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A Mean Value Theorem for Tangentially Convex Functions

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Abstract
The main result is an equality type mean value theorem for tangentially convex functions in terms of tangential subdifferentials, which generalizes the classical one for differentiable functions, as well as Wegge theorem for convex functions. The new mean value theorem is then applied, analogously to what is done in the classical case, to characterize, in the tangentially convex context, Lipschitz functions, increasingness with respect to the ordering induced by a closed convex cone, convexity, and quasiconvexity.

Keywords Mean value theorem · Tangential convexity · Tangential subdifferential · Convexity · Monotonicity · Quasiconvexity · Quasimonotonicity

Mathematics Subject Classification (2010) 26B25 · 49J52 · 47H05

1 Introduction

The class of tangentially convex functions was introduced by Pshenichnyi [24] half a century ago and further studied by Lemaréchal [18], who coined the phrase "tangentially convex." Since then it has received some attention in optimization theory, mainly in connection with optimality conditions [1, 14, 21, 22, 26, 27]. The class of tangentially convex functions at a given point is quite large, as it contains both the class of Gâteaux differentiable functions at that point and that of Clarke regular functions (hence, in particular, the class of convex functions). It is closed under addition and multiplication by nonnegative scalars; it therefore contains a rather large set of nonconvex and nondifferentiable functions (consider, e.g., the sum of a convex function with a Gâteaux differentiable function). Multiplying two nonnegative tangentially convex functions at a point yields a function which is tangentially convex at that point, too.

Dedicated to Miguel Ángel Goberna on the occasion of his 70th birthday

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The aim of this paper is to obtain a mean value theorem for tangentially convex functions. As is well known, the classical mean value theorem is a cornerstone in differential calculus and, as such, it has many applications in mathematical analysis. In view of its importance, in the last decades many generalizations have been obtained in the setting of nosmooth analysis. Some of them are stated as inequalities, but the list of papers dealing with mean value equalities is not so long. To the best of my knowledge, the oldest such result is the mean value theorem for convex functions, in terms of Fenchel subdifferentials, due to L. L. Wegge [30]. This theorem is rather close in spirit to the one presented in this paper, the only essential difference being that the latter deals with tangential subdifferentials instead of Fenchel subdifferentials. In fact, Wegge's theorem is an immediate corollary of Theorem 6 below, but the proofs are of a different nature, that of Theorem 6 relying upon the Hahn-Banach theorem. Other mean value theorems for convex functions, using weaker hypotheses, were obtained by J.-B. Hiriart-Urruty [13]. Soon after the introduction of the notion of generalized gradient in the PhD thesis by F. H. Clarke [5], a mean value theorem for generalized gradients was obtained by G. Lebourg [17], which can also be obtained from Theorem 6, though only in the case of Clarke regular functions. A state of the art in those early days was presented by J.-B. Hiriart-Urruty in [12], where some mean value theorems for a variety of subdifferentials were given, in most cases under some Lipschitz-type assumptions, which are not required in Theorem 6. Mean value theorems for functions defined on infinite dimensional spaces have been obtained, too; the reader may look at [12] and the references therein. Another mean value equality for locally Lipschitz functions was given by J. P. Penot in [23]; in particular, he considered locally Lipschitz tangentially convex functions. There is also an extensive literature on mean value inequalities in nonsmooth analysis; see [28] and its list of references.

The rest of the paper is organized as follows. Section 2 contains some fundamental definitions, including those of tangentially convex function and tangential subdifferential. In Section 3, a mean value theorem for tangentially convex functions, which is the main result of this paper, is presented, together with another mean value theorem of Cauchy type for such functions. Finally, Section 4 uses the mean value theorem of the preceding section to characterize, in the tangentially convex context, Lipschitz functions, increasingness with respect to the ordering induced by a closed convex cone, convexity, and quasiconvexity.

2 Preliminaries

This section recalls the fundamental notions used in the paper, namely those of tangentially convex function and tangential subdifferential.

First, the well known concept of the core (or algebraic interior) of a subset of an Euclidean space, which is required in the definiton of tangential convexity, is recalled.

Definition 1 The core of $X \subseteq \mathbb{R}^n$ is the set

$$core(X) := \{x \in \mathbb{R}^n : \forall d \in \mathbb{R}^n, \exists t_d > 0 \text{ such that } x + td \in X \; \forall t \in [0, t_d]\}.$$

Clearly, $core\,(X)$ contains the topological interior of X.

