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Strong Solutions and Mild Solutions for Sturm-Liouville Differential Inclusions

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Abstract
Existence results for a Cauchy problem driven by a semilinear differential Sturm-Liouville inclusion are achived by proving, in a preliminary way, an existence theorem for a suitable integral inclusion. In order to obtain this proposition we use a recent fixed point theorem that allows us to work with the weak topology and the De Blasi measure of weak noncompactness. So we avoid requests of compactness on the multivalued terms. Then, by requiring different properties on the map p involved in the Sturm-Liouville inclusion, we are able to establish the existence of both mild solutions and strong ones for the problem examinated. Moreover we focus our attention on the study of controllability for a Cauchy problem governed by a suitable Sturm-Liouville equation. Finally we precise that our results are able to study problems involving a more general version of a semilinear differential Sturm-Liouville inclusion.

Keywords Sturm-Liouville differential inclusions · Integral inclusions · Radon-Nikodym property · Measure of weak noncompactness · Controllability · Fixed point theorem

Mathematics Subject Classification 34B24 · 34A60 · 34K09 · 93B05 · 47H08

1 Introduction

Since 1836 Sturm and Liouville has been publishing some papers on problems involving suitable second order linear differential equations. In particular the Authors were interested on founding properties of solutions directly from the equation even when no analytic expressions for solutions were avaible. The influence of these works was so great that this subject became known as Sturm-Liouville theory. The importance of this theory is also linked to the fact that other differential equations can be trasformed in the Sturm-Liouville ones, for

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example Bessel, Hermite, Jacobi, Legendre equations. Moreover the study of differential equations or inclusions involving Sturm-Liouville operators is stimulated by problems related to various areas of applied sciences. For instance, they describe the vibration of a particular system, as the vibration of a plucked string of a guitar. Another example is one-dimensional Schrödinger equation that can represent the motion of a conduction electron in the crystal structure. Also the Airy equations, which described the change of a solution from oscillatory to exponential behaviour, are an example of Sturm-Liouville equations. In particular, these last equations are widely used in quantum mechanics and in the study of the caustics in light reflections, such as that of the rainbow (see [1]).

The fundamental role that this theory plays in the study of the most disparate phenomena, has recently stimulated a great number of results concerning the Sturm-Liouville equations/inclusions (see, for example, [2–11]).

In these papers in finite/infinite dimensional spaces the existence of solutions is obtained through different approaches: by using the theory of monotone operators, the fixed point theory or variational methods (see, for example, [2, 3, 5, 10, 11]). However, the Authors often require strong compactness conditions, which are usually not satisfied in a infinite dimensional framework.

In suitable Banach spaces X we investigate the existence of solutions, both strong and mild ones, for the following Cauchy problem driven by a semilinear differential Sturm-Liouville inclusion

$$\textbf{(SL)}\begin{cases}(p(t)x'(t))' \in \lambda G(t)x(t) + F(t, x(t)), \text{ a.e. } t \in J = [0, a]\\ x(0) = x_0\\ x'(0) = \overline{x}_0,\end{cases}$$

where $p : J \to (0, \infty)$ is a map, $\lambda \in \mathbb{R}$, $x_0, \overline{x}_0 \in X$, $G : J \to \mathcal{P}(\mathbb{R})$, and $F : J \times X \to \mathcal{P}(X)$ are suitable multimaps.

Let us recall that the literature related to Sturm-Liouville equations/inclusions in Banach spaces has been developed by assuming on the map $p : J \to (0, \infty)$ different properties. In this paper we precise which properties are necessary on the map p to introduce the definition of mild solution and strong solution. In particular we note that the Banach spaces having the Radon-Nikodym property are an appropriate framework to have the equivalence between the notion of strong solution and the concept of mild solution (see Sect. 3).

In order to obtain our existence results about the problem **(SL)**, we start to establish the existence of solutions for the following integral inclusion

$$\textbf{(SL-I)} \quad x(t) \in x_0 + p(0)\overline{x}_0 P(t) + \int_0^t (P(t) - P(s))(\lambda G(s)x(s) + F(s, x(s))\, ds,$$

where $P : J \to [0, \infty)$ is the integral map of $\frac{1}{p} \in L^1_+(J)$ (see Theorem 13).

The proof of this theorem, given in the setting of weakly compact generated Banach spaces, is based on a fixed point result, recently proved in [12]. This theorem allows us to work with weak topology and the De Blasi measure of weak noncompactness, so we avoid requests of compactness on the multivalued terms.

This result is central to obtain our objectives:

1. existence of mild solutions for **(SL)**
2. existence of strong solutions for **(SL)**.

We are able to achive the goal **1.** as a consequence of Theorem 13. Indeed, if $p \in \mathcal{C}(J)$ and X is a WCG Banach space, a continuous function $x : J \to X$ is a mild solution for **(SL)** if and only if x is a solution for the integral inclusion **(SL-I)** (see Theorem 14). The idea of this equivalence follows the approach introduced by Cernea in [6].

Then, in the stronger setting of separable Banach spaces we establish the existence of mild solutions for **(SL)** without assumptions about the values of the multimap F (see Theorem 15).

Instead, by using again the relation between strong and mild solutions described in Sect. 3, the goal **2.** is stated adding the Radon-Nikodym property on the Banach spaces and the absolute continuity on the map p (see Theorems 16 and 17).

Obviously all these results are again obtained in the lack of compactness.

The novelty of our paper about the existence of strong solutions for **(SL)** is the fact that we consider a more general setting respect to the ones recently studied. For example in [4] and [5] the Authors consider $X = \mathbb{R}^n$ and $p \in \mathcal{C}^1(J)$. This last setting is again considered in [3] for the study of multeplicity positive classical solutions for a boundary value Sturm-Liouville problem. On the other hand also the study of the existence of mild solutions is developed in literature for the particular space $\mathbb{R}^n$ (see, for example, [11] and [10]) and assuming Lipschitz type hypotheses on the nonlinear multivalued term (see, for example, [6–8] and [9]), which are more restrictive than ours.

Let us say that the key-result on the integral inclusion **(SL-I)** is obtained by assuming (on p and on the multimaps involved) weaker hypotheses than the ones required in the mentioned papers.

Therefore, to the best of our knowledge, all our existence theorems are new in literature.

The paper is organized as follows. In Sect. 2 we collect some background material, as definitions, propositions and theorems known in literature. In Sect. 3 we analyse the relation between strong solutions and mild solutions for Sturm-Liouville problems. In particular we state what conditions must be required on the space X and on the map p for these two concepts to be equivalent. Section 4 is devoted to the existence of continuous solutions for the integral inclusion **(SL-I)** (see Theorem 13). The existence of mild solutions and strong solutions for the Cauchy problem **(SL)** is analysed in Sect. 5, distinguishing two cases. On one hand we require the convexity on the values of the multimap F (see Theorems 14 and 16). On the other we leave this assumption, working with different kind of hypotheses on F (see Theorems 15 and 17). In Sect. 6, by using the obtained multivalued results, we focus our attention on the study of controllability for Cauchy problems driven by Sturm-Liouville equations. In the Sect. 7 we precise that our results are able to study problems involving a more general version of a semilinear differential Sturm-Liouville inclusion

$$(p(t)x'(t))' + q(t)x(t) \in \lambda G(t)x(t) + F(t, x(t)), \text{ a.e. } t \in J.$$

Since the Sturm-Liouville approach offers a valuable tool to study the wave phenomena in physics and engineering, as mentioned recently in [13] and [14], we conclude by noting that our multivalued approach to the second order Sturm-Liouville differential equations can be useful for deeping the controllability of these phenomena and of their practical applications.

2 Preliminaries

We start with the notations used in this article. Let $(X, \|\cdot\|_X)$ be a Banach space, X^* be the dual space of X and τ_w be the weak topology on X. In the sequel we denote with $\overline{B}_X(0, r)$ the closed ball of X centered at the origin and of radius $r > 0$ and the symbol $\overline{A}^w$ stands for the weak closure of a set $A \subset X$. As is well known, a bounded set A of a reflexive space

X is relatively weakly compact. Moreover we recall that a subset A of a Banach space X is said to be *relatively weakly sequentially compact* if any sequence of points in A has a subsequence weakly convergent to a point in X (see [15]). By virtue of Eberlein-Smulian Theorem this property is equivalent to the relative weak compactness ([16], Theorem 3.5.3). In the following, we will use this version of a result of H. Vogt.

Proposition 1 ([17], Theorem 3) *Let A be a relatively weakly compact subset of a Banach space. Then A is weakly closed if and only if A is weakly sequentially closed.*

Moreover, if $S \subset \mathbb{R}$ and $A \subset X$, we use the following notation

$$SA = \bigcup_{\alpha \in S} \alpha A. \tag{1}$$

Now, put $J = [0, a]$ an interval of the real line endowed with the Lebesgue measure μ, we denote by $\mathcal{M}(J)$ the family of all Lebesgue measurable subsets of J, by $\mathcal{C}(J; X)$ the space of all continuous functions $v : J \to X$ provided with the norm $\|v\|_{\mathcal{C}(J;X)} = \max_{t \in J} \|v(t)\|_X$. Moreover $AC(J; X) = \{v : J \to X : \ v \text{ absolutely continuous}\}$ and we use the notation $AC(J)$ if $X = \mathbb{R}$.

Let us note that a sequence $(f_n)_n$ in $\mathcal{C}(J; X)$ weakly converges to $g \in \mathcal{C}(J; X)$ if and only if $(f_n - g)_n$ is uniformly bounded and $f_n(t) \rightharpoonup g(t)$, for every $t \in J$ ([18], Theorem 4).

A function $f : \mathcal{C}(J; X) \to X$ is said to be *(w-w)*sequentially continuous if for every sequence $(x_n)_n$, $x_n \in \mathcal{C}(J; X)$, $x_n \rightharpoonup x$, then $f(x_n) \rightharpoonup f(x)$. A function $f : J \to X$ is said to be $(\mathcal{M}(J), \mathcal{B}(X))$*-measurable* if, for all $A \in \mathcal{B}(X)$, $f^{-1}(A) \in \mathcal{M}(J)$, where $\mathcal{B}(X)$ denotes the Borel σ-field of X (see [16], Definition 2.1.48). A function $f : J \to X$ is said to be *Bochner-measurable* (B-measurable, for short) if there is a sequence of simple functions which converges to f almost everywhere in J (see [16], Definition 3.10.1 (a)) and $f : J \to X$ is said to be *weakly measurable* if for each $l \in X^*$, the real valued function $l(f)$ is measurable (see [16], Definition 3.10.1 (b)). If X is a separable Banach space, these measurability notions are equivalent (see [16], Corollary 3.10.5).

Now, we denote with $L^1(J; X)$ the space of all X-valued Bochner integrable functions on J with norm $\|u\|_{L^1(J;X)} = \int_0^a \|u(t)\|_X \, dt$. For short, if $X = \mathbb{R}$ we name B-measurability as $\mathcal{L}$-measurability and we put $\|\cdot\|_1 = \|\cdot\|_{L^1(J;\mathbb{R})}$. For the set $L^1_+(J) = \{f \in L^1(J; \mathbb{R}) : f(t) \geq 0, \ a.e. \ t \in J\}$ the following result holds

Proposition 2 ([19] Lemma 3.1) *For every $k > 0$, $\nu \in L^1_+(J)$, there exists $n := n(k, \nu) \in \mathbb{N}$ such that*

$$\sup_{t \in J} \int_0^t k\nu(\xi) e^{-n(t-\xi)} \, d\xi < 1.$$

Then, a set $A \subset L^1(J; X)$ has the property of *equi-absolute continuity of the integral* if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for every $E \in \mathcal{M}(J)$, $\mu(E) < \delta_\varepsilon$, we have $\int_E \|f(t)\|_X \, dt < \varepsilon$, whenever $f \in A$, while $A \subset L^1(J; X)$ is *integrably bounded* if there exists $\nu \in L^1_+(J)$ such that $\|f(t)\|_X \leq \nu(t)$, $a.e. \ t \in J$, for every $f \in A$.

Clearly every integrably bounded set has the property of equi-absolute continuity of the integral. We recall that the equi-absolute continuity of the integral is important to characterize the relative weak compactness of bounded sets in $L^1(J; X)$.

Proposition 3 ([20] Corollary 9) *Let A be a bounded subset of $L^1(J;X)$ such that it has the property of equi-absolute continuity of the integral and, for a.e. $t \in J$, the set $A(t) = \{f(t) : f \in A\}$ is relatively weakly compact.*

Then A is relatively weakly compact.

For sake of completeness we report the proof of the following version of Fundamental Theorem of Calculus in Banach spaces.

Proposition 4 *Let X be a Banach space and $f : J \to X$ be a function such that*

i) *$f \in AC(J, X)$;*

ii) *there exists f' a.e. in J;*

iii) *$f' \in L^1(J;X)$.*

Then $\int_0^t f'(s)\,ds = f(t) - f(0)$, $t \in J$.

