# Representative Functions, Variational Convergence and Almost Convexity 

A. Eberhard ${ }^{1}$ (D) R. Wenczel ${ }^{1}$

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#### Abstract

We develop a new epi-convergence based on the use of bounded convergent nets on the product topology of the strong topology on the primal space and weak star topology on the dual space of a general real Banach space. We study the propagation of the associated variational convergences through conjugation of convex functions defined on this product space. These results are then applied to the problem of construction of a bigger-conjugate representative function for the recession operator associated with a maximal monotone operator on this real Banach space. This is then used to study the relationship between the recession operator of a maximal monotone operator and the normal-cone operator associated with the closed, convex hull of the domain of that monotone operator. This allows us to show that the strong closure of the domain of any maximal monotone operator is convex in a general real Banach space.


Keywords Maximal monotone operators • Representative functions • Almost convexity • Recession operators

Mathematics Subject Classification 47H05 • 46N10 • 47H04 • 49J53

## Introduction

Monotone operators have attracted the attention of researchers for many decades due to their important place in the theory of functional analysis and optimisation [2, 26, 30] and [3]. In [18, 21] Martínez-Legaz, Svaiter and Penot pioneered the use of monotone operator theory using representative functions, a tool which has come to be indispensable for the study of this topic. Representative functions are proper convex functions on $X \times X^{*}$ that characterise their associated monotone operator as the set of points of coincidence with the duality pairing. Indeed for any representative function this contact set is always a monotone set and so each representative function represents a given monotone operator. The notion of a representative function was introduced by Fitzpatrick in [14] where he gives an explicit formula for the minimal such function which is now called the Fitzpatrick representative function which may be used to represent any maximal monotone operator.

[^0]When one constructs a monotone operator from other maximal monotone operators the question arises as to how to obtain a representative function for the resultant operator, based on the knowledge of the Fitzpatrick function of the constituent maximal monotone operators. There have been many studies that have covered this issue from the point-of-view of certain binary operations of convex analysis [2,30,33] (and other references contained therein) but we wish, here, to single out constructions where we have a family of monotone operators and take a set-valued limit in the construction [15, 16, 19, 20, 22-24, 27]. The content of the last list of publications is much more varied in the nature of the convergences that are used in these constructions. Moreover, the spaces on which these constructions work are often affected by the convergence used, and the desired interaction of these convergences with conjugation (a necessary tool in the use of representative functions and convex analysis to study monotonicity). This has often resulted at least in one of two restrictive assumptions. The convergence is defined via a strong metric characterization (and possibly other additional assumptions on the family) or the space is assumed to be reflexive where conjugation is characterised in a fashion consistent with imposition of strong topologies on the primal space $X \times X^{*}$ and weak topologies on the dual space $\left(X \times X^{*}\right)^{* \dagger}=X \times X^{*}$ (the conjugate transpose operation).

When $X$ is reflexive, we conveniently find that the closures of a convex set with respect to all combinations of weak and strong product topologies that are possible for a convex set $C \subseteq X \times X^{*}$, all coincide due to the coincidence of the weak and weak* topologies and coincidence of the strong and weak continuous linear forms. Outside of a reflexive space this happy situation is completely missing and we must face the issue of compatibility of convergences and duality (and hence conjugacy) directly. This endeavour constitutes part of the study undertaken in this paper.

In this paper we also continue the study the recession operator [5,13,25] associated with any monotone operator $T$. We will focus on a construction of a bigger-conjugate representative function for the recession operator of a maximal monotone operator $T$.

Definition 1 Let $T: X \rightrightarrows X^{*}$ be an operator. The recession operator, rec $T: X \rightrightarrows X^{*}$ (see [5] for a sequential version)

$$
(\operatorname{rec} T)(z):=\left\{z^{*} \in X^{*} \mid \exists t_{\alpha} \rightarrow 0^{+},\left(z_{\alpha}, z_{\alpha}^{*}\right) \in T \text { such that }\left(z_{\alpha}, t_{\alpha} z_{\alpha}^{*}\right) \rightarrow^{s \times b d w^{*}}\left(z, z^{*}\right)\right\},
$$

where $z_{\alpha} \rightarrow^{s} z$ denote strong (norm) convergence in $X$ along the net and $t_{\alpha} z_{\alpha}^{*} \rightarrow{ }^{\mathrm{w}^{*}} z^{*}$ denotes weak ${ }^{*}$ convergence in the dual space $X^{*}$ and $\left(z_{\alpha}, t_{\alpha} z_{\alpha}^{*}\right) \rightarrow^{s \times b d w^{*}}\left(z, z^{*}\right)$ denotes this joint convergence with norm bounded nets.

One of the reason for interest in this operator is that it provides a natural connection between the domain of the original maximal monotone operator $T$ and its strong closure in that it is shown in [13, Lemma 11] that $\operatorname{dom}(\operatorname{rec} T)=\overline{\operatorname{dom}} T$.

In our study we single out a particular subclass of representative functions for study, those whose conjugate-transpose are pointwise larger that the original representative function. This class of bigger-conjugate representative functions interact with closure operations in interesting ways [13, Theorem 8]. In this paper we show how one may construct biggerconjugate representative functions from the Fitzpatrick function of the original maximal monotone operator $T$ that represents the recession monotone operator rec $T$, in the sense that the set-of-contact with the duality product (the set which it represents) contains the graph of rec $T$. Indeed the monotone operator it represents is $N_{\overline{\text { co }} \operatorname{dom} T}$, where $\overline{\text { co }} \operatorname{dom} T$ denotes the (strong) convex closure of the domain of $T$. Indeed whenever rec $T$ is maximal this result implies $\overline{c o} \operatorname{dom} T=\overline{\operatorname{dom}} T$. Convexity of the closure of the domain, for a monotone
operator, is here referred to as the almost convexity property. We note that the maximality of $\operatorname{rec} T$ for maximal operators $T$ has already been shown to be true in reflexive spaces in [11], providing a another proof of the almost-convexity property in this context. The almostconvexity problem is important for two reasons: 1) its resolution would be helpful as a tool to aid studies of the sum theorem (where domain assumptions can be essential); 2) a counterexample to almost-convexity for a maximal monotone operator is also a counterexample to the sum theorem (it is well known that if the sum theorem holds in a given Banach space, then on this space maximal monotone operators possess the almost-convexity property). We provide necessary and sufficient condition for almost-convexity similar to the necessary conditions used in [30] and the related works of [6] and these conditions also generalise the recent result of [32] (which only applied to bounded domains). We go on to establish the almost-convexity for maximal monotone operators on any real Banach space, establishing a long-held conjecture.

The main obstacle to the program revolves around the study of the variational convergence of convex functions on $X \times X^{*}$ with respect to an epi-convergence based on bounded $s \times \mathrm{w}^{*}$-convergent nets. This being a convergence that is not (in general) induced by a topology, we need to undertake a study of its interaction with conjugation (based on $X \times X^{*}$ endowed with the $s \times \mathrm{w}^{*}$ topology paired with $X^{*} \times X$, endowed with the $\mathrm{w}^{*} \times s$ topology). This requires the building of a theory of epi-convergence based on this new convergence notion. The primary target is the development of some result on the propagation of these variational convergences through conjugation. This allows us to demonstrate that the representative function we construct is indeed bigger-conjugate and also allows us to obtain an explicit formula for it.

The paper is organised as follows: basic definitions are given in Sect. 1, in Sect. 2 we begin our discussion of the "closure operation" induced by the convergence of bounded strong $\times$ weak $^{*}$ convergent nets on $X \times X^{*}$. In Sect. 3 we define the variational $s \times b d \mathrm{w}^{*}$ convergence concept and consider the problem of characterising the convergence for monotone families of convex functions. Section 4 is devoted to the study of the propagation of the conjugation operation through $s \times b d \mathrm{w}^{*}$-epiconvergence of variational convergent families. In Sect. 5 we summarise some tools we use from monotone operator theory. In Sect. 6 we carry the construction of the bigger-conjugate representative function discussed above. Finally in Sects. 7 and 8 we use these tools to study the almost convexity property for maximal monotone operators. In the final Sect. 8 we provide a proof of almost-convexity.

## 1 Preliminaries

We denote by $X$ a real Banach space and $X^{*}$ is its topological dual, paired via the duality product $\left\langle x, x^{*}\right\rangle: X \times X^{*} \rightarrow \mathbb{R}$. In this and the papers [11,13] all topological closures of set in $X \times X^{*}$ are with respect to the $s \times \mathrm{w}^{*}$ topology so as to respect the basic duality relationships for conjugation on $X \times X^{*}$ paired with $X^{*} \times X$, with the latter endowed with the $\mathrm{w}^{*} \times s$ topology. The interior of a set $C \subseteq X$ is denoted by int $C$ and its (strong) closure by $\bar{C}$. The convex hull of a set $T \subseteq X \times X^{*}$ (which is often identified with the graph of the associated operator $T: X \rightrightarrows X^{*}$ taking $x \in X$ to $T(x) \subseteq X^{*}$ ) will be denoted by co $T$ and the convex ( $s \times \mathrm{w}^{*}$ )-closure by $\overline{\mathrm{co}} T$. The complement of set $T$ (in the ambient space) will be denoted $T^{c}$. The indicator function $\delta_{C}(x)$ of a set $C \subseteq X$ takes the value 0 for $x \in C$ and $+\infty$ otherwise. Denote by $\dagger:\left(x^{*}, x\right) \leftrightarrow\left(x, x^{*}\right)$ the transpose operator. We denote both the $s \times \mathrm{w}^{*}$ (resp. $\mathrm{w}^{*} \times s$ )- closed ball of radius $K>0$ by

$$
\overline{B_{K}}(0):=\left\{\left(x, x^{*}\right)\left(\text { resp. }\left(x^{*}, x\right)\right) \mid \max \left\{\|x\|,\left\|x^{*}\right\|_{*}\right\} \leq K\right\} \subseteq X \times X^{*}\left(\text { resp } \subseteq X^{*} \times X\right),
$$

and by $B_{K}(0)$ the corresponding open ball (this is to avoid cumbersome notation like \left.${\overline{B_{K}}}^{\dagger}(0) \subseteq X^{*} \times X\right)$. By $P C\left(X \times X^{*}\right)$ we denote the proper convex functions $f: X \times X^{*} \rightarrow$ $\mathbb{R}_{+\infty}:=\mathbb{R} \cup\{+\infty\}$. Denote by $\Gamma\left(X \times X^{*}\right)$ the set of all ( $\mathrm{w}^{*} \times s$ )-lower-semicontinuous, proper convex functions. When going from sets in $X \times X^{*}$ to ones in $X \times X^{*} \times \mathbb{R}$ (i.e. epi $f$ ) we may use the norm $\left\|\left(x, x^{*}, \alpha\right)\right\|=\max \left\{\left\|\left(x, x^{*}\right)\right\|,|\alpha|\right\}$ i.e. when dealing with epi-graphs we will also use the box norm to extend to the extra single dimension. All closures can be interpreted according to its context, strong in $X$ and $w^{*}$ in $X^{*}$, which closure or ball will be clear from the context. Noting that convergent weak* sequences are necessarily bounded, a slight generalisation on this type of limit involves bounded $s \times \mathrm{w}^{*}$-convergent nets. This is a convergence notion and is not directly associated with a topological convergence (differing in nature from the bounded-weak* topology [17] denoted by bw*).

We can embed a convex set $C \subseteq X \times X^{*}$ into the space $X^{* *} \times X^{*}$ in the usual isometric sense that one usually considers $X \subseteq X^{* *}$. When we do this with epigraphs of functions $f: X \times X^{*} \rightarrow \mathbb{R}_{+\infty}$ : we denote the resulting function by $\widehat{f}: X^{* *} \times X^{*} \rightarrow \mathbb{R}_{+\infty}$. We also denote the conjugation with respect to pairing $\sigma_{w}\left(X \times X^{*}\right)$ with $\sigma_{w^{*}}\left(\left(X \times X^{*}\right)^{*}\right)$ by $f \mapsto$ $f^{\widehat{ }}$. Note that $\widehat{f^{\star}}$ acts on the space $\left(X \times X^{*}\right)^{*}=X^{*} \times X^{* *}$. Denote the transpose operator $\dagger:\left(x^{*}, x\right) \leftrightarrow\left(x, x^{*}\right)$ and the transpose conjugate of $f$ by

$$
\begin{equation*}
f^{* \dagger}\left(x, x^{*}\right):=f^{*}\left(x^{*}, x\right)=\sup _{\left(z, z^{*}\right) \in X \times X^{*}}\left\{\left\langle\left(x, x^{*}\right),\left(z, z^{*}\right)\right\rangle-f\left(z, z^{*}\right)\right\} \tag{1}
\end{equation*}
$$

The conjugate $f^{* \dagger}$ in the sense of (1) and the traditional conjugate $f^{\widehat{\star} \dagger}$ are compatible in the sense that

$$
\begin{equation*}
\left.\left(f^{\widehat{\star \dagger}}\right)\right|_{X \times X^{*}} \equiv f^{* \dagger} \quad \text { on } X \times X^{*} \quad \text { i.e. epi } f^{\widehat{\star} \dagger} \cap\left(X \times X^{*} \times \mathbb{R}\right)=\operatorname{epi} f^{* \dagger} . \tag{2}
\end{equation*}
$$

A representative function for a monotone operator $T$ is a convex function $f: X \times X^{*} \rightarrow$ $\mathbb{R}_{+\infty}$ with $f \geq\langle\cdot, \cdot\rangle$ and $T \subseteq\left\{\left(x, x^{*}\right) \mid f\left(x, x^{*}\right)=\left\langle x, x^{*}\right\rangle\right\}:=M_{f}$. The interest in representative functions stems from that fact that $M_{f}$ is always monotone. Martínez-Legaz and Svaiter [18] also introduced the monotone polar for a monotone set $T \subseteq X \times X^{*}$, by $T^{\mu}:=\left\{\left(x, x^{*}\right) \in X \times X^{*} \mid\left\langle x-y, x^{*}-y^{*}\right\rangle \geq 0, \forall\left(y, y^{*}\right) \in T\right\}$. In [18] it is noted that: $T \subseteq T^{\mu}$ means $T$ is monotone; with $T$ maximal if and only if $T^{\mu}=T$. Related notions are of pre-maximal monotonicity, $T^{\mu \mu}=T^{\mu}$ (i.e. $T^{\mu}$ is maximal and $T$ has a unique maximal extension) and of monotonic closure, $T^{\mu \mu}=T$.

The class of bigger-conjugate representative functions for $T$ is defined as

$$
b R(T):=\left\{f \in P C\left(X \times X^{*}\right) \mid f^{* \dagger} \geq f \geq\langle\cdot, \cdot\rangle, T \subseteq M_{f}\right\} .
$$

The interest in the $f \in b R(T)$ stems from the fact that $M_{f}=M_{f^{*+}}$, with their assured maximality as monotone sets when $X$ is reflexive [10]. In [13] it is shown that in a general real Banach space, representable monotone extensions of $T$, which are given by $M_{f}$ for $f \in b R(T)$, are maximal-like in that they are monotonically closed i.e. $M_{f}^{\mu \mu}=M_{f}$.

Denoting the restriction of a function $F: X^{*} \times X^{* *} \rightarrow \mathbb{R}_{\infty}$ by $\widehat{F}: X^{*} \times X \rightarrow \mathbb{R}_{\infty}$ given by $\widehat{F}\left(x^{*}, x\right)=F\left(x^{*}, x^{* *}\right)$ when $\hat{x}=x^{* *}$ then the Fitzpatrick function

$$
F_{T}\left(x, x^{*}\right)=\left[\left(\left\langle\cdot, \widehat{\cdot\rangle+\delta_{T}}\right)^{\star}\right]^{\dagger}\left(x, x^{*}\right)=\sup _{\left(z, z^{*}\right) \in T}\left\{\left\langle\left(x, x^{*}\right),\left(z, z^{*}\right)\right\rangle-\left\langle z, z^{*}\right\rangle\right\}\right.
$$

is a representative function for $T$, when $T$ is maximal monotone [14]. As the Fitzpatrick function is defined via a conjugate-transpose restricted to $X \times X^{*}$ it is (by definition) $s \times \mathrm{w}^{*}$ closed (and hence strongly closed as well). We note that almost all duality theorems for con-
jugation (i.e. Fenchel duality) are based on the duality pairing that gives rise to the conjugate $f \mapsto f^{\star}$ which unfortunately will not be available when using the transpose conjugate (1).

## 2 A Closure Operation for Convex Subsets in $X \times X^{*}$

When $X$ is not reflexive it is well known that $\left(x, x^{*}\right) \mapsto\left\langle x, x^{*}\right\rangle$ is not continuous under any topology $s \times \tau$ compatible with duality (i.e. $\left.\left(X^{*}, \tau\right)^{*}=X\right)$ unless $X$ is at least reflexive, [33]. Indeed to be continuous with respect to $\tau=\mathrm{bw}^{*}$ we need $X$ finite dimensional [33]. This is why we are interested in $\mathrm{w}^{*}$-convergence of bounded nets as it is well known that the duality product is continuous with respect to this convergence [13].

In the following we study the closure for convex sets $C \subseteq X \times X^{*}$ with respect to bounded $s \times \mathrm{w}^{*}$ convergent nets since in the subsequent analysis we need to construct a biggerconjugate representative function $h \in b R(\operatorname{rec} T)$. The main difficulty is actually showing our construction indeed has a bigger conjugate. This necessitates the introduction of the following closure operation in order to develop an appropriate duality theory. The product topology $s \times \mathrm{w}^{*}$ is problematic to study directly as one member is sequentially determined and the other is not. We will discuss in this section a novel way of characterising this product topology that allows a classical approach to its study.

Definition 2 For $C \subseteq X \times X^{*}$ denote

$$
\begin{equation*}
\bar{C}^{s \times b d \mathrm{w}^{*}}:=\bigcup_{K>0}{\overline{C \cap \bar{B}_{K}(0)}}^{s \times \mathrm{w}^{*}} \tag{3}
\end{equation*}
$$

Clearly $C \subseteq \bar{C}^{s \times b d \mathrm{w}^{*}} \subseteq \bar{C}^{s \times \mathrm{w}^{*}}$ and it is immediate that when $C$ is $s \times \mathrm{w}^{*}$-closed, have $C=\bar{C}^{s \times b d \mathrm{w}^{*}}$. Note that $\bar{C}^{s \times b d \mathrm{w}^{*}}$ consists of all $s \times \mathrm{w}^{*}$ accumulation points of bounded nets from $C$, motivating the notation.

This closure must be strictly stronger than the $s \times \mathrm{w}^{*}$ closure (indeed for the case $C=$ $\{0\} \times C^{*}$, this closure corresponds to that associated with the classical bounded-weak*topology on $X^{*}$ which is itself also strictly stronger than the weak* topology). We say $C$ is $s \times b d \mathrm{w}^{*}$-closed iff $C \supseteq \bar{C}^{s \times b d \mathrm{w}^{*}}$ (and so $C=\bar{C}^{s \times b d \mathrm{w}^{*}}$ ).

Lemma 3 Let $C \subseteq X \times X^{*}$.

1. Then $\overline{C \cap \overline{B_{K}}(0)}{ }^{s \times \mathrm{w}^{*}} \subseteq \bar{C}^{s \times b d \mathrm{w}^{*}} \cap \overline{B_{K}}(0)$ for all $K>0$.
2. If $C$ is convex so is $\bar{C}^{s \times b d \mathrm{w}^{*}}$.
3. If $\left\{C_{i}\right\}_{i \in I}$ are $s \times b d \mathrm{w}^{*}$-closed then so is $\cap_{i \in I} C_{i}$.

Proof The first inclusion follows from definitions. Since $\bar{C}^{s \times b d w^{*}}$ is a union of the nested convex sets $\left\{{\overline{C \cap}{\overline{B_{K}}}^{(0)}}^{s \times \mathrm{w}^{*}}\right\}_{K>0}$ we have $\bar{C}^{s \times b d \mathrm{w}^{*}}$ convex. For the last conclusion, since $C_{i} \supseteq \bar{C}_{i}^{s \times b d \mathrm{w}^{*}}$,

We will now assume $C$ is convex to obtain a much stronger result. We will need to appeal to the following variant of a separation theorem. Denote $Y:=X^{*} \times X$ and consider the canonical embedding of a subset $C \subseteq X \times X^{*}$ into $Y^{*}=\left(X^{*} \times X\right)^{*}=X^{* *} \times X^{*}$ by $J_{Y}(C) \cap\left(X^{* *} \times X^{*}\right)$ (later we will simply write either $\widehat{C}$ or $C \subseteq X^{* *} \times X^{*}$, accepting the abuse of notation). We impose the $\boldsymbol{w}^{*}$-topology on the dual space $Y^{*}:=\left(X^{*} \times X\right)^{*}=$ $X^{* *} \times X^{*}$.

Proposition 4 Suppose $C, D \subseteq X \times X^{*}$ are convex. Denote the $\boldsymbol{w}^{*}$-closure of $J_{X \times X^{*}}(C) \cap$ $\left(X^{* *} \times X^{*}\right)\left(\right.$ and $\left.J_{X \times X^{*}}(D) \cap\left(X^{* *} \times X^{*}\right)\right)$ as a subset of $Y^{*}$ by $\bar{C}^{w^{*}}\left(\right.$ and $\left.\bar{D}^{w^{*}}\right)$.

1. For any convex set $C$,

$$
\begin{equation*}
\left.\bar{C}^{w^{*}}\right|_{X \times X^{*}}=\bar{C}^{s \times \mathrm{w}^{*}} . \tag{4}
\end{equation*}
$$

2. Suppose that $\bar{C}^{w^{*}} \cap \bar{D}^{w^{*}}=\emptyset$ with $D$ bounded. Then there exists $\left(z^{*}, z\right) \in Y\left(=X^{*} \times X\right)$ such that

$$
\begin{equation*}
\delta_{C}^{*}\left(z^{*}, z\right) \leq \alpha<\delta_{D}^{*}\left(z^{*}, z\right) . \tag{5}
\end{equation*}
$$

Conversely if $C, D$ can be strictly separated by a $s \times \mathrm{w}^{*}$-continuous hyperplane, in the sense of (5), then $\bar{C}^{w^{*}} \cap \bar{D}^{w^{*}}=\emptyset$.

Proof Part 1: With $Y=X^{*} \times X$, then from the embedding of $Y$ into $Y^{*}$, we may view $C$ and $D$ as subsets of $Y^{*}=X^{* *} \times X^{*}$. Now $\bar{C}^{s \times \mathrm{w}^{*}}$ is the intersection of enclosing half-spaces in $X \times X^{*}$ formed from $\left(x^{*}, x\right) \in X^{*} \times X$, and $\bar{C}^{w^{*}}$ is the intersection of those in $X^{* *} \times X^{*}$ formed from $\left(x^{*}, x\right) \in X^{*} \times X=Y$. Hence

$$
\bar{C}^{s \times w^{*}}=\bar{C}^{w^{*}} \cap\left(X \times X^{*}\right)=\left.\bar{C}^{w^{*}}\right|_{X \times X^{*}},
$$

which is (4).
Part 2: Now suppose $\bar{C}^{w^{*}} \cap \bar{D}^{w^{*}}=\emptyset$. We then have $\bar{D}^{w^{*}}$ compact in $Y^{*}$ and $\bar{C}^{w^{*}}$ closed. We now invoke Theorem 1.1.5 of [34] to obtain $\left(z^{*}, z\right) \in X^{*} \times X$ satisfying

$$
\left\langle\left(z^{*}, z\right),\left(v^{* *}, v^{*}\right)\right\rangle \leq \alpha_{1}<\alpha_{2} \leq\left\langle\left(z^{*}, z\right),\left(u^{* *}, u^{*}\right)\right\rangle
$$

for all $\left(v^{* *}, v^{*}\right) \in \bar{C}^{w^{*}}$ and $\left(u^{* *}, u^{*}\right) \in \bar{D}^{w^{*}}$. As $C \subseteq \bar{C}^{s \times w^{*}} \subseteq \bar{C}^{w^{*}}$ (and similarly for $D$ ) we have (5). Thus when $\bar{C}^{w^{*}} \cap \bar{D}^{w^{*}}=\emptyset$ we can strictly separate then with a $s \times$ w$^{*}$-continuous hyperplane.

On the other hand whenever this possible, we have $\left(z^{*}, z\right) \in Y:=X^{*} \times X$ which is the pre-dual of $Y^{*}:=\left(X^{*} \times X\right)^{*}=X^{* *} \times X^{*}$ and $\alpha_{1}, \alpha_{2}$ such that for all $\left(v, v^{*}\right) \in C$ and $\left(u, u^{*}\right) \in D$, we have

$$
\left\langle\left(z^{*}, z\right),\left(v, v^{*}\right)\right\rangle \leq \alpha_{1}<\alpha_{2} \leq\left\langle\left(z^{*}, z\right),\left(u, u^{*}\right)\right\rangle .
$$

Hence for $H_{\alpha}:=\left\{\left(y^{* *}, y^{*}\right) \mid\left\langle\left(z^{*}, z\right),\left(y^{* *}, y^{*}\right)\right\rangle \leq \alpha\right\}$ we have

$$
\bar{C}^{w^{*}} \subseteq H_{\alpha_{1}} \text { and } \bar{D}^{w^{*}} \subseteq H_{\alpha_{2}}^{c}
$$

and so $\bar{C}^{w^{*}} \cap \bar{D}^{w^{*}} \subseteq H_{\alpha_{1}} \cap H_{\alpha_{2}}^{c}=\emptyset$.

Corollary 5 Suppose $C \subseteq X \times X^{*}$ is convex. Denote $Y=X^{*} \times X$ and the $\boldsymbol{w}^{*}$-closure of $J_{X}(C) \cap\left(X^{* *} \times X^{*}\right)$ as a subset of $Y^{*}:=\left(X^{*} \times X\right)^{*}$ by $\bar{C}^{w^{*}}$. Then $C$ is $s \times \mathrm{w}^{*}$-closed iff there exists a $\mathrm{w}^{*}$-closed convex set $D \subseteq Y^{*}$ such that $C=\left.D\right|_{X \times X^{*}}$. In particular $C$ is $s \times \mathrm{w}^{*}$-closed iff $C=\bar{C}^{w^{*}} \cap\left(X \times X^{*}\right)$.

Proof We have $C$ is $s \times \mathrm{w}^{*}$-closed iff $C=\bar{C}^{s \times \mathrm{w}^{*}}=\bar{C}^{w^{*}} \cap\left(X \times X^{*}\right)$ and so for $D:=\bar{C}^{w^{*}}$ we have $C=\left.D\right|_{X \times X^{*}}$. On the other hand when $C=\left.D\right|_{X \times X^{*}}$ for $D=\bar{D}^{w^{*}}$ then $C=\bar{D}^{w^{*}} \cap$ $\left(X \times X^{*}\right)$ with this set $s \times \mathrm{w}^{*}$-closed.

Remark 6 Note, that by Corollary 5, a convex set $A$ is $s \times \mathrm{w}^{*}$-closed iff there exists a $\boldsymbol{w}^{*}$ closed set $B$ in $Y^{*}=X^{* *} \times X^{*}$ with $A=B \cap\left(X \times X^{*}\right)$ from which it then follows that $A=\overline{B \cap\left(X \times X^{*}\right)}{ }^{w^{*}} \cap\left(X \times X^{*}\right)$.

## 3 Variational Limits

The recession operator rec $T$ is defined as a Kuratowski-Painlevé limit of a family of sets $\{t T\}_{t>0}$ with respect to the $s \times b d \mathrm{w}^{*}$-convergence. This leads us to study such limits in more detail. A problem that arises often in defining variational limits, in the context of convergences that are not defined by a topology, is that of devising a consistent set of fundamental definitions (this also arises with sequential convergences based on weak topologies). We must do this in our context, and so pursue the framework of a Kuratowski/Painlevétype convergence (see [1, Sect. 5.2]), based on bounded convergent $s \times \mathrm{w}^{*}$ nets. Denote by $\left(s \times \mathrm{w}^{*}\right)-\lim \sup _{t \rightarrow+\infty} A_{t}$ and $\left(s \times \mathrm{w}^{*}\right)-\liminf _{t \rightarrow+\infty} A_{t}$ the usual Kuratowski-Painlevé convergence with respect to the product topology of strong with weak ${ }^{*}$. Denote

$$
\mathcal{N}\left(x, x^{*}\right):=\left\{U \times W \mid x \in U \in \mathcal{N}(x) \text { and } x^{*} \in W \in \mathcal{N}\left(x^{*}\right)\right\}
$$

where $\mathcal{N}(x)$ is the strong neighbourhood basis at $x$ (in $X$ ) and $\mathcal{N}\left(x^{*}\right)$ is a weak* neighbourhood basis at $x^{*}$ (in $X^{*}$ ).