Definition 2 A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called tangentially convex at $x \in core(f^{-1}(\mathbb{R}))$ if, for every $d \in \mathbb{R}^n$, the limit

$$f'(x, d) := \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}$$

exists, is finite, and is a convex function of d. If f is tangentially convex at every point of a given set, one says that f is tangentially convex on that set.

A suitable concept of subdifferential for tangentially convex functions is the following one, which was implicitly given in [24].

Definition 3 The tangemtial subdifferential of a tangentially convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $x \in core(f^{-1}(\mathbb{R}))$ is the set

$$\partial_T f(x) := \{x^* \in \mathbb{R}^n : \langle x^*, d\rangle \le f'(x, d)\ \forall d \in \mathbb{R}^n\}.$$

In the preceding equality and throughout the paper, $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product.

Since $f'(x, \cdot)$ is positively homogeneous, it is sublinear when f is tangentially convex; therefore $\partial_T f(x) \neq 0$. Furthermore, for every $d \in \mathbb{R}^n$ one has

$$f'(x, d) = \max_{x^* \in \partial_T f(x)} \langle x^*, d\rangle,$$

so that $f'(x, \cdot)$ is the support function of $\partial_T f(x)$.

For a convex function f, the tangential subdifferential at x coincides with the Fenchel subdifferential

$$\partial f(x) := \{x^* \in \mathbb{R}^n : f(y) \ge f(x) + \langle x^*, y - x\rangle\ \forall y \in \mathbb{R}^n\},$$

whereas in the case when f is differentiable, one has $\partial_T f(x) = \{\nabla f(x)\}$

The notation used in this paper is rather standard; in particular, the set of extreme points of a convex set C is denoted by $ext\ C$, the closed Euclidean ball centered at the origin with radius $N > 0$ by $B(0; N)$, and the Euclidean norm by $\|\cdot\|$.

3 A Mean Value Theorem

This section contains the main result: a mean value theorem for tangentially convex functions. The starting point is the following Rolle type theorem.

Lemma 4 *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, tangentially convex on $]a, b[$, and $f(a) = 0 = f(b)$, then there exists $c \in]a, b[$ such that $0 \in \partial_T f(c)$.*

Proof If f is constant, then, clearly, $\partial_T f(c) = \{0\}$ for every $c \in]a, b[$. If f is not constant, then either a global maximum or a global minimum of f (there are such points by the Weierstrass extreme value theorem) belongs to $]a, b[$. If $c \in]a, b[$ is a global maximum of f, then $f'(c, -\delta) \le 0$ and $f'(c, \delta) \le 0$ for every $\delta \ge 0$. Hence, 0 is a global maximum of the convex function $f'(c, \cdot)$, which clearly implies that $f'(c, \cdot) \equiv 0$, and this is in turn equivalent to the equality $\partial_T f(c) = \{0\}$. If $c \in \,]a, b[$ is a global minimum of f, then $f'(c, -1) \geq 0$ and $f'(c, 1) \geq 0$; therefore, in view of the equality $\partial_T f(c) = \left[-f'(c, -1), f'(c, 1)\right]$, we deduce that $0 \in \partial_T f(c)$. □

As in the classical case of differentiable functions, from a Rolle type theorem one easily derives a mean value theorem for one variable functions:

Corollary 5 *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and tangentially convex on $]a, b[$, then there exists $c \in \,]a, b[$ such that $\frac{f(b)-f(a)}{b-a} \in \partial_T f(c)$.*

Proof Apply Lemma 4 to the function $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

□

As one of the reviewers of the initial version of this paper kindly pointed out, more general variants of Lemma 4 and Corollary 5 are already available in the literature. Indeed, Theorems 1 and 2 in [13] , which only require continuity, but no differentiability assumption, state the existence of $c \in \,]a, b[$ such that $0 \in \Delta f(c)$ or $\frac{f(b)-f(a)}{b-a} \in \Delta f(c)$, respectively, the set $\Delta f(c)$ being defined in terms of Dini derivatives in such a way that, in the particular case when the one-sided directional derivatives of f at c exist, it reduces to

$$\Delta f(c) := \left[-f'(c, -1), f'(c, 1)\right] \cup \left[f'(c, 1), -f'(c, -1)\right].$$

Since, in the tangentially convex case, one has $-f'(c, -1) \leq f'(c, 1)$, it turns out that, in such a case, $\Delta f(c) = \partial_T f(c)$, and therefore one obtains Lemma 4 and Corollary 5 as immediate corollaries of Theorems 1 and 2 in [13], respectively.