Proof First of all, being f' B-integrable, we can consider $y : J \to X$ so defined

$$y(t) = \int_0^t f'(s)\,ds. \tag{2}$$

We note that, from [21], y is differentiable a.e. on J and the following equality holds

$$y'(t) = f'(t), \text{ a.e. } t \in J. \tag{3}$$

Then, if $H : J \to X$,

$$H(t) = y(t) - f(t), \; t \in J, \tag{4}$$

we have that H is absolutely continuous (see **i)** and (2)) and such that $H'(t) = 0$ a.e. $t \in J$ (see (3)).

Now, we want to prove that H is constant on J. To this aim, let us fix $b \in]0, a]$ and we consider the set $E = \{t \in [0, b] : \; H'(t) = 0\}$ having measure equal to b. Fixed $\varepsilon > 0$, from the absolute continuity of H there exists $\delta_\varepsilon \in]0, \varepsilon[$, such that $\sum_{i=1}^n \|H(b_i) - H(a_i)\|_X < \varepsilon$ for every finite collection $\{]a_i, b_i[\}_{i=0,\dots,n}$ of disjoint intervals with $\sum_{i=1}^n (b_i - a_i) < \delta_\varepsilon$. Next, fixed $\bar{t} \in E$, there exists $\gamma_{\bar{t},\varepsilon} > 0$ such that

$$\frac{\|H(t) - H(\bar{t})\|_X}{|t - \bar{t}|} < \varepsilon, \; t \in E : \; |t - \bar{t}| < \gamma_{\bar{t},\varepsilon}. \tag{5}$$

Now, put $0 < \hat{\gamma} < \gamma_{\bar{t},\varepsilon}$, let us fix $s_{\hat{\gamma}} \in]\bar{t}, \bar{t} + \hat{\gamma}[\cap [0, b]$. Clearly $\{[\bar{t}, s_{\hat{\gamma}}]\}_{\bar{t} \in E, \; \hat{\gamma} < \gamma_{\bar{t},\varepsilon}}$ is a Vitali cover of E (see [22], p. 80). So from Lemma 5.1 of [22] we can find a finite collection $\{I_i^\varepsilon = [t_i, s_{\hat{\gamma}_i}]\}_{i=1,\dots,q_\varepsilon}$, $s_{\hat{\gamma}_0} = 0 < t_1 < s_{\hat{\gamma}_1} < t_2 \dots < s_{\hat{\gamma}_{q_\varepsilon}} < t_{q_\varepsilon+1} = b$, such that

$$\sum_{i=0}^{q_\varepsilon} |s_{\hat{\gamma}_i} - t_{i+1}| = \mu(E \setminus \cup_{i=1}^{q_\varepsilon} I_i^\varepsilon) < \delta_\varepsilon.$$

Hence we have

$$\sum_{i=0}^{q_\varepsilon} \|H(s_{\hat{\gamma}_i}) - H(t_{i+1})\|_X < \varepsilon. \tag{6}$$

On the other hand, recalling that $|s_{\hat{\gamma}_i} - t_i| < \gamma_{\bar{t},\varepsilon}$, $i = 1, \dots, q_\varepsilon$, we can also write (see (5))

$$\sum_{i=1}^{q_\varepsilon} \|H(t_i) - H(s_{\hat{\gamma}_i})\|_X < \varepsilon \sum_{i=1}^{q_\varepsilon} |s_{\hat{\gamma}_i} - t_i| \le \varepsilon b. \tag{7}$$

Thus, from (6) and (7) we can write

$$\|H(b) - H(0)\|_X \le \sum_{i=0}^{q_\varepsilon} \|H(s_{\hat{\gamma}_i}) - H(t_{i+1})\|_X + \sum_{i=1}^{q_\varepsilon} \|H(t_i) - H(s_{\hat{\gamma}_i})\|_X < \varepsilon + \varepsilon b.$$

Since ε is an arbitrary positive number we deduce $H(b) = H(0)$, for every $b \in]0, a]$. Then, put $k \in X$ such that $H(t) = k$, for every $t \in J$, from (4), $y(t) = f(t) + k$, for every $t \in J$. Finally, by using (2) we can write

$$\int_0^t f'(s)\, ds = y(t) - y(0) = f(t) - f(0),\ t \in J,$$

so the thesis holds. □

Further, we recall that a Banach space X is said to be *weakly compactly generated* (WCG, for short) if there exists a weakly compact subset K of X such that $X = \overline{span}(K)$ (see [23]). Let us note that every separable space is weakly compact generated as well as the reflexive ones (see again [23]). On the other hand a Banach space X has the Radon-Nikodym property (RNP, for short) if for every finite measure space (Ω, Σ, μ) and every vector measure $m : \Sigma \to X$ of bounded variation and μ-continuous, there exists $f \in L^1(\Omega, X)$ such that $m(A) = \int_A f\, d\mu$, for all $A \in \Sigma$ (see [24], Definition III.1.3 and Definitions I.1.1, I.1.4, I.2.3). For the sake of completeness we list some spaces that have RNP and do not (see [24], pg. 218). Among the RNP spaces we have l^1, the reflexive ones and separable duals, while $L^\infty[0, 1]$, $L^1([0, 1])$ and $\mathcal{C}(\Omega)$, with Ω infinite compact Hausdorff do not have the Radon-Nikodym property.

The following characterization of RNP-Banach spaces holds

Proposition 5 ([16], Theorem 3.10.36) *A Banach space X is RNP if and only if every absolutely continuous function $f : J \to X$ is differentiable a.e. in J and*

$$f(t) - f(s) = \int_s^t f'(\tau)\, d\tau,\ t, s \in J.$$

Now, if $\mathcal{P}(X)$ is the family of all nonempty subsets of the Banach space X, we denote $\mathcal{P}_b(X) = \{H \in \mathcal{P}(X) :\ H \text{ bounded}\}$, $\mathcal{P}_f(X) = \{H \in \mathcal{P}(X) :\ H \text{ closed}\}$,
$\mathcal{P}_c(X) = \{H \in \mathcal{P}(X) :\ H \text{ convex}\}$ and $\mathcal{P}_{wk}(X) = \{H \in \mathcal{P}(X) :\ H \text{ weakly compact}\}$.

Next for a multimap $F : J \to \mathcal{P}(X)$, put $S^1_{F(\cdot)} = \{f \in L^1(J; X) :\ f(t) \in F(t) \text{ a.e. } t \in J\}$, we call *Aumann integral* of F (see [25], pg. 377) the following subset of X

$$\int_J F(t)\, dt = \left\{ \int_J f(t)\, dt :\ f \in S^1_{F(\cdot)} \right\}.$$

Clearly if $S^1_{F(\cdot)} = \varnothing$ then $\int_J F(t)\, dt = \varnothing$.

Then, for every sequence $(A_n)_n$, $A_n \in X$, the *weak-Kuratowski limit superior* of $(A_n)_n$ is defined as (see [26], Definition 7.1.3)

$$w - \limsup_{n\to\infty} A_n = \{x \in X :\ x_{n_k} \rightharpoonup x,\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < \cdots < n_k < \cdots\}. \tag{8}$$

Proposition 6 ([26], Proposition 7.3.9) *Let X be a Banach space, $1 \le p < \infty$ and $F : J \to \mathcal{P}_{wk}(X)$. If $(f_n)_n$, $f_n \in L^p(J; X)$, is a sequence such that*

i) *there exists $f \in L^p(J; X)$ with $f_n \rightharpoonup f$;*

ii) *$f_n(t) \in F(t)$ a.e. $t \in J$, $n \in \mathbb{N}$,*

then

$$f(t) \in \overline{co}\ w - \limsup_{n\to\infty}\{f_n(t)\}, \quad a.e.\ t \in J,$$

where $\overline{co}$ denotes the closure of the convex hull of a set.

Furthermore a multimap $F : J \to \mathcal{P}(X)$ is said to be *measurable* if for every open set $V \subset X$ one has $F^-(V) = \{t \in J : \ F(t) \cap V \neq \varnothing\} \in \mathcal{M}(J)$ (see [27], Definition 1.3.1). We recall that if a measurable multimap F takes closed values in a separable Banach space X, then F has a $(\mathcal{M}(J), \mathcal{B}(X))$-measurable selection ([16], Theorem 4.3.1), where the selection property holds for every $t \in J$.

Now, if M is a metric space, we say that $F : M \to \mathcal{P}(X)$ has *(s-w)sequentially closed graph* if for every $(x_n)_n$, $x_n \in M$, $x_n \to x$ and for every $(y_n)_n$, $y_n \in F(x_n)$, $y_n \rightharpoonup y$, we have $y \in F(x)$. Moreover if M is equal to the Banach space X and, in the previous definition, we consider $(x_n)_n$ weakly convergent to x, we say that the multimap F has "weakly sequentially closed graph". Analogously the "$(w$-$w)$ sequential continuity" is named "weak sequential continuity".

A multimap $F : J \to \mathcal{P}(X)$ is said to have a *B-measurable selection* if there exists a B-measurable function $f : J \to X$ such that $f(t) \in F(t)$, a.e. $t \in J$. While a multimap $F : J \times X \to \mathcal{P}(X)$ has a *Carathèodory selection* if there exists a function $f : J \times X \to X$ such that

i) for every $t \in J$, $f(t, \cdot)$ is continuous on X;

ii) for every $x \in X$, $f(\cdot, x)$ is $(\mathcal{M}(J), \mathcal{B}(X))$-measurable;

iii) for a.e. $t \in J$ and every $x \in X$, $f(t, x) \in F(t, x)$.

In the sequel we will use the following selection results.

Proposition 7 ([12], Theorem 4.2) *Let M be a metric space, X be a Banach space and $F : J \times M \to \mathcal{P}(X)$ be a multimap such that*

a) *for a.e. $t \in J$, for every $x \in M$, the set $F(t, x)$ is convex;*

b) *for every $x \in M$, $F(\cdot, x) : J \to \mathcal{P}(X)$ has a B-measurable selection;*

c) *for a.e. $t \in J$, $F(t, \cdot) : M \to \mathcal{P}(X)$ has (s-w)sequentially closed graph in $M \times X$;*

d) *for a.e. $t \in J$ and every convergent sequence $(x_n)_n$ in M, the set $\bigcup_n F(t, x_n)$ is relatively weakly compact;*

e) *there exists $\varphi : J \to [0, \infty)$, $\varphi \in L^1_+(J)$, such that*

$$\sup_{z \in F(t,M)} \| z \| \le \varphi(t), \ a.e.\ t \in J.$$

Then, for every B-measurable $v : J \to M$, there is a B-measurable selection for $F(\cdot, v(\cdot))$.

Moreover $F : T \times X \to \mathcal{P}(Y)$, where T, X, Y are topological spaces, is said to be *lower semicontinuous* if for every open set $A \subset Y$ the set $F^-(A) = \{(t, x) \in T \times X : \ F(t, x) \cap A \neq \varnothing\}$ is open (see [27], Definition 1.1.2 and Theorem 1.1.2).

Proposition 8 ([28], Theorem 3.1) *Let T, X and Y be Hausdorff topological spaces, μ be a Radon measure on T and $F : T \times X \to \mathcal{P}(Y)$ be a multimap satisfying the following properties*

(*l-SD*) *for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset T$ such that $\mu(T \setminus K_\varepsilon) < \varepsilon$ and $F_{|K_\varepsilon \times X}$ is lower semicontinuous*;

(*M*) *for every closed set $Z \subset T \times X$ such that $F_{|Z}$ is lower semicontinuous, there exists a continuous selection of $F_{|Z}$, i.e. there exists a continuous function $f : Z \to Y$ such that $f(t, x) \in F(t, x)$, $(t, x) \in Z$.*

Then F has a Carathèodory selection.

Next if K is a subset of X, $x_0 \in K$ and $F : K \to \mathcal{P}(X)$ is a multimap, a closed convex set $M_0 \subset K$ is said to be (x_0, F)*-fundamental*, if $x_0 \in M_0$ and $F(M_0) \subset M_0$ (see [29], p. 620). In this setting we recall the following result which allows to characterize the *smallest* (x_0, F)*-fundamental* set.

Proposition 9 ([29], Theorem 3.1) *Let X be a locally convex Hausdorff space, $K \subset X$, $x_0 \in K$. Let $F : K \to \mathcal{P}(X)$ be a multimap such that $\overline{co}(F(K) \cup \{x_0\}) \subset K$. Then*

1) *$\mathcal{F} = \{H :\ H$ is a $(x_0, F) -$fundamental set$\} \neq \emptyset$;*

2) *put $M_0 = \bigcap_{H \in \mathcal{F}} H$, we have $M_0 \in \mathcal{F}$ and $M_0 = \overline{co}(F(M_0) \cup \{x_0\})$.*

Now we present a fixed point result which will play a key role in our existence theorems.

