Set-Valued and Variational Analysis (2023) 31:20
https://doi.org/10.1007/s11228-023-00681-4

The Radius of Metric Regularity Revisited

Helmut Gfrerer[1] · Alexander Y. Kruger[2]

Received: 25 October 2022 / Accepted: 2 May 2023 / Published online: 1 July 2023

Abstract
The paper extends the radius of metric regularity theorem by Dontchev, Lewis and Rockafellar (2003) by providing an exact formula for the radius with respect to Lipschitz continuous perturbations in general Asplund spaces, thus, answering affirmatively an open question raised twenty years ago by Ioffe. In the non-Asplund case, we give a natural upper bound for the radius complementing the conventional lower bound in the theorem by Dontchev, Lewis and Rockafellar.

Keywords Metric regularity · Strong metric regularity · Radius of regularity · Stability · Lyusternik-Graves theorem

Mathematics Subject Classification (2010) 49J52 · 49J53 · 49K40

1 Introduction

Study of the "radius of good behaviour" was initiated in 2003 by Dontchev, Lewis and Rockafellar [7]. They aimed at quantifying the "distance" from a given *well-posed* problem to the set of ill-posed problems of the same kind. The topic is obviously about stability of problems with respect to perturbations of the problem data, but it goes further than just establishing stability; the goal is to provide quantitative estimates (ideally exact formulas) of how far the problem can be perturbed before well-posedness is lost. This is of significance, e.g., for computational methods.

It is common to describe "good behaviour" of problems in terms of certain regularity properties of (set-valued) mappings involved in modelling of the problems, and talk about the *radius of regularity*. Not surprisingly, the first *radius theorems* were established in [7]

This note is dedicated to the memory of our friend and colleague Professor Asen Dontchev.

✉ Alexander Y. Kruger
alexanderkruger@tdtu.edu.vn

Helmut Gfrerer
helmut.gfrerer@jku.at

1 Institute of Computational Mathematics, Johannes Kepler University Linz, A-4040 Linz, Austria

2 Optimization Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

for the fundamental property of *metric regularity*, followed in [8] by the corresponding statement for the *strong metric regularity*. The definitions of the mentioned properties are collected below (cf. [9, 16, 24]). Here $F : X \rightrightarrows Y$ is a set-valued mapping, and $F^{-1} : Y \rightrightarrows X$ denotes its inverse, i.e., $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$ for all $y \in Y$.

Definition 1.1 Let X and Y be metric spaces, $F : X \rightrightarrows Y$, and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.

(i) F is *metrically regular* at $(\bar{x}, \bar{y})$ if there exist numbers $\alpha > 0$ and $\delta > 0$ such that

$$\alpha d(x, F^{-1}(y)) \leq d(y, F(x)) \quad \text{for all} \quad x \in B_\delta(\bar{x}),\ y \in B_\delta(\bar{y}). \tag{1.1}$$

(ii) F is *strongly metrically regular* at $(\bar{x}, \bar{y})$ if it is metrically regular at $(\bar{x}, \bar{y})$, and F^{-1} has a single-valued localization around $(\bar{y}, \bar{x})$, i.e., there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ and a function $\phi : V \to U$ such that $\operatorname{gph}\phi = \operatorname{gph} F^{-1} \cap (V \times U)$.

The (possibly infinite) supremum of all α satisfying (1.1) for some $\delta > 0$ is called the *regularity modulus* of F at $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{rg} F(\bar{x}, \bar{y})$. It is also known as the *modulus* (or *rate*) of *surjection* [16], and $\operatorname{rg} F(\bar{x}, \bar{y}) = 1/\operatorname{reg} F(\bar{x} \mid \bar{y})$, where $\operatorname{reg} F(\bar{x} \mid \bar{y})$ is the regularity modulus employed in [7–9]. The case $\operatorname{rg} F(\bar{x}, \bar{y}) = 0$ (or equivalently, $\operatorname{reg} F(\bar{x} \mid \bar{y}) = +\infty$) indicates the absence of metric regularity.

In [7, 8], the authors considered perturbations of a set-valued mapping over the classes $\mathcal{F}_{lin}$ of affine (linear) functions and $\mathcal{F}_{lip}$ of single-valued functions $f : X \to Y$, which are Lipschitz continuous near the reference point $\bar{x}$, and used the *Lipschitz modulus*

$$\operatorname{lip} f(\bar{x}) = \limsup_{x,x' \to \bar{x},\, x \neq x'} \frac{d(f(x), f(x'))}{d(x, x')}$$

to measure the size of perturbations.

The next theorem combines [7, Theorem 1.5] and [8, Theorems 4.6]; see also [9, Theorems 6A.7 and 6A.8].

Theorem 1.1 Let X and Y be Banach spaces, $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and $\operatorname{gph} F$ be closed near $(\bar{x}, \bar{y})$. The following estimates hold true:

$$\operatorname{rad}[\mathrm{R}]_{\mathrm{lin}} F(\bar{x}, \bar{y}) \geq \operatorname{rad}[\mathrm{R}]_{\mathrm{lip}} F(\bar{x}, \bar{y}) \geq \operatorname{rg} F(\bar{x}, \bar{y}), \tag{1.2}$$

If F is strongly metrically regular at $(\bar{x}, \bar{y})$, then

$$\operatorname{rad}[\mathrm{sR}]_{\mathrm{lin}} F(\bar{x}, \bar{y}) \geq \operatorname{rad}[\mathrm{sR}]_{\mathrm{lip}} F(\bar{x}, \bar{y}) \geq \operatorname{rg} F(\bar{x}, \bar{y}). \tag{1.3}$$

If $\dim X < \infty$ and $\dim Y < \infty$, then all the inequalities in (1.2) and (1.3) hold as equalities. Moreover, the equalities remain valid if $\mathcal{F}_{lin}$ is restricted to affine functions of rank 1.

In (1.2), $\operatorname{rad}[\mathrm{R}]_{\mathrm{lip}} F(\bar{x}, \bar{y})$ stands for the radius of metric regularity of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ over the class $\mathcal{F}_{lip}$ of Lipschitz continuous perturbations:

$$\operatorname{rad}[\mathrm{R}]_{\mathrm{lip}} F(\bar{x}, \bar{y}) := \inf_{f \in \mathcal{F}_{lip}} \{\operatorname{lip} f(\bar{x}) \mid \quad F + f \text{ is not metrically regular at } (\bar{x}, \bar{y})\}. \tag{1.4}$$

The notation $\operatorname{rad}[\mathrm{R}]_{\mathrm{lin}} F(\bar{x}, \bar{y})$ corresponds to replacing $\mathcal{F}_{lip}$ in (1.4) with the class $\mathcal{F}_{lin}$ of affine perturbations, while $\operatorname{rad}[\mathrm{sR}]_{\mathrm{lip}} F(\bar{x}, \bar{y})$ and $\operatorname{rad}[\mathrm{sR}]_{\mathrm{lin}} F(\bar{x}, \bar{y})$ in (1.3) are defined in a similar way for the property of strong metric regularity ('sR' for brevity). We assume everywhere without loss of generality that perturbation functions f satisfy $f(\bar{x}) = 0$. (Thus, the functions in $\mathcal{F}_{lin}$ are actually linear.) For a more general definition of the radius rad[Property]$_{\mathcal{P}} F(\bar{x},\bar{y})$ allowing for other properties and other classes of perturbations, we refer the readers to [10].