The main result is the folllowing mean value theorem for tangentially convex functions of several variables.

Theorem 6 *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is continuous on a convex set $C \subseteq f^{-1}(\mathbb{R})$ and tangentially convex on $C \setminus ext\ C$, and a and b are two different points in C, then there exists $t_0 \in \,]0, 1[$ such that $f(b) - f(a) \in \langle \partial_T f(a + t_0(b - a)), b - a\rangle$.*

Proof Let $\varphi : [0, 1] \to \mathbb{R}$ be the function defined by $\varphi(t) := f(a + t(b - a))$. Clearly, f is continuous; moreover, for every $t_0 \in \,]0, 1[$ one has $a + t(b - a) \in C \setminus ext\ C$ and $\varphi'(t, \delta) = f'(a + t(b - a), \delta(b - a))$ for every $\delta \in \mathbb{R}$, which shows that φ is tangentially convex at t. Hence, by Corollary 5, there exists $t_0 \in \,]0, 1[$ such that $\varphi(1) - \varphi(0) \in \partial_T \varphi(t_0)$. We thus have

$$(\varphi(1) - \varphi(0))\,\delta \leq \varphi'(t_0, d) = f'(a + t_0(b - a), \delta(b - a)) \qquad \forall \delta \in \mathbb{R}.$$

This means that the linear function l on the one dimensional subspace of $\mathbb{R}^n$ generated by $b - a$ defined by $l(d) := (\varphi(1) - \varphi(0))\delta$, with $d = \delta(b - a)$, is a minorant of the sublinear function $f'(a + t_0(b - a), \cdot)$. By the Hahn-Banach theorem, l admits a linear extension $\mathbb{R}^n \ni d \mapsto \langle x^*, d\rangle$, with $x^* \in \mathbb{R}^n$, that is a minorant of $f'(a + t_0(b - a), \cdot)$ over the whole space, that is, $x^* \in \partial_T f\,(a + t_0\,(b - a))$. Furthermore, since this linear function extends l, we have

$$f(b) - f(a) = \varphi(1) - \varphi(0) = l(b - a) = (x^*, b - a) \in \langle \partial_T f\,(a + t_0\,(b - a))\,, b - a\rangle\,.$$

$\square$

Remark 7 It is implicit in the assumption that f is tangentially convex on $C \setminus ext\ C$ that $C \setminus ext\ C \subseteq core(f^{-1}(\mathbb{R}))$. This remark worths to be taken into account for the rest of the paper.

The classical mean value theorem is an immediate consequence of Theorem 6, since the tangential subdifferential of a differentiable function reduces to the singleton of its gradient. Wegge mean value theorem [30] also follows, because $\partial_T f = \partial f$ if f is convex. In the same way, Lebourg mean value theorem is a consequence of Theorem 6 in the case of Clarke regular functions, because for such functions the tangential subdifferential coincides with the Clarke generalized gradient.

The following result is a generalization of Cauchy mean value theorem.

Corollary 8 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be continuous on a convex set $C \subseteq f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ and tangentially convex on $C \setminus ext\ C$, and a and b be two different points in C. Assume that one of the following conditions hold:*

(i) *One has $(f(b) - f(a)(g(b) - g(a)) \leq 0$.*
(ii) *The function f is differentiable.*

Then there exists $t_0 \in]0, 1[$ such that

$$(f(b) - f(a))\langle \partial_T g\,(a + t_0\,(b - a))\,, b - a\rangle \cap (g(b) - g(a))\langle \partial_T f\,(a + t_0\,(b - a)\,, b - a)\rangle \neq \emptyset. \tag{1}$$

Proof We first observe that, as a direct consequence of Theorem 6, the result holds true when $f(a) = f(b)$. We thus may assume that $f(a) < f(b)$, and then we define $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$h\,(x) := g\,(x) + \frac{g\,(a) - g\,(b)}{f\,(b) - f\,(a)} f\,(x)\,.$$

Applying Theorem 6 to this function, we obtain the existence of $t_0 \in]0, 1[$ such that

$$\begin{aligned} 0 \in \langle \partial_T h\,(a + t_0\,(b - a))\,, b - a\rangle &= \langle \partial_T g\,(a + t_0\,(b - a))\,, b - a\rangle \\ &+ \frac{g\,(a) - g\,(b)}{f\,(b) - f\,(a)} \langle \partial_T f\,(a + t_0\,(b - a)\,, b - a)\rangle. \end{aligned}$$

From this relation, (1) easily follows. $\square$

4 Some Applications

This section presents some applications of Theorem 6. They are all generalizations of standard applications of the mean value theorem for differentiable functions. The first one is a characterization of the Lipschitz property for tangentially convex functions.