Proposition 10 *Let X be a Banach space, $K \subset X$, $x_0 \in K$ and $F : K \to \mathcal{P}(X)$ be a multimap such that*

i) *$\overline{co}(F(K) \cup \{x_0\}) \subset K$;*

ii) *$F(x)$ convex, for every $x \in M_0$;*

iii) *M_0 is weakly compact;*

iv) *$F_{|M_0}$ has weakly sequentially closed graph,*

where M_0 is the smallest (x_0, F)-fundamental set.

Then there exists at least one fixed point for $F_{|M_0}$.

Now, a function $\omega : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be a *measure of weak noncompactness* (M*w*NC, for short) if the following properties are satisfied (see [30], Definition 4.1):

ω_1) ω is a Sadowskii functional, i.e. $\omega(\overline{co}(H)) = \omega(H)$, for every $H \in \mathcal{P}_b(X)$;

ω_2) ω is regular, i.e. $\omega(H) = 0$ if and only if $\overline{H}^w$ is weakly compact.

Further, a MwNC $\omega : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be:

semi-homogeneous if $\omega(\lambda\Omega) = |\lambda|\omega(\Omega)$, for every $\lambda \in \mathbb{R}$, $\Omega \in \mathcal{P}_b(X)$;

monotone if $\Omega_1, \Omega_2 \in \mathcal{P}_b(X) :\ \Omega_1 \subset \Omega_2$ implies $\omega(\Omega_1) \leq \omega(\Omega_2)$;

nonsingular if $\omega(\{x\} \cup \Omega) = \omega(\Omega)$, for every $x \in X$, $\Omega \in \mathcal{P}_b(X)$;

x_0*-stable* if, fixed $x_0 \in X$, $\omega(\{x_0\} \cup \Omega) = \omega(\Omega)$, $\Omega \in \mathcal{P}_b(X)$;

invariant under closure if $\omega(\overline{\Omega}) = \omega(\Omega)$, $\Omega \in \mathcal{P}_b(X)$;

invariant with respect to the union with a compact set if $\omega(\Omega \cup C) = \omega(\Omega)$, for every relatively compact set $C \subset X$ and $\Omega \in \mathcal{P}_b(X)$.

In particular in [31] De Blasi introduces the M*w*NC function $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ so defined

$$\beta(H) = \inf\{\varepsilon \in [0, \infty[: \text{there exists } C \subset X \text{ weakly compact} :\ H \subseteq C + B_X(0, \varepsilon)\},$$

(named in literature *De Blasi*-M*w*NC) and he proves that β has all the properties mentioned above and it is also *algebrically subadditive*, i.e. $\beta\left(\sum_{k=1}^n H_k\right) \leq \sum_{k=1}^n \beta(H_k)$, where $H_k \in \mathcal{P}_b(X)$, $k = 1, \ldots, n$.

We recall the following interesting result for the De Blasi- M*w*NC $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$.

Proposition 11 ([29], Theorem 2.7) *Let (Ω, Σ, μ) be a finite positive measure space and X be a weakly compactly generated Banach space. Then for every countable bounded set $C \subset L^1(J; X)$ having the property of equi-absolute continuity of the integral, the function $\beta(C(\cdot))$ is $(\mathcal{M}(J), \mathcal{M}(\mathbb{R}))$-measurable and*

$$\beta\left(\left\{\int_\Omega x(s)\,ds : x \in C\right\}\right) \le \int_\Omega \beta(C(s))\,ds.$$

In the sequel, fixed $\alpha \in \mathbb{R}$, we use the following Sadowskii functional $\beta_\alpha : \mathcal{P}_b(\mathcal{C}(J; X)) \to \mathbb{R}_0^+$, so defined (see [29], Definition 3.9)

$$\beta_\alpha(M) = \sup_{\substack{C \subset M \\ \text{countable}}} \sup_{t \in J} \beta(C(t))e^{-\alpha t}, \ M \in \mathcal{P}_b(\mathcal{C}(J; X)), \tag{9}$$

where β is the De Blasi-MwNC and, for every $t \in J$, $C(t) = \{x(t) : x \in C\}$. We recall that the Sadowskii functional β_α is x_0-*stable* and *monotone* (see [29], Proposition 3.10) and β_α has the two following properties (see [12], Remark 2.11)

(J) β_α is algebrically subadditive;

(JJ) if $M \subset \mathcal{C}(J; X)$ is relatively weakly compact, then $\beta_\alpha(M) = 0$.

3 Strong and Mild Solutions for Sturm-Liouville Problems

In this section we want to underline the role of the absolute continuity of the function $p : J \to (0, \infty)$ and of the Radon-Nikodym property of the Banach space X in order to have the equivalence between the concept of strong solution and mild solution for the following Sturm-Liouville initial problem

$$\begin{cases} (p(t)x'(t))' = f(t), \ \text{a.e. } t \in J = [0, a] \\ x(0) = x_0 \\ x'(0) = \overline{x}_0 \end{cases} \tag{10}$$

where $f \in L^1(J; X)$, $x_0, \overline{x}_0 \in X$.

Let X be a Banach space and $p : J \to (0, \infty)$ be absolutely continuous on J. For *strong solution* of problem (10) we mean a $\mathcal{C}^1$-function $y : J \to X$ such that $y' \in AC(J; X)$, y' differentiable almost everywhere on J, $y'' \in L^1(J; X)$ and

$$\begin{aligned} &(p(t)y'(t))' = f(t), \ \text{a.e. } t \in J \\ &y(0) = x_0, \qquad y'(0) = \overline{x}_0. \end{aligned} \tag{11}$$

Clearly, to say that every strong solution of (10) is also a mild solution of (10), i.e. $y : J \to X$ satisfies

$$y(t) = x_0 + p(0)\overline{x}_0 \int_0^t \frac{1}{p(s)}\,ds + \int_0^t \frac{1}{p(s)} \int_0^s f(\tau)\,d\tau ds, \ t \in J, \tag{12}$$

it is not necessary the Radon-Nikodym property on X.

Indeed, if $y : J \to X$ is a strong solution, then by integrating the first equality in (11) we obtain

$$\int_0^s (p(\tau)y'(\tau))'\,d\tau = \int_0^s f(\tau)\,ds, \ s \in J.$$

Since $p \in AC(J)$, we have $py' \in AC(J; X)$, there exists $(py')'$ a.e. on J, and $(py')' = f \in L^1(J; X)$. So, by using Proposition 4 we can write

$$(py')(s) - (py')(0) = \int_0^s f(\tau)\, d\tau, \ s \in J,$$

hence, recalling that $y'(0) = \overline{x}_0$ and $p(t) > 0$, $t \in J$, we deduce

$$y'(s) = \frac{p(0)\overline{x}_0}{p(s)} + \frac{1}{p(s)} \int_0^s f(\tau)\, d\tau, \ s \in J.$$

Since $\frac{1}{p} \in L^1(J)$, we can integrate again and, being $y(0) = x_0$, we have (see Proposition 4)

$$y(t) - x_0 = p(0)\overline{x}_0 \int_0^t \frac{1}{p(s)}\, ds + \int_0^t \frac{1}{p(s)} \int_0^s f(\tau)\, d\tau ds, \ t \in J,$$

i.e. $y : J \to X$ satisfies (12). Therefore a strong solution $y : J \to X$ of the Sturm-Liouville initial problem (10) is a mild solution for (10).

On the other hand, if we assume that X is a RNP-Banach space, we have also that every mild solution of (10) is a strong solution.

Indeed, if $y : J \to X$ is a mild solution for (10), recalling the continuity of the function $\frac{1}{p}$, we have that $y \in \mathcal{C}^1(J; X)$. Hence we can derive (12) in J obtaining

$$y'(t) = p(0)\overline{x}_0 \frac{1}{p(t)} + \frac{1}{p(t)} \int_0^t f(\tau)\, d\tau, \ t \in J. \tag{13}$$

Now, taking into account that the functions $\frac{1}{p}$ and $\int_0^{\cdot} f(\tau)\, d\tau$ are absolutely continuous, we have that $y' \in AC(J; X)$. Thanks to the Radon-Nikodym of X, property there exists y'' a.e. on J (see Proposition 5) and $y'' \in L^1(J; X)$. Obviously we can write

$$y(0) = x_0 \qquad y'(0) = \overline{x}_0. \tag{14}$$

Next, multiplying both sides of (13) by $p(t)$, for every $t \in J$, we have

$$p(t)y'(t) = p(0)\overline{x}_0 + \int_0^t f(\tau)\, d\tau, \ t \in J$$

and, being $py' \in AC(J; X)$ and using again Proposition 5, we deduce

$$(p(t)y'(t))' = f(t), \ \text{a.e. } t \in J. \tag{15}$$

Finally, thanks to (14), (15) and the above considerations we can conclude that y is a strong solution for (10).

Remark 1 Let us say that if the positive function p is such that $\frac{1}{p} \in L^1_+(J)$ and X is only a Banach space, the integral equation (12) is well posed, so the notion of mild solution can be introduced.

On the other hand, in order to give the concept of strong solution we have assumed $p \in AC(J)$ instead of p only differentiable a.e. on J (sufficient to introduce the definition). We did that because if the positive function p satisfies the mentioned weaker property it is not possible to say that a strong solution is a mild solution. Indeed, even if $X = \mathbb{R}$, we can not repeat the reasoning above presented since the product between an a.e. differentiable map $p : J \to (0, \infty)$ and a function w, even in $\mathcal{C}^\infty(J; \mathbb{R})$, can be such that $pw' \notin AC(J; \mathbb{R})$.

Example Let us consider the following Sturm-Liouville Cauchy problem

$$\begin{cases} (p(t)w'(t))' = f(t), \ \text{a.e. } t \in J = [0,1] \\ w(0) = 0 \\ w'(0) = 1, \end{cases} \tag{16}$$

where $p : J \to (0, \infty)$ is the Vitali function and $f : J \to \mathbb{R}$ is the null function. Now, we have that the $\mathcal{C}^\infty$ function $w : J \to \mathbb{R}$ so defined

$$w(t) = t, \ t \in J$$

is such that $(p(t)w'(t))' = p'(t) = 0 = f(t)$ a.e. $t \in J$ and $w(0) = 0$, $w'(0) = 1$.

Hence w is a strong solution of (16). However, being $p(t)w'(t) = p(t)$, $t \in J$, pw' is not absolutely continuous on J.

What has been precised above in Remark 1 lead us to define, in the setting of Banach spaces and assuming only $\frac{1}{p} \in L^1_+(J)$, the concept of mild solution for the following Cauchy problem driven by a Sturm-Liouville inclusion

$$\textbf{(SL)}_\Gamma \begin{cases} (p(t)x'(t))' \in \Gamma(t, x(t)), \ \text{a.e. } t \in J = [0,a] \\ x(0) = x_0 \\ x'(0) = \overline{x}_0, \end{cases}$$

where $\Gamma : J \times X \to \mathcal{P}(X)$ and $x_0, \overline{x}_0 \in X$.

Definition 1 A continuous function $x : J \to X$ is a *mild solution* for $\textbf{(SL)}_\Gamma$ if

$$x(t) = x_0 + p(0)\overline{x}_0 \int_0^t \frac{1}{p(s)}\, ds + \int_0^t \frac{1}{p(s)} \int_0^s \gamma(\tau)\, d\tau ds, \ t \in J,$$

where $\gamma \in S^1_{\Gamma(\cdot, x(\cdot))} = \{\gamma \in L^1(J; X) : \ \gamma(t) \in \Gamma(t, x(t)), \ \text{a.e. } t \in J\}$.

While, if the positive function $p \in AC(J)$ and X is a Banach space, we can introduce the following

Definition 2 A $\mathcal{C}^1$-function $x : J \to X$ is a *strong solution* for $\textbf{(SL)}_\Gamma$ if $x' \in AC(J; X)$, x' differentiable almost everywhere on J, $y'' \in L^1(J; X)$, $(p(t)x'(t))' \in \Gamma(t, x(t))$, a.e. $t \in J$, and $x(0) = x_0$, $x'(0) = \overline{x}_0$.

Clearly, if X is a RNP-Banach space the above arguments allow us to claim that Definitions 1 and 2 are equivalent.

4 An Existence Result for Integral Inclusions

To the aim to obtain existence results for the Sturm-Liouville Cauchy problem **(SL)**, in this section we first study the existence of solutions for the integral inclusion **(SL-I)**.

Here we require $p : J \to (0, \infty)$ such that

p1) $\frac{1}{p} \in L^1(J)$

and so the *integral map* $P : J \to [0, \infty)$

$$P(t) = \int_0^t \frac{1}{p(s)}\, ds,\ t \in J \tag{17}$$

is well defined.