The critical inequality $\mathrm{rad}[\mathrm{R}]_{\mathrm{lip}}\, F(\bar{x},\bar{y}) \geq \mathrm{rg}\, F(\bar{x},\bar{y})$ in (1.2) has its roots in the fundamental theorems of Lyusternik and Graves, and is sometimes referred to as the *extended Lyusternik–Graves theorem*. The latter theorem has a long and rather well known history; cf. [4, 5, 7–9, 12, 16, 20]. The version below relates the regularity moduli of set-valued mappings and the Lipschitz modulus of the perturbation function; cf. [8, Corollary 2.4], [4, Theorem 1].

Theorem 1.2 Let X and Y be Banach spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \mathrm{gph} F$, $\mathrm{gph} F$ be closed near $(\bar{x},\bar{y})$, and let $f \in \mathcal{F}_{lip}$. Then

$$\mathrm{rg}\,(F+f)(\bar{x},\bar{y}) \geq \mathrm{rg}\, F(\bar{x},\bar{y}) - \mathrm{lip} f(\bar{x}). \tag{1.5}$$

There have been several attempts to study stability of metric regularity with respect to set-valued perturbations under certain assumptions either of sum-stability or on ways of measuring the distance between set-valued mappings [1, 11–13, 21, 22].

In finite dimensions, Theorem 1.1 gives exact formulas for the radii of the two regularity properties from Definition 1.1 in terms of the regularity modulus, while in general Banach spaces it provides lower bounds for the radii and hence, sufficient conditions for the stability of the properties. This observation naturally raises the following important question:

(A) *Can any of the inequalities in (1.2) and (1.3) hold as equalities in infinite dimensions?*

Closely related is the following question posed by Ioffe [14]:

(B) *Is the bound (1.5) sharp? In other words, in the setting of Theorem 1.2, if* $\mathrm{rg}\, F(\bar{x},\bar{y}) < +\infty$ *and* $r \in [0, \mathrm{rg}\, F(\bar{x},\bar{y})]$, *is there a function* $f \in \mathcal{F}_{lip}$ *such that* $\mathrm{lip} f(\bar{x}) = r$ *and* $\mathrm{rg}\,(F+f)(\bar{x},\bar{y}) = \mathrm{rg}\, F(\bar{x},\bar{y}) - r$?

Note that, from a positive answer to the latter question, the equality $\mathrm{rad}[\mathrm{R}]_{\mathrm{lip}}\, F(\bar{x},\bar{y}) = \mathrm{rg}\, F(\bar{x},\bar{y})$ follows by taking $r = \mathrm{rg}\, F(\bar{x},\bar{y})$.

A partial positive answer to question (A) was given already in [7] (see also [9, Theorems 6A.2]): equalities hold in (1.2) when F is positively homogeneous. For the special case when Y is finite dimensional, equalities in (1.2) were shown for a certain class of mappings in [19]. Next, equalities in (1.2) were established in [3] for a special mapping defined by a semi-infinite system of linear equalities and inequalities. However, this is not the case for general set-valued mappings: as shown in Ioffe [15] (see also [16, Theorem 5.61]), the inequality $\mathrm{rad}[\mathrm{R}]_{\mathrm{lin}}\, F(\bar{x},\bar{y}) \geq \mathrm{rg}\, F(\bar{x},\bar{y})$ in (1.2) can be strict even when $X = Y$ is a Hilbert space, and F is a single-valued function having reasonably good differentiability properties.

Question (B) was positively answered by Ioffe [14] for general set-valued mappings in the finite dimensional setting and for single-valued continuous functions from a metric space X into a Banach space Y. With respect to the latter result, Ioffe stated that "*The question of whether or not a similar fact is valid for set-valued mappings remains open.*"

To the best of our knowledge, there has been no further progress in addressing the two questions stated above. In the current note we make another step to close the gap. We show that in Asplund spaces the bound (1.5) is sharp for general closed graph set-valued mappings, thus, giving a positive answer to question (B) and showing the equality $\mathrm{rad}[\mathrm{R}]_{\mathrm{lip}}\, F(\bar{x},\bar{y}) = \mathrm{rg}\, F(\bar{x},\bar{y})$. We also obtain the relation $\mathrm{rad}[\mathrm{sR}]_{\mathrm{lip}}\, F(\bar{x},\bar{y}) = \mathrm{rg}\, F(\bar{x},\bar{y})$ in the case when F is strongly metrically regular. In the non-Asplund case, we provide natural upper bounds for the radius complementing the conventional lower bound in (1.2) and (1.3).

In the aforementioned paper [8] by Dontchev and Rockafellar, besides metric regularity and strong metric regularity, the properties of metric subregularity and strong metric subregularity were considered. It was shown that the radius of strong metric subregularity (under calm perturbations) in finite dimensions follows the same pattern as that of (strong) metric regularity, i.e., it equals the modulus of metric subregularity. However, the radius of (not strong) metric subregularity fails to satisfy the paradigm promoted in [7, 8], and the property requires new approaches. In the recent papers [6, 10], the radius of metric subregularity has been analyzed for various classes of perturbations in finite and infinite dimensions. Lower and upper bounds for the radius have been established which are different from the modulus of metric subregularity, and can differ from each other by a factor of at most two. The radius of strong metric subregularity has also been examined in [10], and the corresponding result in [8] has been extended to infinite dimensions.

After some preliminaries in the next Section 2, the main results are formulated in Section 3. Theorem 3.1 states that the estimate (1.5) is precise in the Asplund space setting. Theorem 3.2 provides the missing equality $\mathrm{rad}[\mathrm{R}]_{\mathrm{lip}} F(\bar{x}, \bar{y}) = \mathrm{rg}\, F(\bar{x}, \bar{y})$ in the Asplund space setting, and combines it with the other estimates for the radius. The main tools used in the proofs of these theorems are encapsulated in a separate Lemma 3.1. It gives a little more general relations which can be of independent interest. The proof of Lemma 3.1 makes a separate Section 4. It is partially based on our recent work on the radius of (strong) metric subregularity [6, 10].

2 Preliminaries

The note follows the style and (rather self-explanatory) notation of [10]. X and Y are normed spaces. In most statements, they are additionally assumed to be Banach or even Asplund. Their topological duals are denoted by X^* and Y^*, respectively, while $\langle \cdot, \cdot \rangle$ denotes the bilinear form defining the pairing between the spaces. Recall that a Banach space is *Asplund* if every continuous convex function on an open convex set is Fréchet differentiable at all points of a dense subset of its domain, or equivalently, if the dual of each its separable subspace is separable [23]. All reflexive, particularly, all finite dimensional Banach spaces are Asplund.

