixed points. J. Math. Anal. Appl. **222**, 138–150 (1998)
7. Benedetti, I., Martellotti, A.: Multivalued minimax relations with applications. Minimax Theory Appl. **1**, 307–332 (2016)
8. Berenguer, M., Gámez, D., Garralda-Guillem, A.I., Ruiz Galán, M.: Minimax inequalities and variational equations. In: Le Thi, H., Le, H., Pham Dinh, T. (eds.) Optimization of Complex Systems: Theory, Models, Algorithms and Applications. WCGO 2019. Advances in Intelligent Systems and Computing, vol. 991. Springer, Berlin (2020)
9. Border, K.C.: Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University Press, Cambridge (1985)
10. Borwein, J.M., Zhuang, D.: On Fan's minimax theorem. Math. Program. **34**, 232–234 (1986)
11. Brezis, H., Nirenberg, L., Stampacchia, G.: A remark on Ky Fan's minimax principle. Boll. Unione Mat. Ital. **4**(6), 293–300 (1972)
12. Browder, F.E.: The fixed point theory of multi-valued mappings in topological vector spaces. Math. Ann. **177**, 283–301 (1968)
13. Bukhvalov, A.V., Martellotti, A.: On minimax theorems for sets closed in measure. Vladikavkaz. Mat. Zh. **6**(1), 29–36 (2004)
14. De Finetti, B.: Sulle stratificazioni convesse. Ann. Mat. Pura Appl. **30**, 173–183 (1949)
15. Ding, X.P., Park, J.Y.: Generalized vector equilibrium problems in generalized convex spaces. J. Optim. Theory Appl. **120**(2), 327–353 (2004)
16. Ding, X.P., Park, J.Y., Jung, I.H.: Pareto equilibria for multiobjective games in locally L-convex spaces. Comput. Math. Appl. **46**, 1589–1599 (2003)
17. Durea, M.: Variational inclusions for contingent derivative of set-valued maps. J. Math. Anal. Appl. **292**, 351–363 (2004)
18. Eslamizadeh, L., Naraghirad, E.: Existence of solutions of set-valued equilibrium problems in topological vector spaces with applications. Optim. Lett. **14**, 65–83 (2020)
19. Fan, K.: Minimax theorems. Proc. Natl. Acad. Sci. USA **39**, 42–47 (1953)
20. Fan, K.: A minimax inequality and its application. In: Inequalities III (Proc. Third Sympos. Univ. California, Los Angeles, Calif., 1969), pp. 103–113. Academic Press, New York (1972)
21. Fan, K., Glicksberg, I., Hoffmann, A.J.: Systems of inequalities involving convex functions. Proc. Am. Math. Soc. **8**, 617–622 (1957)
22. Fuchssteiner, B., König, H.: New versions of the Hahn-Banach theorem. In: Beckenbach, E.F. (ed.) General Inequalities 2. International Series of Numerical Mathematics (1980)
23. Geraghty, M.A., Lin, B.L.: Minimax theorems without linear structure. Linear Multilinear Algebra **17**, 171–180 (1975)
24. Granas, A., Liu, F.C.: Some minimax theorems without linear structure. In: Nonlinear and Convex Analysis. Proceedings in Honour of Ky Fan. LN in Pure and Appl. Math., vol. 107, pp. 61–75. Dekker, New York (1987)
25. Ha, C.W.: On systems of convex inequalities. J. Math. Anal. Appl. **68**, 25–34 (1979)
26. Horvath, C.: Pointes fixes et coincidences pour les applications multivoques sans convexité. C. R. Acad. Sci. **296**, 403–406 (1983)
27. Kalmoun, E.M.: On some new KKM theorems and related applications. Nonlinear Funct. Anal. Appl. **8**(1), 1–23 (2003)
28. Kassay, G., Radulescu, V.: Equilibrium Problems and Applications. Academic Press, San Diego (2018)
29. Khanh, P.Q., Quan, N.H.: Topologically-based characterizations of the existence of solutions of optimization-related problems. Math. Nachr. **288**(17–18), 2057–2078 (2015)
30. Krystali, A., Varga, C.: Set-valued versions of Ky Fan's inequality with application to variational inclusion theory. J. Math. Anal. Appl. **282**, 8–20 (2003)
31. Lin, L., Ansari, Q., Wu, J.: Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems. J. Optim. Theory Appl. **117**, 121–137 (2003)
32. Martellotti, A., Salvadori, A.: Inequality systems and minimax results without linearstructure. J. Math. Anal. Appl. **148**, 79–86 (1987)
33. Park, S., Kim, H.: Coincidence theorems for admissible multifunctions on generalized convex spaces. J. Math. Anal. Appl. **197**, 173–187 (1996)
34. Pomerol, J.C.: Inequality systems and minimax theorems. J. Math. Anal. Appl. **103**, 263–292 (1984)
35. Pratelli, M.: A minimax theorem without compactness hypothesis. Mediterr. J. Math. **2**, 103–112 (2005)

36. Ruiz Galán, M.: A concave-convex Ky Fan minimax inequality. Minimax Theory Appl. **1**, 111–124 (2016)
37. Scalzo, V.: Remarks on the existence and stability of some relaxed Nash equilibrium in strategic form games. Econ. Theory **61**, 571–586 (2016)
38. Tian, G.: Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity. J. Math. Anal. Appl. **170**, 457–471 (1992)