Now we can introduce, as in [6], the following operator $H_P : L^1(J; X) \to \mathcal{C}(J; X)$, which will play a key role in our existence results

$$H_P\gamma(t) = \int_0^t (P(t) - P(s))\gamma(s)\, ds,\ t \in J,\ \gamma \in L^1(J; X). \tag{18}$$

Remark 2 We observe that the operator H_P is well posed. First of all, put

$$L := \left\| \frac{1}{p} \right\|_1, \tag{19}$$

fixed $\gamma \in L^1(J; X)$ and $t \in J$, we have

$$\|(P(t) - P(s))\gamma(s)\|_X \leq 2L\|\gamma(s)\|_X,\ s \in J, \tag{20}$$

hence, since $2L\|\gamma(\cdot)\|_X \in L^1(J)$, the B-measurable map $(P(t) - P(\cdot))\gamma(\cdot)$ is also B-integrable on J.

Next we prove that, for every $\gamma \in L^1(J; X)$, the function $H_P\gamma \in \mathcal{C}(J; X)$. It is obvious if $\|\gamma\|_{L^1(J;X)} = 0$. On the other hand, fixed $\bar{t} \in J$, if $t \in J$ is such that $t \leq \bar{t}$, by using (20), the following chain of inequalities holds

$$\begin{aligned}
&\|H_P\gamma(\bar{t}) - H_P\gamma(t)\|_X \\
&= \left\| \int_0^{\bar{t}} (P(\bar{t}) - P(s))\gamma(s)\, ds - \int_0^t (P(t) - P(s))\gamma(s)\, ds \right\|_X \\
&= \left\| \int_0^t (P(\bar{t}) - P(s))\gamma(s)\, ds + \int_t^{\bar{t}} (P(\bar{t}) - P(s))\gamma(s)\, ds - \int_0^t (P(t) - P(s))\gamma(s)\, ds \right\|_X \\
&\leq |P(\bar{t}) - P(t)| \left\| \int_0^t \gamma(s)\, ds \right\|_X + \int_t^{\bar{t}} |P(\bar{t}) - P(s)|\, \|\gamma(s)\|_X\, ds \\
&\leq |P(\bar{t}) - P(t)| \|\gamma\|_{L^1(J;X)} + 2L \int_t^{\bar{t}} \|\gamma(s)\|_X\, ds
\end{aligned}$$

Analogously if $t \geq \bar{t}$. Hence, for every $t \in J$, we conclude

$$\|H_P\gamma(\bar{t}) - H_P\gamma(t)\|_X \leq |P(\bar{t}) - P(t)| \|\gamma\|_{L^1(J;X)} + 2L \int_{[\min\{t,\bar{t}\}, \max\{t,\bar{t}\}]} \|\gamma(s)\|_X\, ds. \tag{21}$$

Now, the absolute continuity of the integral and the uniform continuity of P on J imply that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that (see (21))

$$\|H_P\gamma(\bar{t}) - H_P\gamma(t)\|_X < \varepsilon,\ t, \bar{t} \in J : |\bar{t} - t| < \delta(\varepsilon).$$

So $H_P\gamma \in \mathcal{C}(J; X)$.

Moreover, about the operator H_P we have the following

Proposition 12
The operator $H_P : L^1(J;X) \to \mathcal{C}(J;X)$ is linear, bounded, weakly continuous (and so weakly sequentially continuous).

Proof Clearly H_P is linear. Moreover, taking into account of (20) we have that

$$\|H_P\gamma\|_{\mathcal{C}(J;X)} \leq 2L\|\gamma\|_{L^1(J;X)},\ \gamma \in L^1(J;X),$$

i.e. H_P is bounded. Hence we can conclude that H_P is weakly continuous (see [32], Theorem 3.10) and so it is also weakly sequentially continuous (see [33], Definition 1.1). □

Now we present the following existence result of solutions for the integral inclusion **(SL-I)**, where to write some hypotheses we use (1).

Theorem 13
Let X be a weakly compactly generated Banach space, $J = [0,a]$, $\lambda \in \mathbb{R}$ and $x_0, \overline{x}_0 \in X$. Let $p : J \to \mathbb{R}$ be a function satisfying **p1)**, *$F : J \times X \to \mathcal{P}(X)$ and $G : J \to \mathcal{P}(\mathbb{R})$ be two multimaps such that*

F1) *$F(t,x)$ is convex, for every $(t,x) \in J \times X$;*
F2) *for every $x \in X$, $F(\cdot,x) : J \to \mathcal{P}(X)$ has a B-measurable selection;*
F3) *for a.e. $t \in J$, $F(t,\cdot) : X \to X$ has weakly sequentially closed graph;*
FG1) *$\exists(\varphi_n)_n$, $\varphi_n \in L^1_+(J)$ such that*

$$\limsup_{n\to\infty} \frac{2L\|\varphi_n\|_1}{n} < 1, \tag{22}$$

where L is presented in (19) and

$$\|F(t,\overline{B}_X(0,n)) + \lambda G(t)\overline{B}_X(0,n)\| \leq \varphi_n(t),\ a.e.\ t \in J,\ n \in \mathbb{N}; \tag{23}$$

FG2) *there exists $A \subset J$, $\mu(A) = 0$, such that, for all $n \in \mathbb{N}$, there exists $\nu_n \in L^1_+(J)$ with the property*

$$\beta(C_1) \leq \nu_n(t)\beta(C_0),\ t \in J \setminus A$$

for all countable C_0, C_1, with $C_0 \subseteq \overline{B}_X(0,n)$, $C_1 \subseteq F(t,C_0) + \lambda G(t)C_0$, where β is the De Blasi measure of weak noncompactness;
G1) *$G(t)$ is closed, for every $t \in J$;*
G2) *G is measurable.*
Then there exists at least one solution for ***(SL-I)***, *i.e. there exists a continuous function $x : J \to X$ such that*

$$x(t) \in x_0 + p(0)\overline{x}_0 P(t) + \int_0^t (P(t) - P(s))(\lambda G(s)x(s) + F(s,x(s))\,ds,\ t \in J.$$

Proof First of all, since $\mathbb{R}$ is obviously a separable Banach space and **G1)** and **G2)** hold, we have that there exists a $\mathcal{L}$-measurable map $g_\lambda : J \to \mathbb{R}$ such that

$$g_\lambda(t) \in \lambda G(t),\ t \in J. \tag{24}$$

Now we consider the multimap $\Gamma_\lambda : J \times X \to \mathcal{P}(X)$ defined as

$$\Gamma_\lambda(t, x) = F(t, x) + \{g_\lambda(t)x\},\ t \in J,\ x \in X. \tag{25}$$

In a preliminary way we show that Γ_λ satisfies the following properties:

(I) $\Gamma_\lambda(t, \cdot)$ has weakly sequentially closed graph, for a.e. $t \in J$;

(II) $\Gamma_\lambda(t, x)$ is closed, for a.e. $t \in J$, for every $x \in X$.

In order to prove **(I)**, put $N \subset J$ the null measure set for which hypothesis **F3)** holds, we fix $t \in J \setminus N$. Let $(x_n)_n$, $(y_n)_n$ be two sequences in X such that $y_n \in \Gamma_\lambda(t, x_n)$, $n \in \mathbb{N}$ and

$$x_n \rightharpoonup x, \qquad\qquad y_n \rightharpoonup y \tag{26}$$

with $x, y \in X$.

Now, for every $n \in \mathbb{N}$, since $y_n \in \Gamma_\lambda(t, x_n)$, by (25) there exists $z_n \in F(t, x_n)$ such that

$$y_n = z_n + g_\lambda(t)x_n. \tag{27}$$

Clearly, by using (27), (26) and **F3)** we can write

$$z_n = y_n - g_\lambda(t)x_n \rightharpoonup y - g_\lambda(t)x := z \in F(t, x).$$

So, $y = z + g_\lambda(t)x \in \Gamma_\lambda(t, x)$ (see (25)). Then **(I)** is true.

Now we establish **(II)**.

Let $x \in X$ and $t \in J \setminus N$, where N, $\mu(N) = 0$, is such that **(I)** and **FG2**) hold in $J \setminus N$. First we prove that the set $\Gamma_\lambda(t, x)$ is relatively w-compact. To this aim we fix $(y_n)_n$, $y_n \in \Gamma_\lambda(t, x)$. From **FG2)** we have $\beta(C_1) \le \nu_p(t)\beta(C_0) = 0$, being $C_0 = \{x\}$ and $C_1 = \{y_n : n \in \mathbb{N}\}$ such that $C_0 \subset \overline{B}_X(0, p)$, for a suitable $p \in \mathbb{N}$, and $C_1 \subset \Gamma_\lambda(t, C_0) \subset F(t, C_0) + \lambda G(t)C_0$ (see (1)). Hence, being C_1 relatively w-(sequentially)compact, there exist $(y_{n_k})_k \subset (y_n)_n$ and $y \in X$ such that $y_{n_k} \rightharpoonup y$. Then by the arbitrariness of the sequence $(y_n)_n$ we have the relative weak compactness of $\Gamma_\lambda(t, x)$. Since $\Gamma_\lambda(t, x)$ is also weakly sequentially closed (see **(I)**), by using Proposition 1 we deduce that $\Gamma_\lambda(t, x)$ is w-closed. Therefore the convex set $\Gamma_\lambda(t, x)$ is closed too (see **F1)** and (25)).

Now we consider the following integral inclusion

$$\textbf{(SL-I)}_{\Gamma_\lambda} \qquad x(t) \in x_0 + p(0)\overline{x}_0 P(t) + \int_0^t (P(t) - P(s))\Gamma_\lambda(s, x(s))\, ds,\ t \in J.$$

First of all we show that, fixed $u \in \mathcal{C}(J; X)$, the multimap $\Gamma_\lambda(\cdot, u(\cdot))$ is Aumann integrable, i.e.

$$S^1_{\Gamma_\lambda(\cdot, u(\cdot))} = \{\gamma \in L^1(J; X) :\ \gamma(t) \in \Gamma_\lambda(t, u(t)) \text{ a.e. } t \in J\} \neq \varnothing. \tag{28}$$

To this purpose, let us put $M_u = \overline{B}_X(0, n_u)$, where $n_u \in \mathbb{N}$ such that $\|u(t)\|_X \le n_u$, for every $t \in J$.

We will prove that the multimap $\Gamma_{\lambda_{|J \times M_u}} : J \times M_u \to \mathcal{P}(X)$ satisfies all the assumptions of Proposition 7 by considering on M_u the metric induced by that on X.

Clearly the multimap $\Gamma_{\lambda_{|J \times M_u}}$ satisfies hypothesis **a)** of Proposition 7 since its values are traslated of convex sets (see **F1)** and (25)).

Now, in order to prove **b)** of Proposition 7, we fix $x \in M_u$. Next we consider the B-measurable map $\gamma_x : J \to X$, defined $\gamma_x(t) = f_x(t) + g_\lambda(t)x$, $t \in J$, where $f_x : J \to X$ is a

B-measurable map such that $f_x(t) \in F(t,x)$, a.e. $t \in J$ (see **F2)**) and $g_\lambda(\cdot)x : J \to X$ is B-measurable, being g_λ of (24) $\mathcal{L}$-measurable. Then γ_x is a B-measurable selection of $\Gamma_{\lambda_{|J\times M_u}}$ (see (25)). Hence **b)** of Proposition 7 holds for $\Gamma_{\lambda_{|J\times M_u}}$.

In particular, as a consequence of **(I)**, we have that, for a.e. $t \in J$, $\Gamma_{\lambda_{|J\times M_u}}(t,\cdot)$ has (s-w) sequentially closed graph, i.e. **c)** of Proposition 7 holds for the multimap $\Gamma_{\lambda_{|J\times M_u}}$.

Next we demostrate that $\Gamma_{\lambda_{|J\times M_u}}$ satisfies **d)** of Proposition 7.

To this aim, fixed $t \in J \setminus A$, where A is the null measure set presented in **FG2)**, we consider a convergent sequence $(u_n)_n$ in M_u and we fix $(y_p)_p$, $y_p \in \bigcup_n \Gamma_{\lambda_{|J\times M_u}}(t,u_n)$.

By hypothesis **FG2)** there exists $\nu_{n_u} \in L^1_+(J)$ such that

$$\beta(C_1) \le \nu_{n_u}(t)\beta(C_0), \tag{29}$$

where $C_0 = \{u_n,\ n \in \mathbb{N}\} \subset \overline{B}_X(0,n_u) = M_u$ and $C_1 = \{y_p,\ p \in \mathbb{N}\} \subset \Gamma_{\lambda_{|J\times M_u}}(t,C_0) \subset F(t,C_0) + \lambda G(t)C_0$ (see (25), (24) and (1)).

Clearly, being C_0 relatively weakly sequentially compact and so relatively weakly compact, $\beta(C_0) = 0$. Hence (29) implies that C_1 is relatively weakly compact too. Then there exists a subsequence $(y_{p_k})_k$ of $(y_p)_p$ weakly convergent in X. By the arbitrariness of $(y_p)_p$ in $\bigcup_n \Gamma_{\lambda_{|J\times M_u}}(t,u_n)$, we can conclude

$$\bigcup_n \Gamma_{\lambda_{|J\times M_u}}(t,u_n) \text{ is relatively weakly compact.} \tag{30}$$

So, $\Gamma_{\lambda_{|J\times M_u}}$ has property **d)** of Proposition 7.