The *open* unit balls in a normed space and its dual are denoted by $\mathbb{B}$ and $\mathbb{B}^*$, respectively, while $\mathbb{S}$ and $\mathbb{S}^*$ stand for the unit spheres (possibly with a subscript denoting the space). $B_\delta(x)$ and $\overline{B}_\delta(x)$ denote, respectively, the open and closed balls with radius $\delta > 0$ and centre x. Norms and distances (including point-to-set distances) in all spaces are denoted by the same symbols $\|\cdot\|$ and $d(\cdot, \cdot)$, respectively. A subset $\Omega \subset X$ is said to be *closed near a point* $\bar{x} \in \Omega$ if $\Omega \cap U$ is closed for some closed neighbourhood U of $\bar{x}$. Symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of all real numbers, all nonnegative real numbers and all positive integers, respectively, and $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. We use the following conventions: $\inf \emptyset_{\mathbb{R}} = +\infty$ and $\sup \emptyset_{\mathbb{R}_+} = 0$, where $\emptyset$ (possibly with a subscript) denotes the empty subset (of a given set).

Products of primal and dual normed spaces are assumed to be equipped with the sum and maximum norms, respectively:

$$\|(x, y)\| = \|x\| + \|y\|, \quad (x, y) \in X \times Y,$$
$$\|(x^*, y^*)\| = \max\{\|x^*\|, \|y^*\|\}, \quad (x^*, y^*) \in X^* \times Y^*.$$

Given a subset $\Omega \subset X$, a point $\bar{x} \in \Omega$, and a number $\varepsilon > 0$, the set

$$N_{\Omega,\varepsilon}(\bar{x}) := \left\{ x^* \in X^* \mid \limsup_{\Omega \ni x \to \bar{x},\, x \neq \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} < \varepsilon \right\} \tag{2.1}$$

is called the *set of Fréchet ε-normals* to Ω at $\bar{x}$, while $N_\Omega(\bar{x}) = \bigcap_{\varepsilon > 0} N_{\Omega,\varepsilon}(\bar{x})$ is the *Fréchet normal cone* to Ω at $\bar{x}$.

The graph of a set-valued mapping $F : X \rightrightarrows Y$ is defined as $\mathrm{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$. Given a point $(\bar{x}, \bar{y}) \in \mathrm{gph} F$, and a number $\varepsilon \geq 0$, the mapping $D^*_\varepsilon F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ defined for all $y^* \in Y^*$ by

$$D^*_\varepsilon F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_{\mathrm{gph} F,\varepsilon}(\bar{x}, \bar{y})\}, \tag{2.2}$$

is called the *Fréchet ε-coderivative* of F at $(\bar{x}, \bar{y})$. If $\varepsilon = 0$, it reduces to the *Fréchet coderivative* $D^* F(\bar{x}, \bar{y})$ (with the convention in (2.2) that $N_{\mathrm{gph} F,0}(\bar{x}, \bar{y}) := N_{\mathrm{gph} F}(\bar{x}, \bar{y})$).

Remark 2.1 The set of Fréchet ε-normals is often defined [17] with the non-strict inequality in (2.1). Then it allows also for the case $\varepsilon = 0$ in (2.1) directly. Whether the strict or non-strict inequality is used in definition (2.1) does not affect definition (2.3) and the estimates in the current paper, but, as observed by a referee, using the strict inequality in (2.1) leads to slight simplifications in some proofs.

Employing (2.2), we define another nonnegative quantity characterizing the behaviour of F near $(\bar{x}, \bar{y})$ and closely related to the regularity modulus $\mathrm{rg}\, F(\bar{x}, \bar{y})$:

$$\mathrm{rg}^+ F(\bar{x}, \bar{y}) := \liminf_{\substack{\mathrm{gph} F \ni (x,y) \to (\bar{x},\bar{y}),\ \varepsilon \downarrow 0 \\ x^* \in D^*_\varepsilon F(x,y)(S_{Y^*})}} \|x^*\| := \sup_{\varepsilon > 0} \inf_{\substack{(x,y) \in \mathrm{gph} F \cap B_\varepsilon(\bar{x},\bar{y}) \\ x^* \in D^*_\varepsilon F(x,y)(S_{Y^*})}} \|x^*\|. \tag{2.3}$$

Observe that

$$\mathrm{rg}^+ F(\bar{x}, \bar{y}) \leq \liminf_{\substack{\mathrm{gph} F \ni (x,y) \to (\bar{x},\bar{y}) \\ x^* \in D^* F(x,y)(S_{Y^*})}} \|x^*\|. \tag{2.4}$$

Lemma 2.1 Let X and Y be Banach spaces, $F : X \rightrightarrows Y$, and $(\bar{x}, \bar{y}) \in \mathrm{gph} F$. Then $\mathrm{rg}\, F(\bar{x}, \bar{y}) \leq \mathrm{rg}^+ F(\bar{x}, \bar{y})$.

If X and Y are Asplund, and $\mathrm{gph} F$ is closed near $(\bar{x}, \bar{y})$, then $\mathrm{rg}\, F(\bar{x}, \bar{y}) = \mathrm{rg}^+ F(\bar{x}, \bar{y})$.

Proof If $\mathrm{rg}\, F(\bar{x}, \bar{y}) = 0$ or $\mathrm{rg}^+ F(\bar{x}, \bar{y}) = +\infty$, then we trivially have $\mathrm{rg}\, F(\bar{x}, \bar{y}) \leq \mathrm{rg}^+ F(\bar{x}, \bar{y})$. Thus, we may assume that $\mathrm{rg}\, F(\bar{x}, \bar{y}) > 0$ and $\mathrm{rg}^+ F(\bar{x}, \bar{y}) < \infty$. Pick any $\alpha \in (0, \mathrm{rg}\, F(\bar{x}, \bar{y}))$ and $\beta \in (\mathrm{rg}^+ F(\bar{x}, \bar{y}), +\infty)$. Then condition (1.1) is satisfied for some $\delta > 0$, and, given an arbitrary number $\xi \in (0, \delta)$, there exist $(x, y) \in \mathrm{gph} F \cap B_\xi(\bar{x}, \bar{y})$, $y^* \in S_{Y^*}$, and $x^* \in D^*_\xi F(x, y)(y^*)$ such that $\|x^*\| < \beta$. Consider a sequence $\{v_k\}$ in Y such that $y \neq v_k \to y$ and $\langle y^*, \frac{v_k - y}{\|v_k - y\|} \rangle \to -1$ as $k \to \infty$. Then, for each sufficiently large $k \in \mathbb{N}$, we have $v_k \in B_\delta(\bar{y})$ and, by (1.1), one can find a $u_k \in F^{-1}(v_k)$ such that $\alpha\|u_k - x\| < (1 + 1/k)\|v_k - y\|$, and consequently,

$$\|v_k - y\| \leq \|(u_k, v_k) - (x, y)\| \leq ((1 + 1/k)\alpha^{-1} + 1)\|v_k - y\|.$$