Proposition 9 *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be continuous on a convex set $C\subseteq f^{-1}(\mathbb{R})$ and tangentially convex on $C\setminus ext\ C$, and $N>0$. Then*

$$f \text{ is } N\text{-Lipschitz on } C \Leftrightarrow \bigcup_{x\in C\setminus ext\ C} \partial_T f(x)\subseteq B(0;N).$$

Proof If f is N-Lipschitz on C, then, for $x\in C\setminus ext\ C$ and $x^*\in\partial_T f(x)$, we have

$$\left\|x^*\right\| = \sup_{\|d\|=1}\left\langle x^*,d\right\rangle \le \sup_{\|d\|=1} f'(x,d)\le N.$$

This proves the implication $\Rightarrow$.

Conversely, assume that the inclusion on the right hand side of the equivalence holds true, and let x and y be two different points in C. Then, by Theorem 6, there exist $t_0\in\,]0,1[$ and $x^*\in\partial_T f(x+t_0(y-x))$ such that $f(x)-f(y)=(x^*,x-y)$; hence

$$f(x)-f(y)\le\left\|x^*\right\|\|x-y\|\le N\|x-y\|,$$

which proves that f is N-Lipschitz on C. □

It is very well known that continuously differentiable functions are locally Lipschitz. The following proposition is a generalization of this result for tangentially convex functions with upper semicontinuous tangential subdifferential. For the notion of upper semicontinuous set valued mapping, see [4].

Proposition 10 *Let $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be continuous and tangentially convex on an open set $U\subseteq f^{-1}(\mathbb{R})$. If the set valued mapping $\partial_T f:U\rightrightarrows\mathbb{R}^n$ is upper semicontinuous, then f is locally Lipschitz on U.*

Proof Let $x\in U$ and take $\epsilon>0$ such that $x+B(0;\epsilon)\subseteq U$. Since the image of a compact set under an upper semicontinuous set valued mapping is compact [4], the set $\bigcup_{y\in x+B(0;\epsilon)}\partial_T f(y)$ is compact; therefore, for some $N>0$, we have $\bigcup_{y\in x+B(0;\epsilon)}\partial_T f(y)\subseteq B(0;N)$. Thus, applying Proposition 9, we obtain that f is N-Lipschitz on $x+B(0;\epsilon)$, which proves that f is locally Lipschitz on U. □

The ordering $\le_K$ on $\mathbb{R}^n$ induced by a closed convex cone $K\subseteq\mathbb{R}^n$ is defined by

$$x\le_K y\Longleftrightarrow y-x\in K.$$

One says that $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is K-nondecreasing (K-nonincreasing) on $C\subseteq\mathbb{R}^n$ if, for every $x,y\in C$ such that $x\le_K y$, one has $f(x)\le f(y)$ ($f(x)\ge f(y)$, respectively). Recall that the dual cone and the polar cone of K are defined by

$$K^*:=\{x^*\in\mathbb{R}^n:\left\langle x^*,d\right\rangle\ge 0\qquad\forall d\in K\}$$

and

$$K^0:=\{x^*\in\mathbb{R}^n:\left\langle x^*,d\right\rangle\le 0\qquad\forall d\in K\},$$

respectively. A characterization of the monotonicity properties above in terms of tangential subdifferentials is provided next; in this characterization, $aff\ C$ denotes the affine hull of C, that is, the smallest affine manifold that contains C.

Proposition 11 *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is continuous on an open convex set $C \subseteq f^{-1}(\mathbb{R})$ and tangentially convex on C, then the following equivalences hold:*

$$f \text{ is } K\text{-nondecreasing on } C \Leftrightarrow \partial_T f(x) \subseteq K^* \qquad \forall x \in C \setminus ext\ C. \tag{2}$$

$$f \text{ is } K\text{-nonincreasing on } C \Leftrightarrow \partial_T f(x) \subseteq K^0 \qquad \forall x \in C \setminus ext\ C. \tag{3}$$

Proof We first prove (3). If f is K-nonincreasing on C, $x \in C \setminus ext\ C$ and $d \in K$, then, for small enough $t > 0$, one has $x + td \in C$ and $x \leq_K x + td$; hence $f(x + td) \leq f(x)$. From this inequality, it easily follows that $f'(x, d) \leq 0$; hence, since $\langle x^*, d\rangle \leq f'(x, d)$, we obtain $\langle x^*, d\rangle \leq 0$, thus proving that $x^* \in K^0$.