Definition 7 Let $\left\{A_{t}\right\}_{t>0}$ be a family of subsets of $X \times X^{*}$. Then, we define

$$
\begin{align*}
b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} A_{t}:=\left\{\left(x, x^{*}\right) \mid \exists \text { net }\left(x_{\alpha}, x_{\alpha}^{*}, t_{\alpha}\right) \rightarrow^{s \times \mathrm{w}^{*}}\left(x, x^{*},+\infty\right)\right. \\
\text { with } \left.\left(x_{\alpha}, x_{\alpha}^{*}\right) \in A_{t_{\alpha}} \text { and }\left\{x_{\alpha}^{*}\right\} \text { bounded }\right\}, \tag{6}
\end{align*}
$$

$$
\begin{align*}
b d s \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty} A_{t}:= & \left\{\left(x, x^{*}\right) \mid \exists K>0:\right. \\
& \forall V \in \mathcal{N}\left(x, x^{*}\right)\left(\exists t_{V}>0\right)\left(\forall t>t_{V}\right)\left(A_{t} \cap \overline{B_{K}}(0) \cap V \neq \emptyset\right\} . \tag{7}
\end{align*}
$$

We always have $b d s w^{*}-\lim \sup _{t \rightarrow+\infty} A_{t} \supseteq b d s w^{*}-\liminf _{t \rightarrow+\infty} A_{t}$. The set $b d s \mathrm{w}^{*}-\lim \sup _{t \rightarrow+\infty} A_{t}$ may not be $s \times \mathrm{w}^{*}$-closed.

Remark 8 These notions have an obvious extension to subsets of $X \times X^{*} \times \mathbb{R}$ (to include epigraphs of functions on $X \times X^{*}$ ) where $s \times w^{*}$ then stands for $s \times w^{*} \times \tau_{\mathbb{R}}$, the product with the standard topology on the reals.

Remark 9 We note that when $\left(x, x^{*}\right) \in b d s \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty} A_{t}$ and $U \times W \in \mathcal{N}\left(x, x^{*}\right)$ then there exists $K>0$ such that $\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset\right\}$ is residual. Moreover $\left(x, x^{*}\right) \in b d s \mathrm{w}^{*}-\lim \sup _{t \rightarrow+\infty} A_{t}$ then we have $\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset\right\}$ contains a cofinal subset.

Remark 10 Note that if we define $\frac{1}{t} T:=\left\{\left.\left(x, \frac{1}{t} x^{*}\right) \right\rvert\,\left(x, x^{*}\right) \in T\right\}$ then

$$
\begin{equation*}
b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} \frac{1}{t} T=\operatorname{rec} T \tag{8}
\end{equation*}
$$

once again identifying rec $T$ with its graph and rec $T(x)$ with the image of the associated multi-function.

Definition 11 We say that a family $\left\{f_{t}\right\}_{t>0}$ of proper functions $s \times \mathrm{w}^{*}$ - boundedly converges at $\left(x, x^{*}\right)$ as $t \rightarrow \infty$ iff both of the following coincide:

$$
\begin{aligned}
& \left(b d-e-\liminf _{t \rightarrow+\infty} f_{t}\right)\left(x, x^{*}\right):=\inf \left\{\alpha \mid\left(x, x^{*}, \alpha\right) \in b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty}\left(\text { epi } f_{t}\right)\right\} \quad \text { and } \\
& \left(b d-e-\limsup _{t \rightarrow+\infty} f_{t}\right)\left(x, x^{*}\right):=\inf \left\{\alpha \mid\left(x, x^{*}, \alpha\right) \in b d s w^{*}-\liminf _{t \rightarrow+\infty}\left(\operatorname{epi} f_{t}\right)\right\} .
\end{aligned}
$$

When these coincide for all $\left(x, x^{*}\right)$ then we denote the associated function $f$ by:

$$
f=b d-e-\lim _{t \rightarrow+\infty} f_{t} .
$$

We also note in passing that order is preserved when applying these limits, in that $f_{t} \leq g_{t}$ implies $b d-e-\liminf _{t \rightarrow+\infty} f_{t} \leq b d-e-\liminf _{t \rightarrow+\infty} g_{t}$ etc.

This definition is used to take into account the possibility that the infimal value may not actually be in the limit of epigraphs. When $\left\{f_{t}\right\}_{t>0}$ is monotonically non-decreasing this problem does not occur and we can then simply identify epi $\left(b d-e-\lim _{t \rightarrow+\infty} f_{t}\right)=$ $b d s \mathrm{w}^{*}-\lim _{t \rightarrow+\infty}\left(\right.$ epi $\left.f_{t}\right)$ (see later for details).

Clearly

$$
b d-e-\liminf _{t \rightarrow+\infty} f_{t} \leq b d-e-\limsup _{t \rightarrow+\infty} f_{t} .
$$

We may move from a limit with $t \rightarrow+\infty$ to one with $\tau \rightarrow 0^{+}$(as is later done in Sect. 6) via the simple transformation $\tau=\frac{1}{t+\alpha}$ for any $\alpha>0$ and so for now we focus on limits with $t \rightarrow+\infty$. Characterisation similar to those of other epi-limits of functions can be made, but in this case the attainment of the infimum in the following is not assured. We say an epi-limit is attained if there exists a net attaining the infimum in (9). In the case when $X$ is reflexive then we find that weak* and weak topologies coincide and so for convex functions this convergence is characterised sequentially (using Mazur characterisation of weak vs strong closures of convex sets). Hence a diagonalisation argument may be used to assert attainment as weakly convergent sequences are bounded.

Lemma 12 Let $\left\{f_{t}\right\}_{t>0}$ be a family of proper functions. Then

$$
\begin{equation*}
\left(b d-e-\liminf _{t \rightarrow+\infty} f_{t}\right)\left(x, x^{*}\right)=\inf _{\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow s \times b d w^{*}\left(x, x^{*}\right)\right\}} \operatorname{iminf}_{\alpha} f_{\left.t_{\alpha} \rightarrow+\infty\right\}}\left(x_{\alpha}, x_{\alpha}^{*}\right) . \tag{9}
\end{equation*}
$$

Proof We have

$$
\begin{aligned}
\gamma & :=b d-e-\liminf _{t \rightarrow+\infty} f_{t}\left(x, x^{*}\right) \\
& =\inf \left\{\beta \mid \exists \text { net }\left(x_{\alpha}, x_{\alpha}^{*}, t_{\alpha}\right) \rightarrow^{s \times \mathrm{w}^{*}}\left(x, x^{*},+\infty\right) \text { with } \beta=\liminf _{\alpha} f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}
\end{aligned}
$$

and so for any $\varepsilon>0$ we have the existence of $\left(x_{\alpha}, x_{\alpha}^{*}, t_{\alpha}\right) \rightarrow^{s \times \mathrm{w}^{*}}\left(x, x^{*},+\infty\right)$ such that $\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}, \gamma_{\alpha}\right) \in \operatorname{epi} f_{t_{\alpha}}$ with $\gamma_{\alpha} \rightarrow \hat{\gamma} \leq \gamma+\frac{\varepsilon}{2}$. That is, for any $\varepsilon>0$

$$
\begin{aligned}
& \left(b d-e-\liminf _{t \rightarrow+\infty} f_{t}\right)\left(x, x^{*}\right)+\varepsilon \geq \inf _{\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow s \times b d w^{*}\left(x, x^{*}\right)} \inf _{\left\{t_{\alpha} \rightarrow+\infty\right\}} \liminf _{\alpha} f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right) \\
& \text { or } \quad\left(b d-e-\liminf _{t \rightarrow+\infty} f_{t}\right)\left(x, x^{*}\right) \geq \inf _{\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow s \times b d w^{*}\left(x, x^{*}\right)} \inf _{\left\{t_{\alpha} \rightarrow+\infty\right\}} \liminf _{\alpha} f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right) .
\end{aligned}
$$

Take an arbitrary $t_{\alpha} \rightarrow+\infty$ and suppose $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right)$. Place $\gamma^{\prime}:=$ $\liminf _{\alpha} f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)$. Then for $\gamma_{t_{\alpha}}^{\prime}:=f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)$ we have $\left(x_{\alpha}, x_{\alpha}^{*}, \gamma_{t_{\alpha}}^{\prime}\right) \in$ epi $f_{t_{\alpha}}$ and $\liminf _{\alpha} \gamma_{t_{\alpha}}^{\prime}=\gamma^{\prime}$. Then $\left(x, x^{*}, \gamma^{\prime}\right) \in b d \mathrm{w}^{*}-\lim \sup _{t \rightarrow+\infty}$ epi $f_{t}$. Hence

$$
\liminf _{\alpha} f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)=\gamma^{\prime} \geq b d-e-\liminf _{t \rightarrow+\infty} f_{t}\left(x, x^{*}\right) .
$$

As this holds for all $t_{\alpha} \rightarrow+\infty$ and $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times b d w^{*}}\left(x, x^{*}\right)$ we have

$$
\inf _{\left\{\left(x_{t}, x_{t}^{*}\right) \rightarrow s \times b d w^{*}\left(x, x^{*}\right)\right\}} \inf _{\left\{t_{\alpha} \rightarrow+\infty\right\}} \liminf _{\alpha} f_{t_{\alpha}}\left(x_{t_{\alpha}}, x_{t_{a}}^{*}\right) \geq\left(b d-e-\liminf _{t \rightarrow+\infty} f_{t}\right)\left(x, x^{*}\right) .
$$

Note that for any $K>0$ we have $\left(s \times \mathrm{w}^{*}\right)-\lim \sup _{t \rightarrow+\infty}\left(A_{t} \cap \overline{B_{K}}(0)\right)=b d s \mathrm{w}^{*}-$ $\lim \sup _{t \rightarrow+\infty}\left(A_{t} \cap \overline{B_{K}}(0)\right)$ (and similarly for the limit infimum). We may now give a characterisation of these limits, paralleling that given for the Kuratowski-Painlevé limit of variational analysis. Note that the final union means that the limiting sets are not necessarily $s \times \mathrm{w}^{*}$-closed.

Proposition 13 Consider a family $\left\{A_{t}\right\}_{t>0}$ of subsets of $X \times X^{*}$. Then

$$
\begin{align*}
& b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} A_{t}=\bigcup_{K>0} \bigcap_{\eta>0}\left[\left(\bigcup_{t \geq \eta} A_{t}\right) \cap \overline{B_{K}}(0)\right]^{s \times \mathrm{w}^{*}} \text { and }  \tag{10}\\
& b d s \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty} A_{t}=\bigcup_{\substack { K>0  \tag{11}\\
\begin{subarray}{c}{I \subseteq \mathbb{R}_{+} \\
\text {cofinal }{ K > 0 \\
\begin{subarray} { c } { I \subseteq \mathbb { R } _ { + } \\
\text { cofinal } } }\end{subarray}}^{\left[\left(\bigcup_{t \in I} A_{t}\right) \cap \overline{B_{K}}(0)\right]^{s \times \mathrm{w}^{*}}} .
\end{align*}
$$

Moreover we have bdsw*-limsup $\sin _{t \rightarrow+\infty} A_{t}=A$ if $\left(s \times \mathrm{w}^{*}\right)-\lim \sup _{t \rightarrow+\infty}\left(A_{t} \cap \bar{B}_{K}(0)\right)=$ $A \cap \bar{B}_{K}(0)$ for all sufficiently large $K>0$, and bds $\mathrm{w}^{*}-\liminf _{t \rightarrow+\infty} A_{t}=A$ if $\left(s \times \mathrm{w}^{*}\right)$ $\liminf _{t \rightarrow+\infty} A_{t} \cap \bar{B}_{K}(0)=A \cap \bar{B}_{K}(0)$ for all sufficiently large $K>0$. When all $\left\{A_{t}\right\}_{t>0}$ are convex then so is $b d s w^{*}-\liminf _{t \rightarrow+\infty} A_{t}$.

Proof See the Appendix for proof.

We can interpret such limits for subsets $\left\{A_{t}\right\}_{t>0}$ of $Y=X^{*} \times X$ by embedding $\hat{A}_{t}:=$ $J_{X \times X^{*}}\left(A_{t}\right) \cap\left(X^{* *} \times X^{*}\right) \subseteq Y^{*}$ where $Y=X^{*} \times X$.

Definition 14 Suppose $\left\{A_{t}\right\}_{t>0}$ are a family of subsets of $Y:=X^{*} \times X$. Embedding $\hat{A}_{t}:=$ $J_{X \times X^{*}}\left(A_{t}\right) \cap\left(X^{* *} \times X^{*}\right) \subseteq Y^{*}$ denote:

$$
\begin{aligned}
b d\left(w \times \mathrm{w}^{*}\right)-\limsup _{t \rightarrow+\infty} A_{t} & :=\left[\bigcup_{K>0} \bigcap_{\eta>0}\left[\bigcup_{t \geq \eta} \hat{A}_{t} \cap \overline{B_{K}}(0)\right]^{w^{*}}\right] \cap\left(X \times X^{*}\right) \\
& =\left[\bigcup_{K>0} w^{*}-\limsup _{t \rightarrow+\infty}\left[\hat{A}_{t} \cap \overline{B_{K}}(0)\right]\right] \cap\left(X \times X^{*}\right) \text { and } \\
b d\left(w \times \mathrm{w}^{*}\right)-\liminf _{t \rightarrow+\infty} A_{t} & :=\left[\bigcup_{K>0} \bigcap_{\substack{I \subseteq \mathbb{R}_{+} \\
\text {cofinal }}}\left[\bigcup_{t \in I} \hat{A}_{t} \cap \overline{B_{K}}(0)\right]^{w^{*}}\right] \cap\left(X \times X^{*}\right) \\
& =\left[\bigcup_{K>0} w^{*}-\liminf _{t \rightarrow+\infty}\left[\hat{A}_{t} \cap \overline{B_{K}}(0)\right]\right] \cap\left(X \times X^{*}\right) .
\end{aligned}
$$

Remark 15 Note that the limit-infimum and-supremum on the right-hand-side are in the Kuratowski-Painlevé sense.

Definition 16 For $g: X \times X^{*} \rightarrow \mathbb{R}_{+\infty}$ form $\widehat{g}: X^{* *} \times X^{*} \rightarrow \mathbb{R}_{+\infty}$ by $\widehat{g}\left(x, x^{*}\right)=g\left(x, x^{*}\right)$ if $\left(x, x^{*}\right) \in X \times X^{*}$ and $+\infty$ otherwise. Denote, for $\left(x^{* *}, x^{*}\right) \in X^{* *} \times X^{*}$,

$$
\begin{aligned}
& \left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{g_{t}}\right)\left(x^{* *}, x^{*}\right) \\
& \quad:=\inf \left\{\gamma \mid\left(x^{* *}, x^{*}, \gamma\right) \in\left[\bigcup_{K>0} \boldsymbol{w}^{*}-\limsup _{t \rightarrow+\infty}\left[\operatorname{epi} \widehat{g_{t}} \cap \overline{B_{K}}(0)\right]\right]\right\} \text { and } \\
& b d\left(w \times \mathrm{w}^{*}\right)-e-\liminf _{t \rightarrow+\infty} f_{t}^{* *}:=\left.\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f_{t}^{*+}}\right)\right|_{X \times X^{*}} .
\end{aligned}
$$

Note that $\bigcup_{K>0} \boldsymbol{w}^{*}$-limsup $\sup _{t \rightarrow+\infty}\left[\operatorname{epi} \widehat{g_{t}} \cap \overline{B_{K}}(0)\right]$ consists of the limits of all selections of bounded $\boldsymbol{w}^{*}$-convergent subnets. The set epi $\left(b d \boldsymbol{w}^{*}-e\right.$ - $\left.\liminf _{t \rightarrow+\infty} \widehat{g_{t}}\right)$ has the vertical recession direction we associate with an epi-graph but the set may not be closed. Note that $\widehat{f}_{t}^{\star}$ acts on the space $\left(X \times X^{*}\right)^{*}=X^{*} \times X^{* *}$. Note the conjugate in our prior sense $f^{* \dagger}$ and that $f_{t}^{\widehat{\star}}$ which passes from $Y:=X \times X^{*}$ to $Y^{*}=X^{* *} \times X^{*}$ are compatible in the sense that

$$
\begin{equation*}
\left.\left(f_{t}{ }^{\widehat{\dagger} \dagger}\right)\right|_{X \times X^{*}} \equiv f_{t}^{* \dagger} \quad \text { on } X \times X^{*} \quad \text { i.e. epi } f_{t}{ }^{\widehat{\star} \dagger} \cap\left(X \times X^{*} \times \mathbb{R}\right)=\text { epi } f_{t}{ }^{* \dagger} . \tag{12}
\end{equation*}
$$

In general we only have

$$
\left.\left(b d \boldsymbol{w}^{*}-e-\operatorname{limsupepi}_{t \rightarrow+\infty} f_{t}^{\widehat{\star} \dagger}\right)\right|_{X \times X^{*}}=b d\left(w \times \mathrm{w}^{*}\right)-\operatorname{limsupepi}_{t \rightarrow+\infty} f_{t}^{* \dagger}
$$

due the potential failure of the limit set to be convex. Note, that by Corollary 5 , for convex sets $A_{t}$ are $s \times \mathrm{w}^{*}$-closed iff there exists a $\boldsymbol{w}^{*}$-closed set $B_{t}$ in $X^{* *} \times X^{*}$ with $A_{t}=B_{t} \cap$ $\left(X \times X^{*}\right)$ from which it then follows that $A_{t}={\overline{B_{t} \cap\left(X \times X^{*}\right)}}^{w^{*}} \cap\left(X \times X^{*}\right)$.

The following may be proved along similar lines to that in Lemma 12 and so the proof is omitted.

Lemma 17 Let $\left\{f_{t}\right\}_{t>0}$ be a family of proper functions on $X \times X^{*}$. Then

$$
\begin{equation*}
\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f_{t}}\right)\left(x^{* *}, x^{*}\right)=\inf _{\substack{\left.\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow b d w^{*}\left(x^{* *}, x^{*}\right)\right\} \\\left\{t_{\alpha} \rightarrow+\infty\right\}}} \operatorname{iiminf}_{\alpha} \widehat{f_{t_{\alpha}}}\left(x_{\alpha}, x_{\alpha}^{*}\right) . \tag{13}
\end{equation*}
$$

Let us now consider monotonic limit of families of functions.
Proposition 18 Let $\left\{f_{t}\right\}_{t>0}$ be a family of $[-\infty,+\infty]$-valued functions.

1. When $\left\{f_{t}\right\}_{t>0}$ are convex then so is $f:=b d-e-\limsup _{t \rightarrow+\infty} f_{t}$.
2. Assume for each $\left(x, x^{*}\right)$ that $t \mapsto f_{t}\left(x, x^{*}\right)$ is monotonically non-decreasing (as $\left.t \rightarrow \infty\right)$. We have

$$
\begin{align*}
& \bigcap_{t>0}{\overline{\operatorname{epi} f_{t}}}^{s \times b d \mathrm{w}^{*}}=\bigcap_{t>0} \bigcup_{K>0} \overline{\operatorname{epi} f_{t} \cap \overline{B_{K}}(0)}  \tag{14}\\
& \\
& s \times \mathrm{w}^{*} \\
& b d s \mathrm{w}^{*}-\operatorname{limsupepi}_{t \rightarrow+\infty} f_{t} \\
&=\bigcup_{K>0} \bigcap_{t>0}{\overline{\operatorname{epi} f_{t} \cap \overline{B_{K}}(0)}}^{s \times w^{*}}=b d s \mathrm{w}^{*}-\operatorname{liminfepi}_{t \rightarrow+\infty} \supseteq{\overline{\bigcap \operatorname{hepi} f_{t}}}^{s \times b d \mathrm{w}^{*}}
\end{align*}
$$

and so bd-e- $\lim _{t \rightarrow+\infty}$ epi $f_{t}$ exists.
3. When all $f_{t}$ are $s \times b d \mathrm{w}^{*}$ (resp. $s \times \mathrm{w}^{*}$ )-closed then $f:=b d-e-\lim _{t \rightarrow \infty} f_{t}$ is also $s \times b d \mathrm{w}^{*}\left(\right.$ resp. $\left.s \times \mathrm{w}^{*}\right)$-closed and coincides with the pointwise limit (i.e. epi $f=$ $\cap_{t>0}$ epi $\left.f_{t}\right)$, and $\left(s \times \mathrm{w}^{*}\right)-\lim _{t \rightarrow+\infty}\left[\operatorname{epi} f_{t} \cap \overline{B_{K}}(0)\right]=\operatorname{epi} f \cap \overline{B_{K}}(0)$ for each $K>0$.

Proof 1) Convexity of $f:=b d-e$-lim $\sup _{t \rightarrow+\infty} f_{t}$ follows immediately from the last assertion of Proposition 13.
2) Assume $t \mapsto f_{t}\left(x, x^{*}\right)$ is monotonically non-decreasing. As epi $f_{t} \subseteq \operatorname{epi} f_{\tau}$ for all $t \geq \tau$, it follows that $\bigcup_{t \geq \tau}$ epi $f_{t}=\operatorname{epi} f_{\tau}$ so

$$
\begin{equation*}
b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} \operatorname{epi} f_{t}=\bigcup_{K>0} \bigcap_{t>0} \overline{\operatorname{epi} f_{t} \cap \overline{B_{K}}(0)}{ }^{s \times \mathrm{w}^{*}} \subseteq{\overline{\operatorname{epi~} f_{\tau}}}^{s \times b d \mathrm{w}^{*}} \quad \text { for all } \tau>0 \tag{15}
\end{equation*}
$$


To show equality of the limits we need to demonstrate that $b d s \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty}$ epi $f_{t} \supseteq$ $b d s \mathrm{w}^{*}-\lim \sup _{t \rightarrow+\infty}$ epi $f_{t}$. Let $\left(x, x^{*}\right) \in b d \mathrm{w}^{*}-\lim \sup _{t \rightarrow+\infty}$ epi $f_{t}$. Then there exists $K>$ 0 and $\left\{t_{\alpha}\right\}$ with $t_{\alpha} \rightarrow+\infty$ and $\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}$ with $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow\left(x, x^{*}\right)$ for which $\left(x_{\alpha}, x_{\alpha}^{*}\right) \in$ epi $f_{t_{\alpha}} \cap \bar{B}_{K}(0)$ for all $\alpha$. Let $W$ be a neighbourhood of $\left(x, x^{*}\right)$. Then, there exists $\bar{\alpha}$ such that for all $\alpha \geq \bar{\alpha}$ we have $\left(x_{\alpha}, x_{\alpha}^{*}\right) \in \operatorname{epi} f_{t_{\alpha}} \cap \bar{B}_{K}(0) \cap W$. Let $\tau \geq t_{\bar{\alpha}}$. Then (since $t_{\alpha} \rightarrow+\infty$ ) there exists $t_{\alpha^{\prime}} \geq \tau$ with $\alpha^{\prime} \geq \bar{\alpha}$, so epi $f_{\tau} \cap \bar{B}_{K}(0) \cap W \supseteq$ epi $f_{t_{\alpha^{\prime}}} \cap \bar{B}_{K}(0) \cap W$. Therefore $\left(x, x^{*}\right) \in b d \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty}$ epi $f_{t}$.
3) When each epi $f_{\eta}$ is $s \times b d \mathrm{w}^{*}$ (or $s \times \mathrm{w}^{*}$ )-closed, by Lemma 3 part 3 we have $\bigcap_{t>0}$ epi $f_{t}$ is $s \times b d \mathrm{w}^{*}$ (or $s \times \mathrm{w}^{*}$ )-closed, and so, by (14) the limit exists, as epi $f=$
$\bigcap_{\eta>0}$ epi $f_{\eta}$, which is $s \times b d \mathrm{w}^{*}$ (or $s \times \mathrm{w}^{*}$ )-closed. Moreover, as
we have $\left(s \times \mathrm{w}^{*}\right)$-limsup $\sin _{t \rightarrow+\infty}$ epi $f_{t} \cap \overline{B_{K}}(0)=$ epi $f \cap \overline{B_{K}}(0)$. A similar calculation can be used to show $\left(s \times \mathrm{w}^{*}\right)-\liminf _{t \rightarrow \infty}$ epi $f_{t} \cap \overline{B_{K}}(0)=\operatorname{epi} f \cap \overline{B_{K}}(0)$.

We briefly discuss monotonically non-increasing families.
Proposition 19 Assume for each $\left(x, x^{*}\right) \in X \times X^{*}$ we have $t \mapsto g_{t}\left(x, x^{*}\right)$ is monotonically non-increasing (as $t \rightarrow \infty$ ), where the family $\left\{g_{t}\right\}_{t \in \mathbb{R}_{+}}$are $[-\infty,+\infty]$-valued functions.

1. Then

$$
b d s \mathrm{w}^{*}-\liminf _{t \rightarrow \infty} \text { epi } g_{t}=b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} \text { epi } g_{t}={\overline{\left(\bigcup_{t \geq 0} \operatorname{epi} g_{t}\right)^{2}}}^{s \times b d \mathrm{w}^{*}},
$$

and so $g:=b d-e-\lim _{t \rightarrow \infty} g_{t}$ exists.
2. If each $g_{t}$ is convex, then $g:=b d-e-\lim _{t \rightarrow+\infty} g_{t}$ is convex.

Proof As $\left\{g_{t}\right\}_{t>0}$ is monotonically non-increasing, which implies epi $g_{t} \subseteq$ epi $g_{\tau}$ for $t \leq \tau$, we have $\left(\bigcup_{t \in I}\right.$ epi $\left.g_{t}\right)=\left(\bigcup_{t \geq 0}\right.$ epi $\left.g_{t}\right)$ for any residual or cofinal set $I \subseteq \mathbb{R}_{+}$. Let $K>0$. Then for epi $g:=b d \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty}$ epi $g_{t}$, by Proposition 13, for all $\tau>0$,

$$
\begin{aligned}
\text { epi } g & =\bigcup_{H>0} \bigcap_{\eta>0}{\overline{\left(\bigcup_{t \geq \eta} \operatorname{epi} g_{t}\right) \cap \overline{B_{H}}(0)}}^{s \times w^{*}}=\bigcup_{H>0}{\overline{\left(\bigcup_{t \geq 0} \operatorname{epi} g_{t}\right) \cap \overline{B_{H}}(0)}}^{s \times w^{*}} \\
& =\bigcup_{H>0}{\overline{\left(\bigcup_{t \geq \tau}\right.}}^{\left.\operatorname{epi} g_{t}\right) \cap \overline{B_{H}}(0)} \\
& =b d s \mathrm{w}^{*}-\lim _{t \rightarrow \infty} \operatorname{epi} g_{t}={\overline{\left(\bigcup_{t \geq 0} \operatorname{epi} g_{t}\right)}}^{s \times b d \mathrm{w}^{*}}
\end{aligned}
$$

When $\left\{g_{t}\right\}_{t>0}$ is a family of convex functions then $\bigcup_{t \geq 0}$ epi $g_{t}$ is convex (being a monotonically non-decreasing nested set of convex sets) and hence ${\overline{\left(\bigcup_{t \geq 0} \text { epi } g_{t}\right)}}^{s \times b d w^{*}}$ is convex by Lemma 3.

Utilising these observations about monotonic limits we find that the limit-infimum lends itself to another useful interpretation.

Lemma 20 Suppose $\left\{f_{t}\right\}_{t>0}$ is a family of extended-real-valued functions. Then

$$
b d-e-\liminf _{t \rightarrow \infty} f_{t}=b d-e-\lim _{\eta \rightarrow \infty}\left(\inf _{t \geq \eta} f_{t}\right) \geq b d-e-\lim _{\eta \rightarrow \infty}\left(\overline{\inf }_{t \geq \eta}^{s \times b d w^{*}}\right) .
$$

Proof Note that $g_{\eta}:=\overline{\inf _{t \geq \eta} f_{t}} s \times b d w^{*}$ is a monotonically non-decreasing family of $s \times$ $b d \mathrm{w}^{*}$-closed functions and so by Proposition 18 we have existence of the epi-limits $b d-e-\lim _{t \rightarrow \infty} g_{t}=\sup _{\eta} g_{\eta}$ and $b d-e-\lim _{\eta \rightarrow \infty}\left(\inf _{t \geq \eta} f_{t}\right)$. By Proposition 13 with $A_{t}:=$ epi $f_{t}$

$$
\begin{aligned}
& b d s \text { w. }^{*}-\operatorname{limsupepi}_{t \rightarrow+\infty} f_{t} \\
& \quad=\bigcup_{K>0} \bigcap_{\eta>0} \overline{\left[\left(\bigcup_{t \geq \eta} \operatorname{epi} f_{t}\right) \cap \overline{B_{K}}(0)\right]^{s \times \mathrm{w}^{*}}=\bigcup_{K>0} \bigcap_{\eta>0} \overline{\left(\operatorname{epi}\left(\inf _{t \geq \eta} f_{t}\right)\right) \cap \overline{B_{K}}(0)} s \times \mathrm{w}^{*}} \\
& \quad=b d s \mathrm{w}^{*}-\lim _{\eta \rightarrow+\infty} \operatorname{epi}\left(\inf _{t \geq \eta} f_{t}\right) \subseteq \bigcup_{K>0}\left(\operatorname{epi} g_{\eta}\right) \cap \overline{B_{K}}(0)=\bigcap_{\eta>0} \operatorname{epi} g_{\eta} \\
& \quad=\operatorname{epi}\left(\sup _{\eta} g_{\eta}\right)=\operatorname{epi}\left(b d-e-\lim _{\eta \rightarrow \infty} g_{t}\right)=\operatorname{epi}\left[b d-e-\lim _{\eta \rightarrow \infty}\left(\overline{\overline{\inf }_{t \geq \eta} f_{t}} s \times \mathrm{w}^{*}\right)\right]
\end{aligned}
$$

## 4 The $s \times b d w^{*}$-Convergence and Conjugation

We will need to understand how the limits used in construction of a representative function for the monotone operator rec $T$, interact with conjugacy, in order to show that such a function is bigger-conjugate. In this section we explore the interaction of conjugation with this new convergence notion.