By definition of the Fréchet ε-coderivative, using the notation $\mu_+ := \max\{\mu, 0\}$, we obtain:

$$\begin{aligned}\xi &> \limsup_{k \to \infty} \frac{(\langle x^*, u_k - x \rangle - \langle y^*, v_k - y \rangle)_+}{\|(u_k, v_k) - (x, y)\|} \geq \limsup_{k \to \infty} \frac{(\|v_k - y\| - \|x^*\|\|u_k - x\|)_+}{((1 + 1/k)\alpha^{-1} + 1)\|v_k - y\|} \\ &\geq \limsup_{k \to \infty} \frac{1 - \|x^*\|(1 + 1/k)\alpha^{-1}}{(1 + 1/k)\alpha^{-1} + 1} = \frac{1 - \|x^*\|\alpha^{-1}}{\alpha^{-1} + 1} > \frac{\alpha - \beta}{\alpha + 1}.\end{aligned}$$

Since $\xi > 0$ can be arbitrarily small, it follows that $\alpha \le \beta$. Letting $\alpha \uparrow \operatorname{rg} F(\bar{x}, \bar{y})$ and $\beta \downarrow \operatorname{rg}^+ F(\bar{x}, \bar{y})$, we arrive at $\operatorname{rg} F(\bar{x}, \bar{y}) \le \operatorname{rg}^+ F(\bar{x}, \bar{y})$. The equality $\operatorname{rg} F(\bar{x}, \bar{y}) = \operatorname{rg}^+ F(\bar{x}, \bar{y})$ in the Asplund space setting is a consequence of [20, Theorem 4.5] and (2.4). $\square$

Remark 2.2 The above proof of inequality $\operatorname{rg} F(\bar{x}, \bar{y}) \le \operatorname{rg}^+ F(\bar{x}, \bar{y})$ is a modification of the corresponding parts of the proofs of [18, Theorem 3.1 and Theorem 5.1 (i)]. It can also be deduced from [20, Theorem 1.43(i)]. In view of [20, Theorem 4.5], it follows from Lemma 2.1 that (2.4) becomes equality if the spaces are Asplund.

The next statement from [10] extends [20, Theorem 1.62 (i)] which addresses the case $\varepsilon = 0$.

Lemma 2.2 Let $F : X \rightrightarrows Y$, $f : X \to Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and $\varepsilon \ge 0$. Suppose that f is Fréchet differentiable at $\bar{x}$. Set $\varepsilon_1 := (\|\nabla f(\bar{x})\| + 1)^{-1}\varepsilon$ and $\varepsilon_2 := (\|\nabla f(\bar{x})\| + 1)\varepsilon$. Then, for all $y^* \in Y^*$, it holds

$$D^*_{\varepsilon_1} F(\bar{x}, \bar{y})(y^*) \subset D^*_{\varepsilon}(F + f)(\bar{x}, \bar{y} + f(\bar{x}))(y^*) - \nabla f(\bar{x})^* y^* \subset D^*_{\varepsilon_2} F(\bar{x}, \bar{y})(y^*).$$

3 Radius of Metric Regularity

We start this section with a lemma which constitutes a key ingredient for the proofs of our main results. Recall from [14, p. 552] that a function $f : X \to Y$ between Banach spaces is *Lipschitz rank one* on an open subset $U \subset X$ if, for any $x \in U$, there is a neighborhood $U_x \subset U$ of x, on which f can be represented in the form

$$f(u) = \xi(u)y \quad (u \in U_x),$$

where $\xi : U_x \to \mathbb{R}$ is Lipschitz continuous and $y \in Y$.

Lemma 3.1 Let X and Y be Banach spaces, $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and $\operatorname{gph} F$ be closed near $(\bar{x}, \bar{y})$. Suppose that $\operatorname{rg}^+ F(\bar{x}, \bar{y}) < +\infty$. Then there exists a function $f \in \mathcal{F}_{lip}$, Lipschitz rank one on $X \setminus \{\bar{x}\}$, such that $\operatorname{lip} f(\bar{x}) \le \operatorname{rg}^+ F(\bar{x}, \bar{y})$, and

$$\operatorname{rg}^+(F + \alpha f)(\bar{x}, \bar{y}) \le (1 - \alpha)\operatorname{rg}^+ F(\bar{x}, \bar{y}) \quad \text{for all} \quad \alpha \in [0, 1];$$

in particular, $\operatorname{rg}^+(F + f)(\bar{x}, \bar{y}) = 0$.

The proof of Lemma 3.1 is given in the next section. We are now in a position to extend [14, Theorem 4.1] to set-valued mappings and, thus, to answer positively question (B) in Section 1 in the Asplund space setting.

Theorem 3.1 Let X and Y be Asplund spaces, $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, and $\operatorname{gph} F$ be closed near $(\bar{x}, \bar{y})$. If $\operatorname{rg} F(\bar{x}, \bar{y}) < +\infty$, then, for every real $r \in [0, \operatorname{rg} F(\bar{x}, \bar{y})]$, there is a function $f \in \mathcal{F}_{lip}$, Lipschitz rank one on $X \setminus \{\bar{x}\}$, such that $\operatorname{lip} f(\bar{x}) = r$ and

$$\operatorname{rg}(F + f)(\bar{x}, \bar{y}) = \operatorname{rg} F(\bar{x}, \bar{y}) - r.$$

Proof If $\operatorname{rg} F(\bar{x}, \bar{y}) = 0$, then $r = 0$, and we take $f \equiv 0$. Let $\operatorname{rg} F(\bar{x}, \bar{y}) < +\infty$ and $r \in [0, \operatorname{rg} F(\bar{x}, \bar{y})]$. Set $\alpha := r/\operatorname{rg} F(\bar{x}, \bar{y})$. Thus, $\alpha \in [0, 1]$. By Lemma 2.1, $\operatorname{rg}^+ F(\bar{x}, \bar{y}) = \operatorname{rg} F(\bar{x}, \bar{y}) < +\infty$, and by Lemma 3.1, there is a function $\tilde{f} \in \mathcal{F}_{lip}$, Lipschitz rank one on $X \setminus \{\bar{x}\}$, such that $\operatorname{lip} \tilde{f}(\bar{x}) \le \operatorname{rg}^+ F(\bar{x}, \bar{y})$, and

$$\operatorname{rg}^+(F + \alpha \tilde{f})(\bar{x}, \bar{y}) \le (1 - \alpha)\operatorname{rg}^+ F(\bar{x}, \bar{y}) = \operatorname{rg} F(\bar{x}, \bar{y}) - r.$$

The function $f := \alpha \tilde{f}$ is also Lipschitz rank one on $X \setminus \{\bar{x}\}$, and

$$\mathrm{lip} f(\bar{x}) = \alpha \mathrm{lip} \tilde{f}(\bar{x}) \le \alpha \mathrm{rg}^+ F(\bar{x}, \bar{y}) = \alpha \mathrm{rg}\, F(\bar{x}, \bar{y}) = r.$$

Since f is continuous near $\bar{x}$, the graph of $F + f$ is closed near $(\bar{x}, \bar{y})$ and, by Lemma 2.1, $\mathrm{rg}^+(F + f)(\bar{x}, \bar{y}) = \mathrm{rg}\,(F + f)(\bar{x}, \bar{y})$. Thus,

$$\mathrm{rg}\,(F + f)(\bar{x}, \bar{y}) = \mathrm{rg}^+(F + \alpha \tilde{f})(\bar{x}, \bar{y}) \le \mathrm{rg}\, F(\bar{x}, \bar{y}) - r.$$

On the other hand, by Theorem 1.2, $\mathrm{rg}\,(F+f)(\bar{x}, \bar{y}) \ge \mathrm{rg}\, F(\bar{x}, \bar{y}) - \mathrm{lip} f(\bar{x}) \ge \mathrm{rg}\, F(\bar{x}, \bar{y}) - r$. Hence, $\mathrm{lip} f(\bar{x}) = r$ and $\mathrm{rg}\,(F + f)(\bar{x}, \bar{y}) = \mathrm{rg}\, F(\bar{x}, \bar{y}) - r$. □

Remark 3.1 The single-valued version of Theorem 3.1 in [14, Theorem 4.1] claims the existence of a perturbation function with the properties as in Theorem 3.1 and being Lipschitz rank one on the whole space X. However, a close look at the proof of [14, Theorem 4.1] shows that the function constructed there is proved to be Lipschitz rank one only on $X \setminus \{\bar{x}\}$.