Finally also **e)** of Proposition 7 is true, since in correspondence of $\overline{B}_X(0,n_u) = M_u$, there exists $\varphi_{n_u} \in L^1_+(J)$ such that (see **FG1)**, (25) and (24))

$$\sup_{z\in\Gamma_\lambda(t,M_u)} \|z\|_X \le \|F(t,M_u) + \lambda G(t)M_u\| \le \varphi_{n_u}(t),\ \text{a.e. } t \in J.$$

Therefore, by virtue of Proposition 7, in correspondence of the B-measurable map u, there exists $\gamma_u : J \to X$ B-measurable such that

$$\gamma_u(t) \in \Gamma_\lambda(t,u(t)),\ \text{a.e. } t \in J. \tag{31}$$

Since $u(t) \in M_u = \overline{B}_X(0,n_u)$, $t \in J$, by **FG1)** γ_u is B-integrable on J.

Then by (31) we can conclude that $\gamma_u \in S^1_{\Gamma_\lambda(\cdot,u(\cdot))}$, hence (28) holds.

Now we prove the existence of a solution for **(SL-I)**$_{\Gamma_\lambda}$, showing that the multioperator $\Upsilon_\lambda : \mathcal{C}(J;X) \to \mathcal{P}_c(\mathcal{C}(J;X))$, so defined (see (17), (18) and (28))

$$\Upsilon_\lambda u = \{y \in \mathcal{C}(J;X) :\ y(t) = x_0 + p(0)\overline{x}_0 P(t) + H_P\gamma(t),\ \gamma \in S^1_{\Gamma_\lambda(\cdot,u(\cdot))}\},\ u \in \mathcal{C}(J;X), \tag{32}$$

has at least one fixed point.

Clearly Υ_λ is well defined (see (28) and **F1)**).

Now prove that Υ_λ satisfies all the hypotheses of Proposition 10 through the following steps.

Step 1. The multioperator Υ_λ has weakly sequentially closed graph.

Let us fix two sequences $(u_n)_n$, $(x_n)_n$ in $\mathcal{C}(J;X)$ such that $x_n \in \Upsilon_\lambda u_n$, $n \in \mathbb{N}$, and

$$u_n \rightharpoonup u, \quad x_n \rightharpoonup x, \tag{33}$$

where $u, x \in \mathcal{C}(J;X)$. We have to prove that $x \in \Upsilon_\lambda u$.

First of all, the weak convergence of $(u_n)_n$ implies

$$u_n(t) \rightharpoonup u(t), \ t \in J \tag{34}$$

and the existence of $\overline{n} \in \mathbb{N}$ such that

$$\|u_n\|_{\mathcal{C}(J;X)} \leq \overline{n}, \ n \in \mathbb{N}. \tag{35}$$

Then, since $x_n \in \Upsilon_\lambda u_n$, $n \in \mathbb{N}$, there exists $\gamma_n \in S^1_{\Gamma_\lambda(\cdot, u_n(\cdot))}$ such that (see (18))

$$x_n(t) = x_0 + p(0)\overline{x}_0 P(t) + H_P \gamma_n(t), \ t \in J. \tag{36}$$

Now we show that the sequence $(\gamma_n)_n$ has a subsequence weakly convergent in $L^1(J; X)$. To this aim we prove that $B = \{\gamma_n : \ n \in \mathbb{N}\} \subset L^1(J; X)$ is relatively weakly compact by using Proposition 3. First, we note that (see (35))

$$\gamma_n(t) \in \Gamma_\lambda(t, u_n(t)) \subset \Gamma_\lambda(t, \overline{B}_X(0, \overline{n})), \ \text{a.e. } t \in J, \tag{37}$$

so, by (23) there exists $\varphi_{\overline{n}} \in L^1_+(J)$ such that (see (37), (25), and (24))

$$\|\gamma_n(t)\|_X \leq \varphi_{\overline{n}}(t), \ \text{a.e. } t \in J.$$

Therefore B is bounded in $L^1(J; X)$ and it has the property of equi-absolute continuity of the integral. Then, to show that $B(t)$ is relatively weakly compact for a.e. $t \in J$, let us fix $t \in J \setminus \overline{N}$, where $\overline{N}$ is the null measure set for which **FG2)** and (37) hold. Since the countable sets $C_0 = \{u_n(t), \ n \in \mathbb{N}\}$ and $C_1 = B(t)$ satisfy the inclusions (see (35) and (37))

$$C_0 \subset \overline{B}_X(0, \overline{n}), \qquad C_1 \subset \Gamma_\lambda(t, C_0) \subset F(t, C_0) + \lambda G(t) C_0, \tag{38}$$

by **FG2)** there exists $\nu_{\overline{n}} \in L^1_+(J)$ such that

$$\beta(C_1) \leq \nu_{\overline{n}}(t)\beta(C_0) = 0,$$

being C_0 relatively weakly compact (see (34)). Therefore, C_1 is relatively weakly compact too.

Hence

$$B(t) \text{ is relatively weakly compact, for a.e. } t \in J. \tag{39}$$

Thanks to Proposition 3 we have that B is relatively weakly compact in $L^1(J; X)$. So there exist a subsequence $(\gamma_{n_k})_k$ of $(\gamma_n)_n$ and $\gamma \in L^1(J; X)$ such that

$$\gamma_{n_k} \rightharpoonup \gamma. \tag{40}$$

Now we show that the map γ is a selection of $\Gamma_\lambda(\cdot, u(\cdot))$. We obtain that by applying Proposition 6 to the sequence $(\gamma_{n_k})_k$ and to the multimap $\Psi_\lambda : J \to \mathcal{P}(X)$, defined as

$$\Psi_\lambda(t) = \begin{cases} \overline{\bigcup_k \Gamma_\lambda(t, u_{n_k}(t))}^w, \ t \in J \setminus \hat{N} \\ \{0\}, \ t \in \hat{N} \end{cases} \tag{41}$$

where $\hat{N}$ is the null measure set for which **FG2)** and (37) hold.

First we note that Ψ_λ assumes weakly compact values. Indeed fixed $t \in J \setminus \hat{N}$ and $(y_p)_p$, $y_p \in \bigcup_k \Gamma_\lambda(t, u_{n_k}(t))$, put $C_0 = \{u_{n_k}(t) : \ k \in \mathbb{N}\} \subset \overline{B}_X(0, \overline{n})$ (see (35)) and $C_1 = \{y_p : \ p \in \mathbb{N}\}$, thanks to **FG2)** and analogous arguments to the ones above presented in (29)-(30), we deduce that $\overline{\bigcup_k \Gamma_\lambda(t, u_{n_k}(t))}^w$ is weakly compact. Hence $\Psi_\lambda(t) \in \mathcal{P}_{wk}(X)$, $t \in J$. Moreover hypotheses **i)** and **ii)** of Proposition 6 are obviously satified by $(\gamma_{n_k})_k$ and Ψ_λ (see (40) and (37), (41) respectively). Therefore, applying Proposition 6 we conclude

$$\gamma(t) \in \overline{co}\ w - \limsup_{k\to\infty}\{\gamma_{n_k}(t)\}, \ \text{a.e. } t \in J. \tag{42}$$

Next, if $\tilde{N} \subset J$ is a null measure set such that, for every $t \in J \setminus \tilde{N}$, (37), (42), **(I)** and **(II)** hold, we claim (see (37))

$$\overline{co}\ w - \limsup_{k\to\infty}\{\gamma_{n_k}(t)\} \subset \overline{co}\ w - \limsup_{k\to\infty} \Gamma_\lambda(t, u_{n_k}(t)), \ t \in J \setminus \tilde{N}. \tag{43}$$

Now, we note that (see (8), (34), and **(I)**)

$$w - \limsup_{k\to\infty} \Gamma_\lambda(t, u_{n_k}(t)) \subset \Gamma_\lambda(t, u(t)), \ t \in J \setminus \tilde{N}.$$

Then in virtue of **F1)** and **(II)**, we can write

$$\overline{co}\ w - \limsup_{k\to\infty} \Gamma_\lambda(t, u_{n_k}(t)) \subset \Gamma_\lambda(t, u(t)), \ t \in J \setminus \tilde{N}. \tag{44}$$

Finally, thanks to (42), (43), (44), $\gamma(t) \in \Gamma_\lambda(t, u(t))$ a.e. $t \in J$. So we can conclude

$$\gamma \in S^1_{\Gamma_\lambda(\cdot, u(\cdot))}. \tag{45}$$

Now, being $H_P\gamma_{n_k} \rightharpoonup H_P\gamma$ (see (40) and Proposition 12) we have $H_P\gamma_{n_k}(t) \rightharpoonup H_P\gamma(t)$, for every $t \in J$. Then by (36) we deduce

$$x_{n_k}(t) \rightharpoonup x_0 + p(0)\overline{x}_0 P(t) + H_P\gamma(t) =: \tilde{x}(t), \ t \in J. \tag{46}$$

From the uniqueness of the limit we can conclude that the functions $x, \tilde{x} : J \to X$, respectively defined in (33) and in (46), are the same. Recalling the definition of Υ_λ (see (32)), from (45) and (46) we obtain that $x \in \Upsilon_\lambda u$. Therefore Υ_λ has a weakly sequentially closed graph.

Step 2. There exists $\overline{p} \in \mathbb{N}$ such that the ball $\overline{B}_{\mathcal{C}(J;X)}(0_{\mathcal{C}}, \overline{p})$, where $0_{\mathcal{C}}$ denote the null function on J, is invariant under the action of the multioperator Υ_λ.

Assume by contradiction that, for every $n \in \mathbb{N}$, there exists $u_n \in \mathcal{C}(J; X)$, with $\|u_n\|_{\mathcal{C}(J;X)} \leq n$, such that there exists $x_{u_n} \in \Upsilon_\lambda u_n$, $\|x_{u_n}\|_{\mathcal{C}(J;X)} > n$.

Now, by the fact that, for every $n \in \mathbb{N}$, $\|x_{u_n}\|_{\mathcal{C}(J;X)} > n$, there exists $t_n \in J$ such that $\|x_{u_n}(t_n)\|_X > n$. Taking into account (20) we can write

$$\begin{aligned} n < \|x_{u_n}(t_n)\|_X &\leq \|x_0\|_X + \|p(0)\overline{x}_0 P(t_n)\|_X + \int_0^{t_n} \|(P(t_n) - P(s))\gamma_{u_n}(s)\|_X\, ds \\ &\leq \|x_0\|_X + L|p(0)|\|\overline{x}_0\|_X + 2L \int_0^a \|\gamma_{u_n}(s)\|_X\, ds, \end{aligned}$$

where $\gamma_{u_n} \in S^1_{\Gamma_\lambda(\cdot,u_n(\cdot))}$. Next, since $\|u_n\|_{\mathcal{C}(J;X)} \leq n$, by **FG1)** there exists $\varphi_n \in L^1_+(J)$ such that (see (28), (25), (24) and (23))

$$\|\gamma_{u_n}(t)\|_X \leq \varphi_n(t),\ a.e.\ t \in J.$$

Then we have

$$n < \|x_{u_n}(t_n)\|_X \leq \|x_0\|_X + L|p(0)|\|\overline{x}_0\|_X + 2L\|\varphi_n\|_1.$$

Hence, passing to the superior limit, remembering (22) we obtain the following contradiction

$$1 \leq \limsup_{n\to\infty} \left(\frac{\|x_0\|_X + L|p(0)|\|\overline{x}_0\|_X}{n} + \frac{2L\|\varphi_n\|_1}{n} \right) < 1.$$

Therefore the existence of $\overline{p} \in \mathbb{N}$ such that $\overline{B}_{\mathcal{C}(J;X)}(0_\mathcal{C}, \overline{p})$ is invariant under the action of the operator Υ_λ is proved.

Step 3. There exists the smallest $(0_\mathcal{C}, \Upsilon_\lambda)$-fundamental set which is weakly compact.

First of all, considering $\overline{p}$ fixed in **Step 2**, we have that $\Upsilon_\lambda(K) \subset K$, where $K = \overline{B}_{\mathcal{C}(J;X)}(0_\mathcal{C}, \overline{p})$ is a subset of the locally convex Hausdorff space $\mathcal{C}(J;X)$ equipped with the weak topology. Since $\overline{co}(\Upsilon_\lambda(K) \cup \{0_\mathcal{C}\}) \subset K$, by Proposition 9 we have that there exists M_0 the smallest $(0_\mathcal{C}, \Upsilon_\lambda)$-fundamental set such that

$$M_0 \subset \overline{B}_{\mathcal{C}(J;X)}(0_\mathcal{C}, \overline{p}) = K \tag{47}$$

$$M_0 = \overline{co}(\Upsilon(M_0) \cup \{0_\mathcal{C}\}) \tag{48}$$

Now, we will prove that M_0 is weakly compact.