The following has been observed for almost all viable epi-limits (see [21, Lemma 1]). We provide a proof for our context.

Proposition 21 Let $\left\{f_{t}\right\}_{t>0}$ be a family of functions. Then

$$
\begin{equation*}
\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}^{*}\right)^{*} \leq b d-e-\limsup _{t \rightarrow \infty} f_{t} \tag{16}
\end{equation*}
$$

If $\left\{f_{t}\right\}_{t>0}$ are proper convex, so is bd-e $-\lim \sup _{t \rightarrow \infty} f_{t}$.
Proof By Proposition 18, the convexity of $f_{t}$ yields same for $f:=b d-e-\limsup _{t \rightarrow \infty} f_{t}$. Let $H:=b d-e-\liminf _{t \rightarrow+\infty} f_{t}^{*}$ and we will show $f \geq H^{*}$. If $f\left(x, x^{*}\right)=+\infty$ or $H \equiv+\infty$ (so $\left.H^{*} \equiv-\infty\right)$ there is nothing to prove. We may now assume $f\left(x, x^{*}\right)<+\infty$ and $H$ is not identically $+\infty$. Let $+\infty>\gamma>f\left(x, x^{*}\right)$. As $\left(x, x^{*}\right) \in b d s w^{*}-\liminf _{t \rightarrow+\infty}$ epi $f_{t}$, then for some $K>0$, we have for all $U \times W \in \mathcal{N}\left(x, x^{*}\right)$ there exists $t_{V}>0$ such that for $t>t_{V}$ we have (epi $\left.f_{t}\right) \cap[U \times W] \cap \overline{B_{K}}(0) \neq \emptyset$, and so there exists $\left(x_{t}, x_{t}^{*}\right) \in \operatorname{epi} f_{t} \cap \overline{B_{K}}(0) \cap$ $[U \times W]$ so $f_{t}\left(x_{t}, x_{t}^{*}\right) \leq \gamma_{t}$ with $\left(x_{t}, x_{t}^{*}\right) \in U \times W$ and $\left\|\left(x_{t}, x_{t}^{*}\right)\right\| \leq K$. Now for $+\infty>$ $\beta>H\left(y, y^{*}\right)$ we have $\left(y, y^{*}\right) \in b d s \mathrm{w}^{*}-\lim \sup _{t \rightarrow+\infty}$ epi $f_{t}^{*}$ so there exists $\left(y_{t_{\alpha}}, y_{t_{\alpha}}^{*}, \beta_{t_{\alpha}}\right) \rightarrow$ $\left(y, y^{*}, \beta\right)$ as $t \rightarrow \infty$ with $\left\|y_{t_{\alpha}}^{*}\right\| \leq K^{\prime}<+\infty$ bounded and $+\infty>\beta_{t_{\alpha}} \geq f_{t_{\alpha}}^{*}\left(y_{t_{\alpha}}^{*}, y_{t_{\alpha}}\right)$. We know that $f_{t_{\alpha}}^{*}\left(y_{t_{\alpha}}^{*}, y_{t_{\alpha}}\right)>-\infty$ since the presumption that $f_{t_{\alpha}}^{*}\left(y_{t_{\alpha}}^{*}, y_{t_{\alpha}}\right)=-\infty$ would imply $f_{t_{\alpha}} \geq f_{t_{\alpha} *}^{* *} \equiv+\infty$ and we know that $f\left(x, x^{*}\right)<+\infty$. Similarly if it was the case that $f_{t_{\alpha}}\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right)=-\infty$ then $f_{t_{\alpha}}^{*} \equiv+\infty$, implying $\beta_{t_{\alpha}}=+\infty$, counter to construction. By the Fenchel inequality for all $\alpha$ we have,

$$
\gamma_{t_{\alpha}}+\beta_{t_{\alpha}} \geq f_{t_{\alpha}}\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right)+f_{t_{\alpha}}^{*}\left(y_{t_{\alpha}}^{*}, y_{t_{\alpha}}\right) \geq\left\langle\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right),\left(y_{t_{a}}^{*}, y_{t_{\alpha}}\right)\right\rangle .
$$

For each $\varepsilon>0$ we may take $W \in \mathcal{N}\left(x^{*}\right)$ such that $\sup _{u^{*} \in W}\left\langle y, u^{*}-x^{*}\right\rangle \leq \varepsilon$ where $B_{\varepsilon}(x) \times W \times I_{\varepsilon}(\gamma) \in \mathcal{N}\left(x, x^{*}, \gamma\right)$ has $\left(x, x^{*}\right) \in B_{\varepsilon}(x) \times W \in \mathcal{N}\left(x, x^{*}\right)$ and $\gamma_{t_{\alpha}} \leq \gamma+\varepsilon$ with $\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right) \in B_{\varepsilon}(x) \times W$. In this case we observe that $\lim \sup _{\alpha}\left\langle\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right),\left(y_{t_{\alpha}}^{*}, y_{t_{\alpha}}\right)\right\rangle \leq$ $\gamma+\beta+\varepsilon$. Now

$$
\begin{array}{r}
\left\langle\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right),\left(y_{t_{\alpha}}^{*}, y_{t_{\alpha}}\right)\right\rangle=\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle+\left\langle x, y_{t_{\alpha}}^{*}-y^{*}\right\rangle+\left\langle y, x_{t_{\alpha}}^{*}-x^{*}\right\rangle \\
\quad+\left\langle x_{t_{\alpha}}-x, y_{t_{\alpha}}^{*}\right\rangle+\left\langle y_{t_{\alpha}}-y, x_{t_{\alpha}}^{*}\right\rangle \\
\geq\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle+\left\langle x, y_{t_{\alpha}}^{*}-y^{*}\right\rangle+\left\langle y, x_{t_{\alpha}}^{*}-x^{*}\right\rangle \\
\quad-K^{\prime}\left\|x_{t_{\alpha}}-x\right\|+K\left\|y_{t_{\alpha}}-y\right\| .
\end{array}
$$

Since $y_{t_{\alpha}}^{*} \rightarrow{ }^{\mathrm{w}^{*}} y^{*}, x_{t_{\alpha}}^{*} \rightarrow \rightarrow^{\mathrm{w}^{*}} x^{*}, x_{t_{\alpha}} \rightarrow^{s} x$ it follows that

$$
\begin{aligned}
\gamma+\beta+\varepsilon & \geq\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle-\varepsilon\left(K+K^{\prime}\right)+\underset{\alpha}{\lim \sup }\left\langle y, x_{t_{\alpha}}^{*}-x^{*}\right\rangle \\
& \geq\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle-\varepsilon\left(K+K^{\prime}+1\right) .
\end{aligned}
$$

As $\varepsilon>0$ was arbitrary we have $\gamma+\beta \geq\left\langle\left(x, x^{*}\right),\left(y^{*}, y\right)\right\rangle$ and so $\gamma \geq\left\langle\left(x, x^{*}\right),\left(y^{*}, y\right)\right\rangle-\beta$. As $\beta>H\left(y^{*}, y\right)$ and $\gamma>f\left(x, x^{*}\right)$ are arbitrary we get (for any $\left(y^{*}, y\right)$ with $H\left(y^{*}, y\right)<$ $+\infty$ )

$$
f\left(x, x^{*}\right) \geq\left\langle\left(x, x^{*}\right),\left(y^{*}, y\right)\right\rangle-H\left(y^{*}, y\right), \text { for all }\left(y, y^{*}\right) \text { so } f\left(x, x^{*}\right) \geq H^{*}\left(x, x^{*}\right) .
$$

Then $b d-e-\lim \sup _{t \rightarrow \infty} f_{t}=f \geq H^{*}=\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}^{*}\right)^{*}$.
We may leverage this result for non-decreasing nets to obtain a continuity result for conjugates.

Proposition 22 Let $\left\{f_{t}\right\}_{t>0}$ be a family of functions with $t \mapsto f_{t}\left(x, x^{*}\right)$ monotonically nondecreasing (as $t \rightarrow \infty$ ). Then

$$
\begin{equation*}
\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{* *}\right)^{*}=\overline{b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}}{ }^{w^{*} \times s} . \tag{17}
\end{equation*}
$$

When all $f_{t}$ are $s \times w^{*}$-closed, convex with $f_{t}>-\infty$ then $b d-e-\lim _{t \rightarrow \infty} f_{t}$ exists and is also $s \times w^{*}$-closed and convex with

$$
\begin{equation*}
b d-e-\lim _{t \rightarrow \infty} f_{t}=\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}\right)^{*} \quad \text { and } \quad\left(b d-e-\lim _{t \rightarrow \infty} f_{t}\right)^{*}=\overline{b d-e-\lim _{t \rightarrow \infty} f_{t}^{*} \mathrm{w}^{*} \times s} . \tag{18}
\end{equation*}
$$

Proof Let $g_{t}:=f_{t}^{* \dagger}$ then $\left\{g_{t}\right\}_{t>0}$ is monotonically non-increasing. Hence by Proposition 19 we have $g_{t} \geq g:=b d-e-\lim _{t \rightarrow \infty} g_{t}$ for all $t>0$ and so $g_{t}^{*} \leq g^{*}$, implying $b d-e-\lim _{t \rightarrow \infty} g_{t}^{*} \leq$ ${\overline{g^{*}}}^{b d w^{*} \times s}=g^{*}$ and hence

$$
\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{* *}\right)^{* \dagger}=\left(b d-e-\lim _{t \rightarrow \infty} g_{t}^{*}\right)^{*} \geq g^{* *}=\overline{b d-e-\lim _{t \rightarrow \infty} f_{t}^{* \dagger}}{ }^{s \times \mathrm{w}^{*}}
$$

Now apply Lemma 21 to $g_{t}$ to get

$$
\left(b d-e-\liminf g_{t \rightarrow \infty}^{*}\right)^{*} \leq b d-e-\limsup _{t \rightarrow \infty} g_{t} \quad \text { or }
$$

$$
\begin{gathered}
\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{* *}\right)^{* \dagger} \leq b d-e-\lim _{t \rightarrow \infty} f_{t}^{* \dagger}, \\
\text { implying }\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{* *}\right)^{*} \leq \overline{b d-e-\liminf _{t \rightarrow \infty} f_{t}^{*} \mathrm{w}^{*} \times s} \leq\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{* *}\right)^{*} .
\end{gathered}
$$

For the case when $f_{t}$ is convex and $s \times \mathrm{w}^{*}$-closed with $f_{t}>-\infty$, then $f_{t}^{* *}=f_{t}$, so $\left(b d-e-\lim _{t \rightarrow \infty} f_{t}\right)^{*}=\overline{b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}}{ }^{s \times w^{*}}$ and in particular, due to Proposition 18 then $b d-e-\lim _{t \rightarrow \infty} f_{t}=\sup _{t} f_{t}>-\infty$ is closed and convex so $b d-e-\lim _{t \rightarrow \infty} f_{t}=$ $\left(b d-e-\lim _{t \rightarrow \infty} f_{t}\right)^{* *}$ along with $\left(\overline{b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}}{ }^{s \times w^{*}}\right)^{*}=\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}\right)^{*}$, via the fact that our conjugation is based on the pairing of $X \times X^{*}$ with $X^{*} \times X$ and associated $s \times \mathrm{w}^{*}$ continuous linear functions on $X \times X^{*}$ (so the $s \times \mathrm{w}^{*}$-closure does not affect the value of the conjugate).

Again we may utilise the symmetry that is present in the pairing of $\sigma_{s \times \mathrm{w}^{*}}\left(X \times X^{*}\right)$ with $\sigma_{\mathrm{w}^{*} \times s}\left(X^{*} \times X\right)$. A result that uses the passage of the conjugate $*: \sigma_{s \times \mathrm{w}^{*}}\left(X \times X^{*}\right) \rightarrow$ $\sigma_{\mathrm{w}^{*} \times s}\left(X^{*} \times X\right)$ is also applicable to inverse $*: \sigma_{\mathrm{w}^{*} \times s}\left(X^{*} \times X\right) \rightarrow \sigma_{s \times \mathrm{w}^{*}}\left(X \times X^{*}\right)$ (using the transpose operator).

Proposition 23 Let $\left\{f_{t}\right\}_{t>0}$ be a family of convex functions, $f_{t} \not \equiv+\infty$ with $t \mapsto f_{t}\left(x, x^{*}\right)$ monotonically non-increasing (as $t \rightarrow \infty)$. Then bd-e- $\lim _{t \rightarrow \infty} f_{t}^{*}$ is $\left(\mathrm{w}^{*} \times s\right)$-closed and

$$
\begin{equation*}
b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}=\overline{b d-e-\lim _{t \rightarrow \infty} f_{t}^{*^{*}}}{ }^{w^{*} \times s}=\left(b d-e-\lim _{t \rightarrow \infty} f_{t}\right)^{*} . \tag{19}
\end{equation*}
$$

Proof If $\left\{f_{t}\right\}_{t>0}$ is monotonically non-increasing then for $g_{t}:=f_{t}^{*}$ we have $\left\{g_{t}\right\}_{t>0}$ is monotonically non-decreasing and closed. Hence applying Proposition 22 we have (noting that either $g_{t}>-\infty$ or $\left.f_{t} \geq f_{t}^{* *}=g_{t}^{*} \equiv+\infty\right)$

$$
\begin{align*}
b d-e-\lim _{t \rightarrow \infty} f_{t}^{*} & =b d-e-\lim _{t \rightarrow \infty} g_{t}=\left(b d-e-\limsup g_{t \rightarrow \infty}^{*}\right)^{*} \\
& =\left(b d-e-\limsup _{t \rightarrow \infty} f_{t}^{* *}\right)^{*}=\left(b d-e-\limsup _{t \rightarrow \infty} \bar{f}_{t}^{s \times \mathrm{w}^{*}}\right)^{*} . \tag{20}
\end{align*}
$$

In the following, by Proposition 19 we have the first equality, with the inequality following from $f_{t} \geq \inf _{t>0} f_{t}$ so

Thus, as the conjugate is not affected by the $s \times \mathrm{w}^{*}$-closure, and by Proposition 19 we have $b d-e-\lim _{t \rightarrow \infty} f_{t}=\overline{\inf _{t>0} f_{t}} s \times b d \mathrm{w}^{*}$, it follows from (20)-(21) that:

$$
\begin{equation*}
b d-e-\lim _{t \rightarrow \infty} f_{t}^{*} \leq\left({\overline{\left(\overline{\inf }_{t>0}\right.} s \times b d \mathrm{w}^{*}}^{s \times \mathrm{w}^{*}}\right)^{*}=\left(\overline{\inf }_{t>0}^{s \times b d \mathrm{w}^{*}}\right)^{*}=\left(b d-e-\lim _{t \rightarrow \infty} f_{t}\right)^{*} . \tag{22}
\end{equation*}
$$

By Proposition 21 we have $\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}\right)^{*} \leq b d-e-\lim _{t \rightarrow \infty} f_{t}$ so

$$
\begin{equation*}
b d-e-\lim _{t \rightarrow \infty} f_{t}^{*} \geq \overline{b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}}{ }^{w^{*} \times s}=\left(b d-e-\lim _{t \rightarrow \infty} f_{t}^{*}\right)^{* *} \geq\left(b d-e-\lim _{t \rightarrow \infty} f_{t}\right)^{*} . \tag{23}
\end{equation*}
$$

Combining (22) and (23) we get (19).

We note the following for later use when further investigating the continuity of conjugation in relation to these variational limits.

Lemma 24 Let $\left\{f_{t}\right\}_{t>0}$ be a family of $s \times b d \mathrm{w}^{*}$-(resp. $s \times \mathrm{w}^{*}$-)closed functions on $X \times X^{*}$ for which $\left(\inf _{t} f_{t}\right)\left(x, x^{*}\right)>-\infty$ for all $\left(x, x^{*}\right)$. Recalling that $\widehat{f}_{t}$ denotes the embedding of $f_{t}$ within the conjugate space $\left(X \times X^{*}\right)^{*}=X^{*} \times X^{* *}$, (by setting $\widehat{f}_{t}=f_{t}$ on the subspace $X \times X^{*}$ and $+\infty$ otherwise) with the associated conjugation $\widehat{\star}: X \times X^{*} \rightarrow\left(X \times X^{*}\right)^{*}=$ $X^{*} \times X^{* *}$. Then on $X \times X^{*}$

$$
\begin{equation*}
\overline{\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f}_{t}\right)^{\widehat{\kappa \dagger}}}{ }^{s \times b d w^{*}}=\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}\right)^{* \dagger} \tag{24}
\end{equation*}
$$

Proof We have, using the $\boldsymbol{w}^{*}$-continuity of $\left(x^{* *}, x^{*}\right) \mapsto\left\langle\left(x^{* *}, x^{*}\right),\left(y^{*}, y\right)\right\rangle=\left\langle x^{* *}, y^{*}\right\rangle+$ $\left\langle x^{*}, y\right\rangle$, the Fenchel inequality $\left\langle\left(x_{\alpha}, x_{\alpha}^{*}\right),\left(y^{*}, y\right)\right\rangle-f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right) \leq f_{t}^{*}\left(y^{*}, y\right)$ and Lemma 17 that

$$
\begin{aligned}
& \left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f}_{t}\right)^{\widehat{x \dagger}}\left(y, y^{*}\right) \\
& =\sup _{\left(x^{*}, x^{* *}\right)}\left\{\left\langle\left(x^{* *}, x^{*}\right),\left(y^{*}, y\right)\right\rangle-\inf _{\substack{\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow b d w^{*}\left(x^{* *}, x^{*}\right)\right\} \\
\left\{t_{\alpha} \rightarrow+\infty\right\}}} \liminf _{\alpha} \widehat{f}_{t}\left(x_{t}, x_{t}^{*}\right)\right\} \\
& =\sup _{\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow b w^{*}\left(x^{* *}, x^{*}\right) \in X^{* *} \times X^{*}\right\}}^{\substack{\left\{t_{\alpha} \rightarrow+\infty\right\}}} \limsup _{\alpha}\left\{\left\{\left\langle\left(x_{\alpha}, x_{\alpha}^{*}\right),\left(y^{*}, y\right)\right\rangle-f_{t_{\alpha}}\left(\left(x_{\alpha}, x_{\alpha}^{*}\right)\right)\right\}\right] \\
& \quad \leq \sup _{\left\{t_{\alpha} \rightarrow+\infty\right\}} \limsup _{\alpha} f_{t_{\alpha}}^{* \dagger}\left(y, y^{*}\right)=\lim _{\eta \rightarrow+\infty} \sup _{t \geq \eta} f_{t}^{* \dagger}\left(y, y^{*}\right) \\
& \quad=\lim _{\eta \rightarrow+\infty}\left(\inf _{t \geq \eta} f_{t}\right)^{* \dagger}\left(y, y^{*}\right) .
\end{aligned}
$$

That is, we have a monotonically nondecreasing family $\left\{g_{\eta}=\inf _{t \geq \eta} f_{t}\right\}$ where

$$
\begin{equation*}
\left.\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f_{t}}\right)^{\widehat{x} \dagger}\right|_{X \times X^{*}} \leq \lim _{\eta \rightarrow+\infty}\left(g_{\eta}\right)^{* \dagger} . \tag{25}
\end{equation*}
$$

As $\left\{g_{\eta}^{*}\right\}$ is closed and monotonically non-increasing, Proposition 19 implies

$$
b d-e-\lim _{\eta \rightarrow \infty}\left(g_{\eta}\right)^{* \dagger}=\overline{\inf _{\eta}\left(g_{\eta}\right)^{* \dagger}} s \times b d \mathrm{w}^{*} \quad=\overline{\lim _{\eta \rightarrow+\infty}\left(g_{\eta}\right)^{* \dagger}} s \times b d \mathrm{w}^{*}
$$

Using this observation and Proposition 22 (18), we have

$$
\begin{aligned}
\overline{\lim }_{\eta \rightarrow+\infty}\left(g_{\eta}\right)^{* \dagger} & s \times b d \mathrm{w}^{*} \\
& =b d-e-\lim _{\eta \rightarrow \infty}\left(g_{\eta}\right)^{* \dagger}=\overline{b d-e-\lim _{\eta \rightarrow \infty}\left(g_{\eta}\right)^{* \dagger}} s \times \mathrm{w}^{*} \\
& =\left(b d-e-\lim _{\eta \rightarrow \infty} g_{\eta}\right)^{* \dagger} .
\end{aligned}
$$

Combining this with (25) and Lemma 20 we have

$$
\begin{aligned}
& \overline{\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f}_{t}\right)^{\hat{\star \dagger}}} \mathrm{s} \mathrm{\times bd} \mathrm{\mathbf{w}}^{*} \leq \overline{\lim _{\eta \rightarrow+\infty}\left(g_{\eta}\right)^{* \dagger}} s \times b d \mathrm{w}^{*} \\
&=\left(b d-e-\lim _{\eta \rightarrow \infty} g_{\eta}\right)^{* \dagger} \\
&=\left(b d-e-\lim _{\eta \rightarrow \infty}\left(\inf _{t \geq \eta} f_{t}\right)\right)^{* \dagger}=\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}\right)^{* \dagger} .
\end{aligned}
$$

Hence

$$
\overline{\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f}_{t}\right)^{\hat{x}^{\dagger}}} \mathrm{s} \mathrm{\times bdw}^{*} \leq\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}\right)^{*} .
$$

The reverse inequality is immediate from epi $\left(b d w^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f}_{t}\right) \supseteq$ epi $\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}\right)$ as then, when restricting to $X^{*} \times X$, we have

$$
\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f_{t}}\right)^{\widehat{\star}} \geq\left(b d-\widehat{e-\liminf _{t \rightarrow \infty}} f_{t}\right)^{\widehat{ }}=\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}\right)^{*} .
$$

The last inequality is preserved on taking the ( $b d \mathrm{w}^{*} \times s$ ) -closure as the conjugate $\left(b d-e \text { - } \lim _{t \rightarrow \infty} f_{t}\right)^{*}$ is actually $\left(\mathrm{w}^{*} \times s\right)$-closed. This gives the reverse inequality.

We will strengthen our conjugation results by adapting Theorem 2 of [21]. We first show the following result.

Proposition 25 Assume that $\left\{f_{t}\right\}_{t \in \mathbb{R}_{+}}$is a family of ( $s \times \mathrm{w}^{*}$ )-closed, proper, convex functions. Assume in addition that for any cofinal subset $I_{1} \subseteq \mathbb{R}_{+}$there exists a cofinal subset $I_{2}$ of $I_{1}$ and a bounded net $\left\{\left(x_{\alpha}^{*}, x_{\alpha}\right)\right\}_{t_{\alpha} \in I_{2}}$ such that $\left\{f_{t_{\alpha}}^{*}\left(x_{\alpha}^{*}, x_{\alpha}\right)\right\}_{t_{\alpha} \in I_{2}}$ is bounded above. Denote $f:=b d-e-\lim \sup _{t \rightarrow \infty} f_{t}$. Then, on $X \times X^{*}$ we have

$$
\begin{equation*}
\bar{f}^{s \times b d \mathbf{w}^{*}} \leq\left(b d \boldsymbol{w}-e-\liminf _{t \rightarrow+\infty} \widehat{f_{t}^{\star}}\right)^{\widehat{\star}} \tag{26}
\end{equation*}
$$

Furthermore, if $\inf _{t} f_{t}^{*}\left(x^{*}, x\right)>-\infty$ for all $\left(x^{*}, x\right) \in X^{*} \times X$ then on $X \times X^{*}$ :

$$
\begin{equation*}
\overline{b d-e-\limsup _{t \rightarrow \infty} f_{t}}{ }^{s \times b d w^{*}}=\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}^{*}\right)^{*} . \tag{27}
\end{equation*}
$$

Proof Given $\left(\bar{x}, \bar{x}^{*}\right) \in X \times X^{*}$ and $\bar{\gamma}<\bar{f}^{s \times b d w^{*}}\left(\bar{x}, \bar{x}^{*}\right)$ we note that $\left(\bar{x}, \bar{x}^{*}, \bar{\gamma}\right) \notin$ $\overline{b d s \mathrm{w}^{*}-\liminf }{ }_{t \rightarrow \infty}\left(\right.$ epi $\left.f_{t}\right) ~=~ s o d \mathrm{w}^{*}$, so any $K>\max \left\{|\bar{\gamma}|,\left\|\left(\bar{x}, \bar{x}^{*}\right)\right\|\right\} \geq 0$ we can find a cofinal subset $I_{1}$ of $\mathbb{R}_{+}$, a norm ball $U \ni \bar{x}$, a weak* neighbourhood $W \ni \bar{x}^{*}$ and $\rho>0$ for which

$$
\begin{aligned}
& {[[(U \times W)] \times[\bar{\gamma}+(-\rho, \rho)]] \cap \text { epi } f_{t_{\alpha}} \cap \bar{B}_{K}(0)} \\
& =\left(\left[\left[(U \times W) \cap \bar{B}_{K}(0)\right] \times[\bar{\gamma}+[-\rho, \rho]] \cap[-K, K]\right]\right) \cap \text { epi } f_{t_{\alpha}} \\
& =\emptyset
\end{aligned}
$$

for each $t_{\alpha} \in I_{1}$. After adjusting $\rho>0$ accordingly we may claim

$$
\begin{aligned}
& {\left[\left[\left(\{\bar{x}\} \times{\overline{W \cap B_{K}(0)}}^{\mathrm{w}^{*}}\right)\right] \times[\bar{\gamma}+[-\rho, \rho]]\right] \cap \text { epi } f_{t_{\alpha}}} \\
& \subseteq([(U \times W)] \times[\bar{\gamma}+[-\rho, \rho]]) \cap \operatorname{epi} f_{t_{\alpha}} \cap \bar{B}_{K}(0)=\emptyset .
\end{aligned}
$$

The Hahn-Banach separation theorem holds (for the separation of the ( $s \times \mathrm{w}^{*}$ )-closed convex set epi $f_{t_{\alpha}}$ and the $\left(s \times \mathrm{w}^{*}\right)$-compact convex set $\left.\{\bar{x}\} \times{\overline{W \cap B_{K}(0)}}^{\mathrm{w}^{*}}\right)$ within a locally convex linear topological space [34, Theorem 1.1.3] and $X \times X^{*}$ is made so by endowing (as usual) $X$ with the strong topology and $X^{*}$ with the weak ${ }^{*}$ topology. Hence we have $\left(y_{\alpha}^{*}, y_{\alpha},-\lambda_{\alpha}\right) \in$ $X^{*} \times X \times \mathbb{R}$ of unit norm (the $\lambda_{\alpha} \geq 0$ implied by the epigraphical recession direction in epi $f_{t_{\alpha}}$ ) for which

$$
\begin{aligned}
\left\langle\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(x, x^{*}\right)\right\rangle-\lambda_{\alpha} \gamma & \leq\left\langle\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(z, z^{*}\right)\right\rangle-\lambda_{\alpha} r \quad \text { for all }\left(x, x^{*}, \gamma\right) \in \operatorname{epi} f_{t_{\alpha}} \\
\text { and }\left(z, z^{*}, r\right) & \in\left[\left(\{\bar{x}\} \times W \cap \overline{B_{K}}(0)\right)\right] \times[\bar{\gamma}+(-\rho, \rho)] .
\end{aligned}
$$

Using the fact this holds for all $r \in \bar{\gamma}+(-\rho, \rho)$ we have

$$
\begin{align*}
& \left\langle\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(x, x^{*}\right)\right\rangle-\lambda_{\alpha} \gamma \leq\left\langle\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\lambda_{\alpha}(\bar{\gamma}-\rho) \\
& \quad \text { for all }\left(x, x^{*}, \gamma\right) \in \operatorname{epi} f_{t_{\alpha}} . \tag{28}
\end{align*}
$$

Now let $I_{2}$ be cofinal in $I_{1}$ and take a bounded $\left\{\left(x_{\alpha}^{*}, x_{\alpha}\right)\right\}_{t_{\alpha} \in I_{2}}$ be such that $f^{*}\left(x_{\alpha}^{*}, x_{\alpha}\right)$ is bounded above i.e. $f_{t_{\alpha}}^{*}\left(x_{\alpha}^{*}, x_{\alpha}\right)<b$ and $\left\|\left(x_{\alpha}^{*}, x_{\alpha}\right)\right\| \leq H$ (some $H$ ). Then for each $t_{\alpha} \in I_{2}$ we have

$$
\begin{equation*}
\left\langle\left(x_{\alpha}^{*}, x_{\alpha}\right),\left(x, x^{*}\right)\right\rangle-\gamma \leq b \quad \text { for all }\left(\left(x, x^{*}\right), \gamma\right) \in \operatorname{epi} f_{t_{\alpha}} . \tag{29}
\end{equation*}
$$