Below is our main radius theorem for metric regularity. It employs regularity constant (2.3) as an upper bound for the radius of metric regularity and extends Theorem 1.1.

Theorem 3.2 Let X and Y be Banach spaces, $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \mathrm{gph} F$, and $\mathrm{gph} F$ be closed near $(\bar{x}, \bar{y})$. Then

$$\mathrm{rg}\, F(\bar{x}, \bar{y}) \le \mathrm{rad[R]}_{\mathrm{lip}}\, F(\bar{x}, \bar{y}) \le \mathrm{rg}^+ F(\bar{x}, \bar{y}). \tag{3.1}$$

If X and Y are Asplund, then all inequalities in (3.1) hold as equalities. If, in addition, F is strongly metrically regular near $(\bar{x}, \bar{y})$ then we also have

$$\mathrm{rad[sR]}_{\mathrm{lip}}\, F(\bar{x}, \bar{y}) = \mathrm{rg}\, F(\bar{x}, \bar{y}) = \mathrm{rg}^+.$$

Proof The first inequality in (3.1) is the second inequality in (1.2) in Theorem 1.1. The equalities in Asplund spaces are direct consequences of Lemma 2.1. The remaining second inequality in (3.1) follows from Lemma 3.1 together with the definition (1.4) of the radius of metric regularity. Indeed, by Lemma 3.1, there is a function $f \in \mathcal{F}_{lip}$ such that $\mathrm{lip} f(\bar{x}) \le \mathrm{rg}^+ F(\bar{x}, \bar{y})$, and $0 \le \mathrm{rg}\,(F + f)(\bar{x}, \bar{y}) \le \mathrm{rg}^+(F + f)(\bar{x}, \bar{y}) = 0$, i.e., $F + f$ is not metrically regular at $(\bar{x}, \bar{y})$, and therefore $\mathrm{rad[R]}_{\mathrm{lip}}\, F(\bar{x}, \bar{y}) \le \mathrm{lip} f(\bar{x}) \le \mathrm{rg}^+ F(\bar{x}, \bar{y})$. Finally, the last assertion follows from $\mathrm{rad[sR]}_{\mathrm{lip}}\, F(\bar{x}, \bar{y}) \le \mathrm{rad[R]}_{\mathrm{lip}}\, F(\bar{x}, \bar{y}) = \mathrm{rg}\, F(\bar{x}, \bar{y})$ together with the bound $\mathrm{rad[sR]}_{\mathrm{lip}}\, F(\bar{x}, \bar{y}) \ge \mathrm{rg}\, F(\bar{x}, \bar{y})$ in (1.3) in Theorem 1.1. □

4 Proof of Lemma 3.1

We assume that $\mathrm{rg}^+ F(\bar{x}, \bar{y}) > 0$; otherwise the statement trivially holds with $f \equiv 0$. By definition (2.3), there exist sequences $\mathrm{gph} F \ni (x_k, y_k) \to (\bar{x}, \bar{y})$, $\varepsilon_k \downarrow 0$, $y_k^* \in \mathbb{S}_{Y^*}$, and $x_k^* \in D^*_{\varepsilon_k} F(x_k, y_k)(y_k^*)$ such that

$$\mathrm{rg}^+ F(\bar{x}, \bar{y}) = \lim_{k \to \infty} \|x_k^*\|. \tag{4.1}$$

Consider any real $\gamma > \mathrm{rg}^+ F(\bar{x}, \bar{y})$. Without loss of generality, we assume that

$$\inf_{k \in \mathbb{N}} \|x_k^*\| > 0 \quad \text{and} \quad \sup_{k \in \mathbb{N}} \|x_k^*\| < \gamma. \tag{4.2}$$

We have to consider two cases.

Case 1: $x_k \neq \bar{x}$ for infinitely many $k \in \mathbb{N}$. Denote $t_k := \|x_k - \bar{x}\|$. Thus, $t_k \to 0$ as $k \to \infty$. By passing to subsequences and then relabeling appropriately, we can ensure that $0 < t_{k+1} < t_k/2$ for all $k \in \mathbb{N}$. As a consequence, $t_k - t_{k+1} > t_{k+1} > t_{k+1} - t_{k+2}$. Hence, $\{t_k\}$ and $\{t_k - t_{k+1}\}$ are both strictly decreasing sequences of positive numbers. For each $k \in \mathbb{N}$, define $\rho_k := (t_k - t_{k+1})/2$. Thus, $\{\rho_k\}$ is also a strictly decreasing sequence of positive numbers. Moreover,

$$\overline{B}_{t_k+\rho_k}(\bar{x}) \cap \overline{B}_{\rho_i}(x_i) = \emptyset \quad \text{for all} \quad i, k \in \mathbb{N},\ i < k. \tag{4.3}$$

Indeed, let $i, k \in \mathbb{N}$ and $i < k$. Then

$$\begin{aligned} \inf_{x \in \overline{B}_{t_k+\rho_k}(\bar{x}),\ x' \in \overline{B}_{\rho_i}(x_i)} \|x - x'\| &\geq \|x_i - \bar{x}\| - \sup_{x \in \overline{B}_{t_k+\rho_k}(\bar{x})} \|x - \bar{x}\| - \sup_{x' \in \overline{B}_{\rho_i}(x_i)} \|x' - x_i\| \\ &\geq t_i - (t_k + \rho_k) - \rho_i \geq t_i - (t_{i+1} + \rho_{i+1}) - \rho_i = \rho_i - \rho_{i+1} > 0. \end{aligned}$$

Since $\overline{B}_{\rho_k}(x_k) \subset \overline{B}_{t_k+\rho_k}(\bar{x})$, it follows from (4.3) that

$$\overline{B}_{\rho_k}(x_k) \cap \overline{B}_{\rho_i}(x_i) = \emptyset \quad \text{for all} \quad i \neq k. \tag{4.4}$$