To this aim, we establish the relative weak sequential compactness of $S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$. At first we show that the set $M_0(t)$ is relatively weakly compact, for every $t \in J$.

Let us consider the Sadovskij functional β_α, $\alpha \in \mathbb{R}^+$, defined in (9). Since β_α is 0-stable and it satisfies **(J)** and **(JJ)**, from (48), (32) and (18) we have

$$\begin{aligned}
\beta_\alpha(M_0) &= \beta_\alpha(\Upsilon_\lambda(M_0)) \\
&= \beta_\alpha\left(\{x_0 + p(0)\overline{x}_0 P(\cdot) + H_P\gamma :\ \gamma \in S^1_{\Gamma_\lambda(\cdot,u(\cdot))},\ u \in M_0\}\right) \\
&\leq \beta_\alpha(\{x_0\}) + \beta_\alpha(\{p(0)\overline{x}_0 P(\cdot)\}) + \beta_\alpha(\{H_P\gamma :\ \gamma \in S^1_{\Gamma_\lambda(\cdot,u(\cdot))},\ u \in M_0\}) \\
&= \beta_\alpha(\{H_P\gamma :\ \gamma \in S^1_{\Gamma_\lambda(\cdot,u(\cdot))},\ u \in M_0\}) \\
&= \sup_{\substack{C\subset S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))} \\ C \text{ countable}}} \ \sup_{t\in J} \beta\left(\left\{\int_0^t (P(t)-P(s))\gamma(s)\,ds :\ \gamma \in C\right\}\right) e^{-\alpha t},
\end{aligned} \tag{49}$$

where the last equality is deduced by the definition of the set $\{H_P\gamma :\ \gamma \in S^1_{\Gamma_\lambda(\cdot,u(\cdot))},\ u \in M_0\}$.

Now, fixed $t \in J$ and a countable set $C \subset S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$, since X is a weakly compactly generated Banach space, we can apply Proposition 11 to the countable set

$$C_t^C = \{(P(t) - P(\cdot))\gamma(\cdot),\ \gamma \in C\}, \tag{50}$$

being C_t^C bounded in $L^1(J;X)$ and having the property of equi-absolute continuity of the integral (see **FG1)** and (20)), and so we have

$$\beta\left(\left\{\int_0^t (P(t)-P(s))\gamma(s)\,ds:\ \gamma\in C\right\}\right)\le\int_0^t \beta(C_t^C(s))\,ds. \tag{51}$$

Next, by the Axiom of Choice, for every $\gamma\in C$, we can consider a continuous function $u_\gamma\in M_0$ such that $\gamma(s)\in\Gamma_\lambda(s,u_\gamma(s))$ a.e. $s\in J$. So we construct the countable subset of M_0

$$C_0^C=\{u_\gamma\in M_0:\ \gamma\in C\} \tag{52}$$

Obviously, since C is countable, there exists a null measure subset $V\subset J$, containing the set A defined in **FG2**), such that

$$\gamma(s)\in\Gamma_\lambda(s,u_\gamma(s)),\ s\in J\setminus V,\ \gamma\in C,$$

where $u_\gamma\in C_0^C$. Hence, fixed $s\in J\setminus V$, by (52) we deduce (see (47) and (25))

$$C_0^C(s)\subset M_0(s)\subset\overline{B}_X(0,\overline{p})$$

$$C(s)\subset F(s,C_0^C(s))+\lambda G(s)C_0^C(s).$$

Hence, by hypothesis **FG2)**, there exists $\nu_{\overline{p}}\in L^1_+(J)$ such that

$$\beta(C(s))\le\nu_{\overline{p}}(s)\beta(C_0^C(s)). \tag{53}$$

Since (53) holds for a.e. $s\in J$, for the fixed t we deduce the following inequalities

$$\int_0^t\beta(C(s))\,d\xi\le\int_0^t\nu_{\overline{p}}(s)\beta(C_0^C(s))\,d\xi\le\int_0^t\nu_{\overline{p}}(s)\sup_{\substack{C_0\subset M_0\\ C_0\ \text{countable}}}\beta(C_0(s))\,ds. \tag{54}$$

Now, recalling (50) and the semi-homogeneity of β, we have

$$\beta(C_t^C(s))=\beta((P(t)-P(s))C(s))=|P(t)-P(s)|\beta(C(s)),\ \text{a.e. } s\in[0,t],$$

and so we can write

$$\int_0^t\beta(C_t^C(s))\,ds=\int_0^t|P(t)-P(s)|\beta(C(s))\,ds. \tag{55}$$

The above considerations allow us to claim that for every countable set $C\subset S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$ there exists a countable subset $C_0^C\subset M_0$ such that (51), (54) and (55) are true.

Now from (49), by using (51), (55), (19) and (54) we deduce

$$\begin{aligned}\beta_\alpha(M_0)&\le\sup_{\substack{C\subset S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}\\ C\ \text{countable}}}\ \sup_{t\in J}\left(\int_0^t\beta(C_t^C(s))\,ds\right)e^{-\alpha t}\\ &=\sup_{\substack{C\subset S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}\\ C\ \text{countable}}}\ \sup_{t\in J}\left(\int_0^t|P(t)-P(s)|\beta(C(s))\,ds\right)e^{-\alpha t}\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\substack{C \subset S^1_{\Gamma_\lambda(\cdot, M_0(\cdot))} \\ C \text{ countable}}} \sup_{t \in J} \left(2L \int_0^t \beta(C(s))\, ds \right) e^{-\alpha t} \\
&\leq \sup_{\substack{C \subset S^1_{\Gamma_\lambda(\cdot, M_0(\cdot))} \\ C \text{ countable}}} \sup_{t \in J} \left(2L \int_0^t \nu_{\overline{p}}(s) \sup_{\substack{C_0 \subset M_0 \\ C_0 \text{ countable}}} \beta(C_0(s))\, ds \right) e^{-\alpha t} \\
&\leq \sup_{t \in J} \left(2L \int_0^t e^{-\alpha(t-s)} \nu_{\overline{p}}(s) \sup_{\substack{C_0 \subset M_0 \\ C_0 \text{ countable}}} \sup_{s \in J} e^{-\alpha s} \beta(C_0(s))\, ds \right) \\
&= \beta_\alpha(M_0) \sup_{t \in J} \int_0^t 2L e^{-\alpha(t-s)} \nu_{\overline{p}}(s)\, ds. \qquad (56)
\end{aligned}
$$

By virtue of Proposition 2 we can say that there exists $m \in \mathbb{N}$ such that

$$
\sup_{t \in J} \int_0^t 2L e^{-m(t-s)} \nu_{\overline{p}}(s)\, ds < 1. \qquad (57)
$$

By considering (56) for $\alpha = m$, taking into account (57), we deduce

$$
\beta_m(M_0) = 0. \qquad (58)
$$

Then, for every $t \in J$, by definition of $\beta_m(M_0)$, we have that the set $M_0(t)$ is relatively weakly sequentially compact. Indeed, fixed a sequence $(z_n)_n$, $z_n \in M_0(t)$, $n \in \mathbb{N}$, we consider the countable set $\tilde{C}_t = \{z_n : n \in \mathbb{N}\}$. By (58) we have $\beta(\tilde{C}_t) = 0$ and, since β is regular the set $\tilde{C}_t$ is relatively weakly compact. Hence there exists a subsequence $(z_{n_k})_k$ of $(z_n)_n$ such that $z_{n_k} \rightharpoonup z \in X$. Therefore, by the arbitrariness of the sequence $(z_n)_n$, we can claim the relative weak sequential compactness of $M_0(t)$.

Finally, we are in a position to show that the set $S^1_{\Gamma_\lambda(\cdot, M_0(\cdot))}$ is relatively weakly compact in $L^1(J; X)$.

First of all, since (47) holds, by **FG1)** there exist $\varphi_{\overline{p}} \in L^1_+(J)$ and a null measure set $N \subset J$ such that

$$
\|\gamma(t)\|_X \leq \varphi_{\overline{p}}(t), \; t \in J \setminus N, \; \gamma \in S^1_{\Gamma_\lambda(\cdot, M_0(\cdot))}. \qquad (59)
$$

Therefore $S^1_{\Gamma_\lambda(\cdot, M_0(\cdot))}$ has the property of equi-absolute continuity of the integral and it is bounded in $L^1(J; X)$.

Then we note that, for a.e. $t \in J$, the set $S^1_{\Gamma_\lambda(t, M_0(t))}$ is relatively weakly compact in X.

Let us consider the null measure set $N^* \supset A \cup N$, where A and N are presented in **FG2)** and (59) respectively. Now, fixed $t \in J \setminus N^*$ we note that $S^1_{\Gamma_\lambda(t, M_0(t))}$ is norm bounded in X by the constant $\varphi_{\overline{p}}(t)$.

Next, we consider a sequence $(y_n)_n$, $y_n \in S^1_{\Gamma_\lambda(t, M_0(t))}$, $n \in \mathbb{N}$. Then there exists a sequence $(g_n)_n$, where g_n is the representative of an element in $S^1_{\Gamma_\lambda(\cdot, M_0(\cdot))}$, satisfying $y_n = g_n(t) \in \Gamma_\lambda(t, M_0(t))$, $n \in \mathbb{N}$. So, for every $n \in \mathbb{N}$, there exists $u_n \in M_0$ such that $y_n \in \Gamma_\lambda(t, u_n(t))$.

Now, by considering the two countable sets $C_0 = \{u_n(t) : n \in \mathbb{N}\} \subset \overline{B}_X(0, \overline{p})$ (see (47)) and $C_1 = \{y_n : n \in \mathbb{N}\}$ and reasoning as in **Step 1** (see from (38) to (39)) we have that C_1 is relatively weakly compact, hence there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ such that $(y_{n_k})_k$ is weakly convergent.

By the arbitrariness of $(y_n)_n$ in $S^1_{\Gamma_\lambda(t,M_0(t))}$, we can conclude that $S^1_{\Gamma_\lambda(t,M_0(t))}$ is relatively weakly sequentially compact. So, by using Eberlein Smulian Theorem the set $S^1_{\Gamma_\lambda(t,M_0(t))}$ is relatively weakly compact.

Therefore, since $S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$ is bounded in $L^1(J;X)$ and it has the equi-absolute continuity of the integral property, we have that $S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$ is relatively weakly compact in $L^1(J;X)$ (see Proposition 3) and so $S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$ is relatively weakly sequentially compact too.

Now, in order to obtain the weak compactness of M_0, by (48) it is sufficient to show that $\Upsilon_\lambda(M_0)$ is relatively weakly compact in $\mathcal{C}(J;X)$.

To this aim we fix a sequence $(x_n)_n$, $x_n \in \Upsilon_\lambda(M_0)$. Then there exists $(u_n)_n$, $u_n \in M_0$, such that, for every $n \in \mathbb{N}$, $x_n \in \Upsilon_\lambda u_n$, hence

$$x_n(t) = x_0 + p(0)\overline{x}_0 P(t) + \int_0^t (P(t) - P(s))\gamma_n(s)\,ds,\ t \in J,\ n \in \mathbb{N} \tag{60}$$

where $\gamma_n \in S^1_{\Gamma_\lambda(\cdot,u_n(\cdot))} \subset S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$.

By the relative weak sequential compactness of $S^1_{\Gamma_\lambda(\cdot,M_0(\cdot))}$ in $L^1(J;X)$ we can find a subsequence $(\gamma_{n_k})_k$ of $(\gamma_n)_n$ and $\gamma \in L^1(J;X)$, such that $\gamma_{n_k} \rightharpoonup \gamma$. Then, being $H_P\gamma_{n_k} \rightharpoonup H_P\gamma$ (see Proposition 12), we have

$$H_P\gamma_{n_k}(t) \rightharpoonup H_P\gamma(t),\ t \in J. \tag{61}$$

Now, let us consider the subsequence $(x_{n_k})_k$ of $(x_n)_n$ defined in (60). By using (61) we can write

$$x_{n_k}(t) \rightharpoonup x_0 + p(0)\overline{x}_0 P(t) + H_P\gamma(t) := \overline{x}(t),\ t \in J,$$

where $\overline{x} : J \to X$ is a continuous map. Next, recalling that $x_{n_k} \in \Upsilon_\lambda(M_0)$, for every n_k, we have (see (48) and (47))

$$\|x_{n_k} - \overline{x}\|_{\mathcal{C}(J;X)} \le \overline{p} + \|\overline{x}\|_{\mathcal{C}(J;X)},$$

i.e. the sequence $(x_{n_k} - \overline{x})_k$ is uniformly bounded on J. Then we can say that $x_{n_k} \rightharpoonup \overline{x}$.

Hence, by the arbitrariness of $(x_n)_n$ in $\Upsilon_\lambda(M_0)$, we can claim that $\Upsilon_\lambda(M_0)$ is relatively weakly sequentially compact and so, the Eberlein Smulian Theorem allows us to state that $\Upsilon_\lambda(M_0)$ is relatively weakly compact.