Taking (28) along $I_{2} \subseteq I_{1}$ we may multiply (28) by $q>0$ and add to (29) to get

$$
\begin{align*}
& \left\langle\left(x_{\alpha}^{*}, x_{\alpha}\right)+q\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(x, x^{*}\right)\right\rangle-\left(1+\lambda_{\alpha} q\right) \gamma \\
& \leq\left\langle\left(x_{\alpha}^{*}, x_{\alpha}\right)+q\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\left\langle\left(x_{\alpha}^{*}, x_{\alpha}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle+b-\lambda_{\alpha} q \bar{\gamma}-\lambda_{\alpha} q \rho \\
& \leq\left\langle\left(x_{\alpha}^{*}, x_{\alpha}\right)+q\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle \\
& \quad-\left(1+\lambda_{\alpha} q\right) \bar{\gamma}+\bar{\gamma}+\left\|x_{\alpha}\right\|\left\|\bar{x}^{*}\right\|+\left\|x_{\alpha}^{*}\right\|\|\bar{x}\|+b-\lambda_{\alpha} q \rho \\
& \leq\left\langle\left(x_{\alpha}^{*}, x_{\alpha}\right)+q\left(y_{\alpha}^{*}, y_{\alpha}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\left(1+\lambda_{\alpha} q\right) \bar{\gamma} \\
& \quad+\left[\bar{\gamma}+\left(\left\|\bar{x}^{*}\right\|+\|\bar{x}\|\right) H+b-\lambda_{\alpha} q \rho\right] . \tag{30}
\end{align*}
$$

Now choose $\bar{q}>0$ sufficiently large so that $\left[\bar{\gamma}+\left[\left\|\bar{x}^{*}\right\|+\|\bar{x}\|\right] H+b-q \rho\right] \leq 0$ for all $q \geq \bar{q}$. Indeed we may take

$$
\bar{q}=\frac{1}{\rho}\left\{\bar{\gamma}+\left[\left\|\bar{x}^{*}\right\|+\|\bar{x}\|\right] H+b\right\} .
$$

Let $q=2 \bar{q}$ and set $\left(z_{\alpha}^{*}, z_{\alpha}\right):=\left(1+2 \lambda_{\alpha} q\right)^{-1}\left[\left(x_{\alpha}^{*}, x_{\alpha}\right)+2 q\left(y_{\alpha}^{*}, y_{\alpha}\right)\right]$; on division of (30) by $\left(1+2 \lambda_{\alpha} q\right)$ we obtain

$$
\left\langle\left(z_{\alpha}^{*}, z_{\alpha}\right),\left(x, x^{*}\right)\right\rangle-\gamma \leq\left\langle\left(z_{\alpha}^{*}, z_{\alpha}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\bar{\gamma} \quad \text { for all }\left(\left(x, x^{*}\right), \gamma\right) \in \operatorname{epi} f_{t_{\alpha}} .
$$

Thus $f_{t_{\alpha}}^{*}\left(z_{\alpha}^{*}, z_{\alpha}\right) \leq \gamma_{\alpha}:=\left\langle\left(z_{\alpha}^{*}, z_{\alpha}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\bar{\gamma}$ for $t_{\alpha} \in I_{2}$. As $\left\{\left(x_{\alpha}^{*}, x_{\alpha}\right)\right\}_{t_{\alpha} \in I_{2}}$ and $\left\{\left(y_{\alpha}^{*}, y_{\alpha}\right)\right\}_{t_{\alpha} \in I_{2}}$ are bounded, as $q=2 \bar{q}$ we have $\left\{\left(z_{\alpha}^{*}, z_{\alpha}, \gamma_{\alpha}\right)\right\}_{t_{\alpha} \in I_{2}}$ bounded for all $\alpha$ (in norm by, say, $\bar{K}=\left(\left\|\left(\bar{x}, \bar{x}^{*}\right)\right\|+\bar{\gamma}\right)(H+2 \bar{q})$ ) and so may extract a $\boldsymbol{w}^{*}$-convergent (in $\left.X^{*} \times X^{* *}=\left(X \times X^{*}\right)^{*}\right)$ subnet $\left\{\left(z_{\alpha}^{*}, z_{\alpha}\right)\right\}_{t_{\alpha} \in I_{3} \subseteq I_{2}}$ with

$$
\left(z_{\alpha}^{*}, z_{\alpha}\right):=\left(1+\lambda_{\alpha} q\right)^{-1}\left[\left(x_{\alpha}^{*}, x_{\alpha}\right)+q\left(y_{\alpha}^{*}, y_{\alpha}\right)\right] \rightarrow_{\alpha \in I_{3}}^{w^{*}}\left(y^{*}, y^{* *}\right) \in \overline{B_{\bar{K}}}(0) .
$$

Also as $\gamma_{\alpha} \rightarrow\left\langle\left(y^{*}, y^{* *}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\bar{\gamma}(\leq \bar{K})$ we have

$$
\begin{aligned}
\left(y^{*}, y^{* *},\left\langle\left(y^{*}, y^{* *}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\bar{\gamma}\right) & \in \boldsymbol{w}^{*}-\limsup _{t \rightarrow+\infty}\left(\operatorname{epi} \widehat{f_{t}^{*}} \cap \overline{B_{\bar{K}}}(0)\right) \\
& \subseteq \bigcup_{K>0} \boldsymbol{w}^{*}-\limsup _{t \rightarrow+\infty}\left(\operatorname{epi} \widehat{f_{t}^{*}} \cap \overline{B_{K}}(0)\right) .
\end{aligned}
$$

That is, $\left(b \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f_{t}^{*}}\right)\left(y^{*}, y^{* *}\right) \leq\left\langle\left(y^{*}, y^{* *}\right),\left(\bar{x}, \bar{x}^{*}\right)\right\rangle-\bar{\gamma}$, which implies

$$
\bar{\gamma} \leq\left(b d \boldsymbol{w}^{*}-e-\liminf _{t \rightarrow+\infty} \widehat{f_{t}^{*}}\right)^{\widehat{*}}\left(\bar{x}, \bar{x}^{*}\right),
$$

 arbitrary ( $\bar{x}, \bar{x}^{*}$ ) we get (26). Using (24), it follows that on $X \times X^{*}$, have $\bar{f}^{s \times b d w^{*}} \leq$ $\overline{\left(b d \boldsymbol{w}^{*}-e-\lim \inf _{t \rightarrow+\infty} \widehat{f_{t}^{* \dagger}}\right)^{\Delta \star} \times b d \mathrm{w}^{*}}=\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}^{* \dagger}\right)^{* \dagger}$. To get (27), using (16) we have $f \geq\left(b d-e-\liminf f_{t \rightarrow \infty} f_{t}^{* \dagger}\right)^{* \dagger}$ with the latter $\left(s \times w^{*}\right)$-closed and hence also for $s \times b d \mathrm{w}^{*}$. Taking a $s \times b d \mathrm{w}^{*}$-closure we have the reverse inequality, giving the results.

Again we may utilise the symmetry that is present in the pairing of $\sigma_{s \times \mathrm{w}^{*}}\left(X \times X^{*}\right)$ with $\sigma_{\mathrm{w}^{*} \times s}\left(X^{*} \times X\right)$.

Corollary 26 Assume that $\left\{f_{t}\right\}_{t \in \mathbb{R}_{+}}$is a family of ( $s \times \mathrm{w}^{*}$ )-closed, proper, convex functions for which $\left(\inf _{t} f_{t}\right)\left(x, x^{*}\right)>-\infty$ for all $\left(x, x^{*}\right)$, and that for any cofinal subset $I_{1}$ of $\mathbb{R}_{+}$ there is a cofinal $I_{2} \subseteq I_{1}$ and a bounded net $\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}_{t_{\alpha} \in I_{2}}$ such that $\left\{f_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}_{t_{\alpha} \in I_{2}}$ is bounded above. Then we have

$$
\begin{equation*}
\overline{b d-e-\limsup _{t \rightarrow \infty} f_{t}^{*} \times x d \mathrm{w}^{*}}=\left(b d-e-\liminf _{t \rightarrow \infty} f_{t}\right)^{*} . \tag{31}
\end{equation*}
$$

Proof We apply (27) to the family of convex functions $\left\{f_{t}^{* \dagger}\right\}_{t>0}$ that are well defined on $X \times X^{*}$. We then we need to assume that for any cofinal subset $I_{2}$ of $I_{1}$ there is a bounded net $\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}_{t_{\alpha} \in I_{2}}$ such that $\left\{\left(f_{t_{\alpha}}^{*}\right)^{*}\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}_{t_{\alpha} \in I_{2}}$ is bounded above. As $\left(f_{t_{\alpha}}^{*}\right)^{*}=f_{t_{\alpha}}$ this corresponds to our stated assumption. The result now follows.

## 5 Some Tools from Monotone Operator Theory

When $h \in P C\left(X \times X^{*}\right)$ may not be representative (i.e. $h$ need not majorise the duality product on $\left.X \times X^{*}\right)$, we form $M_{h}^{\leq}:=\left\{\left(x, x^{*}\right) \in X \times X^{*} \mid h\left(x, x^{*}\right) \leq\left\langle x, x^{*}\right\rangle\right\}$, and note that
when, instead, $h \geq\langle\cdot, \cdot\rangle$ (i.e. $h$ is representative) it is well known that we have $M_{h}^{\leq}=M_{h}$, a monotone set. Denote $R(T):=\left\{h \in P C\left(X \times X^{*}\right) \mid h \geq\langle\cdot, \cdot\rangle\right.$ and $\left.T \subseteq M_{h}\right\}$. When $F_{T}=$ $\langle\cdot, \cdot\rangle-\inf _{\left(y, y^{*}\right) \in T}\left\langle\cdot-y, \cdot-y^{*}\right\rangle$ is not representative we may study the set $T^{\mu}=M_{F_{T}}^{\leq}$. One always has $T^{\mu \mu \mu}=T^{\mu}$ and if $T$ is monotone then $T \subseteq T^{\mu}$. In [18] $T \mapsto T^{\mu}$ is shown to be a polarity and as a consequence $A \subseteq B$ implies $A^{\mu} \supseteq B^{\mu}, T \subseteq T^{\mu \mu}$ and $(A \cup B)^{\mu}=A^{\mu} \cap B^{\mu}$ (for any sets $A, B \subseteq X \times X^{*}$ ). From definitions it is clear that $T^{\mu}$ always has $w^{*}$-closed convex images. If $T$ is monotone but not maximal, the Fitzpatrick function $F_{T}$ may not be representative. On the other hand the Penot/Svaiter function $P_{T}:=F_{T}^{* \dagger} \geq\langle\cdot, \cdot\rangle$, does indeed represent $T$ in that $T \subseteq M_{P_{T}}$. Following [18] we say $T$ is representable when there exists $h \in R(T)$ with $T=M_{h}$.

Recall $b R(T):=\left\{h \in P C\left(X \times X^{*}\right) \mid P_{T} \geq h^{* \dagger} \geq h \geq\langle\cdot, \cdot\rangle\right\}$ are the bigger-conjugate representative functions with $T \subseteq M_{h} \subseteq T^{\mu}$. It is known, $[18,30]$ that when $h \in R(T)$ is closed or if $h \in b R(T)$, then $h \in\left[F_{T}, P_{T}\right]$ where $\left[F_{T}, P_{T}\right]=\left\{g \in P C\left(X, X^{*}\right) \mid F_{T} \leq g \leq\right.$ $\left.P_{T}\right\}$, in the pointwise partial order [12, Lemma 2.1]. Note that if $h \in b R(T)$ then we have $T \subseteq M_{h}$ and $h \in b R\left(M_{h}\right)$ (see [13, Lemma 3]). When $h \in b R(T)$ then $h \geq F_{M_{h}}$, a detailed proof of which may be found in [12, Lemma 2.6B] or [13, Lemma 3].

Proposition 27 ([13, Theorem 8]) (Monotonic Closure Theorem) Let $h \in b R(T)$. Then $M_{h}$ is monotonically closed, i.e. $M_{h}^{\mu \mu}=M_{h}$.

Moreover this set is unique in the following sense.
Lemma 28 [12, Lemma 2.11], [30] Let $T: X \rightrightarrows X^{*}$ be a monotone operator, let $k, h \in$ $b R(T)$ for which $h \leq k$. Then $M_{k}=M_{h} \supseteq T$.

We note that in [31, Theorem 11.2] are examples of representable operators that are not monotonically closed and so the use of bigger-conjugate representative functions plays an important role.

Remark 29 As noted in [13] if $F_{M_{h}} \geq\langle\cdot, \cdot\rangle$, then $\bar{h}^{s \times w^{*}} \in b R(T)$ which indicates that we only really need to consider $\left(s \times w^{*}\right)$-closed representative functions when seeking to characterise maximality but this not mandatory and may be difficult to enforce in some constructions. Of course when $h \in b R(T)$ and $h \geq \bar{h}^{s \times \mathrm{w}^{*}} \in b R(T)$ then $M_{\bar{h}^{s \times w^{*}}}=M_{h}$.

We will summarise some results we require that appear in [13]. First note that in [7] it is shown that not every maximal monotone operator has a $\left(s \times w^{*}\right)$-closed graph (even for the case of a subdifferential of a convex function). As we do not a priori assume any closure property for elements of $R(T)$ or $b R(T)$ we study an appropriate closure.

Lemma 30 ([13, Lemma 5]) Suppose $T$ is monotone, and $\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}_{\alpha} \subseteq T$ is a bounded net converging in the $\left(s \times \mathrm{w}^{*}\right)$-topology to $\left(x, x^{*}\right)$. Then $\left(x, x^{*}\right) \in T^{\mu \mu}$ i.e. $T \subseteq \bar{T}^{s \times b d \mathrm{w}^{*}} \subseteq T^{\mu \mu}$. In particular when $T$ is monotonically closed (as is the case when $h \in b R(T)$ and $T=M_{h}$ ) we have $T\left(=T^{\mu \mu}\right)=\bar{T}^{s \times b d w^{*}}$.

Proof Let $\left(y, y^{*}\right) \in T^{\mu}$. Then as $\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}_{\alpha} \subseteq T$ is bounded, there is $K>0$ such that $\left\|x_{\alpha}^{*}\right\| \leq K$ for all $\alpha$ so that

$$
\begin{align*}
0 & \leq\left\langle y-x_{\alpha}, y^{*}-x_{\alpha}^{*}\right\rangle \\
& =\left\langle y-x, y^{*}-x_{\alpha}^{*}\right\rangle+\left\langle x-x_{\alpha}, y^{*}-x_{\alpha}^{*}\right\rangle \\
& \leq\left\langle y-x, y^{*}-x_{\alpha}^{*}\right\rangle+\left[\left\|y^{*}\right\|+K\right]\left\|x-x_{\alpha}\right\| . \tag{32}
\end{align*}
$$

As $x_{\alpha} \rightarrow x$ in the norm topology and $x_{\alpha}^{*} \rightarrow x^{*}$ in the weak* topology we have, on taking the limit,

$$
0 \leq\left\langle y-x, y^{*}-x^{*}\right\rangle \quad \text { for all }\left(y, y^{*}\right) \in T^{\mu} .
$$

Hence $\left(x, x^{*}\right) \in T^{\mu \mu}$ and so $T \subseteq \bar{T}^{s \times b d w^{*}} \subseteq T^{\mu \mu}$.
This closure is particularly well suited to the study of representative functions. Denote by $\bar{h}^{s \times b d w^{*}}$ the convex function with epigraph $\overline{\operatorname{epi} h}{ }^{s \times b d w^{*}}$.

Lemma 31 Let $h \in R(T)$.

1. Then $\bar{h}^{s \times b d \mathrm{w}^{*}} \in R(T)$ and $M_{\bar{h}^{s \times b d w^{*}}} \supseteq{\overline{M_{h}}}^{s \times b d \mathrm{w}^{*}} \supseteq M_{h} \supseteq T$, and so when $\bar{h}^{s \times b d \mathrm{w}^{*}}=h$ we have $M_{h}={\overline{M_{h}}}^{s \times b d w^{*}}$.
2. When $h \in b R(T)$ then $\bar{h}^{s \times b d \mathrm{w}^{*}} \in b R(T)$ and $M_{\bar{h}^{s \times b d w^{*}}}={\overline{M_{h}}}^{s \times b d \mathrm{w}^{*}}=\left(M_{h}\right)^{\mu \mu}=M_{h} \supseteq$ $T$.

Proof Part 1) First we note that $\bar{h}^{s \times b d w^{*}}$ majorises the duality product. Indeed, as $h \geq\langle\cdot, \cdot\rangle$ we have, from (32) (with $\left(y, y^{*}\right)=\left(x, x^{*}\right)$ ), that there is $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times b d w^{*}}\left(x, x^{*}\right)$ for which

$$
\bar{h}^{s \times b d w^{*}}\left(x, x^{*}\right)=\lim _{\alpha} h\left(x_{\alpha}, x_{\alpha}^{*}\right) \geq \lim _{\alpha}\left\langle x_{\alpha}, x_{\alpha}^{*}\right\rangle=\left\langle x, x^{*}\right\rangle,
$$

giving the inequality for the closure.
On the other hand, if $\left(x, x^{*}\right) \in{\overline{M_{h}}}^{s \times b d \mathrm{w}^{*}}$, and $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right)$ with $\left(x_{\alpha}, x_{\alpha}^{*}\right) \in$ $M_{h}$ we have

$$
\left\langle x, x^{*}\right\rangle=\lim _{\alpha} h\left(x_{\alpha}, x_{\alpha}^{*}\right) \geq \bar{h}^{s \times b d w^{*}}\left(x, x^{*}\right) \geq\left\langle x, x^{*}\right\rangle
$$

and so $\left(x, x^{*}\right) \in M_{\bar{h}^{s \times b d w^{*}}}$. It follows that $M_{h} \subseteq{\overline{M_{h}}}^{s \times b d w^{*}} \subseteq M_{\bar{h}^{s \times b d w^{*}}}$.
Part 2) When $h \in b R(T)$ then $h \leq h^{* \dagger}$, so from the first part, $\langle\cdot, \cdot\rangle \leq \bar{h}^{s \times b d w^{*}} \leq h \leq$ $h^{* \dagger} \leq\left(\bar{h}^{s \times b d w^{*}}\right)^{* \dagger}$ and so $\bar{h}^{s \times b d w^{*}} \in b R(T)$. This implies, using [30], that $M_{\bar{h}^{s \times b d w^{*}}}=M_{h}$ $\left(={\overline{M_{h}}}^{s \times b d w^{*}}\right)$.

We make note of the following for later use. Recall Definition 1 for the recession operator $\operatorname{rec} M$.

Proposition 32 ([13, Proposition 5]) Suppose $T: X \rightrightarrows X^{*}$ is a monotone operator. If $x, y \in$ $\overline{\operatorname{dom}} T$ then for any $\lambda \in[0,1]$ we have $(\lambda x+(1-\lambda) y, 0) \in(\operatorname{rec} T)^{\mu \mu}$ and so

$$
\begin{equation*}
\cos \overline{\operatorname{dom}} T \subseteq P_{X}\left[(\operatorname{rec} T)^{\mu \mu}\right] \tag{33}
\end{equation*}
$$

Moreover when rec $T$ is monotonically closed then $\overline{\operatorname{dom}} T$ is convex.
Remark 33 When $X$ is reflexive it is well known that for maximal monotone $T$ the strong closure of the domain $\overline{\operatorname{dom}} T$ is convex. When $X$ is reflexive and $T$ is maximal monotone, then it is shown in [13] that $(\operatorname{rec} T)(x)=N_{\overline{\text { codom }} T}(x)$ for all $x \in \overline{\operatorname{dom}} T$. Consequently
 imally monotone.

## 6 A Bigger-Conjugate Representative Function for rec $\boldsymbol{T}$

The objective of the section is to construct a concrete representative $h \in b R(\operatorname{rec} T)$ using a variational limit. In doing so we further connect to this operator, the normal cone operator associated with the convex closure of the domain of $T$. The construction we use is the following.

Definition 34 Let $X$ be an arbitrary Banach space. Suppose $T: X \rightrightarrows X^{*}$ is maximal monotone and denote its Fitzpatrick function by $F_{T}$. Then we form

$$
F_{T} 0_{2}^{+}:={\overline{b d-e-\liminf _{t \rightarrow+0} F_{t T}}}^{s \times b d \mathrm{w}^{*}}
$$

where $(t T)(x):=t T(x)$.

We begin by taking $\left(z, z^{*}\right) \in T$ and translating this point to $(0,0)$ noting that as

$$
\begin{equation*}
F_{T-\left(z, z^{*}\right)}\left(x, x^{*}\right)=F_{T}\left(x+z, x^{*}+z^{*}\right)-\left(\left\langle x+z, x^{*}+z^{*}\right\rangle-\left\langle x, x^{*}\right\rangle\right), \tag{34}
\end{equation*}
$$

any analysis based on $s \times \mathrm{w}^{*}$ convergent bounded nets will be unaffected by such a translation, which is the case for the arguments to follow. Thus without loss of generality we assume $(0,0) \in T$, so that $F_{T}, F_{T} 0^{+} \geq 0$ on $X \times X^{*}$.

We will build up the properties of this function in a series of Lemmas and Propositions.
Lemma 35 Let $X$ be an arbitrary real Banach space. Suppose $T: X \rightrightarrows X^{*}$ is monotone, with Fitzpatrick function $F_{T}$. Suppose also that $(0,0) \in T$. Then:

1. $O n X \times X^{*}$,

$$
F_{T} 0_{2}^{+} \geq 0
$$

Moreover, denoting $g_{t}\left(x, x^{*}\right):=F_{t T}\left(x, x^{*}\right)=t F_{T}\left(x, \frac{x^{*}}{t}\right)$ then for all $\left(y, y^{*}\right) \in X \times$ $X^{*}$,

$$
g_{t}^{* \dagger}\left(y, y^{*}\right)=t P_{T}\left(y, \frac{y^{*}}{t}\right)=P_{t T}\left(y, y^{*}\right) \geq F_{t T}\left(y, y^{*}\right)=t F_{T}\left(y, \frac{y^{*}}{t}\right)=g_{t}\left(y, y^{*}\right) .
$$

2. We have

$$
\begin{align*}
& \operatorname{Pr}_{X} \operatorname{dom} F_{T} 0_{2}^{+} \subseteq \overline{\operatorname{Pr}_{X} \operatorname{dom} F_{T}}, \quad \text { and, when } T \text { maximal, }  \tag{35}\\
& \operatorname{Pr}_{X} \operatorname{dom} F_{T} 0_{2}^{+} \subseteq \overline{\operatorname{co}} \operatorname{dom} T .
\end{align*}
$$

Furthermore when $T$ is maximal, then $F_{T} 0_{2}^{+} \geq\langle\cdot, \cdot\rangle$ on $X \times X^{*}$, with also $\left(F_{T} 0_{2}^{+}\right)(x$, $\left.x^{*}\right)=\left\langle x, x^{*}\right\rangle$ for all $\left(x, x^{*}\right) \in \operatorname{rec} T$, so $F_{T} 0_{2}^{+} \in R(\operatorname{rec} T)\left(i . e . \operatorname{rec} T \subseteq M_{F_{T} 0_{2}^{+}}\right)$.
3. We have $\left(F_{T} 0_{2}^{+}\right)^{* \dagger} \geq F_{T} 0_{2}^{+}$, and so if $F_{T} 0_{2}^{+}$is convex and $T$ maximal, then $F_{T} 0_{2}^{+} \in$ $b R(\operatorname{rec} T)$. Furthermore, if $F_{T} 0_{2}^{+}=\overline{b d-e-\lim _{t \rightarrow+0} F_{t T}}{ }^{s \times b d w^{*}}$ (i.e. exists as an epi-limit) then also $F_{T} 0_{2}^{+}$is convex with $F_{T} 0_{2}^{+} \geq F_{\mathrm{rec} T}$ and so

$$
\left(F_{\mathrm{rec} T}\right)^{* \dagger}=P_{\mathrm{rec} T} \geq\left(F_{T} 0_{2}^{+}\right)^{* \dagger} \quad \text { on } \quad X \times X^{*} .
$$

Proof Part 1: As $(0,0) \in T$ it is immediate that for all $\left(x, x^{*}\right)$ we have $F_{T}\left(x, x^{*}\right) \geq\langle x, 0\rangle+$ $\left\langle x^{*}, 0\right\rangle-\langle 0,0\rangle=0$, so clearly $F_{T} 0_{2}^{+} \geq 0$. Next note that

$$
t F_{T}\left(x, \frac{x^{*}}{t}\right)=\left\langle x, x^{*}\right\rangle \quad \text { iff } \quad F_{T}\left(x, \frac{x^{*}}{t}\right)=\left\langle x, \frac{x^{*}}{t}\right\rangle \quad \text { iff } \quad x^{*} \in t T(x) .
$$

Indeed

$$
\begin{aligned}
F_{t T}\left(x, x^{*}\right) & =\sup _{\left(y, t y^{*}\right) \in t T}\left(\left\langle x, t y^{*}\right\rangle+\left\langle y, x^{*}\right\rangle-t\left\langle y, y^{*}\right\rangle\right) \\
& =t \sup _{\left(y, y^{*}\right) \in T}\left(\left\langle x, y^{*}\right\rangle+\left\langle y, \frac{x^{*}}{t}\right\rangle-\left\langle y, y^{*}\right\rangle\right)=t F_{T}\left(x, \frac{x^{*}}{t}\right)=g_{t}\left(x, x^{*}\right) .
\end{aligned}
$$

Hence $g_{t}^{* \dagger}=F_{t T}^{* \dagger}=P_{t T}$. Furthermore, we have (as $t>0$ ) and $P_{T}:=F_{T}^{* \dagger}$ that

$$
\begin{align*}
g_{t}^{* \dagger}\left(y, y^{*}\right) & =\sup _{\left(x, x^{*}\right)}\left\{\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle-t F_{T}\left(x, \frac{x^{*}}{t}\right)\right\} \\
& =t \sup _{\left(x, x^{*}\right)}\left\{\left\langle\left(x, \frac{x^{*}}{t}\right),\left(y, \frac{y^{*}}{t}\right)\right\rangle-F_{T}\left(x, \frac{x^{*}}{t}\right)\right\} \\
& =t \sup _{\left(x, x^{*}\right)}\left\{\left\langle\left(x, x^{*}\right),\left(y, \frac{y^{*}}{t}\right)\right\rangle-F_{T}\left(x, x^{*}\right)\right\} \\
& =t P_{T}\left(y, \frac{y^{*}}{t}\right) \geq t F_{T}\left(y, \frac{y^{*}}{t}\right)=g_{t}\left(y, y^{*}\right) . \tag{36}
\end{align*}
$$

Part 2: Clearly we always have $\operatorname{Pr}_{X}$ dom $F_{T} 0_{2}^{+} \subseteq \overline{\operatorname{Pr}_{X} \operatorname{dom} F_{T}}$ while $\overline{\text { co }} \operatorname{dom} T=$ $\overline{\operatorname{Pr}_{X} \operatorname{dom} F_{T}}$, when $T$ is maximal [6, Theorem 3.6] which gives (35). Now assume $T$ is maximal monotone. Take $\left(x, x^{*}\right) \in \operatorname{rec} T$ so there exist nets $t_{\alpha} \rightarrow 0$ and $x_{\alpha}^{*} \in t_{\alpha} T\left(x_{\alpha}\right)$, with $\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right)\right\}$ bounded and $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times \mathrm{w}^{*}}\left(x, x^{*}\right)$. That $x_{\alpha}^{*} \in t_{\alpha} T\left(x_{\alpha}\right)$, implies (for all $\alpha$ ) that $F_{T}\left(x_{\alpha}, \frac{x_{\alpha}^{*}}{t_{\alpha}}\right)=\left\langle x_{\alpha}, \frac{x_{\alpha}^{*}}{t_{\alpha}}\right\rangle$ or $t_{\alpha} F_{T}\left(x_{\alpha}, \frac{x_{\alpha}^{*}}{t_{\alpha}}\right)=\left\langle x_{\alpha}, x_{\alpha}^{*}\right\rangle$. Thus

$$
\begin{equation*}
\liminf _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, \frac{x_{\alpha}^{*}}{t_{\alpha}}\right)=\lim _{\alpha}\left\langle x_{\alpha}, x_{\alpha}^{*}\right\rangle=\left\langle x, x^{*}\right\rangle . \tag{37}
\end{equation*}
$$

Furthermore for arbitrary $\left(x, x^{*}\right) \in X \times X^{*}$, and any net $\left(w_{\alpha}, w_{\alpha}^{*}\right) \rightarrow\left(x, x^{*}\right)$ with $\left\{w_{\alpha}^{*}\right\}$ bounded and $t_{\alpha} \rightarrow_{+} 0$, we have

$$
\underset{\alpha}{\liminf t_{\alpha}} F_{T}\left(w_{\alpha}, \frac{w_{\alpha}^{*}}{t_{\alpha}}\right) \geq \liminf _{\alpha}\left\langle t_{\alpha}, \frac{w_{\alpha}^{*}}{t_{\alpha}}\right\rangle=\lim _{\alpha}\left\langle w_{\alpha}, w_{\alpha}^{*}\right\rangle=\left\langle x, x^{*}\right\rangle
$$