For each $k \in \mathbb{N}$, choose a point $v_k \in \mathbb{S}_Y$ such that

$$\langle y_k^*, v_k \rangle > 1 - 1/k. \tag{4.5}$$

For all $k \in \mathbb{N}$ and $x \in X$, set

$$s_k(x) := \max\left\{1 - (\|x - x_k\|/\rho_k)^{1+\frac{1}{k}}, 0\right\}, \tag{4.6a}$$

$$g_k(x) := \langle x_k^*, x - x_k \rangle v_k, \tag{4.6b}$$

$$f_k(x) := s_k(x) g_k(x). \tag{4.6c}$$

Observe that $s_k(x) = 0$ and $f_k(x) = 0$ for all $x \notin B_{\rho_k}(x_k)$. In view of (4.4), the function

$$f(x) := -\sum_{k=1}^{\infty} f_k(x), \quad x \in X, \tag{4.7}$$

is well defined, $f(x) = -f_k(x)$ for all $x \in B_{\rho_k}(x_k)$ and all $k \in \mathbb{N}$, and $f(x) = 0$ for all $x \notin \cup_{k=1}^{\infty} B_{\rho_k}(x_k)$. In particular, $f(\bar{x}) = 0$. Observing that $s_k(x_k) = 1$, $g_k(x_k) = 0$, and the function s_k is differentiable at x_k with $\nabla s_k(x_k) = 0$, we have

$$f(x_k) = 0 \quad \text{and} \quad \nabla f(x_k) = -\langle x_k^*, \cdot \rangle v_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.8}$$

Given any $x, x' \in \overline{B}_{\rho_k}(x_k)$, by the mean-value theorem applied to the function $t \mapsto t^{1+\frac{1}{k}}$ on $\mathbb{R}_+$, there is a number $\theta \in [0, 1]$ such that

$$\begin{aligned} s_k(x) - s_k(x') = &-(1 + 1/k)\, \rho_k^{-(1+\frac{1}{k})} \big(\theta \|x - x_k\| \\ &+ (1 - \theta)\|x' - x_k\|\big)^{\frac{1}{k}} (\|x - x_k\| - \|x' - x_k\|), \end{aligned}$$

and consequently, assuming without loss of generality that $\|x - x_k\| \leq \|x' - x_k\|$, we have

$$|s_k(x) - s_k(x')| \leq (1 + 1/k)\, \rho_k^{-(1+\frac{1}{k})} \|x' - x_k\|^{\frac{1}{k}} \|x - x'\|.$$

If $x \neq x'$, then

$$\begin{aligned}\frac{\|f_k(x)-f_k(x')\|}{\|x-x'\|} &\le \frac{|s_k(x)-s_k(x')|}{\|x-x'\|}\|g_k(x)\| + s_k(x')\frac{\|g_k(x)-g_k(x')\|}{\|x-x'\|} \\ &\le (1{+}1/k)\left(\|x'{-}x_k\|/\rho_k\right)^{1+\frac{1}{k}}\|x_k^*\| + \left(1-\left(\|x'-x_k\|/\rho_k\right)^{1+\frac{1}{k}}\right)\|x_k^*\| \\ &= \left(\frac{1}{k}\left(\|x'-x_k\|/\rho_k\right)^{1+\frac{1}{k}}+1\right)\|x_k^*\| \le (1+1/k)\,\|x_k^*\|. \end{aligned} \tag{4.9}$$

By (4.2), there is a $\hat{k} \in \mathbb{N}$ such that $(1+1/k)\,\|x_k^*\| < \gamma$ for all $k > \hat{k}$. Thus, for any $k > \hat{k}$, the function f_k is Lipschitz continuous on $\overline{B}_{\rho_k}(x_k)$ with modulus less than γ. As a consequence, we also have

$$\|f_k(x)\| < \gamma(\rho_k - \|x-x_k\|) \quad \text{for all} \quad x \in B_{\rho_k}(x_k). \tag{4.10}$$

Indeed, given any $x \in B_{\rho_k}(x_k)$ with $x \neq x_k$, we set $x' := x_k + \rho_k \frac{x-x_k}{\|x-x_k\|} \in \overline{B}_{\rho_k}(x_k)$. Then $f_k(x') = 0$, and consequently, $\|f_k(x)\| < \gamma\|x'-x\| = \gamma(\rho_k - \|x-x_k\|)$. If $x = x_k$, then, in view of (4.8), inequality (4.10) holds true trivially.

Claim 1: For any $k > \hat{k}$, the function f is Lipschitz continuous on $B_{t_k+\rho_k}(\bar{x})$ with modulus γ.

Indeed, let $k > \hat{k}$, $x, x' \in B_{t_k+\rho_k}(\bar{x})$ and $x \neq x'$. 1) If $x, x' \in B_{\rho_i}(x_i)$ for some $i \ge k$, then $\|f(x)-f(x')\| < \gamma\|x-x'\|$ since $f = -f_i$ on $B_{\rho_i}(x_i)$. 2) If $x \in B_{\rho_i}(x_i)$ and $x' \in B_{\rho_j}(x_j)$ for some $i, j \ge k$ with $i \neq j$, then, thanks to (4.4), we have $\|x_i - x_j\| \ge \rho_i + \rho_j$, and using (4.10), we obtain

$$\begin{aligned}\|f(x)-f(x')\| &\le \|f(x)\| + \|f(x')\| < \gamma(\rho_i+\rho_j - \|x-x_i\| - \|x'-x_j\|) \\ &\le \gamma(\|x_i-x_j\| - \|x-x_i\| - \|x'-x_j\|) \le \gamma\|x-x'\|.\end{aligned}$$

3) If $x \in B_{\rho_i}(x_i)$ for some $i \ge k$, and $x' \notin \bigcup_{j=k}^{\infty} B_{\rho_j}(x_j)$, then $\|x'-x_i\| \ge \rho_i$, and, thanks to (4.3), we also have $x' \notin \bigcup_{j=1}^{k-1} B_{\rho_j}(x_j)$, implying $f(x') = 0$. Using (4.10), we obtain

$$\|f(x)-f(x')\| < \gamma(\|x'-x_i\| - \|x-x_i\|) \le \gamma\|x-x'\|.$$

4) If $x, x' \notin \cup_{j=k}^{\infty} B_{\rho_j}(x_j)$, then $f(x) = f(x') = 0$. Thus, in all cases, $\|f(x)-f(x')\| < \gamma\|x-x'\|$. This completes the proof of Claim 1. $\lhd$

Hence, $f \in \mathcal{F}_{lip}$ with $\operatorname{lip} f(\bar{x}) \le \gamma$ and, since γ can be chosen arbitrarily close to $\operatorname{rg}^+ F(\bar{x},\bar{y})$, we have $\operatorname{lip} f(\bar{x}) \le \operatorname{rg}^+ F(\bar{x},\bar{y})$.

Claim 2: The function f is Lipschitz rank one on $X \setminus \{\bar{x}\}$.