Finally, since $M_0 = \overline{co}(\Upsilon_\lambda(M_0) \cup \{0\})$, we are able to conclude the weak compactness of M_0.

From the above considerations we are in a position to apply Proposition 10 to the multi-operator $\Upsilon_\lambda : M_0 \to \mathcal{P}_c(M_0)$. Hence there exists $x \in M_0$ such that

$$x(t) = x_0 + p(0)\overline{x}_0 P(t) + \int_0^t (P(t) - P(s))\gamma(s)\,ds,\ t \in J$$

where $\gamma \in S^1_{\Gamma_\lambda(\cdot,x(\cdot))}$. Of course, x is a solution for the integral inclusion **(SL-I)**$_{\Gamma_\lambda}$. Now, since $\gamma \in L^1(J;X)$ and $\gamma(t) \in \Gamma_\lambda(t,x(t))$, a.e. $t \in J$, by (25) and (24) we deduce $\gamma \in S^1_{F(\cdot,x(\cdot))+\lambda G(\cdot)x(\cdot)}$. Therefore we have that x is a solution for **(SL-I)**. □

5 Existence Results for a Sturm-Liouville Problem

In this section we achive the existence of mild solutions and strong solutions for the Cauchy problem **(SL)**. We start with results on the existence of mild ones.

Theorem 14
Let X be a weakly compactly generated Banach space, $\lambda \in \mathbb{R}$ and $x_0, \overline{x}_0 \in X$. Let $p : J \to (0, \infty)$ be a function satisfying
p2) $p \in \mathcal{C}(J)$,
$F : J \times X \to \mathcal{P}(X)$ and $G : J \to \mathcal{P}(\mathbb{R})$ be two multimaps having the properties ***F1)**, **F2)**, **F3)**, **G1)**, **G2)**, **FG1)**, and **FG2)** of Theorem 13.*
Then there exists at least one mild solution for problem **(SL)**.

Proof First of all, we note that a mild solution $x : J \to X$ for problem **(SL)** satisfying

$$x(t) = x_0 + p(0)\overline{x}_0 \int_0^t \frac{1}{p(s)}\, ds + \int_0^t \frac{1}{p(s)} \int_0^s \gamma_\lambda(\tau)\, d\tau ds,\ t \in J,$$

where $\gamma_\lambda \in S^1_{\lambda G(\cdot)x(\cdot)+F(\cdot,x(\cdot))}$, can be rewritten as

$$x(t) = x_0 + p(0)\overline{x}_0 P(t) + H_P \gamma_\lambda(t),\ t \in J, \tag{62}$$

where $P : J \to [0, \infty)$ and $H_P : L^1(J; X) \to \mathcal{C}(J; X)$ are defined as in (17) and in (18) respectively.

To prove that, we have only to show

$$\int_0^t \frac{1}{p(s)} \int_0^s \gamma_\lambda(\tau)\, d\tau ds = H_P \gamma_\lambda(t),\ t \in J, \tag{63}$$

where $H_p \gamma_\lambda(t) = \int_0^t (P(t) - P(s))\gamma_\lambda(s)\, ds, t \in J$. First of all, from (17), since p is a positive and continuous function, we have

$$P'(t) = \frac{1}{p(t)},\ t \in J.$$

Hence, the request of continuity on p allows us to apply the formula of integration by parts (see [34], Theorem 3.3) and so we can write (see (18))

$$\begin{aligned}\int_0^t \frac{1}{p(s)} \int_0^s \gamma_\lambda(\tau)\, d\tau ds &= \int_0^t P'(s) \int_0^s \gamma_\lambda(\tau)\, d\tau ds \\ &= P(t) \int_0^t \gamma_\lambda(s)\, ds - \int_0^t P(s)\gamma_\lambda(s)\, ds \\ &= \int_0^t (P(t) - P(s))\gamma_\lambda(s)\, ds = H_P \gamma_\lambda(t),\ t \in J,\end{aligned}$$

therefore (63) is true.

By the above consideration we can claim that the existence of a mild solution for the Cauchy problem **(SL)** is equivalent to prove the existence of a solution for the integral inclusion **(SL-I)**.

Observing now that the continuity and positivity of the map p allow us to say that $\frac{1}{p} \in L^1(J)$, we have that all the hypotheses of Theorem 13 are satisfied. Then the existence of a mild solution for the Sturm-Liouville Cauchy problem **(SL)** follows from Theorem 13. □

Now we present, in the setting of separable Banach spaces, the following result in which we establish the existence of mild solutions for the Cauchy problem **(SL)** without assumptions about the values of the multimap F.

Theorem 15

Let X be a separable Banach space, $\lambda \in \mathbb{R}$ and $x_0, \overline{x}_0 \in X$. Let $p : J \to (0, \infty)$ be a function satisfying **p2**), *$F : J \times X \to \mathcal{P}(X)$, $G : J \to \mathcal{P}(\mathbb{R})$ be two multimaps having the properties*

(l-SD)$_F$ *$\forall \varepsilon > 0$ there exists a compact $K_\varepsilon \subset J$ such that $\mu(J \setminus K_\varepsilon) < \varepsilon$ and $F_{|K_\varepsilon \times X}$ is weakly lower semicontinuous*;

(M)$_F$ *for every weakly closed set $Z \subset J \times X$ such that $F_{|Z}$ is weakly lower semicontinuous, there exists a weakly continuous selection of F on Z, i.e. there exists a weakly continuous function $f : Z \to X$ such that $f(t, x) \in F(t, x)$, $(t, x) \in Z$;*

and ***G1***), ***G2***), ***FG1***) *and* ***FG2***) *of Theorem 13.*

Then there exists at least one mild solution for the Cauchy problem **(*SL*)**.

Proof First we consider (X, τ_w) the topological Hausdorff space, where τ_w is the weak topology relatively to the Banach space X. Hence, taking into account of **(l-SD)**$_F$ and **(M)**$_F$, in virtue of Proposition 8 we have that there exists $f : J \times X \to X$ such that

$$\text{for every } t \in J,\ \ f(t, \cdot) \text{ is weakly continuous;} \tag{64}$$

$$\text{for every } x \in X,\ \ f(\cdot, x) \text{ is weakly measurable;} \tag{65}$$

$$f(t, x) \in F(t, x),\ \text{ a.e. } t \in J,\ \forall x \in X. \tag{66}$$

Now, put $\tilde{F} : J \times X \to \mathcal{P}(X)$ so defined

$$\tilde{F}(t, x) = \{f(t, x)\},\ \ (t, x) \in J \times X, \tag{67}$$

we show that Theorem 14 allows us to prove the existence of a mild solution for the following Cauchy problem

$$\begin{cases} (p(t)x'(t))' \in \lambda G(t)x(t) + \tilde{F}(t, x(t)),\ t \in J \\ x(0) = x_0 \\ x'(0) = \overline{x}_0. \end{cases} \tag{68}$$

For the sake of clarity, let us denote with $\tilde{\mathbf{F}}\mathbf{1}$), $\tilde{\mathbf{F}}\mathbf{2}$) and so on the hypotheses of Theorem 14 referring to the multimap $\tilde{F}$.

First we say that the multimap $\tilde{F}$ obviously satisfies $\tilde{\mathbf{F}}\mathbf{1}$) of Theorem 14. Then, by using the separability of X and (65) we have that, for every $x \in X$, the function $f(\cdot, x) : J \to X$ is B-measurable. So $\tilde{\mathbf{F}}\mathbf{2}$) of Theorem 14 holds.

Now we prove that $\tilde{F}$ has property $\tilde{\mathbf{F}}\mathbf{3}$) of Theorem 14.

To this aim, fixed $t \in J$, we consider $(x_n)_n$ and $(y_n)_n$, $y_n \in \tilde{F}(t, x_n)$, two arbitrary sequences in X such that

$$x_n \rightharpoonup x \tag{69}$$

and

$$y_n \rightharpoonup y \tag{70}$$

where $x, y \in X$. We want to show that $y \in \tilde{F}(t, x)$.

Now, from (64) we deduce that $f(t, \cdot)$ is weakly sequentially continuous. So, from (69) we have that

$$y_n = f(t, x_n) \rightharpoonup f(t, x). \tag{71}$$

Because of the uniqueness of the weak limit, from (70) and (71) we infer that $y = f(t, x) \in \tilde{F}(t, x)$ (see (67)). Thus also $\tilde{\mathbf{F}}\mathbf{3}$) of Theorem 14 is true.

Now, in order to prove $\tilde{\mathbf{F}}\mathbf{G1}$) of Theorem 14, let us fix $t \in J \setminus H^*$, where H^* is the null measure set for which (66) and **FG1)** hold. Thanks to **FG1)** we know that there exists $(\varphi_n)_n$, $\varphi_n \in L^1_+(J)$, satisfying (22) and such that (see (67) and (66))

$$\|\tilde{F}(t, \overline{B}_X(0,n)) + \lambda G(t)\overline{B}_X(0,n)\| \leq \|F(t, \overline{B}_X(0,n)) + \lambda G(t)\overline{B}_X(0,n)\| \leq \varphi_n(t),\ \ n \in \mathbb{N}.$$

Therefore $\tilde{F}$ and G have the property $\tilde{\mathbf{F}}\mathbf{G1}$) of Theorem 14.

Finally we will show that also property $\tilde{\mathbf{F}}\mathbf{G2}$) of Theorem 14 holds. Indeed, put A the set presented in **FG2)**, let $\tilde{H}$ be the null measure set for which (66) is true and $A \subset \tilde{H}$.

Fixed $t \in J \setminus \tilde{H}$ and $n \in \mathbb{N}$, let $\tilde{C}_0$ be a countable subset of $\overline{B}_X(0,n)$ and $\tilde{C}_1$ a countable subset of $\tilde{F}(t, \tilde{C}_0) + \lambda G(t)\tilde{C}_0$. Now, by virtue of (66) and (67) we have

$$\tilde{C}_1 \subset f(t, \tilde{C}_0) + \lambda G(t)\tilde{C}_0 \subset F(t, \tilde{C}_0) + \lambda G(t)\tilde{C}_0$$

and so, thanks to **FG2)**, there exists $\nu_n \in L^1_+(J)$ such that

$$\beta(\tilde{C}_1) \leq \nu_n(t)\beta(\tilde{C}_0).$$

This proves that $\tilde{\mathbf{F}}\mathbf{G2}$) of Theorem 14 is true too.

In conclusion, recalling that the separable Banach space X is WCG, all the hypotheses of Theorem 14 are fulfilled. So there exists at least one mild solution for (68), which also is a mild solution for the problem **(SL)**. □

Finally, in the setting on RNP-Banach spaces, taking into account of the relations between strong and mild solutions described in Sect. 3, thanks to Theorem 14 and Theorem 15 we are in a position to enunciate the following two results that ensure the existence of strong solutions for the problem **(SL)**.

Theorem 16
Let X be a weakly compactly generated RNP-Banach space, $\lambda \in \mathbb{R}$ and $x_0, \overline{x}_0 \in X$. Let $p : J \to (0, \infty)$ be a function satisfying

***p*3)** $p \in AC(J)$,

and $F : J \times X \to \mathcal{P}(X)$, $G : J \to \mathcal{P}(\mathbb{R})$ be two multimaps having the properties ***F*1)**, ***F*2)**, ***F*3)**, ***G*1)**, ***G*2)**, ***FG*1)** *and* ***FG*2)** *of Theorem 13.*

Then there exists at least one function $x : J \to X$ defined

$$x(t) = x_0 + p(0)\overline{x}_0 \int_0^t \frac{1}{p(s)}\, ds + \int_0^t \frac{1}{p(s)} \int_0^s \gamma(\tau)\, d\tau ds,\ t \in J, \tag{72}$$

where $\gamma \in S^1_{F(\cdot,x(\cdot))+\lambda G(\cdot)x(\cdot)}$, strong solution for problem **(*SL*)**.

Theorem 17
Let X be a separable RNP-Banach space, $\lambda \in \mathbb{R}$ and $x_0, \overline{x}_0 \in X$. Let $p : J \to (0, \infty)$ be a function satisfying ***p*3)**, *$F : J \times X \to \mathcal{P}(X)$, $G : J \to \mathcal{P}(\mathbb{R})$ be two multimaps having the properties* $(\boldsymbol{l}\text{-}\boldsymbol{SD})_F$ *and* $(\boldsymbol{M})_F$ *of Theorem 15,* ***G*1)**, ***G*2)**, ***FG*1)** *and* ***FG*2)** *of Theorem 13.*

Then there exists at least one strong solution of the type (72) for the Cauchy problem **(*SL*)**.