Hence

$$
\begin{align*}
& b d \mathrm{w}^{*}-\limsup _{t \rightarrow+0}\left(\mathrm{epi} g_{t}\right) \subseteq \operatorname{epi}\langle\cdot, \cdot \cdot\rangle \quad \text { or } \quad b d-e-\liminf _{t \rightarrow+0} g_{t} \geq\langle\cdot, \cdot\rangle, \quad \text { giving }  \tag{38}\\
& F_{T} 0_{2}^{+} \geq{\overline{b d-e-\liminf _{t \rightarrow+0}} g_{t}}_{s \times b d \mathrm{w}^{*}}^{\geq\langle\cdot, \cdot\rangle, \quad \text { on } X \times X^{*},}
\end{align*}
$$

the last inequality following from the ( $s \times b d \mathrm{w}^{*}$ )-continuity of the duality product. Moreover by (37) and Lemma 12 we have

$$
\begin{aligned}
F_{T} 0_{2}^{+}\left(x . x^{*}\right) & ={\overline{b d-e-\liminf _{t \rightarrow+0} g_{t}}}^{s \times b d w^{*}}\left(x, x^{*}\right) \leq \liminf _{\alpha} F_{t_{\alpha} T}\left(x_{\alpha}, x_{\alpha}^{*}\right) \\
& =\liminf _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, \frac{x_{\alpha}^{*}}{t_{\alpha}}\right)=\left\langle x, x^{*}\right\rangle,
\end{aligned}
$$

for any $\left(x, x^{*}\right) \in \operatorname{rec} T$. Thus $F_{T} 0_{2}^{+}=\langle\cdot, \cdot\rangle$ on rec $T$.
Part 3: We now show that $h:=F_{T} 0_{2}^{+}$satisfies $h \leq h^{* \dagger}$. In order to apply Proposition 25 to $g_{t}=F_{t T}$ we need to supply a bounded family $\left\{\left(x_{\tau}, x_{\tau}^{*}\right)\right\}_{\tau>0}$ with $\left\{g_{1 / \tau}^{* \dagger}\left(x_{\tau}, x_{\tau}^{*}\right)\right\}$ bounded (then given a cofinal subnet $I_{1}$ we have same holding on $I_{2}=I_{1}$ ). To this end, note that as $(0,0) \in T$ then $P_{T}(0,0)=\langle 0,0\rangle=0$ where $P_{T}$ is strongly-closed convex on $X \times X^{*}$. Let $\left(x_{t}, x_{t}^{*}\right) \in B_{t}(0,0) \cap \operatorname{dom} P_{T}$ be such that $\lim _{t \rightarrow 0} P_{T}\left(x_{t}, x_{t}^{*}\right)=$ $\liminf _{\left(y, y^{*}\right) \rightarrow(0,0)} P_{T}\left(y, y^{*}\right)=0$. Then for $t \in(0,1]$, we have $\left\{\left(x_{t}, t x_{t}^{*}\right)\right\}_{t \in(0,1]}$ a bounded net, for which $g_{t}^{* \dagger}\left(x_{t}, t x_{t}^{*}\right)=t P_{T}\left(x_{t}, \frac{t x_{t}^{*}}{t}\right)=t P_{T}\left(x_{t}, x_{t}^{*}\right) \rightarrow_{t \rightarrow 0} 0$. Using the convention $\tau=\frac{1}{t}$ we have provided the desired bounded family $\left\{\left(x_{\tau}, x_{\tau}^{*}\right)\right\}_{\tau>0}$ with $\left\{g_{1 / \tau}^{* \dagger}\left(x_{\tau}, x_{\tau}^{*}\right)\right\}$ bounded. We note that the family $\left\{g_{t}^{* \dagger}:=P_{t T}\right\}_{t>0}$ consists of positive functions. Thus, Proposition 25 yields

$$
\begin{equation*}
\overline{b d-e-\limsup _{\tau \rightarrow+\infty} g_{1 / \tau}} s \times b d \mathrm{w}^{*}=\left(b d-e-\liminf _{\tau \rightarrow+\infty} g_{1 / \tau}^{*}\right)^{*} \tag{39}
\end{equation*}
$$

and by (36) we have $g_{1 / \tau} \leq g_{1 / \tau}^{* \dagger}$. Using these facts we obtain

$$
\left.\begin{array}{rl}
h & =F_{T} 0_{2}^{+}={\overline{b d-e-} \liminf _{t \rightarrow+0} F_{t T}}_{s \times b d \mathrm{w}^{*}}^{\leq{\overline{b d-e-} \limsup _{\tau \rightarrow+\infty} g_{1 / \tau}}^{5 \times b d \mathrm{w}^{*}}=\left(b d-e-\liminf _{\tau \rightarrow+\infty} g_{1 / \tau}^{* \dagger}\right)^{* \dagger}} \\
& \leq\left(b d-e-\liminf _{\tau \rightarrow+\infty} g_{1 / \tau}\right)^{* \dagger}=\left(\overline{b d-e-\liminf _{t \rightarrow+0} g_{t}} s \times b d w^{*}\right.
\end{array}\right)^{* \dagger}=h^{* \dagger} .
$$

Thus if $F_{T} 0_{2}^{+}$is convex, then it is bigger-conjugate convex. In particular this holds when $F_{T} 0_{2}^{+}$exists as an epi-limit. Assume now this epi-limit condition on $F_{T} 0_{2}^{+}$. Consider the Fitzpatrick function for rec $T$ i.e.

$$
\begin{aligned}
& F_{\mathrm{rec} T}\left(x, x^{*}\right)=\left(\langle\cdot, \cdot\rangle+\delta_{\mathrm{rec} T}\right)^{*}\left(x, x^{*}\right) \\
&=\left(\langle\cdot, \cdot\rangle+\delta_{b d \mathrm{w}^{*}-\lim \sup _{t \rightarrow+0} t T}\right)^{*}\left(x, x^{*}\right) \\
&=\left(b d-e-\liminf _{t \rightarrow+0}\left(\langle\cdot, \cdot\rangle+\delta_{t T}\right)\right)^{*}\left(x, x^{*}\right) \\
& \leq\left(b d-e-\liminf _{t \rightarrow+0} \overline{c o s^{s \times w^{*}}}\left(\langle\cdot, \cdot\rangle+\delta_{t T}\right)\right)^{*}\left(x, x^{*}\right) \\
&=\overline{b d-e-\limsup _{t \rightarrow+0}\left(\langle\cdot, \cdot\rangle+\delta_{t T}\right)^{*}} s \times b d \mathrm{w}^{*} \\
&\left(x, x^{*}\right)=\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right),
\end{aligned}
$$

where the third equality follows easily from definitions, the inequality from the order reversal of conjugation and the following equality from an application of Proposition 25. For Proposition 25 to be applicable we observed that from the above $g_{t}^{* \dagger}=P_{t T}=$ $\overline{\mathrm{co}^{s \times \mathrm{w}^{*}}}\left(\langle\cdot, \cdot\rangle+\delta_{t T}\right) \geq 0$ has already been shown to satisfy the necessary boundedness preconditions for the application of Proposition 25.

Finally we note that we have already shown that when $T$ is maximal, $\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=$ $F_{\text {rec } T}\left(x, x^{*}\right)=\left\langle x, x^{*}\right\rangle$ for $\left(x, x^{*}\right) \in \operatorname{rec} T$ always i.e. $M_{F_{T} 0_{2}^{+}} \supseteq \operatorname{rec} T$.

We recall the following results from [28] and [11, Corollary 23].
Remark 36 If $C \subseteq X \times X^{*}$ is closed convex, then the usual concept of recession corresponds to the "asymptotic cone" $0^{+} C$ defined by: $\left(y, y^{*}\right) \in 0^{+} C$ iff for some $c \in C$ (and hence for any $c \in C) c+\mathbb{R}_{+}\left(y, y^{*}\right) \subseteq C$ or $C+t\left(y, y^{*}\right) \subseteq C$ for all $t>0$. When $C$ is not closed, $0^{+} C$ may itself not be closed. When $C$ is not closed, $0^{+} C$ can be defined by a limiting process. In [28] we have $0^{+} C=\lim \sup _{\lambda \downarrow 0} \lambda C:=\bigcap_{\varepsilon>0}\left[\overline{\bigcup_{0<\lambda<\varepsilon} \lambda C}\right]$ (where the limit supremum is taken with respect to an appropriate topology associated with the duality pairing).

Lemma 37 ([11, Lemma 22]) Let $f \in \Gamma\left(X \times X^{*}\right)$. The polar of the convex cone generated by $\operatorname{dom} f^{*}$ i.e.

$$
\left(\text { cone } \operatorname{dom} f^{*}\right)^{\circ}:=\left\{\left(x, x^{*}\right) \mid \delta_{\operatorname{dom} f^{*}}^{*}\left(x^{*}, x\right) \leq 0\right\}
$$

is the same as the asymptotic cone $0^{+}[f \leq \alpha]$ for any $\alpha>\inf f$.
The transpose operator can take a polar set (which resides in the dual $X^{*} \times X$ ) back into the primal space $X \times X^{*}$. This permits statements which otherwise would not be possible in general Banach space theory.

Corollary 38 ([11, Corollary 23]) Suppose $C \subseteq X \times X^{*}$ is a $s \times \mathrm{w}^{*}$-closed convex set. Then

$$
\left(\operatorname{dom} \delta_{C}^{*}\right)^{\circ}=\left\{\left(w, w^{*}\right) \mid \delta_{\mathrm{dom} \delta_{C}^{*}}^{*}\left(w^{*}, w\right) \leq 0\right\}=0^{+} C .
$$

Thus for $h \in b R(T)$ we have

$$
\begin{equation*}
\overline{\operatorname{dom}}^{s \times \mathrm{w}^{*}} \delta_{\overline{\mathrm{co}} M_{h}}^{*}=\left(0^{+}{\left.\overline{\mathrm{co}} M_{h}\right)^{\circ} \quad \text { and } \quad \overline{\operatorname{dom}}^{s \times \mathrm{w}^{*}} \delta_{\mathrm{dom} F_{M_{h}}}^{*}=\left(0^{+}{\overline{\operatorname{dom} F_{M_{h}}}}^{s \times \mathrm{w}^{*}}\right)^{\circ} . . . . . .}\right. \tag{40}
\end{equation*}
$$

Recall that the asymptotic function $f 0^{+}$of a proper, $\left(s \times \mathrm{w}^{*}\right)$-closed, convex function $f$ defined on $X \times X^{*}$ is given by

$$
\begin{aligned}
\left(f 0^{+}\right)\left(x, x^{*}\right) & :=\lim _{\tau \rightarrow \infty} \frac{1}{\tau}\left[f\left(\left(y, y^{*}\right)+\tau\left(x, x^{*}\right)\right)-f\left(y, y^{*}\right)\right] \\
& =\sup _{\tau>0} \frac{1}{\tau}\left(f\left(\left(y, y^{*}\right)+\tau\left(x, x^{*}\right)\right)-f\left(y, y^{*}\right)\right),
\end{aligned}
$$

for any (or all) $\left(y, y^{*}\right) \in \operatorname{dom} f$, noting that the above expression is independent of the choice of $\left(y, y^{*}\right) \in \operatorname{dom} f$.

Remark 39 Recall from [28, Theorem 3B] that $\left(f 0^{+}\right)\left(x, x^{*}\right) \leq \mu$ equivalent to the following:

1. $f\left(\left(y, y^{*}\right)+\tau\left(x, x^{*}\right)\right) \leq f\left(y, y^{*}\right)+\tau \mu$ for some (and equivalently for all) $\left(y, y^{*}\right) \in$ $\operatorname{dom} f$ and $\tau \geq 0$.
2. $f\left(\left(y, y^{*}\right)+\left(x, x^{*}\right)\right)-f\left(y, y^{*}\right) \leq \mu$ for all $\left(y, y^{*}\right) \in \operatorname{dom} f$.
3. There exists a net $\left(x_{i}, x_{i}^{*}\right)$ and $t_{i}>0$ such that $\lim _{i} t_{i}=0$ and $\lim _{i}\left(x_{i}, x_{i}^{*}\right)=\left(x, x^{*}\right)$ with

$$
\lim _{i} t_{i} f\left(\frac{1}{t_{i}}\left(x_{i}, x_{i}^{*}\right)\right) \leq \mu .
$$

4. $\left\langle\left(x, x^{*}\right),\left(w, w^{*}\right)\right\rangle \leq \mu$ for all $\left(w, w^{*}\right) \in \operatorname{dom} f^{* \dagger}$.

Remark 40 We note that $0^{+}{\overline{\operatorname{dom} F_{M_{h}}}}^{s \times \mathrm{w}^{*}}$ is closed while $0^{+} \operatorname{dom} F_{M_{h}}$ may not be.
Proposition 41 ([11, Proposition 26]) Suppose $h \in b R(T)$. Then

$$
\begin{equation*}
F_{M_{h}} 0^{+}=\delta_{\overline{c o}^{s} \times w^{*}}^{* \dagger} M_{h} \tag{41}
\end{equation*}
$$

Furthermore, for any $\alpha>\inf \mathcal{F}_{M_{h}}$,

$$
\begin{equation*}
0^{+}\left[F_{M_{h}} \leq \alpha\right] \subseteq \operatorname{dom} \delta_{{\overline{\cos ^{s} \times \mathrm{w}^{*}}{ }_{M_{h}}}_{* \dagger} \subseteq 0^{+} \operatorname{dom} F_{M_{h}} . . . . ~}^{\text {. }} \tag{42}
\end{equation*}
$$

We have seen that the problem of constructing a member of $b R(\operatorname{rec} T)$ for any monotone operator $T$, has been reduced to the problem of showing the existence of a particular epilimit. We will now consider the recession function associated with the Fitzpatrick function.

Proposition 42 Let $X$ be an arbitrary real Banach space. Suppose $T: X \rightrightarrows X^{*}$ is monotone and $(0,0) \in T$. Denote its Fitzpatrick function by $F_{T}$. For $\left(x, x^{*}\right),\left(y, y^{*}\right) \in X \times X^{*}$, define:

$$
f_{\tau}\left(\left(y, y^{*}\right),\left(x, x^{*}\right)\right):=\left\{\begin{array}{cl}
\frac{1}{\tau}\left[F_{T}\left(\left(y, y^{*}\right)+\tau\left(x, x^{*}\right)\right)-F_{T}\left(y, y^{*}\right)\right] & \text { if }\left(y, y^{*}\right) \in \operatorname{dom} F_{T}  \tag{43}\\
+\infty & \text { otherwise }
\end{array}\right.
$$

Then $\tau \mapsto f_{\tau}$ is monotonically nondecreasing as $\tau \rightarrow \infty$, and convex in $\left(x, x^{*}\right)$. Also the pointwise limit of the family $\left.\tau \mapsto f_{\tau}\left(\left(y, y^{*}\right),(\cdot, \cdot)\right)\right)$ is independent of the choice of $\left(y, y^{*}\right) \in$ $\operatorname{dom} F_{T}$, and exists as a convex function coinciding with $F_{T} 0^{+}$on $X \times X^{*}$, with $\operatorname{dom} F_{T} 0^{+} \subseteq$ $\operatorname{dom} F_{T}$. Moreover for any $\left(x, x^{*}\right) \in \operatorname{dom} F_{T} 0_{2}^{+} \subseteq{\overline{\operatorname{dom} F_{T}}} \times b d \mathrm{w}^{*}$ we have

$$
\begin{aligned}
& \left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=\overline{b d-e-\lim _{\tau \rightarrow+\infty} f_{\tau}\left(\left(x, x^{*}\right), \cdot\right)} \\
& \text { sith also, } \quad\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)={\overline{\left(w_{T} 0^{+}(0, \cdot)\right)}}^{b d \mathrm{w}^{*}}\left(x^{*}\right) \quad \text { with the epi-limit existing, } \\
& \quad \text { with } \\
& \quad \text { for all }\left(x, x^{*}\right) \in X \times X^{*} .
\end{aligned}
$$

Furthermore, when $T$ is maximal

$$
\begin{equation*}
\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)={\overline{\left(F_{T} 0^{+}(0, \cdot)\right)}}^{b d \mathrm{w}^{*}}\left(x^{*}\right)+\delta_{\overline{\mathrm{co}} \mathrm{dom} T}(x) \quad \text { for all }\left(x, x^{*}\right) \in X \times X^{*} . \tag{44}
\end{equation*}
$$

(In particular, when $T$ maximal, then $F_{T} 0_{2}^{+}$is convex.)
Proof Due to the convexity of $F_{T}$, [29, Theorems 23.1 or 8.5$]$ we have $\tau \mapsto f_{\tau}$ (pointwise) monotonically nondecreasing (on $\operatorname{dom} F_{T}$ ) as $\tau \rightarrow \infty$, and infinite when $\left(y, y^{*}\right) \notin$
$\operatorname{dom} F_{T}$. So, $\tau \mapsto f_{\tau}$ is pointwise monotonically nondecreasing as $\tau \rightarrow \infty$. Moreover, $\sup _{\tau>0} f_{\tau}\left(\left(y, y^{*}\right),\left(x, x^{*}\right)\right) \leq \mu$ is equivalent to

$$
\begin{gathered}
\left(\left(y, y^{*}\right), F_{T}\left(y, y^{*}\right)\right)+\tau\left(\left(x, x^{*}\right), \mu\right) \in \text { epi } F_{T} \quad \text { for all } \tau>0 \\
\text { or } \quad\left(\left(x, x^{*}\right), \mu\right) \in 0^{+} \text {epi } F_{T},
\end{gathered}
$$

using the fact that epi $F_{T}$ is $\left(s \times \mathrm{w}^{*}\right)$-closed. This means the bound of $\mu$ holds for all choices of $\left(y, y^{*}\right) \in \operatorname{dom} F_{T}$ (including $(0,0) \in \operatorname{dom} F_{T}$, and $\left(y, y^{*}\right)=\left(x, x^{*}\right)$, when $\left.\left(x, x^{*}\right) \in \operatorname{dom} F_{T}\right)$. Note that there exists $\mu<+\infty$ with $\left(\left(x, x^{*}\right), \mu\right) \in 0^{+}$epi $F_{T}$ iff $\left(F_{T} 0^{+}\right)\left(x, x^{*}\right)<+\infty$ iff $\left(x, x^{*}\right) \in \operatorname{dom} F_{T} 0^{+}$. As $(0,0) \in \operatorname{dom} F_{T}$ and $F_{T}(0,0)=0$ we have $\left(\left(x, x^{*}\right), \mu\right) \in 0^{+}$epi $F_{T}$ with $\mu<+\infty$ iff $\tau\left(\left(x, x^{*}\right), \mu\right) \in$ epi $F_{T}$ for all $\tau>0$ and hence $\left(x, x^{*}\right) \in \operatorname{dom} F_{T}$ follows (take $\tau=1$ ). Thus

$$
\begin{equation*}
\operatorname{dom} F_{T} 0^{+} \subseteq \operatorname{dom} F_{T} . \tag{45}
\end{equation*}
$$

Moreover it then follows that $\left(\left(x, x^{*}\right), \mu\right) \in 0^{+}$epi $F_{T}$ iff $\left(\left(x, x^{*}\right), F_{T}\left(x, x^{*}\right)\right)+\tau\left(\left(x, x^{*}\right)\right.$, $\mu) \in \operatorname{epi} F_{T}$ for all $\tau>0$ iff $f_{\tau}\left(\left(x, x^{*}\right),\left(x, x^{*}\right)\right) \leq \mu$ for all $\tau>0$ iff $\left(F_{T} 0^{+}\right)\left(x, x^{*}\right)=$ $\sup _{\tau} f_{\tau}\left(\left(x, x^{*}\right),\left(x, x^{*}\right)\right) \leq \mu$.

Define $h: X \times X^{*} \rightarrow \mathbb{R}_{+\infty}$ as follows:

$$
\begin{equation*}
h\left(x, x^{*}\right):=\sup _{\tau} f_{\tau}\left(\left(x, y^{*}\right),\left(0, x^{*}\right)\right)=\left(F_{T} 0^{+}\right)\left(0, x^{*}\right), \tag{46}
\end{equation*}
$$

for any $y^{*}$ such that $\left(x, y^{*}\right) \in \operatorname{dom} F_{T}$, if such $y^{*}$ exists; with $h\left(x, x^{*}\right):=+\infty$ otherwise. Such a $y^{*}$ exists iff $x \in \operatorname{Pr}_{X}$ dom $F_{T}$. Utilising these observations we note $h\left(x, x^{*}\right)=$ $\sup _{\tau} f_{\tau}\left(\left(x, x^{*}\right),\left(0, x^{*}\right)\right)$ is clearly finite if $\left(x, x^{*}\right) \in \operatorname{dom} F_{T} 0^{+} \subseteq \operatorname{dom} F_{T}$. In summary:

$$
h\left(x, x^{*}\right)=\left\{\begin{array}{cc}
\left(F_{T} 0^{+}\right)\left(0, x^{*}\right) & \text { if } x \in \operatorname{Pr}_{X} \operatorname{dom} F_{T}  \tag{47}\\
+\infty & \text { otherwise }
\end{array}\right.
$$

Note that (47) implies $h \geq 0$, and $h(x, 0)=0$ for $x \in \operatorname{Pr}_{X}$ dom $F_{T}$, together with

$$
\begin{align*}
\operatorname{dom}\left(F_{T} 0^{+}(0, \cdot)\right) & \cap\left(\operatorname{Pr}_{X} \operatorname{dom} F_{T} \times X^{*}\right)  \tag{48}\\
& \subseteq \operatorname{dom} h \cap\left(\operatorname{Pr}_{X} \operatorname{dom} F_{T} \times X^{*}\right)=\operatorname{dom} h
\end{align*}
$$

By definitions, on $X \times X^{*}$,

$$
h\left(x, x^{*}\right) \geq\left(F_{T} 0^{+}\right)\left(0, x^{*}\right) .
$$

Thus $\operatorname{dom} h \subseteq \operatorname{dom}\left(F_{T} 0^{+}(0, \cdot)\right)$ and utilising (48)

$$
\begin{equation*}
\operatorname{dom}\left(F_{T} 0^{+}(0, \cdot)\right) \cap\left(\operatorname{Pr}_{X} \operatorname{dom} F_{T} \times X^{*}\right)=\operatorname{dom} h \tag{49}
\end{equation*}
$$

Denote by $\bar{h}$ the function with the epigraph $\overline{\operatorname{epi} h}^{s \times b d w^{*}}$ and similarly for ${\overline{\left(F_{T} 0^{+}(0, \cdot)\right)}}^{s \times b d w^{*}}=$ $\overline{\left(F_{T} 0^{+}(0, \cdot)\right)}{ }^{b d \mathrm{w}^{*}}$. Using the identity (47) and inclusion (49), on taking lower closures we have

$$
\begin{align*}
\bar{h}\left(x, x^{*}\right) & ={\overline{\left(F_{T} 0^{+}(0, \cdot)\right)+\delta_{\mathrm{Pr}_{X} \operatorname{dom} F_{T} \times X^{*}}(\cdot)}}^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right) \\
& ={\overline{\left(F_{T} 0^{+}(0, \cdot)\right)}}^{b \mathrm{w}^{*}}\left(x^{*}\right)+\delta_{\overline{\mathrm{Pr}_{X} \operatorname{dom} F_{T}}}(x) . \tag{50}
\end{align*}
$$

Now we address the relationship of these quantities to $F_{T} 0_{2}^{+}$. By Proposition 18, part 3 , and the $\left(s \times b d \mathrm{w}^{*}\right)$-closedness of $\left.f_{\tau}\left(\left(y, y^{*}\right),(\cdot, \cdot)\right)\right)$ (for any fixed $\left.\left(y, y^{*}\right) \in \operatorname{dom} F_{T}\right)$ we have $b d-e-\lim _{\tau \rightarrow+\infty} f_{\tau}\left(\left(y, y^{*}\right), \cdot\right)=F_{T} 0^{+}$on $X \times X^{*}$ existing as an epi-limit, and is also a $\left(s \times b d \mathrm{w}^{*}\right)$-closed convex function. Consequently the epi-limit is independent of the choice of $\left(y, y^{*}\right) \in \operatorname{dom} F_{T}$.

Taking $\left(y, y^{*}\right)=(0,0) \in \operatorname{dom} F_{T}$ then $f_{\tau}((0,0), \cdot)=\frac{1}{\tau} F_{T}(\tau \cdot)$ and for all $\left(x, x^{*}\right) \in X \times$ $X^{*}$

$$
\begin{aligned}
\left(F_{T} 0^{+}\right)\left(x, x^{*}\right) & =\sup _{\tau} f_{\tau}\left((0,0),\left(x, x^{*}\right)\right) \\
& =\left(b d-e-\lim _{t \rightarrow+0} t F_{T}\left(\frac{(\cdot, \cdot)}{t}\right)\right)\left(x, x^{*}\right)
\end{aligned}
$$

Let $\left(x, x^{*}\right) \in X \times X^{*}$. For any $\epsilon>0$, suppose the net $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow_{s \times b d w^{*}}\left(x, x^{*}\right)$ is chosen so that $\lim _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, \frac{x_{\alpha}^{*}}{t_{\alpha}}\right) \leq\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right)+\epsilon$ whenever the latter is finite. Then $\left(t_{\alpha} x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow_{t_{\alpha} \rightarrow 0}\left(0, x^{*}\right)$ so that

$$
\begin{aligned}
\left(F_{T} 0^{+}\right)\left(0, x^{*}\right) & =\sup _{\tau>0} f_{\tau}\left((0,0),\left(0, x^{*}\right)\right) \leq \liminf _{\alpha} t_{\alpha} F_{T}\left(\frac{1}{t_{\alpha}}\left(t_{\alpha} x_{\alpha}, x_{\alpha}^{*}\right)\right) \\
& =\liminf _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, \frac{x_{\alpha}^{*}}{t_{\alpha}}\right) \leq\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right)+\epsilon
\end{aligned}
$$

As $\epsilon>0$ is arbitrary we conclude, for all $\left(x, x^{*}\right) \in X \times X^{*}$,

$$
\begin{equation*}
\left(F_{T} 0^{+}\right)\left(0, x^{*}\right)=\sup _{\tau>0} f_{\tau}\left((0,0),\left(0, x^{*}\right)\right) \leq\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right) \tag{51}
\end{equation*}
$$

and so

$$
\begin{equation*}
{\overline{\left(F_{T} 0^{+}(0, \cdot)\right)}}^{b d \mathrm{w}^{*}}\left(x^{*}\right) \leq{\overline{b d-e-\liminf _{t \rightarrow+0} F_{t T}}}^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right)=\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right) . \tag{52}
\end{equation*}
$$

On the other hand, $F_{T}(0,0)=0$ so

$$
\begin{aligned}
\left(F_{T} 0^{+}\right)\left(0, x^{*}\right) & =\sup _{\tau>0} \frac{1}{\tau}\left[F_{T}\left((0,0)+\tau\left(0, x^{*}\right)\right)-F_{T}(0,0)\right]=\sup _{\tau>0} f_{\tau}\left((0,0),\left(0, x^{*}\right)\right) \\
& =\sup _{\tau>0} \frac{1}{\tau} F_{T}\left(\tau\left(0, x^{*}\right)\right)=\lim _{t \rightarrow+0} F_{t T}\left(0, x^{*}\right) \\
& \geq \inf _{\left\{\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow s \times b d w^{*}\left(0, x^{*}\right)\right\}} \liminf _{\alpha} F_{\left.t_{\alpha} \rightarrow+\infty\right\}}\left(\left(x_{\alpha}, x_{\alpha}^{*}\right)\right) \\
& =\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(0, x^{*}\right) \geq\left(F_{T} 0_{2}^{+}\right)\left(0, x^{*}\right) .
\end{aligned}
$$

Combining this with (51) we have

$$
\begin{equation*}
\left(F_{T} 0^{+}\right)\left(0, x^{*}\right)=\sup _{\tau>0} f_{\tau}\left((0,0),\left(0, x^{*}\right)\right)=\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(0, x^{*}\right) \tag{53}
\end{equation*}
$$

and, combining instead with (52), gives $\overline{\left(F_{T} 0^{+}(0, \cdot)\right)}$ bdw* $\left(x^{*}\right)=\left(F_{T} 0_{2}^{+}\right)\left(0, x^{*}\right)$, for all $x^{*} \in$ $X^{*}$.