Indeed, let $x \in X \setminus \{\bar{x}\}$. If $x \in \overline{B}_{\rho_k}(x_k)$ for some $k \in \mathbb{N}$, then, thanks to (4.4), (4.6b), (4.6c) and (4.7), there is a $\tilde{\rho}_k > \rho_k$ such that $f(u) = -s_k(u)\langle x_k^*, u - x_k\rangle v_k$ for all $u \in B_{\tilde{\rho}_k}(x_k)$. Note that $B_{\tilde{\rho}_k}(x_k)$ is a neighbourhood of x. If $x \notin \cup_{k\in\mathbb{N}}\overline{B}_{\rho_k}(x_k)$, then, thanks to (4.4), $f(u) = 0$ for all u in a sufficiently small neighborhood of x. Hence, f is Lipschitz rank one on $X \setminus \{\bar{x}\}$. This completes the proof of Claim 2. $\lhd$

Next, consider any $\alpha \in [0,1]$. In view of (4.8), for all $k \in \mathbb{N}$, we have $D^* f(x_k)(y_k^*) = -\langle y_k^*, v_k\rangle x_k^*$. Thanks to the first inclusion in Lemma 2.2, we have

$$\left(1-\alpha\langle y_k^*, v_k\rangle\right) x_k^* \in D^*_{\varepsilon_k} F(x_k, y_k)(y_k^*) + D^*(\alpha f)(x_k)(y_k^*) \subset D^*_{\varepsilon_k'}(F+\alpha f)(x_k,\bar{y})(y_k^*),$$

where $\varepsilon_k' := (\alpha\|\nabla f(x_k)\|+1)\varepsilon_k$. Thanks to (4.8) and (4.2), $\varepsilon_k' = (\alpha\|x_k^*\|+1)\varepsilon_k < (\gamma+1)\varepsilon_k$. Thus, $\varepsilon_k' \to 0$ as $k \to \infty$ and, in view of (4.5), we conclude that

$$\operatorname{rg}^+(F+\alpha f)(\bar{x},\bar{y}) \le \lim_{k\to\infty} (1-\alpha(1-1/k))\,\|x_k^*\| = (1-\alpha)\operatorname{rg}^+ F(\bar{x},\bar{y}).$$

This completes the proof of Case 1.

Case 2: $x_k \neq \bar{x}$ for not more than finitely many $k \in \mathbb{N}$. Then $x_k = \bar{x}$ for all sufficiently large k and, replacing the sequences by their tails, we can assume that $x_k = \bar{x}$ for all $k \in \mathbb{N}$. We are going to reduce this case to the previous one. For that, we now construct new sequences $\operatorname{gph} F \ni (\tilde{x}_k, \tilde{y}_k) \to (\bar{x}, \bar{y})$ and $\tilde{\varepsilon}_k \downarrow 0$ such that $\tilde{x}_k \neq \bar{x}$ and $x_k^* \in D^*_{\tilde{\varepsilon}_k}(\tilde{x}_k, \tilde{y}_k)(y_k^*)$ for all $k \in \mathbb{N}$.

Claim 3: There are not more than finitely many $k \in \mathbb{N}$ with the property

$$\exists \rho > 0 : \quad \operatorname{gph} F \cap B_\rho(\bar{x}, y_k) \subset \{\bar{x}\} \times Y. \tag{4.11}$$

Indeed, if (4.11) is true with some $\rho > 0$ for some $k \in \mathbb{N}$, then, by the definition of the ε-coderivative, we can find a $\tilde{\rho} \in (0, \rho)$ such that, for all $(x, y) \in \operatorname{gph} F \cap B_{\tilde{\rho}}(\bar{x}, y_k) \setminus \{(\bar{x}, y_k)\}$, we have

$$-\langle y_k^*, y - y_k \rangle = \langle x_k^*, x - \bar{x} \rangle - \langle y_k^*, y - y_k \rangle < \varepsilon_k(\|x - \bar{x}\| + \|y - y_k\|),$$

verifying that $0 \in D^*_{\varepsilon_k} F(\bar{x}, y_k)(y_k^*)$. If there were infinitely many $k \in \mathbb{N}$ fulfilling (4.11), this would contradict the assumption that $\operatorname{rg}^+ F(\bar{x}, \bar{y}) > 0$. This completes the proof of Claim 3. $\lhd$

Thus, we can assume that (4.11) fails for all $k \in \mathbb{N}$. Choose any number $k \in \mathbb{N}$ and find a number $\rho_k \in (0, 1/k)$ such that, for all $(x, y) \in \operatorname{gph} F \cap \overline{B}_{\rho_k}(\bar{x}, y_k) \setminus \{(\bar{x}, y_k)\}$, it holds

$$\langle x_k^*, x - \bar{x} \rangle - \langle y_k^*, y - y_k \rangle < \varepsilon_k(\|x - \bar{x}\| + \|y - y_k\|). \tag{4.12}$$

Next, we define the function $\varphi_k : (0, \rho_k] \to \mathbb{R}$ by

$$\varphi_k(\rho) := \sup \left\{ \frac{\langle x_k^*, x - \bar{x} \rangle - \langle y_k^*, y - y_k \rangle - \varepsilon_k \|y - y_k\|}{\|x - \bar{x}\|} \,\middle|\, \begin{array}{l} (x, y) \in B_\rho(\bar{x}, y_k) \\ x \neq \bar{x}, \ \ y \in F(x) \end{array} \right\}. \tag{4.13}$$

By the assumption, $\varphi_k(\rho) > -\infty$ for all $\rho \in (0, \rho_k]$. On the other hand, condition (4.12) implies $\varphi_k(\rho_k) \leq \varepsilon_k$. Since φ_k is nondecreasing on $(0, \rho_k]$, we have

$$\varsigma_k := \inf_{\rho \in (0, \rho_k]} \varphi_k(\rho) = \lim_{\rho \downarrow 0} \varphi_k(\rho) < \varepsilon_k. \tag{4.14}$$

Claim 4: $\lim_{k \to \infty} \varsigma_k = 0$.

Since $\varepsilon_k \downarrow 0$, in view of (4.14), it remains to show that $\liminf_{k \to \infty} \varsigma_k \geq 0$. Assume on the contrary that $\liminf_{k \to \infty} \varsigma_k < 0$, i.e., for some number $\eta > 0$ and a subsequence of ς_k, it holds, without relabeling, that $\varsigma_k < -\eta$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, by (4.14), we can find a $\tilde{\rho}_k \in (0, \rho_k]$ with $\varphi_k(\tilde{\rho}_k) < -\eta$, i.e., by definition (4.13), for all $(x, y) \in \operatorname{gph} F \cap B_{\tilde{\rho}_k}(\bar{x}, y_k)$ with $x \neq \bar{x}$, we have

$$\langle x_k^*, x - \bar{x} \rangle - \langle y_k^*, y - y_k \rangle < \varepsilon_k \|y - y_k\| - \eta \|x - \bar{x}\|. \tag{4.15}$$

Moreover, thanks to (4.12), condition (4.15) is satisfied also for all $(x, y) \in \operatorname{gph} F \cap B_{\tilde{\rho}_k}(\bar{x}, y_k)$ with $x = \bar{x}$ and $y \neq y_k$. Hence, in view of the first inequality in (4.2), we have for all $(x, y) \in \operatorname{gph} F \cap B_{\tilde{\rho}_k}(\bar{x}, y_k) \setminus \{(\bar{x}, y_k)\}$:

$$\langle \left(1 - \eta/\|x_k^*\|\right) x_k^*, x - \bar{x} \rangle \leq \langle x_k^*, x - \bar{x} \rangle + \eta \|x - \bar{x}\| < \langle y_k^*, y - y_k \rangle + \varepsilon_k \|y - y_k\|.$$