6 Application: Study of Controllability

Now we turn our attention to establish the controllability for the following Cauchy problem driven by a Sturm-Liouville equation

$$\begin{cases} ((t^2+1)x'(t))' = \lambda \sin(t)\, x(t) + h(t) + u(t), \text{ a.e. } t \in J = [0, \frac{\sqrt{3}}{3}] \\ x(0) = x_0 \\ x'(0) = \overline{x}_0 \\ u(t) \in U(t, x(t)), \text{ a.e. } t \in J \end{cases} \tag{73}$$

where $x_0, \overline{x}_0 \in L^2(J)$, $\lambda \in [0, 1]$, $h : J \to L^2(J)$ is a function such that

h1) h is B-measurable;
h2) $\|h(\cdot)\|_2 \in L^1_+(J)$,

and $U : J \times L^2(J) \to \mathcal{P}(L^2(J))$ is the multimap so defined

$$U(t, x) = \{y \in L^2(J) : \exists b \in L^2(J), \|b\|_2 \le r(t), \text{ such that } y = x + b\}, \tag{74}$$

with $r : J \to \mathbb{R}^+$ a fixed map.

The approach that we shall follow is to rewrite problem (73) in the form of problem **(SL)**, which we study using our Theorem 16.

First of all, we recall that the reflexive space $X = L^2(J)$ is a weakly compactly generated and RNP-Banach space.

Now, we rewrite problem (73) in the form

$$\begin{cases} ((t^2+1)x'(t))' \in \lambda G(t)x(t) + F(t, x(t)), \text{ a.e. } t \in J \\ x(0) = x_0 \\ x'(0) = \overline{x}_0, \end{cases} \tag{75}$$

where $G : J \to \mathcal{P}(\mathbb{R})$ and $F : J \times L^2(J) \to \mathcal{P}(L^2(J))$ are so defined

$$G(t) = \{\sin(t)\}, \ t \in J$$

$$F(t, x) = U(t, x) + \{h(t)\} \tag{76}$$

and we prove that G and F satisfy all the hypotheses of Theorem 16.

Obviously F assumes nonempty values since, fixed $(t, x) \in J \times L^2(J)$, we have that $x + h(t) \in F(t, x)$, being $x + 0_{L^2(J)} \in U(t, x)$.

Moreover, for every $(t, x) \in J \times L^2(J)$, $F(t, x)$ is convex, being $F(t, x) = \{h(t) + x\} + \overline{B}_{L^2(J)}(0, r(t))$. So **F1)** of Theorem 16 is true.

On the other hand, fixed $x \in L^2(J)$ and $f : J \to L^2(J)$ as

$$f(t) = h(t) + x, \ t \in J,$$

by (76), since $x \in U(t, x)$, we can say

$$f(t) \in F(t, x).$$

Moreover f is B-measurable (see **h1)**), so **F2)** of Theorem 16 holds.

To prove **F3)**, let us fix $t \in J$, $(x_n)_n$, $x_n \in L^2(J)$, such that $x_n \rightharpoonup x \in L^2(J)$ and $(y_n)_n$, $y_n \in F(t, x_n)$, such that $y_n \rightharpoonup y \in L^2(J)$.

Since, for every $n \in \mathbb{N}$, $y_n \in F(t, x_n)$, from (76) there exists $v_n \in U(t, x_n)$ such that

$$y_n = v_n + h(t).$$

Then, by $y_n \rightharpoonup y$ we have

$$v_n = y_n - h(t) \rightharpoonup y - h(t) := v,$$

and so we can write $y = v + h(t)$. Now, we note that $v \in U(t, x)$. Indeed, for every $n \in \mathbb{N}$, being $v_n \in U(t, x_n)$, there exists $b_n \in L^2(J)$, $\|b_n\|_2 \leq r(t)$, such that (see (74))

$$v_n = x_n + b_n.$$

Then, thanks to the weak convergences $x_n \rightharpoonup x$ and $v_n \rightharpoonup v$, we have

$$b_n = v_n - x_n \rightharpoonup v - x := b. \tag{77}$$

Clearly $b \in L^2(J)$. Moreover, as a consequence of (77), we say (see [32], Proposition 3.5)

$$\|b\|_2 \leq \liminf_{n\to\infty} \|b_n\|_2 \leq r(t).$$

Hence $v = x + b \in U(t, x)$ (see (74)) and so $y = v + h(t) \in F(t, x)$ (see (76)).

Therefore also **F3)** of Theorem 16 is satisfied.

About the multimap G we immediately note that $G(t)$ is closed, for every $t \in J$. Moreover G is obviously measurable.

So G has the properties **G1)** and **G2)** of Theorem 16.

Now, in order to establish **FG1)** and **FG2)** of Theorem 16 we recall that, being $p(t) = t^2 + 1$, $t \in J = [0, \frac{\sqrt{3}}{3}]$, $L = \frac{\pi}{6}$ (see (19)). Then we can consider the sequence $(\varphi_n)_n$, where $\varphi_n : J \to \mathbb{R}_0^+$ is defined as

$$\varphi_n(t) = n(1 + \sin(t)) + \|b\|_2 + \|h(t)\|_2, \;\; t \in J, \;\; n \in \mathbb{N}. \tag{78}$$

Clearly, from **h2)**, $\varphi_n \in L^1_+(J)$, $n \in \mathbb{N}$. To show that (23) is true by using this sequence $(\varphi_n)_n$, let us fix $n \in \mathbb{N}$, $t \in J$ and $x, x^* \in \overline{B}_{L^2(J)}(0, n)$. Now, for every $y \in F(t, x) + \lambda G(t)x^*$ there exists $b \in L^2(J)$: $\|b\|_2 \leq r(t)$, such that (see (74) and (76))

$$y = x + b + h(t) + \lambda x^* \sin(t).$$

Since $\lambda \in [0, 1]$, we have (see (78))

$$\|y\|_2 \leq \|x\|_2 + \|b\|_2 + \|h(t)\|_2 + \lambda \sin(t)\|x^*\|_2 \leq \varphi_n(t).$$

By the arbitrariness of $y \in F(t, x) + \lambda G(t)x^*$ and $x, x^* \in \overline{B}_{L^2(J)}(0, n)$, we deduce that

$$\|F(t, \overline{B}_{L^2(J)}(0, n)) + \lambda G(t)\overline{B}_{L^2(J)}(0, n)\| \leq \varphi_n(t), \;\; t \in J, \;\; n \in \mathbb{N}, \tag{79}$$

i.e. (23) holds.

Moreover we note that

$$\limsup_{n\to\infty} \frac{2L\int_0^{\frac{\sqrt{3}}{3}} \varphi_n(s)\,ds}{n} = \frac{\pi}{3}\left(\frac{\sqrt{3}}{3}+1-\cos\left(\frac{\sqrt{3}}{3}\right)\right) < 1,$$

and so we can conclude that **FG1)** is true.

Finally to claim that **FG2)** holds we recall the reflexivity of the Banach space $L^2(J)$.

Now, fixed $n \in \mathbb{N}$, for every countable $C_0 \subset \overline{B}_{L^2(J)}(0,n)$ we have $\beta(C_0) = 0$, being β the De Blasi measure of weak noncompactness. Moreover, for every $t \in J$ and every countable $C_1 \subset F(t, C_0) + \lambda G(t)C_0$, taking into account (79) we deduce the relative weak compactness of the set C_1 in $L^2(J)$ and so $\beta(C_1) = 0$. Hence being $\beta(C_1) = \beta(C_0) = 0$, we have **FG2)** of Theorem 16 true too (even assuming $A = \emptyset$ and $(\nu_n)_n$, $\nu_n = 0_{L^1_+(J)}$, $n \in \mathbb{N}$).

By means the arguments above presented, we are in a position to apply Theorem 16 to problem (75). Then there exists a $\mathcal{C}^1$-function $x : J \to L^2(J)$ such that (see (62), (18) and (17))

$$x(t) = x_0 + \overline{x}_0 \arctan(t) + \int_0^t (\arctan(t) - \arctan(s))\gamma(s)\,ds,\ t \in J,$$

where $\gamma \in S^1_{F(\cdot,x(\cdot))+\lambda G(\cdot)x(\cdot)}$, which is a strong solution for (75).

Next we consider $u_x : J \to L^2(J)$ so defined

$$u_x(t) = \gamma(t) - \lambda \sin(t)x(t) - h(t),\ t \in J, \tag{80}$$

which is obviously B-measurable being the difference of B-measurable maps (see **h1)**).

Now, to show that u_x is a selection of $U(\cdot, x(\cdot))$, let us fix $t \in J \setminus N$ such that $\gamma(t) \in F(t, x(t)) + \lambda G(t)x(t)$, where N is a null measure subset of J. Then by (76) there exists $v_t \in U(t, x(t))$ such that

$$\gamma(t) = v_t + h(t) + \lambda \sin(t)x(t).$$

On the other hand, from (80) we have $\gamma(t) = u_x(t) + h(t) + \lambda \sin(t)x(t)$ and so $u_x(t) = v_t \in U(t, x(t))$. Hence $u_x(t) \in U(t, x(t))$ for a.e. $t \in J$.

For the above considerations we conclude that $\{x, u_x\}$ is an admissible strong-pair for problem (73), where $x : J \to L^2(J)$ is a $\mathcal{C}^1$-function such that $x' \in AC(J; L^2(J))$, x' differentiable a.e. on J, $x'' \in L^1(J; L^2(J))$, and $u_x : J \to L^2(J)$ is a B-measurable map.

Remark 3

Let us note that if we study the controllability of (73) in the WCG Banach space $X = \mathcal{C}(J)$, which is not RNP, the controllability could be given by the multimap $V : J \times X \to \mathcal{P}(X)$ so defined $V(t, x) = \{y \in \mathcal{C}(J) : \exists b \in \mathcal{C}(J), \|b\|_\infty \le q(t),\ y = x + b\}$, $(t, x) \in J \times \mathcal{C}(J)$. We observe that, by considering the same function r, fixed $(t, x) \in J \times \mathcal{C}(J)$, $V(t, x)$ is strictly included in the control set $U(t, x)$ (see (74)). Indeed, we have that if $y \in V(t, x)$ we deduce that $y \in L^2(J)$ and there exists $b \in \mathcal{C}(J) \subset L^2(J)$ such that $y = x + b$ satisfying

$$\|b\|_2 \le \left\{\int_0^{\frac{\sqrt{3}}{3}} \|b\|_\infty^2\,ds\right\}^{\frac{1}{2}} = \|b\|_\infty \sqrt{\frac{\sqrt{3}}{3}} \le q(t).$$

So we can say that $y \in U(t, x)$.

Therefore, in the setting $\mathcal{C}(J)$ we analyse a different situation respect to the one in $L^2(J)$, in which we are more constrained in controlling the system. In this case we are no longer able to use Theorem 16, but we can only establish the existence of an admissible mild-pair for problem (73), by applying Theorem 14.

7 Conclusions

In this paper, the existence of mild solutions and strong solutions to the Cauchy problem governed by a semilinear differential Sturm-Liouville inclusion **(SL)** is studied. Let us note that the existence results obtained in Sect. 5 for **(SL)** allow us to deduce analogous existence theorems for the more general problem

$$\textbf{(SL-q)}\begin{cases}(p(t)x'(t))' + q(t)x(t) \in \lambda G(t)x(t) + F(t, x(t)), \text{ a.e. } t \in J\\ x(0) = x_0\\ x'(0) = \overline{x}_0,\end{cases}$$

where $q : J \to \mathbb{R}$ is a nonnegative $\mathcal{L}$-measurable function.

Indeed we can rewrite problem **(SL-q)** in the form of **(SL)** introducing the multimap $G_q : J \to \mathcal{P}(\mathbb{R})$ so defined

$$G_q(t) = \{-q(t)\} + \lambda G(t)$$

By doing so we are able to obtain the analogous of Theorems 14-17 for **(SL-q)** assuming, instead of **FG1)** and **FG2)**, the following

FGq1) there exists $(\varphi_n)_n$, $\varphi_n \in L^1_+(J)$ such that

$$\limsup_{n\to\infty} \frac{2L\|\varphi_n\|_1}{n} < 1,$$

where L is the positive costant presented in (19), and

$$\|F(t, \overline{B}_X(0, n)) + (\{-q(t)\} + \lambda G(t))\overline{B}_X(0, n)\| \le \varphi_n(t), \ a.e.\ t \in J, \ n \in \mathbb{N};$$

FGq2) there exists $A \subset J$, $\mu(A) = 0$, such that, for all $n \in \mathbb{N}$, there exists $\nu_n \in L^1_+(J)$ with the property

$$\beta(C_1) \le \nu_n(t)\beta(C_0), \ t \in J \setminus A$$

for all countable C_0, C_1, with $C_0 \subseteq \overline{B}_X(0, n)$, $C_1 \subseteq F(t, C_0) + (\{-q(t)\} + \lambda G(t))C_0$.

Indeed, in this setting the multimap G_q clearly assumes closed values and it is measurable (see [26], Proposition 2.2.57).

Finally, we hope that our study can be used to investigate physically important problems that can be modeled using Sturm-Liouville second order differential equations.

With the aim of examining controllability of these phenomena can be advantageous our approach: the multimap involved in the Sturm-Liouville inclusion can be used to describe the control action.

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