Recall that $\bar{h}$ denotes the closure $\bar{h}^{s \times b d \mathrm{w}^{*}}$. Then for each $\left(x, x^{*}\right) \in X \times X^{*}$ there exists a bounded net $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times b d w^{*}}\left(x, x^{*}\right)$ such that

$$
\bar{h}\left(x, x^{*}\right)=\lim _{\alpha} h\left(x_{\alpha}, x_{\alpha}^{*}\right) .
$$

Suppose $\bar{h}\left(x, x^{*}\right)<+\infty$ (noting that we have $\bar{h} \geq 0$ ). Let $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right)$ be any fixed bounded net with $\lim _{\alpha} h\left(x_{\alpha}, x_{\alpha}^{*}\right)=\bar{h}\left(x, x^{*}\right)$. Then, for $\left(x_{\alpha}, y_{\alpha}^{*}\right) \in \operatorname{dom} F_{T}$, as $-\infty<$ $\widehat{f_{\tau}}\left(x_{\alpha}, x_{\alpha}^{*}\right):=f_{\tau}\left(\left(x_{\alpha}, y_{\alpha}^{*}\right),\left(0, x_{\alpha}^{*}\right)\right) \leq h\left(x_{\alpha}, x_{\alpha}^{*}\right)$, we have for any $\varepsilon>0$, and $\tau_{\alpha} \rightarrow+\infty$ that eventually,

$$
\widehat{\hat{f}_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)-\varepsilon \leq \bar{h}\left(x, x^{*}\right) .
$$

Due to (43), with $\tau=\frac{1}{t}$, in the expression for $\widehat{f_{\tau}}\left(w, w^{*}\right):=f_{\tau}\left(\left(w, y^{*}\right),\left(0, w^{*}\right)\right) \quad(w \in$ $\operatorname{Pr}_{X} \operatorname{dom} F_{T}$ ) we have, recalling (for any $y^{*}$ for which $\left(w, y^{*}\right) \in \operatorname{dom} F_{T}$ ),

$$
\begin{align*}
& g_{t}\left(w, w^{*}\right)=\frac{1}{\tau} F_{T}\left(w, y^{*}+\tau w^{*}\right), \quad \text { that } \\
& g_{t}\left(w, w^{*}\right)=\widehat{f_{\tau}}\left(w, w^{*}\right)+t F_{T}\left(w, y^{*}\right) . \tag{54}
\end{align*}
$$

As we have chosen a fixed bounded net $\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow^{s \times w^{*}}\left(x, x^{*}\right)$ so for any $t_{\alpha} \downarrow 0^{+}$and associated $\tau_{\alpha} \rightarrow \infty$ we will have $-\infty<\lim \sup _{\alpha}{\widehat{f_{\tau_{\alpha}}}}\left(x_{\alpha}, x_{\alpha}^{*}\right) \leq \lim \sup _{\alpha} h\left(x_{\alpha}, x_{\alpha}^{*}\right) \leq \bar{h}\left(x, x^{*}\right)<$ $+\infty$. Thus we eventually have $x_{\alpha} \in \operatorname{Pr}_{X} \operatorname{dom} F_{T}$ and for any $\left(x_{\alpha}, y_{\alpha}^{*}\right) \in \operatorname{dom} F_{T}$ with $\frac{y_{\alpha}^{*}}{\tau_{\alpha}} \rightarrow 0$ (which we achieve via our selection of $t_{\alpha}$ ) we have a bounded net $\left(x_{\alpha}, x_{\alpha}^{*}+\frac{y_{\alpha}^{*}}{\tau_{\alpha}}\right) \rightarrow_{\alpha}^{s \times b d \mathrm{w}^{*}}$ ( $x, x^{*}$ ) with

$$
\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right) \leq \liminf _{\alpha} g_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)=\liminf _{\alpha} \frac{1}{\tau_{\alpha}} F_{T}\left(x_{\alpha}, \tau_{\alpha}\left(x_{\alpha}^{*}+\frac{y_{\alpha}^{*}}{\tau_{\alpha}}\right)\right),
$$

noting that by Lemma $35, \operatorname{Pr}_{X}$ dom $F_{T} 0_{2}^{+} \subseteq \overline{\operatorname{Pr}_{X} \text { dom } F_{T}}$. Then from

$$
\begin{align*}
g_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)-\widehat{f_{\tau_{\alpha}}}\left(x_{\alpha}, x_{\alpha}^{*}\right) & =t_{\alpha} F_{T}\left(x_{\alpha}, y_{\alpha}^{*}\right) \quad \text { we get } \\
\liminf _{\alpha} g_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)-\limsup _{\alpha} \widehat{f}_{\tau_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right) & \leq \liminf _{\alpha}\left[g_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)-\widehat{f}_{\tau_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)\right] \\
& =\liminf _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, y_{\alpha}^{*}\right) \\
\text { and so } \quad\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right) & \leq \liminf _{\alpha} g_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right) \\
& \leq \limsup _{\alpha} \widehat{f}_{\tau_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right)+\liminf _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, y_{\alpha}^{*}\right) \\
& \leq \limsup _{\alpha} h\left(x_{\alpha}, x_{\alpha}^{*}\right)+\liminf _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, y_{\alpha}^{*}\right) \\
& \leq \bar{h}\left(x, x^{*}\right)+\liminf _{\alpha} t_{\alpha}\left(x_{\alpha}, y_{\alpha}^{*}\right) \tag{55}
\end{align*}
$$

where the fourth inequality follows from the fact that $\widehat{\hat{\tau}_{\alpha}} \leq h$. Note that $F_{T}\left(x_{\alpha}, y_{\alpha}^{*}\right)$ is finite for all $\alpha$. Since this inequality holds for all $t_{\alpha} \rightarrow 0$, (with $t_{\alpha} y_{\alpha}^{*}=\frac{y_{\alpha}^{*}}{\tau_{\alpha}} \rightarrow 0$ ) we select $t_{\alpha}:=$ $\min \left\{\frac{\epsilon_{\alpha}}{\left|F_{T}\left(x_{\alpha}, y_{\alpha}^{*}\right)\right|+1}, \frac{\epsilon_{\alpha}}{\left\|y_{\alpha}^{*}\right\|}\right\}$ where $\epsilon_{\alpha} \rightarrow 0$ so that $t_{\alpha} \rightarrow 0$. Then $0 \leq \liminf _{\alpha} t_{\alpha} F_{T}\left(x_{\alpha}, y_{\alpha}^{*}\right) \leq$
$\epsilon_{\alpha} \rightarrow 0$ and $\frac{y_{\alpha}^{*}}{\tau_{\alpha}}=t_{\alpha} y_{\alpha}^{*} \rightarrow 0$. Thus

$$
\begin{aligned}
\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right) & \leq\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right) \leq \liminf _{\alpha} g_{t_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right) \\
& \left.\leq \limsup _{\alpha} f_{\tau_{\alpha}}\left(x_{\alpha}, x_{\alpha}^{*}\right),\left(0, x_{\alpha}^{*}\right)\right) \leq \bar{h}\left(x, x^{*}\right) .
\end{aligned}
$$

In particular we have established

$$
\begin{equation*}
\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right) \leq \bar{h}\left(x, x^{*}\right) \leq h\left(x, x^{*}\right) \quad \text { for all }\left(x, x^{*}\right) \in X \times X^{*}, \tag{56}
\end{equation*}
$$

and so $F_{T} 0_{2}^{+} \leq \bar{h}$ on $X \times X^{*}$. For the reverse inequality, we note, for $\left(x, x^{*}\right) \in \operatorname{dom} F_{T} 0_{2}^{+}$, that from (35), have $x \in \overline{\operatorname{Pr}_{X} \text { dom } F_{T}}$ so that using (50) and (52), it follows that

$$
\begin{equation*}
\bar{h}\left(x, x^{*}\right)={\overline{\left(F_{T} 0^{+}(0, \cdot)\right)}}^{b d w^{*}}\left(x^{*}\right)+\delta_{\overline{\mathrm{Pr}_{X} \operatorname{dom} F_{T}}}(x) \leq\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right), \tag{57}
\end{equation*}
$$

forcing equality on $X \times X^{*}$ and verifying the result.
Note that when maximal, the relation (44) obtains from use of [6, Thm 3.6].

Corollary 43 Let $X$ be an arbitrary real Banach space. Suppose $T: X \rightrightarrows X^{*}$ is maximal monotone and $(0,0) \in T$. Denote its Fitzpatrick function by $F_{T}$. Then there exists a convex function h on $X \times X^{*}$ for which: (i)

$$
\begin{equation*}
h=b d-e-\liminf _{t \rightarrow+0} F_{t T} \quad \text { on } \quad \operatorname{Pr}_{X} \operatorname{dom} F_{T} \times X^{*} \tag{58}
\end{equation*}
$$

(ii) $\bar{h}^{s \times b d w^{*}}=F_{T} 0_{2}^{+}$; (iii) $\operatorname{dom} h=\operatorname{Pr}_{X} \operatorname{dom} F_{T} \times \operatorname{dom} F_{T} 0^{+}(0, \cdot)$ where $\operatorname{dom} F_{T} 0^{+}(0, \cdot)$ is a convex cone in $X^{*}$. Furthermore for any $x \in \operatorname{Pr}_{X}$ dom $F_{T}$ we have the epi-limit (58) attained in that there exists $y^{*}$ such that $\left(x, y^{*}\right) \in \operatorname{dom} F_{T}$ and

$$
h\left(x, x^{*}\right)=\lim _{t \rightarrow 0} F_{t T}\left(x, x^{*}+t y^{*}\right) .
$$

Proof We have already defined a convex function via (46) which we will show satisfies the assertions of this Corollary. First we note that from (56) it follows that $h\left(x, x^{*}\right) \geq$ $\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right)$. Then, from the representation (54) of $g_{t}$, the fact that when $x \in \operatorname{Pr}_{X} \operatorname{dom} F_{T}$ there exists $y^{*} \in X^{*}$ such that $\left(x, y^{*}\right) \in \operatorname{dom} F_{T}$ and $F_{T} \geq 0$ we have: (recalling the functions $\widehat{f}$ from the preceding proof)

$$
\begin{aligned}
\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right) & =\left(b d-e-\liminf _{t \rightarrow+0} g_{t}\right)\left(x, x^{*}\right) \\
& =b d-e-\liminf _{t \rightarrow+0}\left(\widehat{f_{t}}\left(x, x^{*}\right)+t F_{T}\left(x, y^{*}\right)\right) \\
& \geq b d-e-\lim _{\tau \rightarrow+\infty} \widehat{f_{\tau}}\left(x, x^{*}\right)=h\left(x, x^{*}\right)
\end{aligned}
$$

where we have used Proposition 18 part 3 and the pointwise monotonicity of $\tau \mapsto \widehat{f_{\tau}}(\cdot)$ to obtain that last equality. This implies, for $x \in \operatorname{Pr}_{X} \operatorname{dom} F_{T}$ that we have equality:

$$
\begin{equation*}
h\left(x, x^{*}\right)=\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right)=\sup _{\tau>0} f_{\tau}\left(\left(x, y^{*}\right),\left(0, x^{*}\right)\right)=\left(F_{T} 0^{+}\right)\left(0, x^{*}\right), \tag{59}
\end{equation*}
$$

where we have used (53) and the independence of the recession function on the choice of base-point $\left(x, y^{*}\right) \in \operatorname{dom} F_{T}$.

As $\sup _{\tau>0} f_{\tau}\left(\left(x, y^{*}\right),\left(0, x^{*}\right)\right)=\lim _{t \rightarrow 0}\left(t F_{T}\left(x, \frac{x^{*}+t y^{*}}{t}\right)-t F_{T}\left(x, y^{*}\right)\right)=\lim _{t \rightarrow 0} F_{t T}(x$, $x^{*}+t y^{*}$ ) (on placing $\tau=\frac{1}{t}$ ) it is evident that the associated epi-limit is actually attained when $x \in \operatorname{Pr}_{X}$ dom $F_{T}$ i.e. $\left(b d-e-\liminf _{t \rightarrow+0} F_{t T}\right)\left(x, x^{*}\right)=\lim _{t \rightarrow 0} F_{t T}\left(x, x^{*}+t y^{*}\right)$. From (49) and (50) it is evident that $\operatorname{dom} h=\operatorname{Pr}_{X} \operatorname{dom} F_{T} \times \operatorname{dom} F_{T} 0^{+}(0, \cdot)$. Finally we note that positive homogeneity implies $\operatorname{dom} F_{T} 0^{+}(0, \cdot)$ is a cone in $X^{*}$ and the convexity of $x^{*} \mapsto F_{T}\left(0, x^{*}\right)$ implies that this (fixed) cone is convex i.e. the domain of $h$ is rectangular.

The next result augments that of [25, Lemma 2] which deals with a parallel characterisation of the representative function of the range recession operator in a reflexive space.

Corollary 44 Let $g \in b R(T)$ and assume $(0,0) \in M_{g}$. Then we have the following.

1. When $M_{g}$ is maximal, then $\left(F_{M_{g}} 0_{2}^{+}\right)\left(x, x^{*}\right)=\delta_{\text {codom } M_{g}}^{*}\left(x^{*}\right)+\delta_{\overline{\text { codom } M_{g}}}(x)$ for all $\left(x, x^{*}\right) \in X \times X^{*}$.
2. When $\overline{\operatorname{co}}$ dom $M_{g}=\overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}}$ (which is true when $M_{g}$ is maximal, [6]) we have

$$
M_{F_{M_{g}} 0_{2}^{+}}=N_{\overline{\text { codom } M_{g}}},
$$

which is maximal.
3. When a monotone operator $T \ni(0,0)$ is maximal, then $M_{F_{T} 0_{2}^{+}}=N_{\overline{\text { codom } T}}$ is also maximal and $F_{T} 0_{2}^{+}=\delta_{\overline{\mathrm{co}} \mathrm{dom} T}^{*}+\delta_{\overline{\mathrm{co}} \mathrm{dom} T} \geq\langle\cdot, \cdot\rangle$.

Proof We use Proposition 41 and 42 to note that

$$
\left(F_{M_{g}} 0^{+}\right)(0, \cdot)=\delta_{\overline{c o} M_{g}}^{* \dagger}\left(0, x^{*}\right)=\delta_{\overline{\operatorname{cod} d o m} M_{g}}^{*}\left(x^{*}\right)
$$

which is w*-lower-semicontinuous. Moreover, codom $M_{g} \subseteq P_{X} \operatorname{dom} P_{M_{g}} \subseteq P_{X} \operatorname{dom} F_{M_{g}}$ with $\overline{\operatorname{co}} \operatorname{dom} M_{g}=\overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}}$ (when $M_{g}$ is maximal) so $\overline{\left(F_{M_{g}} 0^{+}(0, \cdot)\right)}{ }^{b d w^{*}}\left(x^{*}\right)=$ $\delta_{\text {codom } M_{g}}^{*}\left(x^{*}\right)$ and

$$
\begin{aligned}
\left(F_{M_{g}} 0_{2}^{+}\right)\left(x, x^{*}\right) & ={\overline{\left(F_{M_{g}} 0^{+}(0, \cdot)\right)}}^{b d \mathrm{w}^{*}}\left(x^{*}\right)+\delta_{\overline{\overline{\mathrm{P}}_{X} \operatorname{dom} F_{M_{g}}}}(x) \\
& =\delta_{\overline{\mathrm{co}} \operatorname{dom} M_{g}}^{*}\left(x^{*}\right)+\delta_{\overline{\mathrm{Pr}_{X} \operatorname{dom} F_{M_{g}}}}(x) \\
& =\delta_{\overline{\mathrm{co}} \operatorname{dom} M_{g}}^{*}\left(x^{*}\right)+\delta_{\overline{\mathrm{co}} \operatorname{dom} M_{g}}(x) \text { when } M_{g} \text { is maximal. }
\end{aligned}
$$

Thus whenever $\overline{\operatorname{co}}$ dom $M_{g}=\overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}}$ we have

$$
\begin{align*}
M_{F_{M_{g}} 0_{2}^{+}}^{\leq} & =\left\{\left(x, x^{*}\right) \mid\left\langle x, x^{*}\right\rangle=\delta_{\overline{\mathrm{codom}} M_{g}}^{*}\left(x^{*}\right)+\delta_{\overline{\mathrm{Pr}_{X} \operatorname{dom} F_{M_{g}}}}(x)\right\} \\
& =\left\{\left(x, x^{*}\right) \mid x \in \overline{\operatorname{co}} \operatorname{dom} M_{g} \text { and }\left\langle y-x, x^{*}\right\rangle \leq 0 \text { for all } y \in \overline{\operatorname{co}} \operatorname{dom} M_{g}\right\} \\
& =N_{\overline{\operatorname{co}} \operatorname{dom} M_{g}} . \tag{60}
\end{align*}
$$

When $T$ is maximal, so $g:=F_{T} \in b R(T)$, then $M_{g}=T$ is established, so by Part 2, $M_{F_{T} 0_{2}^{+}}=N_{\overline{\text { co dom }} T}$ which is maximal.

Recall that we call a monotone operator $T$ of type "Brøndsted-Rockafellar" (BR) [4], if, whenever $\left(x, x^{*}\right) \in X \times X^{*}$ and $\alpha, \beta>0$ satisfy

$$
\inf _{\left(y, y^{*}\right) \in T}\left\langle x-y, x^{*}-y^{*}\right\rangle \geq-\alpha \beta,
$$

then there exists $\left(w, w^{*}\right) \in T$ with $\|x-w\| \leq \alpha$ and $\left\|x^{*}-w^{*}\right\| \leq \beta$.
Remark 45 Note that when $\left(x, x^{*}\right) \in \operatorname{dom} F_{T}$ we have $+\infty>F_{T}\left(x, x^{*}\right)=\left\langle x, x^{*}\right\rangle-$ $\inf _{\left(y, y^{*}\right) \in T}\left\langle x-y, x^{*}-y^{*}\right\rangle$ so

$$
\inf _{\left(y, y^{*}\right) \in T}\left\langle x-y, x^{*}-y^{*}\right\rangle=\left\langle x, x^{*}\right\rangle-F_{T}\left(x, x^{*}\right) \geq-b>-\infty
$$

and if $T$ is of BR-type then hence for any $\alpha, \beta>0$ such that $\alpha \beta=b$ we have existence of $\left(w, w^{*}\right) \in T$ with $\|x-w\| \leq \alpha$ and $\left\|x^{*}-w^{*}\right\| \leq \beta$. Hence (using the $\alpha=\frac{1}{n}$ and $\beta=$ $b n$ ) we have existence of $w_{n} \in \operatorname{dom} T$ such that $w_{n} \rightarrow x$ and hence $x \in \overline{\operatorname{dom}} T$. That is, $\operatorname{Pr}_{X} \operatorname{dom} F_{T} \subseteq \overline{\operatorname{dom}} T\left(\subseteq \overline{\operatorname{Pr}_{X} \operatorname{dom} F_{T}}\right)$. Hence $\overline{\operatorname{dom}} T=\overline{\operatorname{Pr}_{X} \operatorname{dom} F_{T}}$ is convex.

The following obtains by a similar argument as given in [13, Proposition 13].
Lemma 46 Suppose $T: X \rightrightarrows X^{*}$ is a monotone operator of type $(B R)$ where $X$ is an arbitrary real Banach space. Then for all $x \in \overline{\operatorname{dom}} T$ ( $=\overline{\operatorname{co}} \operatorname{dom} T$ from remark 45) we have $N_{\overline{\operatorname{dom}} T}(x)=(\operatorname{rec} T)(x)=\operatorname{rec}\left(T^{\mu}\right)(x)$. Thus $\operatorname{rec} T=N_{\overline{\operatorname{dom}} T}$ and hence $\operatorname{rec} T$ is maximal.

Proof See appendix for proof.

## 7 Conditions for the Almost-Convexity Property

It is conjectured that the domain of a maximal monotone operator $T$ has the almost convex property.

Definition 47 We say that the almost convex property (ACP) holds for a monotone operator $T$ iff we have $\overline{\mathrm{dom}} T$ convex.

This property has been studied by a number of authors and good summary of the best results to date may be found in [6]. Also see [32] for some recent insights. Note that by definition $\operatorname{dom}(\operatorname{rec} T) \subseteq \overline{\operatorname{dom}} T$-indeed, in [13] equality is shown by demonstrating that $x \in \overline{\operatorname{dom}} T$ implies $(x, 0) \in \operatorname{rec} T$. Thus maximality of rec $T$ immediately delivers the almost-convexity property. We note that the maximality of $\operatorname{rec} T$ has already been shown to be true in reflexive spaces in [11], providing another proof of the almost convexity property in reflexive spaces. In non-reflexive spaces we know that all monotone operators that are subdifferentials and those with non-empty interiors in their domains also possess the almost convexity property.

Proposition 48 Let $X$ be an arbitrary real Banach space and $T: X \rightrightarrows X^{*}$ be monotone. Suppose $g \in b R(T)$ is such that $\overline{\operatorname{co}}$ dom $M_{g}=\overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}}$ (which is true when $M_{g}$ is maximal [6]). Then rec $M_{g}$ is a "pre-maximal monotone operator", in the sense that

$$
\left(\operatorname{rec} M_{g}\right)^{\mu}=M_{\bar{F}_{\text {rec } M_{g}}^{\leq}}^{\leq}=M_{F_{M_{g}} 0_{2}^{+}}=N_{\overline{\text { codom } M_{g}}} \quad \text { iff } \quad \operatorname{dom}\left(\operatorname{rec} M_{g}\right)^{\mu} \subseteq \overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}} .
$$

Thus when $T$ and $\operatorname{rec} T$ are maximal, we have $\operatorname{rec} T=N_{\overline{\text { cod }} \overline{\operatorname{dom} T}}$.

Proof First we note that by [13, Lemma 11], $\operatorname{dom}\left(\operatorname{rec} M_{g}\right)=\overline{\operatorname{dom}} M_{g}$ so we have $0 \in$ $\operatorname{rec} M_{g}(x)$ for all $x \in \overline{\operatorname{dom}} M_{g}$ and

$$
\begin{aligned}
F_{\mathrm{rec} M_{g}}\left(x, x^{*}\right) & =\sup _{\left(y, y^{*}\right) \in \operatorname{rec} M_{g}}\left\langle x, y^{*}\right\rangle+\left\langle y, x^{*}\right\rangle-\left\langle y, y^{*}\right\rangle \\
& \geq \sup _{y \in \frac{\sup M_{g}}{}\langle x, 0\rangle+\left\langle y, x^{*}\right\rangle-\langle y, 0\rangle=\delta_{\text {codom } M_{g}}^{*}\left(x^{*}\right)} .\left\{\begin{array}{l}
\text { dom }
\end{array}\right)
\end{aligned}
$$

and by Lemma 35 part $3, F_{\text {rec } M_{g}} \leq F_{M_{g}} 0_{2}^{+}$and so we have from Corollary 44 part 2 that

$$
M_{F_{M_{g}} 0_{2}^{+}}=N_{\overline{\text { codom } M_{g}}} \subseteq\left(\operatorname{rec} M_{g}\right)^{\mu}=M_{F_{\text {rec } M_{g}}^{\leq}}^{\leq} \subseteq\left\{\left(x, x^{*}\right) \mid \delta_{\text {codom } M_{g}}^{*}\left(x^{*}\right) \leq\left\langle x^{*}, x\right\rangle\right\} .
$$

So, we have dom $\left(\operatorname{rec} M_{g}\right)^{\mu} \subseteq \overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}}=\overline{\operatorname{co}} \operatorname{dom} M_{g}$ iff

$$
\left(\operatorname{rec} M_{g}\right)^{\mu}=M_{\bar{F}_{\text {rec }} M_{g}}^{\leq} \subseteq\left\{\left(x, x^{*}\right) \mid \delta_{\overline{\text { co }} \operatorname{dom} M_{g}}^{*}\left(x^{*}\right) \leq\left\langle x^{*}, x\right\rangle, x \in \overline{\operatorname{co}} \operatorname{dom} M_{g}\right\}=N_{\overline{\text { cod dom } M_{g}}} .
$$

This implies $\left(\operatorname{rec} M_{g}\right)^{\mu}$ is maximal (confirming pre-maximality). Furthermore when $\left(\operatorname{rec} M_{g}\right)^{\mu}=N_{\overline{\text { cod }}} \operatorname{dom} M_{g}$ we have dom $\left(\operatorname{rec} M_{g}\right)^{\mu}=\overline{\operatorname{co}} \operatorname{dom} M_{g}=\overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}}$.

When $T$ and $\operatorname{rec} T$ are maximal we may choose $g=F_{T}$ and get dom $\left(\operatorname{rec} M_{g}\right)^{\mu}=$ $\operatorname{dom}\left(\operatorname{rec} T^{\mu}\right)^{\mu}=\operatorname{domrec} T \subseteq \overline{\operatorname{Pr}_{X} \operatorname{dom} F_{M_{g}}}$ and so $\left(\operatorname{rec} M_{g}\right)^{\mu}=\operatorname{rec} T=N_{\overline{\text { codom }} \overline{ } \text {. }}$.

We now focus on studying when $F_{M} 0_{2}^{+}$is a bigger-conjugate representative function for $\operatorname{rec} T$ with the domain $\overline{\operatorname{dom}} T$. In the following the condition (61) has a striking similarity to that used in [30] and related works [6].

Proposition 49 Let $X$ be an arbitrary Banach space. Suppose $T: X \rightrightarrows X^{*}$ is maximal monotone with Fitzpatrick function $F_{T}$. Denote $g:=F_{T} 0_{2}^{+}$. Suppose also that

$$
\begin{equation*}
\sup _{\left(y, y^{*}\right) \in T}\left\langle x-y, y^{*}\right\rangle=+\infty \quad \text { for all } x \notin \overline{\operatorname{dom} T} . \tag{61}
\end{equation*}
$$

Then $\operatorname{rec} T \subseteq M_{g}$ and $\operatorname{dom}(\operatorname{rec} T)^{\mu \mu} \subseteq \operatorname{dom} M_{g} \subseteq \overline{\operatorname{dom}} T$.
Proof Via the maximality of $T$ the Fitzpatrick function $F_{T}$ represents $T$ in that $T=M_{F_{T}}$. We note that by Lemma 35, $g \in b R(\operatorname{rec} T)$ and $M_{g}=N_{\overline{\text { codom } T}}$ for $g=F_{T} 0_{2}^{+}$. We will make use of the following inequality: for any $\left(x, x^{*}\right) \in X \times X^{*}$ and $\left(y, y^{*}\right) \in T$ and $\tau>0$ we have from the convexity of $F_{T} \in b R(T)$ that

$$
\begin{aligned}
\frac{1}{\tau+1} F_{T}\left(x,(1+\tau) x^{*}\right)+\frac{\tau}{1+\tau} F_{T}\left(y, y^{*}\right) & \geq F_{T}\left(\frac{1}{\tau+1} x+\frac{\tau}{1+\tau} y, x^{*}+\frac{\tau}{1+\tau} y^{*}\right) \\
& \geq\left\langle\frac{1}{\tau+1} x+\frac{\tau}{1+\tau} y, x^{*}+\frac{\tau}{1+\tau} y^{*}\right\rangle
\end{aligned}
$$

and when $\left(y, y^{*}\right) \in T$ we have $F_{T}\left(y, y^{*}\right)=\left\langle y, y^{*}\right\rangle$ and there follows:

$$
\begin{aligned}
\frac{1}{\tau+1} F_{T}\left(x,(1+\tau) x^{*}\right) & \geq\left(\frac{1}{\tau+1}\right)^{2}\left\langle x+\tau y,(1+\tau) x^{*}+\tau y^{*}\right\rangle-\frac{\tau}{1+\tau}\left\langle y, y^{*}\right\rangle \\
& =\left\langle x, x^{*}\right\rangle-\frac{\tau}{(\tau+1)^{2}}\left\langle x-y, x^{*}-y^{*}\right\rangle-\left(\frac{\tau}{1+\tau}\right)^{2}\left\langle x-y, x^{*}\right\rangle
\end{aligned}
$$

$$
\begin{equation*}
=\left\langle x, x^{*}\right\rangle+\frac{\tau(\tau-1)}{(\tau+1)^{2}}\left\langle x-y, x^{*}\right\rangle+\frac{\tau}{(\tau+1)^{2}}\left\langle x-y, y^{*}\right\rangle . \tag{62}
\end{equation*}
$$

We next show that dom $M_{g} \subseteq \overline{\operatorname{dom}} T$. We assume $x \notin \overline{\operatorname{dom}} T$. Thus there exists $\delta>0$ such that $B_{\delta}(x) \cap \overline{\operatorname{dom}} T=\emptyset$. We may also assume that $x \in \overline{\operatorname{co}}$ dom $T$, otherwise by (44) we have $\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=+\infty>\left\langle x, x^{*}\right\rangle$ and $x \notin M_{g}$ is deduced immediately.

Taking the supremum over all $\left(y, y^{*}\right) \in T$ in (62) we obtain

$$
\begin{aligned}
g_{\frac{1}{\tau+1}}\left(x, x^{*}\right) & =\frac{1}{\tau+1} F_{T}\left(x,(1+\tau) x^{*}\right) \\
& \geq\left\langle x, x^{*}\right\rangle+\frac{\tau(\tau-1)}{(\tau+1)^{2}} \sup _{y \in \operatorname{dom} T}\left\langle x-y, x^{*}\right\rangle+\frac{\tau}{(\tau+1)^{2}} \sup _{\left(y, y^{*}\right) \in T}\left\langle x-y, y^{*}\right\rangle=+\infty .
\end{aligned}
$$

As this holds also for all $B_{\delta}(x) \times X^{*}$ we have $b d-e-\liminf _{t \rightarrow+0} g_{t} \geq+\infty$ on $B_{\delta}(x) \times X^{*}$ for some $\delta>0$. Hence

$$
\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=\left(\overline{b d-e-\liminf _{t \rightarrow+0} g_{t}} s \times b d \mathrm{w}^{*}\right)\left(x, x^{*}\right)=+\infty>\left\langle x, x^{*}\right\rangle \text { for all } x^{*} \in X^{*} .
$$

Hence $\left(x, x^{*}\right) \notin \operatorname{dom} M_{F_{T} 0_{2}^{+}}$and so $\operatorname{dom} M_{F_{T} 0_{2}^{+}} \subseteq \overline{\operatorname{dom}} T$. As $\operatorname{dom} M_{g} \subseteq \overline{\operatorname{dom}} T$ it immediately follows, since $(\operatorname{rec} T) \subseteq M_{g}$ by Lemma 35, that we have $(\operatorname{rec} T)^{\mu \mu} \subseteq\left(M_{g}\right)^{\mu \mu}=M_{g}$ (due to Proposition 27) so dom $(\operatorname{rec} T)^{\mu \mu} \subseteq \operatorname{dom} M_{g} \subseteq \overline{\operatorname{dom}} T$.