Thus, $\left(1 - \eta/\|x_k^*\|\right) x_k^* \in D^*_{\varepsilon_k} F(\bar{x}, y_k)(y_k^*)$ for all $k \in \mathbb{N}$, and, by the definition, we obtain

$$\operatorname{rg}^+ F(\bar{x}, \bar{y}) \leq \liminf_{k \to \infty} \left(1 - \eta/\|x_k^*\|\right) \|x_k^*\| = \liminf_{k \to \infty} \|x_k^*\| - \eta,$$

which contradicts (4.1). This completes the proof of Claim 4. $\lhd$

Fix any number $k \in \mathbb{N}$. Next, we choose a number $\tilde{\rho}_k \in (0, \rho_k/2)$, and then some $(\hat{x}_k, \hat{y}_k) \in \operatorname{gph} F \cap B_{\tilde{\rho}_k}(\bar{x}, y_k)$ with $\hat{x}_k \neq \bar{x}$ and

$$\frac{\langle x_k^*, \hat{x}_k - \bar{x}\rangle - \langle y_k^*, \hat{y}_k - y_k\rangle - \varepsilon_k \|\hat{y}_k - y_k\|}{\|\hat{x}_k - \bar{x}\|} > \varphi_k(\tilde{\rho}_k) - \frac{1}{k} \geq \varsigma_k - \frac{1}{k}. \tag{4.16}$$

Consider the (continuous) function $\psi_k : \operatorname{gph} F \cap \overline{B}_{\rho_k}(\bar{x}, y_k) \mapsto \mathbb{R}$ given by

$$\psi_k(x, y) := \varepsilon_k \|(x, y) - (\bar{x}, y_k)\| - \langle x_k^*, x - \bar{x}\rangle + \langle y_k^*, y - y_k\rangle. \tag{4.17}$$

By (4.12), $\psi_k(x, y) \geq 0$ for all $(x, y) \in \operatorname{gph} F \cap \overline{B}_{\rho_k}(\bar{x}, y_k)$, while from (4.16) we obtain:

$$\psi_k(\hat{x}_k, \hat{y}_k) < \varepsilon_k' \|\hat{x}_k - \bar{x}\|,$$

where $\varepsilon_k' := \varepsilon_k + 1/k - \varsigma_k$. Observe that $\varepsilon_k' > 0$, and $\lim_{k\to\infty} \varepsilon_k' = 0$ by Claim 4. By Ekeland's variational principle (see, e.g., [20, Theorem 2.26]), we find a point $(\tilde{x}_k, \tilde{y}_k) \in \operatorname{gph} F \cap \overline{B}_{\rho_k}(\bar{x}, y_k)$ such that $\psi_k(\tilde{x}_k, \tilde{y}_k) \leq \psi_k(\hat{x}_k, \hat{y}_k)$, and

$$\|(\tilde{x}_k, \tilde{y}_k) - (\hat{x}_k, \hat{y}_k)\| \leq \|\hat{x}_k - \bar{x}\|, \tag{4.18}$$

$$\psi_k(x, y) + \varepsilon_k' \|(x, y) - (\tilde{x}_k, \tilde{y}_k)\| > \psi_k(\tilde{x}_k, \tilde{y}_k) \tag{4.19}$$

for all $(x, y) \in \operatorname{gph} F \cap \overline{B}_{\rho_k}(\bar{x}, y_k) \setminus \{(\tilde{x}_k, \tilde{y}_k)\}$. From (4.18), we obtain:

$$\begin{aligned}
\|(\tilde{x}_k, \tilde{y}_k) - (\bar{x}, y_k)\| &\leq \|(\hat{x}_k, \hat{y}_k) - (\bar{x}, y_k)\| + \|(\tilde{x}_k, \tilde{y}_k) - (\hat{x}_k, \hat{y}_k)\| \\
&\leq \|(\hat{x}_k, \hat{y}_k) - (\bar{x}, y_k)\| + \|\hat{x}_k - \bar{x}\| \\
&\leq 2\|(\hat{x}_k, \hat{y}_k) - (\bar{x}, y_k)\| \leq 2\tilde{\rho}_k < \rho_k.
\end{aligned}$$

Finally, using (4.17) and (4.19), we have for all $(x, y) \in \operatorname{gph} F \cap \overline{B}_{\rho_k}(\bar{x}, y_k) \setminus \{(\tilde{x}_k, \tilde{y}_k)\}$:

$$\begin{aligned}
\langle x_k^*, x - \tilde{x}_k\rangle - \langle y_k^*, y - \tilde{y}_k\rangle &= \langle x_k^*, x - \bar{x}\rangle - \langle y_k^*, y - y_k\rangle - \langle x_k^*, \tilde{x}_k - \bar{x}\rangle + \langle y_k^*, \tilde{y}_k - y_k\rangle \\
&= \varepsilon_k \left(\|(x, y) - (\bar{x}, y_k)\| - \|(\tilde{x}_k, \tilde{y}_k) - (\bar{x}, y_k)\|\right) - \psi_k(x, y) + \psi_k(\tilde{x}, \tilde{y}) \\
&< (\varepsilon_k + \varepsilon_k')\|(x, y) - (\tilde{x}_k, \tilde{y}_k)\|.
\end{aligned}$$

Since $(\tilde{x}_k, \tilde{y}_k) \in \operatorname{int} \overline{B}_{\rho_k}(\bar{x}, y_k)$, we have $x_k^* \in D^*_{\tilde{\varepsilon}_k} F(\tilde{x}_k, \tilde{y}_k)(y_k^*)$, where $\tilde{\varepsilon}_k := \varepsilon_k + \varepsilon_k' \to 0$ as $k \to \infty$. Thus, we have constructed the desired sequences $(\tilde{x}_k, \tilde{y}_k)$ and $\tilde{\varepsilon}_k$; hence, Case 2 reduces to Case 1, and the proof of Lemma 3.1 is complete. □

Remark 4.1 The above proof is constructive. The function f with the desirable properties is defined by formulas (4.6) and (4.7) which involve special sequences $\{x_k\}$, $\{\rho_k\}$ and $\{v_k\}$. The procedure adopted here follows that used in [10]. It is not unique. One could try to adjust the techniques used, e.g., in [2, Proof of Lemma 2].

Acknowledgements The authors wish to thank the referees for their unique dedication and hard work reading the manuscript, checking every detail, and making impressive effort to help us improve the text. The paper has indeed strongly benefited from the comments and suggestions of the referees.
Many thanks to the Editor-in-Chief for the perfect choice of the referees and overall handling of our manuscript.

Funding The second author benefited from the support of the Australian Research Council, project DP160100854, and the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska–Curie Grant Agreement No. 823731 CONMECH. Open Access funding enabled and organized by CAUL and its Member Institutions

Data availability Data sharing is not applicable to this article as no datasets have been generated or analysed during the current study.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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