We will link the almost-convexity to the recession operator and also generalise the recent result of [32].

Theorem 50 Let $X$ be an arbitrary Banach space. Suppose $T: X \rightrightarrows X^{*}$ is maximal monotone with Fitzpatrick function $F_{T}$ and denote $g:=F_{T} 0_{2}^{+}$. When $\operatorname{dom} M_{g} \subseteq \overline{\operatorname{dom}} T$ (as is the case under the assumptions of Proposition 49) then it follows that $T$ is almost-convex. Moreover the condition (61) of Proposition 49 is necessary and sufficient for almost-convexity of $T$.

Proof By Proposition 42, $g$ is convex, so Lemma 35 gives $g \in b R(\operatorname{rec} T)$ with rec $T \subseteq M_{g}$. Since $M_{g}=\left(M_{g}\right)^{\mu \mu}$ (thanks to Proposition 27), then Proposition 32 yields

$$
\operatorname{co} \overline{\operatorname{dom}} T \subseteq P_{X}\left[(\operatorname{rec} T)^{\mu \mu}\right]=\operatorname{dom}(\operatorname{rec} T)^{\mu \mu} \subseteq \operatorname{dom} M_{g} \subseteq \overline{\operatorname{dom}} T
$$

whence $\overline{\operatorname{dom}} T=\operatorname{co} \overline{\operatorname{dom}} T$, implying convexity of the closure of the domain. Thus (61) of Proposition 49 is sufficient for almost-convexity of $T$.

For the converse, suppose dom $T$ has convex closure. Then by Corollary 44 (part 3) we have $\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=\delta_{\overline{\mathrm{dom} T} T}^{*}\left(x^{*}\right)+\delta_{\overline{\mathrm{dom} T}}(x)$ and so $x \notin \overline{\operatorname{dom} T} \operatorname{iff}\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=+\infty$. For any $t_{\beta} \rightarrow 0^{+},\left(x_{\beta}, x_{\beta}^{*}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right)$ with $x_{\beta}^{*} \in(\operatorname{dom} T)^{\perp}$ we have from the identity $g_{t_{\beta}}=F_{t_{\beta} T}$, the definition of the Fitzpatrick function $F_{t_{\beta} T}$, (and direct calculation) that

$$
\liminf _{\beta} g_{t_{\beta}}\left(x_{\beta}, x_{\beta}^{*}\right)=\liminf _{\beta} \sup _{\left(y, y^{*}\right) \in T} t_{\beta}\left\langle x_{\beta}-y, y^{*}\right\rangle \geq\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=+\infty .
$$

Hence $\sup _{\left(y, y^{*}\right) \in T}\left\langle x_{\beta}-y, y^{*}\right\rangle=+\infty$ eventually. In particular we have $\sup _{\left(y, y^{*}\right) \in T}\langle x-$ $\left.y, y^{*}\right\rangle=+\infty$ for $x \notin \overline{\operatorname{dom} T}$.

We have $g=F_{M_{g}} 0_{2}^{+} \in b R(\operatorname{rec} T)$ but do not yet know in general that $\overline{\operatorname{rec} M_{g}}{ }^{s b d \mathrm{w}^{*}}=$ $\left(M_{g}\right)^{\mu \mu}$.

Lemma 51 Suppose $X$ be an arbitrary real Banach space and $T: X \rightrightarrows X^{*}$ is a maximal monotone operator. Then $F_{T} 0_{2}^{+} \in b R(\operatorname{rec} T)$ with

$$
\overline{\operatorname{rec} T}^{s \times b d \mathrm{w}^{*}}={\overline{b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+0}(t T)^{\mu}}}^{s \times b d \mathrm{w}^{*}} \subseteq M_{F_{T} 0_{2}^{+}}=N_{\overline{\mathrm{co}} \operatorname{dom} T}
$$

Proof The first equality holds since maximality of $T$ implies $(t T)^{\mu}=t T$, so we may apply (8) to obtain rec $T=b d s w^{*}-\lim \sup _{t \rightarrow+0}(t T)^{\mu}$. We next investigate

$$
\begin{aligned}
& \operatorname{rec} T= b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+0}(t T)^{\mu}=\left\{\left(x, x^{*}\right) \mid \exists\left(x_{\alpha}, x_{\alpha}^{*}, t_{\alpha}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}, 0\right)\right. \\
&\left.\quad \text { with }\left(x_{\alpha}, x_{\alpha}^{*}\right) \in M_{F_{t_{\alpha} T}}=\left(t_{\alpha} T\right)^{\mu}\right\} \\
&=\left\{\left(x, x^{*}\right) \mid \exists\left(x_{\alpha}, x_{\alpha}^{*}, t_{\alpha}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}, 0\right) \text { with } F_{t_{\alpha} T}\left(x_{\alpha}, x_{\alpha}^{*}\right) \leq\left\langle x_{\alpha}, x_{\alpha}^{*}\right\rangle\right\} \\
&=\left\{\left(x, x^{*}\right) \mid \exists\left(x_{\alpha}, x_{\alpha}^{*}, t_{\alpha}, \beta_{\alpha}\right) \rightarrow^{s \times b d w^{*}}\left(x, x^{*}, 0, \beta\right)\right. \\
&\text { with } \left.\left(x_{\alpha}, x_{\alpha}^{*}, \beta_{\alpha}\right) \in \operatorname{epi} F_{t_{\alpha} T} \text { and } \beta_{\alpha} \leq\left\langle x_{\alpha}, x_{\alpha}^{*}\right\rangle\right\} \\
& \subseteq\left\{\left(x, x^{*}\right) \mid \exists \beta:\left(x, x^{*}, \beta\right) \in b d s w^{*}-\operatorname{limsupepi}_{t \rightarrow 0} F_{t T} \text { with } \beta \leq\left\langle x, x^{*}\right\rangle\right\} .
\end{aligned}
$$

Absorbing the extra dimension into the closure of the epi-graph, applying Lemma 31 part 2 (since $F_{T} 0_{2}^{+} \in b R(\operatorname{rec} T)$, by Proposition 42)

$$
\begin{aligned}
\overline{\operatorname{rec} T}^{s \times b d \mathrm{w}^{*}} & \subseteq\left\{\left(x, x^{*}\right) \mid \exists \beta:\left(x, x^{*}, \beta\right) \in{\overline{b d s \mathrm{w}^{*}-{\operatorname{limsupepi} F_{t T}}^{s \times 0}}{ }^{s} d \mathrm{w}^{*}} \text { with } \beta \leq\left\langle x, x^{*}\right\rangle\right\} \\
& =\left\{\left(x, x^{*}\right) \mid \exists \beta:\left(x, x^{*}, \beta\right) \in \operatorname{epi}\left({\overline{b d-e-\liminf _{t \rightarrow+0}}}^{s \times b d \mathrm{w}^{*}}\right) \text { and } \beta \leq\left\langle x, x^{*}\right\rangle\right\} \\
& =\left\{\left(x, x^{*}\right) \mid \exists \beta:\left(x, x^{*}, \beta\right) \in \operatorname{epi} F_{T} 0_{2}^{+} \text {and } \beta \leq\left\langle x, x^{*}\right\rangle\right\} \\
& =\left\{\left(x, x^{*}\right) \mid\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right) \leq\left\langle x, x^{*}\right\rangle\right\}=M_{F_{T} 0_{2}^{+}}=N_{\overline{\text { codom }},}
\end{aligned}
$$

Definition 52 Suppose $X$ be an arbitrary real Banach space and $T: X \rightrightarrows X^{*}$ is a maximal monotone operator. We define the $\varepsilon$-maximal enlargement of $T$ as

$$
\begin{aligned}
T^{\mu[\varepsilon]} & :=\left\{\left(x, x^{*}\right) \mid F_{T}\left(x, x^{*}\right) \leq\left\langle x, x^{*}\right\rangle+\varepsilon\right\} \\
& \equiv\left\{\left(x, x^{*}\right) \mid\left\langle x-y, x^{*}-y^{*}\right\rangle \geq-\varepsilon \text { for all }\left(y, y^{*}\right) \in T\right\} .
\end{aligned}
$$

Note that this set is clearly $\left(s \times b d \mathrm{w}^{*}\right)$-closed (but not necessarily $\left(s \times \mathrm{w}^{*}\right)$-closed). Note that as $t T=\left\{\left(x, t x^{*}\right) \mid\left(x, x^{*}\right) \in T\right\}$ then if follows that $t\left(T^{\mu[\varepsilon]}\right)=(t T)^{\mu[t \varepsilon]}$.

Proposition 53 Suppose $X$ be an arbitrary real Banach space and $T: X \rightrightarrows X^{*}$ is a maximal monotone operator. Then for $\operatorname{rec}\left(T^{\mu_{0}}\right):=b d s \mathrm{w}^{*}-\lim \sup _{\substack{t \rightarrow+0 \\ \varepsilon \rightarrow+0}}(t T)^{\mu[\varepsilon]}$ we have

$$
{\overline{\operatorname{rec}\left(T^{\mu_{0}}\right)}}^{s \times b d \mathrm{w}^{*}}=M_{F_{T} 0_{2}^{+}}=N_{\overline{\mathrm{co}} \mathrm{dom} t} \supseteq \overline{\operatorname{rec} T}^{s \times b d \mathrm{w}^{*}}
$$

Proof Consider $\varepsilon>0$ and

$$
\begin{aligned}
& \left\{\left(x, x^{*}\right) \mid \exists \beta:\left(x, x^{*}, \beta\right) \in \overline{b d s \mathrm{w}^{*}-\limsup _{t \rightarrow 0}} \begin{array}{l}
\left.s \times b d \mathrm{w}^{*}, \text { with } \beta \leq\left\langle x, x^{*}\right\rangle\right\} \\
=\left\{\left(x, x^{*}\right) \mid \exists\left(y_{\gamma}, y_{\gamma}^{*}, t_{\gamma}, \beta_{\gamma}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x_{\alpha}, x_{\alpha}^{*}, 0, \beta_{\alpha}\right) \text { and }\left(x_{\alpha}, x_{\alpha}^{*}, \beta_{\alpha}\right) \rightarrow_{\alpha}^{s \times b d w^{*}}\left(x, x^{*}, \beta\right)\right. \\
\left.\varepsilon_{\gamma} \rightarrow^{+} 0, \text { with }\left(y_{\gamma}, y_{\gamma}^{*}, \beta_{\gamma}\right) \in \operatorname{epi} F_{t_{\gamma} T} \text { and } \beta_{\gamma}-\varepsilon_{\gamma} \leq\left\langle y_{\gamma}, y_{\gamma}^{*}\right\rangle\right\}
\end{array}\right. \\
& =\left\{\left(x, x^{*}\right) \mid \exists\left(y_{\gamma}, y_{\gamma}^{*}, t_{\gamma}, \varepsilon_{\gamma}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x_{\alpha}, x_{\alpha}^{*}, 0,0\right)\right. \\
& \text { with } \left.F_{t_{\gamma} T}\left(y_{\gamma}, y_{\gamma}^{*}\right) \leq\left\langle y_{\gamma}, y_{\gamma}^{*}\right\rangle+\varepsilon_{\gamma},\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow_{\alpha}^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right)\right\} \\
& =\left\{\left(x, x^{*}\right) \mid \exists\left(y_{\gamma}, y_{\gamma}^{*}, t_{\gamma}\right) \rightarrow^{s \times b d \mathrm{w}^{*}}\left(x_{\alpha}, x_{\alpha}^{*}, 0\right), \varepsilon_{\gamma} \rightarrow^{+} 0,\right. \\
& =\frac{\text { with } \left.\left(y_{\gamma}, y_{\gamma}^{*}\right) \in\left(t_{\gamma} T\right)^{\mu_{\varepsilon_{\gamma}}},\left(x_{\alpha}, x_{\alpha}^{*}\right) \rightarrow_{\alpha}^{s \times b d \mathrm{w}^{*}}\left(x, x^{*}\right)\right\}}{=\frac{b d s \mathrm{w}^{*}-\limsup _{\substack{t \rightarrow+0 \\
\varepsilon \rightarrow+0}}(t T)^{\mu[\varepsilon]}}{s \times b d \mathrm{w}^{*}} .} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \overline{\operatorname{rec}\left(T^{\mu_{0}}\right)} \\
& \qquad=\left\{\left(x d \mathrm{w}^{*}\right.\right. \\
& \quad=\left\{\left(x, x^{*}\right) \mid\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=\left(\overline{b d-e-\liminf _{t \rightarrow+0} F_{t T}} s \times b d \mathrm{w}^{*}\right)\left(x, x^{*}\right) \leq\left\langle x, x^{*}\right\rangle\right\} \\
& \quad=\left\{\left(x, x^{*}\right) \mid\left(F_{T} 0_{2}^{+}\right)\left(x, x^{*}\right)=\left\langle x, x^{*}\right\rangle\right\}=M_{F_{T} 0_{2}^{+}}=N_{\overline{\operatorname{co}} \operatorname{dom} T} .
\end{aligned}
$$

Again the condition (63) is known to hold in reflexive space [9, Corollary 3.8].
Theorem 54 Suppose $X$ is an arbitrary real Banach space and $T: X \rightrightarrows X^{*}$ is a maximal monotone operator. Then $\operatorname{dom} \operatorname{rec}\left(T^{\mu_{0}}\right) \subseteq \overline{\operatorname{dom}} T$ is a sufficient condition for the domain of $T$ to be almost convex.

Proof We have this condition implying

$$
\begin{equation*}
\operatorname{dom} \overline{\operatorname{rec}\left(T^{\mu_{0}}\right)}{ }^{s \times b d w^{*}}=\overline{\operatorname{dom}} \operatorname{rec} T, \tag{63}
\end{equation*}
$$

which is a sufficient condition for the domain of $T$ to be almost convex since by Proposition
 inclusion. It is clear that dom $\overline{\operatorname{rec}} T^{s \times b d \mathrm{w}^{*}}=\overline{\operatorname{dom}} T$ so $\overline{\operatorname{dom}} T=\overline{\operatorname{co}} \operatorname{dom} T$.

Once again we note that it has already been shown in [8, Corollary 5.3.16] that in a reflexive Banach space one has $d\left(T^{\mu_{\varepsilon}}, T\right) \leq \sqrt{2 \varepsilon}$ and dom $T^{\mu_{\varepsilon}} \subseteq \overline{\operatorname{dom}} T$ for any maximal monotone operator. Consequently condition (63) may be verified in a reflexive space as $t\left(T^{\mu_{\varepsilon}}\right)=(t T)^{\mu_{\varepsilon t}}$ where $t T$ is maximal when $T$ is maximal.

## 8 Almost Convexity

We will need to utilise the duality mapping which can be viewed as the subdifferential of convex functions:

$$
\mathcal{J}_{X}(x)=\partial\left(\frac{1}{2}\|\cdot\|^{2}\right)(x) \subseteq X^{*} \quad \text { and } \quad \mathcal{J}_{X^{*}}\left(x^{*}\right)=\partial\left(\frac{1}{2}\|\cdot\|_{*}^{2}\right)\left(x^{*}\right) \subseteq X^{* *}
$$

When $X$ is reflexive we may assume $x^{*} \mapsto \frac{1}{2}\left\|x^{*}\right\|_{*}^{2}$ is Fréchet differential and then $\mathcal{J}_{X^{*}}\left(x^{*}\right) \subseteq X$. In the context of non-reflexive spaces it is unclear whether either $\mathcal{J}_{X}$ (resp. $\mathcal{J}_{X^{*}}$ ) are onto $X^{*}$ (resp. $X$ ). We will need to consider the (approximate) minimisers for the problem for a maximal monotone operator $T$ : given $\left(z, z^{*}\right) \in T^{\mu[\varepsilon / 2]}$ solve for the minimum (for $\lambda>0$ ) of:

$$
\begin{equation*}
G_{\lambda}\left(y, y^{*}\right):=F_{T}\left(y, y^{*}\right)-\left\langle y, y^{*}\right\rangle+\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(y, y^{*}\right)\right\|^{2} . \tag{64}
\end{equation*}
$$

Note that $\frac{\varepsilon}{2} \geq F_{T}\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle=G_{\lambda}\left(z, z^{*}\right) \geq \inf _{X \times X^{*}} G_{\lambda}$. Note also that any point satisfying $\frac{\varepsilon}{2} \geq G_{\lambda}\left(y, y^{*}\right) \geq 0$ satisfies $\left(y, y^{*}\right) \in T^{\mu[\varepsilon / 2]}$. When $X$ is reflexive we may seek a solution of $\inf _{\left(y, y^{*}\right) \in X \times X^{*}} G_{\lambda}\left(y, y^{*}\right)$ by considering the optimality condition for the problem (note that in the Fréchet sense $\left.D_{\left(u, u^{*}\right)}\left\langle u, u^{*}\right\rangle(\cdot)=\left\langle\left(u^{*}, u\right), \cdot\right\rangle\right)$ :

$$
(0,0) \in \partial F_{T}\left(y, y^{*}\right)-\left(y^{*}, y\right)+\frac{1}{\lambda}\left(-\mathcal{J}_{X \times X^{*}}\right)\left(\left(z, z^{*}\right)-\left(y, y^{*}\right)\right) .
$$

To ensure a solution we need the onto property, so in absence of this we will focus on approximate minimisers and variational principles. This is partly motivated by the following observation.

Proposition 55 Suppose $X$ is an arbitrary real Banach space and $T: X \rightrightarrows X^{*}$ is a monotone operator. Then $\left\{\left(x, x^{*}\right) \mid\left(x^{*}, x\right) \in \partial_{\varepsilon} F_{T}\left(x, x^{*}\right)\right\} \subseteq T^{\mu[\varepsilon / 2]}$.

When $M$ is maximal we have $T^{\mu[\varepsilon / 2]}=\left\{\left(x, x^{*}\right) \mid\left(x^{*}, x\right) \in \partial_{\varepsilon} F_{T}\left(x, x^{*}\right)\right\}$, and hence

$$
(0,0) \in \partial_{\varepsilon} F_{T}\left(z, z^{*}\right)-\left(z^{*}, z\right)+\frac{1}{\lambda}\left(-\mathcal{J}_{X \times X^{*}}\right)\left(\left(z, z^{*}\right)-\left(z, z^{*}\right)\right)
$$

for any $\left(z, z^{*}\right) \in T^{\mu[\varepsilon / 2]}$.
Proof See the Appendix.
Let us return to the problem (64). The following is a consequence of the Ekeland variational principle.

Proposition 56 Suppose $X$ is an arbitrary real Banach space, $T: X \rightrightarrows X^{*}$ a maximal monotone operator. Then for any $\left(z, z^{*}\right) \in X \times X^{*}$, there exists a constant $\Lambda=\Lambda\left(z, z^{*}\right)>0$ such that for all $0<\lambda<\Lambda$ :

$$
\begin{equation*}
F_{T}\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle \geq \frac{\lambda}{2(1+\lambda)^{2}}\left[d\left(\left(z, z^{*}\right), T\right)\right]^{2} . \tag{65}
\end{equation*}
$$

Moreover for any $\left(z, z^{*}\right)$ we have:

$$
\begin{equation*}
\left(F_{T} 0_{2}^{+}\right)\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle \geq \frac{1}{2}[d(z, \overline{\operatorname{dom}} T)]^{2} . \tag{66}
\end{equation*}
$$

Proof If $\left(z, z^{*}\right) \in T$ or $\left(z, z^{*}\right) \notin \operatorname{dom} F_{T}$, we may take $\Lambda\left(z, z^{*}\right)=1$, as a trivial case. Hereafter, consider $\left(z, z^{*}\right) \in\left(\operatorname{dom} F_{T}\right) \backslash T$. Place $\varepsilon:=F_{T}\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle$, so $\varepsilon>0$.

As $\left(z, z^{*}\right) \notin T$ there exist $\left(x, x^{*}\right) \in T$ and $\delta>0$ such that $\left\langle x-z, x^{*}-z^{*}\right\rangle=-2 \delta<0$ (with also $\frac{\varepsilon}{2} \geq G_{\lambda}\left(z, z^{*}\right) \geq 2 \delta>0$ for all $\lambda>0$ ). Because $T$ is maximal we have $\bar{T}^{s \times s^{*}}=T$ and so we may assume $\left\|\left(z, z^{*}\right)-\left(x, x^{*}\right)\right\|=\max \left\{\|x-z\|,\left\|x^{*}-z^{*}\right\|\right\}=N \bar{\delta}>0$ where $\bar{\delta}, N>0$ are chosen so that $B_{\bar{\delta}}\left(z, z^{*}\right) \cap T=\emptyset$ and $\left\langle x-u, x^{*}-u^{*}\right\rangle \leq-\delta<0$ for all $\left(u, u^{*}\right) \in$ $B_{\bar{\delta}}\left(z, z^{*}\right)$. Note that since $\left(x, x^{*}\right) \in T$ we have $\left(x^{*}, x\right) \in \partial F_{T}\left(x, x^{*}\right)$ so for all $\left(u, u^{*}\right) \in$ $X \times X^{*}$,

$$
\begin{aligned}
G_{\lambda}\left(u, u^{*}\right) \geq & F_{M}\left(x, x^{*}\right)+\left\langle\left(x, x^{*}\right),\left(u, u^{*}\right)-\left(x, x^{*}\right)\right\rangle-\left\langle u, u^{*}\right\rangle \\
& +\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(u, u^{*}\right)\right\|^{2} \\
= & \left\langle\left(x, x^{*}\right),\left(u, u^{*}\right)\right\rangle-\left\langle x, x^{*}\right\rangle-\left\langle u, u^{*}\right\rangle+\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(u, u^{*}\right)\right\|^{2} \\
= & -\left\langle x-u, x^{*}-u^{*}\right\rangle+\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(u, u^{*}\right)\right\|^{2}=: g_{\lambda}\left(u, u^{*}\right) .
\end{aligned}
$$

Then $g_{\lambda}\left(z, z^{*}\right)=-\left\langle x-z, x^{*}-z^{*}\right\rangle=2 \delta$ for all $\lambda>0$. Now for $\left(u, u^{*}\right) \in B_{\bar{\delta}}\left(z, z^{*}\right)$ we have $g_{\lambda}\left(u, u^{*}\right) \geq \delta+\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(u, u^{*}\right)\right\|^{2} \geq \delta>2 \delta-\frac{\varepsilon}{2} \geq g_{\lambda}\left(z, z^{*}\right)-\varepsilon$ for any $\lambda>0$. Moreover, for all $\left(u, u^{*}\right) \in X \times X^{*}$, we have

$$
\begin{align*}
& g_{\lambda}\left(u, u^{*}\right) \geq \geq \\
& \text { i.e. } \quad g_{\lambda}\left(u, u^{*}\right)+\|x-z\| x^{*}-z^{*} \|-\max \left\{\|x-z\|,\left\|x^{*}-z^{*}\right\|\right\}\left[\|z-u\|+\left\|z^{*}-u^{*}\right\|\right] \\
& \quad-\|z-u\|\left\|z^{*}-u^{*}\right\|+\frac{1}{2 \lambda}\|z-u\|^{2}+\frac{1}{2 \lambda}\left\|z^{*}-u^{*}\right\|^{2} \\
&=-N \bar{\delta}[\alpha+\beta]+\left(\frac{1}{2 \lambda}-\frac{1}{2}\right)\left[\alpha^{2}+\beta^{2}\right]+(\alpha-\beta)^{2} \\
& \geq-N \bar{\delta}[\alpha+\beta]+\frac{1}{2 \lambda}(1-\lambda)\left[\alpha^{2}+\beta^{2}\right]  \tag{67}\\
& \geq
\end{align*}
$$

where $N \bar{\delta}=\max \left\{\|x-z\|,\left\|x^{*}-z^{*}\right\|\right\}$ and $\alpha:=\|z-u\|, \beta:=\left\|z^{*}-u^{*}\right\|$. Next, note for all $\left(u, u^{*}\right) \notin B_{\bar{\delta}}\left(z, z^{*}\right)$ that $\max \{\alpha, \beta\} \geq \bar{\delta}$. As the right hand side of (67) tends to infinity as $\lambda \rightarrow$ 0 , we have the existence of $\lambda_{1}>0$ for which $0<\lambda \leq \lambda_{1}$ implies for all $\left(u, u^{*}\right) \notin B_{\bar{\delta}}\left(z, z^{*}\right)$ that $g_{\lambda}\left(u, u^{*}\right) \geq 2 \delta$, and so $g_{\lambda}\left(u, u^{*}\right) \geq 2 \delta=g_{\lambda}\left(z, z^{*}\right) \geq g_{\lambda}\left(z, z^{*}\right)-\varepsilon$ for $0<\lambda \leq \lambda_{1}$. Thus for $0<\lambda \leq \lambda_{1}$ we have $g_{\lambda}\left(u, u^{*}\right) \geq g_{\lambda}\left(z, z^{*}\right)-\varepsilon$ for all $\left(u, u^{*}\right) \in X \times X^{*}$.

As $\left(u, u^{*}\right) \mapsto g_{\lambda}\left(u, u^{*}\right)=-\left\langle x-u, x^{*}-u^{*}\right\rangle+\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(u, u^{*}\right)\right\|^{2}$ is jointly lowersemicontinuous with respect to the strong (norm) topologies, and bounded below, we are able to apply the Ekeland variational principle for any fixed $0<\lambda \leq \lambda_{1}$. Thus for any $\eta>0$ there exists a point $\left(y, y^{*}\right)$ such that i) $g_{\lambda}\left(y, y^{*}\right)+\frac{\eta}{2}\left\|\left(y, y^{*}\right)-\left(z, z^{*}\right)\right\| \leq g_{\lambda}\left(z, z^{*}\right)$; ii) $\left\|\left(y, y^{*}\right)-\left(z, z^{*}\right)\right\| \leq \frac{\varepsilon}{\eta}$; iii) $g_{\lambda}\left(u, u^{*}\right)+\frac{\eta}{2}\left\|\left(y, y^{*}\right)-\left(u, u^{*}\right)\right\|>g_{\lambda}\left(y, y^{*}\right)$ for all $\left(u, u^{*}\right) \neq$ $\left(y, y^{*}\right)$ i.e. $\left(y, y^{*}\right)$ is a strict minimiser. Suppose $\left(y, y^{*}\right) \notin B_{\bar{\delta}}\left(z, z^{*}\right)$ then $g_{\lambda}\left(y, y^{*}\right) \geq 2 \delta$ and so i) implies $2 \delta+\frac{\eta}{2} \bar{\delta} \leq g_{\lambda}\left(y, y^{*}\right)+\frac{\eta}{2}\left\|\left(y, y^{*}\right)-\left(z, z^{*}\right)\right\| \leq g_{\lambda}\left(z, z^{*}\right)=2 \delta$, a contradiction. Thus $\left(y, y^{*}\right) \in B_{\bar{\delta}}\left(z, z^{*}\right)$ and $-\left\langle x-y, x^{*}-y^{*}\right\rangle \geq \delta>0$ and hence i) implies (for any fixed
$0<\lambda \leq \lambda_{1}$ and any given $\eta>0$ )

$$
\begin{aligned}
F_{T}\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle & =G\left(z, z^{*}\right) \geq g_{\lambda}\left(z, z^{*}\right) \geq g_{\lambda}\left(y, y^{*}\right) \\
& =-\left\langle x-y, x^{*}-y^{*}\right\rangle+\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(y, y^{*}\right)\right\|^{2} \\
& \geq \frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(y, y^{*}\right)\right\|^{2} .
\end{aligned}
$$

Now we consider the necessary optimality condition at $\left(y, y^{*}\right)$ for $\left(u, u^{*}\right) \mapsto g_{\lambda}\left(u, u^{*}\right)+$ $\frac{\eta}{2}\left\|\left(u, u^{*}\right)-\left(y, y^{*}\right)\right\|$ i.e.

$$
\left(u, u^{*}\right) \mapsto-\left\langle x-u, x^{*}-u^{*}\right\rangle+\frac{1}{2 \lambda}\left\|\left(z, z^{*}\right)-\left(u, u^{*}\right)\right\|^{2}+\frac{\eta}{2}\left\|\left(u, u^{*}\right)-\left(y, y^{*}\right)\right\| .
$$

Applying the Clarke subdifferential and its calculus, noting that $D_{\left(u, u^{*}\right)}\left\langle u, u^{*}\right\rangle(\cdot)=$ $\left\langle\left(u^{*}, u\right), \cdot\right\rangle$ in the Fréchet sense and that both $\frac{1}{2}\|\cdot\|^{2}$ and $\|\cdot\|$ are finite, convex and hence also regular, we obtain

$$
(0,0) \in\left(x^{*}, x\right)-\left(y^{*}, y\right)+\frac{1}{\lambda}(-\mathcal{J})_{X \times X^{*}}\left(\left(z, z^{*}\right)-\left(y, y^{*}\right)\right)+\frac{\eta}{2} \bar{B}_{1}(0)
$$

and so $\left(z, z^{*}\right)-\left(x, x^{*}\right) \in\left(z, z^{*}\right)-\left(y, y^{*}\right)+\frac{1}{\lambda}(-\mathcal{J})_{X \times X^{*}}\left(\left(z, z^{*}\right)-\left(y, y^{*}\right)\right)+\frac{\eta}{2} \bar{B}_{1}(0)$.
That is

$$
d\left(\left(z, z^{*}\right), T\right) \leq\left\|\left(z, z^{*}\right)-\left(x, x^{*}\right)\right\| \leq\left(1+\frac{1}{\lambda}\right)\left\|\left(z, z^{*}\right)-\left(y, y^{*}\right)\right\|+\frac{\eta}{2} .
$$

Hence for a given $0<\lambda \leq \lambda_{1}$ and any given $\eta>0$

$$
F_{T}\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle \geq \frac{1}{2 \lambda}\left(1+\frac{1}{\lambda}\right)^{-2}\left[d\left(\left(z, z^{*}\right), T\right)-\frac{\eta}{2}\right]^{2}
$$

As $\eta$ is arbitrary we have $F_{T}\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle \geq \frac{1}{2} \frac{\lambda}{(1+\lambda)^{2}} d^{2}\left(\left(z, z^{*}\right), T\right)$ as long as $0<\lambda \leq \lambda_{1}$ and so we may take $\Lambda=\lambda_{1}$, and we have (65) following.

Proof of (66): As done earlier (without loss of generality) we will translate the graph of $T$ so that we may assume $(0,0) \in T$. By Corollary 43 we have a convex function $h$ for which $F_{T} 0_{2}^{+}=\bar{h}^{s \times b d \mathrm{w}^{*}}$ with dom $h=\operatorname{Pr}_{X} \operatorname{dom} F_{T} \times \operatorname{dom} F_{T} 0^{+}(0, \cdot)$ and that for $\left(z, z^{*}\right) \in \operatorname{dom} h$ there exists a net attaining the epi-limit defining $h$ i.e. there exists $y^{*}$ such that $h\left(z, z^{*}\right)=$ $\lim _{\tau \rightarrow+0} F_{\tau T}\left(z, z^{*}+\tau y^{*}\right)$. From the positive homogeneity of $h$ we have additionally that for any $\lambda^{\prime}>0$ :

$$
\begin{aligned}
h\left(x, x^{*}\right) & =\frac{1}{\lambda^{\prime}} h\left(z, \lambda^{\prime} z^{*}\right)=\lim _{t \rightarrow+0} \frac{1}{\lambda^{\prime}} F_{t T}\left(z, \lambda^{\prime} z^{*}+t y^{*}\right) \\
& =\lim _{t \rightarrow+0}\left(\frac{t}{\lambda^{\prime}}\right) F_{T}\left(z, y^{*}+\left(\frac{\lambda^{\prime}}{t}\right) z^{*}\right)=\lim _{\tau \rightarrow+0} \tau F_{T}\left(z, y^{*}+\frac{1}{\tau^{\prime}} z^{*}\right)
\end{aligned}
$$

and so we may take a convergent sequence $\tau_{n} \rightarrow 0$ and place $t_{n}\left(\lambda^{\prime}\right)=\lambda^{\prime} \tau_{n}$ and get

$$
h\left(x, x^{*}\right)=\lim _{n} \frac{1}{\lambda^{\prime}} F_{t_{n}\left(\lambda^{\prime}\right) T}\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right) .
$$

Note that when $0<\lambda<\lambda^{\prime}$ we have $t_{n}(\lambda)<t_{n}\left(\lambda^{\prime}\right)$. Now for each $n$, there exists a $\Lambda_{n} \rightarrow 0$ such that (65) holds for the maximal monotone operator $t_{n}\left(\lambda^{\prime}\right) T$ and the point $\left(z, \lambda^{*}+t_{n}\left(\lambda^{* *}\right)\right.$ for $0<\lambda<\Lambda_{n}$. In the following we will use the fact that for each $n$ we have $\sup _{0<\lambda^{\prime}<\Gamma^{\prime}}(\cdot) \geq \inf _{0<\lambda^{\prime}<\Lambda_{n}}(\cdot) \geq \inf _{\substack{0<\lambda^{\prime}<\Lambda_{n} \\ n \geq \Gamma}}(\cdot)$ irrespective of the relative sizes of $\Lambda_{n}$ and $\Gamma^{\prime}$. Indeed suppose $\Gamma^{\prime} \geq \Lambda_{n}$ then $\sup _{0<\lambda^{\prime}<\Gamma^{\prime}}(\cdot) \geq \sup _{0<\lambda^{\prime}<\Lambda_{n}}(\cdot) \geq \inf _{0<\lambda^{\prime}<\Lambda_{n}}(\cdot)$. On the other hand when $\Gamma^{\prime}<\Lambda_{n}$ we have $\sup _{0<\lambda^{\prime}<\Gamma^{\prime}}(\cdot) \geq \inf _{0<\lambda^{\prime}<\Gamma^{\prime}}(\cdot) \geq \inf _{0<\lambda^{\prime}<\Lambda_{n}}(\cdot)$ again.

Then as $F_{T}(0,0)=0$ implies $h(z, 0)=0$ for all $z$, we have, for any $z \in \operatorname{codom} T$ (using the fact that a limit can be written as a limit supremum) that:

$$
\begin{align*}
&(h(z, \cdot))_{x^{*}}^{\prime}\left(0 ; z^{*}\right) \\
& \quad=\inf _{\lambda^{\prime} \geq 0} \frac{1}{\lambda^{\prime}}\left(h\left(z, 0+\lambda^{\prime *}\right)-h(z, 0)\right)=\lim _{\lambda^{\prime} \rightarrow 0} \frac{1}{\lambda^{\prime}}\left(h\left(z, 0+\lambda^{\prime *}\right)-h(z, 0)\right) \\
& \quad=\lim _{\lambda^{\prime} \rightarrow 0} \lim _{n} \frac{1}{\lambda^{\prime}} F_{t_{n}\left(\lambda^{\prime}\right) T}\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right) \\
& \quad=\inf _{\Gamma^{\prime}} \sup _{0<\lambda^{\prime}<\Gamma^{\prime}} \sup _{\Gamma} \inf _{n \geq \Gamma} F_{t_{n}\left(\lambda^{\prime}\right) T}\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right) \\
& \quad=\inf _{\Gamma^{\prime}} \sup _{\Gamma} \sup _{0<\lambda^{\prime}<\Gamma^{\prime}} \inf _{n \geq \Gamma} \frac{1}{\lambda} F_{t_{n}\left(\lambda^{\prime}\right) T}\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right) \\
& \quad \geq \sup _{\Gamma} \inf _{0<\lambda^{\prime}<\Lambda_{n}} \inf _{n \geq \Gamma} \frac{1}{\lambda^{\prime}} F_{t_{n}\left(\lambda^{\prime}\right) T}\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right) \\
& \quad \geq \sup _{\Gamma} \inf _{n \geq \Gamma} \inf _{0<\lambda<\Lambda_{n}} \frac{1}{\lambda^{\prime}} F_{t_{n}\left(\lambda^{\prime}\right) T}\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right) \quad\left(\text { and as } t_{n}\left(\lambda^{\prime}\right)=\lambda^{\prime} \tau_{n}\right) \\
& \quad \geq \liminf _{n} \inf _{0<\lambda^{\prime} \leq \Lambda_{n}}\left(\left\langle z, z^{*}+\tau_{n} y^{*}\right\rangle+\frac{1}{2}\left(\frac{1}{1+\lambda^{\prime}}\right)^{2} d^{2}\left(\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right), t_{n}\left(\lambda^{\prime}\right) T\right)\right) \\
& \quad \geq\left\langle z, z^{*}\right\rangle+\liminf _{n} \frac{1}{2}\left(\frac{1}{1+\lambda_{n}}\right)^{2} \inf _{0<\lambda \leq \Lambda_{n}} d^{2}\left(\left(z, \lambda^{\prime} z^{*}+t_{n}\left(\lambda^{\prime}\right) y^{*}\right), t(\lambda)_{n} T\right) \\
&\left.\quad \geq\left\langle z, z^{*}\right\rangle+\frac{1}{2} \liminf _{n} d^{2}\left(z, \operatorname{Pr}_{X}\left[\bigcup_{0<\lambda \leq \Lambda_{n}} t_{n}(\lambda) T\right)\right]\right) \geq\left\langle z, z^{*}\right\rangle+\frac{1}{2} d^{2}(z, \overline{\operatorname{dom} T), \quad(()} \tag{68}
\end{align*}
$$

where we have used the fact that for all $t>0$ we have $\operatorname{Pr}_{X}(t T) \subseteq \overline{\operatorname{dom}} T$.
As the $\lambda \mapsto \frac{1}{\lambda}\left(h\left(z, 0+\lambda z^{*}\right)-h(z, 0)\right)$ non-decreasing we have (as $\left.h(z, 0)=0\right)$

$$
h\left(z, z^{*}\right)=h\left(z, 0+z^{*}\right)-h(z, 0) \geq(h(z, \cdot))_{x^{*}}^{\prime}\left(0 ; z^{*}\right)
$$

from which (66) follows from (68) when $z \in \overline{\operatorname{co}} \operatorname{dom} T=\overline{\operatorname{Pr}_{X} \operatorname{dom}} F_{T}$, since $F_{T} 0_{2}^{+}=$ $\bar{h}^{s \times b d w^{*}}$. When $z \notin \overline{\operatorname{co}} \operatorname{dom} T$ then by Proposition 42, the left hand side of (66) is $+\infty$.

We may now provide a proof of the conjectured almost-convexity for general monotone operators.

Theorem 57 Suppose $X$ is an arbitrary real Banach space and $T: X \rightrightarrows X^{*}$ is a maximal monotone operator. Then $T$ is almost-convex, in that $\overline{\operatorname{dom}} T=\overline{\operatorname{co}} \operatorname{dom} T$, which is convex.

Proof Suppose $z \in \overline{\operatorname{co}} \operatorname{dom} T$. Using (66) we find that these exists $\left(z, z^{*}\right) \in M_{F_{T} 0_{2}^{+}}=$ $N_{\overline{\text { cod }}}{ }^{\text {dom } T}$ and we have

$$
0=\left(F_{T} 0_{2}^{+}\right)\left(z, z^{*}\right)-\left\langle z, z^{*}\right\rangle \geq \frac{1}{2}[d(z, \overline{\operatorname{dom} T})]^{2} .
$$

Hence $\overline{\operatorname{co}} \operatorname{dom} T \subseteq \overline{\operatorname{dom} T}$.

## Appendix

Proof of Proposition 13 (Proof of (10)): When $\left(x, x^{*}\right) \in b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} A_{t}$ then there exists $I=\left\{t_{\alpha}\right\}$ cofinal in $\mathbb{R}_{+}$and $\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right) \rightarrow\left(x, x^{*}\right)$ as $t_{\alpha} \rightarrow+\infty$ with $\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right) \in A_{t_{\alpha}}$ and $\left\|x_{t_{\alpha}}^{*}\right\| \leq K$ (for some fixed bound $K>0$ ). Let $\eta \in \mathbb{R}_{+}$and $U$ a norm-open ball around $x$ and $W$ a weak* neighbourhood of $x^{*}$. There exists $t_{\alpha} \in I$ with $t_{\alpha} \geq \eta$ so that $\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t_{\alpha}} \neq \emptyset$ and so $\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset$ for some $t \geq \eta$. Thus $\left(x, x^{*}\right) \in \overline{\left[\left(\bigcup_{t \geq \eta} A_{t_{\alpha}}\right) \cap \overline{B_{K}}(0)\right]^{s \times \mathrm{w}^{*}}}$ from which the inclusion

$$
b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} A_{t} \subseteq \bigcup_{K>0} \bigcap_{\eta>0} \overline{\left[\left(\bigcup_{t \geq \eta} A_{t}\right) \cap \overline{B_{K}}(0)\right]^{s \times \mathrm{w}^{*}}}
$$

follows. For the inverse inclusion, suppose $\left(x, x^{*}\right) \notin b d s w^{*}-\lim \sup _{t \rightarrow+\infty} A_{t}$. We claim that (for each $K>0$ ) there exists $U$ a norm-open ball around $x$ and $W$ a weak* neighbourhood of $x^{*}$ such that $\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset\right\}$ fails to be cofinal:
(To prove this claim, presume otherwise. Then for some $K>0$ and any neighbourhood $U \times W$, we have $\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset\right\}$ cofinal. Then

$$
(U \times W)_{\eta}:=\bigcup_{\substack{t \geq \eta \\ t \in I}}\left\{\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t}\right\} \neq \emptyset,
$$

since $I$ is cofinal. Then we note that $\mathcal{F}_{I}:=\left\{(U \times W)_{\eta} \neq \emptyset \mid U \times W\right.$ nbhd of $\left.\left(x, x^{*}\right), \eta>0\right\}$ is a filterbase in that for any pair $(U \times W)_{\eta_{i}} \in \mathcal{F}$ for $i=1,2$ we have for all $\eta \geq \max \left\{\eta_{1}, \eta_{2}\right\}$ that

$$
\begin{aligned}
\left(U_{1} \times W_{1}\right)_{\eta_{1}} \cap\left(U_{2} \times W_{2}\right)_{\eta_{2}} & \supseteq \bigcup_{\substack{t \geq \eta \\
t \in T}}\left\{\left[\left[\left(U_{1} \cap U_{2}\right) \times\left(W_{1} \cap W_{2}\right)\right] \cap \overline{B_{K}}(0)\right] \cap A_{t}\right\} \\
& =\left(\left(U_{1} \cap U_{2}\right) \times\left(W_{1} \cap W_{2}\right)\right)_{\eta} .
\end{aligned}
$$

As we have $U \times W \supseteq(U \times W)_{\eta} \in \mathcal{F}_{I}$ it follows that $\mathcal{F}_{I} \rightarrow\left(x, x^{*}\right)$. As we may construct a net from a filter basis (via selection $\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right) \in(U \times W)_{t_{\alpha}}$, for $t_{\alpha} \in I$ where $\alpha \equiv U \times W \in$ $\mathcal{N}\left(x, x^{*}\right)$ partially ordered via set-inclusion). Indeed for any $U \times W \ni\left(x, x^{*}\right)$ and $\eta>0$ there exists $t_{\alpha} \in I, t_{\alpha} \geq \eta$ such that $\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right) \in\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t_{\alpha}}$. Thus there exists a net $\left(x_{t_{\alpha}}, x_{t_{\alpha}}^{*}\right) \in A_{t_{\alpha}} \cap \overline{B_{K}}(0)$, with $t_{\alpha} \in I\left(t_{\alpha} \rightarrow \infty\right)$, converges to ( $x, x^{*}$ ). Hence $\left(x, x^{*}\right) \in$ $b d s \mathrm{w}^{*}-\lim \sup _{t \rightarrow+\infty} A_{t}$, counter to assumption. This establishes the claim.)

Hence the complement

$$
\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t}=\emptyset\right\}
$$

contains a residual set, implying $\left(x, x^{*}\right) \notin\left[\left(\bigcup_{t \geq \eta} A_{t}\right) \cap \overline{B_{K}}(0)\right]^{s \times w^{*}}$, for some $\eta>0$, irrespective of the size of $K>0$.
(Proof of (11)): Consider $\left(x, x^{*}\right) \in b d s w^{*}-\liminf _{t \rightarrow+\infty} A_{t}$ first. Let $I \subseteq \mathbb{R}_{+}$be a cofinal set and let $V \in \mathcal{N}\left(x, x^{*}\right)$. Then $V \cap \cup_{t \in T} A_{t} \cap \overline{B_{K}}(0) \supseteq V \cap A_{t_{V}^{\prime}} \cap \overline{B_{K}}(0) \neq \emptyset$ for some $t_{V}^{\prime}>$ $t_{V}$ with $t_{V}^{\prime} \in I$. Then $\left(x, x^{*}\right) \in{\overline{\bigcup_{t \in T} A_{t} \cap \overline{B_{K}}(0)}}^{s \times \mathrm{w}^{*}}$. As $I$ was arbitrary, $\left(x, x^{*}\right) \in \bigcap_{\Gamma \subseteq \mathbb{R}_{+}}$


Conversely, suppose $\left(x, x^{*}\right) \notin b d s \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty} A_{t}$. Then we claim that for each $K>0$ there exists $U \times W \in \mathcal{N}\left(x, x^{*}\right)$ such that $\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset\right\}$ is not residual. (Indeed, if this claim is false, then $\exists K>0: \forall U \times W \in \mathcal{N}\left(x, x^{*}\right)$, $\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset\right\}$ is residual so there exists $t_{V}$ with $[[U \times W] \cap$ $\left.\overline{B_{K}}(0)\right] \cap A_{t} \neq \emptyset$ for all $t>t_{V}$ which means that $\left(x, x^{*}\right) \in b d s \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty} A_{t}$ counter to assumption.) Hence $I:=\left\{t \in \mathbb{R}_{+} \mid\left[[U \times W] \cap \overline{B_{K}}(0)\right] \cap A_{t}=\emptyset\right\}$ is cofinal, so

$$
\left(x, x^{*}\right) \notin\left[\left(\bigcup_{t \in I} A_{t}\right) \cap \bar{B}_{K}(0)\right]^{s \times w^{*}} \supseteq \bigcap_{\substack{I \subseteq \mathbb{R}_{+} \\ \text {cofinal }}}\left[\left(\bigcup_{t \in I} A_{t}\right) \cap \bar{B}_{K}(0)\right]^{s \times w^{*}}
$$

 establishes (11).

When $\left(s \times \mathrm{w}^{*}\right)$-limsup $\operatorname{sut}_{t \rightarrow+\infty}\left(A_{t} \cap \bar{B}_{K}(0)\right)=A \cap \bar{B}_{K}(0)$ for all $K>0$ sufficiently large, then taking the union across $K>0$ in the last identity

$$
A=\bigcup_{K>0}\left[A \cap \bar{B}_{K}(0)\right]=\bigcup_{K>0} \bigcap_{\eta>0} \overline{\left.\bigcup_{t \geq \eta} A_{t} \cap \overline{B_{K}}(0)\right]^{s \times \mathrm{w}^{*}}}=b d s \mathrm{w}^{*}-\limsup _{t \rightarrow+\infty} A_{t} .
$$

A similar argument covers the other case.
Finally assume all $\left\{A_{t}\right\}_{t>0}$ are convex. Let $\left(x, x^{*}\right),\left(y, y^{*}\right) \in b d s \mathrm{w}^{*}-\liminf _{t \rightarrow+\infty} A_{t}$ and $W \in \mathcal{N}(0,0)$ such that $W+W^{\prime} \subseteq V \in \mathcal{N}(0,0)$. Then there exist $K^{\prime}, K^{\prime \prime}>0$ such that for $t>t_{W}^{1}$ there exists $\left(x_{t}, x_{t}^{*}\right) \in A_{t} \cap \bar{B}_{K^{\prime}}(0) \cap\left[\left(x, x^{*}\right)+W\right]$ and for $t>t_{W}^{2}$ we have $\left(y_{t}, y_{t}^{*}\right) \in A_{t} \cap \bar{B}_{K^{\prime \prime}}(0) \cap\left[\left(y, y^{*}\right)+W\right]$ so for $\lambda \in[0,1], t>t_{V}:=\max \left\{t_{W}^{1}, t_{W}^{2}\right\}$ and $K:=\max \left\{K^{\prime}, K^{\prime \prime}\right\}$

$$
\begin{aligned}
& \lambda\left(x_{t}, x_{t}^{*}\right)+(1-\lambda)\left(y_{t}, y_{t}^{*}\right) \\
& \in A_{t} \cap \bar{B}_{K}(0) \cap\left[\lambda\left[\left(x, x^{*}\right)+W\right]+(1-\lambda)\left[\left(y, y^{*}\right)+W\right]\right] \\
& \quad=A_{t} \cap \bar{B}_{K}(0) \cap\left[\lambda\left(x, x^{*}\right)+(1-\lambda)\left(y, y^{*}\right)+\lambda W+(1-\lambda) W\right] \\
& \quad \subseteq A_{t} \cap \bar{B}_{K}(0) \cap\left[\lambda\left(x, x^{*}\right)+(1-\lambda)\left(y, y^{*}\right)+V\right]
\end{aligned}
$$

and so $A_{t} \cap \bar{B}_{K}(0) \cap\left[\lambda\left(x, x^{*}\right)+(1-\lambda)\left(y, y^{*}\right)+V\right] \neq \emptyset$ for $t>t_{V}$. So $\lambda\left(x, x^{*}\right)+$ $(1-\lambda)\left(y, y^{*}\right) \in b d s w^{*}-\liminf _{t \rightarrow+\infty} A_{t}$.

Proof of Lemma 46 We show that $N_{\overline{\operatorname{cod} d o m} T}(x) \subseteq(\operatorname{rec} T)(x)=\operatorname{rec}\left(T^{\mu}\right)(x)$ for any $x \in$ $\overline{\operatorname{dom}} T$ and as a consequence of the inclusion $\left(\operatorname{rec} T^{\mu}\right)(x) \subseteq N_{\overline{\text { codom }} T}(x)$ of [13, Proposition 13] gives the desired equality. Take arbitrary $\left(x, x^{*}\right) \in \operatorname{rec}\left(T^{\mu}\right),\left(y, y^{*}\right) \in T$ and
$z^{*} \in N_{\overline{\text { codom }} T}(x)$. Then there exists $t_{\alpha} \downarrow 0$ and $x_{\alpha}^{*} \in T^{\mu}\left(x_{\alpha}\right)$ with $x_{\alpha} \rightarrow x$ and a bounded net $t_{\alpha} x_{\alpha}^{*} \rightarrow x^{*}$. Consider

$$
\begin{aligned}
&\left\langle y-x_{\alpha}, y^{*}-\left(x_{\alpha}^{*}+\frac{1}{t_{\alpha}} z^{*}\right)\right\rangle=\left\langle y-x_{\alpha}, y^{*}-x_{\alpha}^{*}\right\rangle-\frac{1}{t_{\alpha}}\left\langle y-x_{\alpha}, z^{*}\right\rangle \\
&=\frac{1}{t_{\alpha}}\left\langle x-y, z^{*}\right\rangle-\frac{1}{t_{\alpha}}\left\langle x-x_{\alpha}, z^{*}\right\rangle \geq-\frac{1}{t_{\alpha}}\left\|x-x_{\alpha}\right\|\left\|z^{*}\right\|:=-\varepsilon_{\alpha} .
\end{aligned}
$$

As $\left(y, y^{*}\right) \in T$ was arbitrary we have

$$
\left(x_{\alpha}, x_{\alpha}^{*}+\frac{1}{t_{\alpha}} z^{*}\right) \in\left\{\left(w, w^{*}\right) \mid\left\langle w-y, w^{*}-y^{*}\right\rangle \geq-\varepsilon_{\alpha}, \forall\left(y, y^{*}\right) \in T\right\} .
$$

We may now apply the ( BR ) property to assert that for any $\eta>0$ there exists $\left(w_{\alpha}, w_{\alpha}^{*}\right) \in T$ such that $\left\|w_{\alpha}-x_{\alpha}\right\| \leq \frac{\varepsilon_{\alpha}}{\eta}$ and $\left\|w_{\alpha}^{*}-\left(x_{\alpha}^{*}+\frac{1}{t_{\alpha}} z^{*}\right)\right\| \leq \eta$. Consequently we have

$$
\begin{align*}
& \left\|w_{\alpha}-x\right\| \leq\left\|w_{\alpha}-x_{\alpha}\right\|+\left\|x-x_{\alpha}\right\| \\
& \qquad \leq \frac{1}{\eta t_{\alpha}}\left\|x-x_{\alpha}\right\|\left\|z^{*}\right\|+\left\|x-x_{\alpha}\right\|=\left\|x-x_{\alpha}\right\|\left[\frac{\left\|z^{*}\right\|}{\eta t_{\alpha}}+1\right] \\
& \text { and } \quad\left\|t_{\alpha} w_{\alpha}^{*}-\left(t_{\alpha} x_{\alpha}^{*}+z^{*}\right)\right\| \leq \eta t_{\alpha} \tag{69}
\end{align*}
$$

Using the freedom we have on the choice of $\eta$ for each given $\alpha$, we take $\eta=\eta_{\alpha}:=\frac{\left\|x-x_{\alpha}\right\|^{\frac{1}{2}}}{t_{\alpha}}>$ 0 ; then we find that $\frac{\left\|x-x_{\alpha}\right\|}{\eta_{\alpha} t_{\alpha}}=\left\|x-x_{\alpha}\right\|^{\frac{1}{2}} \rightarrow 0$ and $\eta_{\alpha} t_{\alpha}=\left\|x-x_{\alpha}\right\|^{\frac{1}{2}} \rightarrow 0$. As we have a bounded net $t_{\alpha} x_{\alpha}^{*} \rightarrow^{w^{*}} x^{*}$ it follows for (69) that $\left\{t_{\alpha} w_{\alpha}^{*}\right\}_{\alpha}$ is bounded with $t_{\alpha} w_{\alpha}^{*} \rightarrow^{w^{*}}$ $x^{*}+z^{*}$ and as $\left(w_{\alpha}, w_{\alpha}^{*}\right) \in T$ with $w_{\alpha} \rightarrow x$ we have $x^{*}+z^{*} \in(\operatorname{rec} T)(x)$. As $x^{*}=0 \in$
 $z^{*}=0 \in N_{\overline{\text { codom }} T}(x)$ we also have $\operatorname{rec}\left(T^{\mu}\right)(x) \subseteq(\operatorname{rec} T)(x)$, forcing equality. As $\overline{\operatorname{dom}} T$ is convex we have $N_{\overline{\operatorname{dom} T}}(x)=(\operatorname{rec} T)(x)$ for all $x \in \overline{\operatorname{dom}} T=\overline{\operatorname{co}} \operatorname{dom} T$ and hence $N_{\overline{\operatorname{dom} T}}=$ $\operatorname{rec} T$.

Proof of Proposition 55 When $\left(x, x^{*}\right) \in \partial_{\varepsilon} F_{T}\left(x, x^{*}\right)$ then $F_{T}\left(y, y^{*}\right)-F_{T}\left(x, x^{*}\right) \geq\left\langle\left(x, x^{*}\right)\right.$, $\left.\left(y, y^{*}\right)-\left(x, x^{*}\right)\right\rangle-\varepsilon$ for all $\left(y, y^{*}\right)$ so

$$
\begin{aligned}
\left\langle\left(x, x^{*}\right),\left(x, x^{*}\right)\right\rangle-F_{T}\left(x, x^{*}\right) & \geq \sup _{\left(y, y^{*}\right)}\left[\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle-F_{T}\left(y, y^{*}\right)\right] \\
& =F_{T}^{* \dagger}\left(x, x^{*}\right)-\varepsilon \\
& =P_{T}\left(y, y^{*}\right)-\varepsilon \geq F_{T}\left(x, x^{*}\right)-\varepsilon
\end{aligned}
$$

and hence

$$
2\left\langle x, x^{*}\right\rangle-2 F_{T}\left(x, x^{*}\right) \geq-\varepsilon \quad \text { or } \quad \frac{\varepsilon}{2}+\left\langle x, x^{*}\right\rangle \geq F_{T}\left(x, x^{*}\right) .
$$

Thus $\left(x, x^{*}\right) \in T^{\mu[\varepsilon / 2]}$. On the other hand when $T$ is maximal $F_{T}\left(y, y^{*}\right) \geq\left\langle y, y^{*}\right\rangle$ for all $\left(y, y^{*}\right)$ and we have $\left(x, x^{*}\right) \in T^{\mu[\varepsilon / 2]}$, implying

$$
\begin{aligned}
\left\langle x, x^{*}\right\rangle-\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle+\left\langle y, y^{*}\right\rangle & =\left\langle x-y, x^{*}-y^{*}\right\rangle \geq-\frac{\varepsilon}{2} \\
\text { so } \quad F_{T}\left(y, y^{*}\right)-F_{T}\left(x, x^{*}\right) & \geq\left\langle y, y^{*}\right\rangle-F_{T}\left(x, x^{*}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \geq\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle-\left\langle x, x^{*}\right\rangle-\frac{\varepsilon}{2}-F_{T}\left(x, x^{*}\right) \\
& \geq\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)\right\rangle-2\left\langle x, x^{*}\right\rangle-\varepsilon
\end{aligned}
$$

using $F_{T}\left(x, x^{*}\right) \leq\left\langle x, x^{*}\right\rangle+\frac{\varepsilon}{2}\left(\right.$ as $\left.\left(x, x^{*}\right) \in T^{\mu[\varepsilon / 2]}\right)$ in the last inequality. This implies

$$
F_{T}\left(y, y^{*}\right)-F_{T}\left(x, x^{*}\right) \geq\left\langle\left(x, x^{*}\right),\left(y, y^{*}\right)-\left(x, x^{*}\right)\right\rangle-\varepsilon \text { for all }\left(y, y^{*}\right)
$$

so $\left(x, x^{*}\right) \in \partial_{\varepsilon} F_{T}\left(x, x^{*}\right)$.

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## Declarations

Competing Interests The authors declare no competing interests.

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[^0]:    A. Eberhard
    andy.eberhard@rmit.edu.au

    1 Mathematical Sciences, RMIT, GPO Box 2476V, Melbourne, Victoria 3001, Australia