Conversely, let x and y be two different points in C such that $x \leq_K y$. By Theorem 6, there exist $t_0 \in]0, 1[$ and $x^* \in \partial_T f(y + t_0(x - y))$ such that $f(x) - f(y) = \langle x^*, x - y\rangle$. Since $\langle x^*, x - y\rangle \leq 0$, we immediately deduce that $f(x) \geq f(y)$, which proves that f is K-nonincreasing on C.

Equivalence (2) is nothing but (3) with K replaced with $-K$. □

A fundamental notion in nonlinear analysis is that of monotonicity. A set valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be monotone if, for every $x, y \in \mathbb{R}^n$, $x^* \in F(x)$ and $y^* \in F(y)$, one has $\langle x^* - y^*, x - y\rangle \geq 0$. It is well known and easy to prove that the Fenchel subdifferential operator is monotone; therefore, a natural question is whether the tangential subdifferential operator is monotone, too. The following two results establish that this is the case only for convex functions, in which case the tangential subdifferential and the Fenchel subdifferential agree. Other characterizations of convexity in terms of monotonicity of various subdifferentials were obtained by D. Aussel, J. N. Corvellec and M. Lassonde [2] and R. Correa, A. Jofré and L. Thibault [6, 7].

Proposition 12 *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is continuous on $f^{-1}(\mathbb{R})$, the set $f^{-1}(\mathbb{R})$ is convex and open, f is tangentially convex on $f^{-1}(\mathbb{R})$, and $\partial_T f$ is monotone, then $\partial_T f = \partial f$.*

Proof Let $x \in f^{-1}(\mathbb{R})$, $x^* \in \partial_T f(x)$, and $y \in f^{-1}(\mathbb{R}) \setminus \{x\}$. By Theorem 6, there exist $t_0 \in]0, 1[$ and $z^* \in \partial_T f(x + t_0(y - x))$ such that $f(y) - f(x) = \langle z^*, y - x\rangle$. Since $\partial_T f$ is monotone, from the preceding equality we obtain $f(y) - f(x) \geq \langle x^*, y - x\rangle$, which shows that $x^* \in \partial f(x)$. Hence $\partial_T f(x) \subseteq \partial f(x)$. Since the opposite inclusion is an easy consequence of the tangential convexity of f, we conclude that $\partial_T f(x) = \partial f(x)$. □

Corollary 13 *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is continuous on $f^{-1}(\mathbb{R})$, the set $f^{-1}(\mathbb{R})$ is convex and open, f is tangentially convex on $f^{-1}(\mathbb{R})$, and $\partial_T f$ is monotone, then f is convex.*

Proof By Proposition 12, the function f is subdifferentiable on its domain and is therefore convex. □

In a similar way as monotonicity of the tangential subdifferential is closely related to convexity, as we have just seen, quasimonotonicity (a notion weaker than monotonicity) of the tangential subdifferential is closely related to quasiconvexity (a notion weaker than convexity). A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be quasiconvex if all its sublevel sets $f^{-1}(]-\infty, \lambda])$, $\lambda \in \mathbb{R}$, are convex. Clearly, every convex function is quasiconvex. Quasiconvex functions are not so well known as convex functions, nevertheless they are important in optimization and, maybe even more, in economic theory, mainly in microeconomics. They were introduced by B. de Finetti [8]; for an introduction to quasiconvex functions, see [3], and for applications to economics, as well as other important developments on quasiconvexity, see [20, 25]. As for the notion of quasimonotonicity, wich will be introduced next, the interested reader may consult [10]; some applications to economics are discussed in [15, 20].

Definition 14 [11, 16] A set valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be quasimonotone if, for every $x, y \in \mathbb{R}^n$, $x^* \in F(x)$ and $y^* \in F(y)$, one has

$$\min\left\{\left\langle x^*, y-x\right\rangle, \left\langle y^*, x-y\right\rangle\right\} \leq 0.$$

It is easy to see that every monotone mapping is quasimonotone.

Many notions of subdifferential for quasiconvex functions have been proposed in the literature. The oldest one is the Greenberg-Pierskalla subdifferential, which is the largest of all of them. If a function has a nonempty Greenberg-Pierskalla subdifferential at every point of its domain, then it is quasiconvex, but the converse does not hold (in the same way as a convex function does not necessarily have a nonempty Fenchel subdifferential everywhere). Conditions for the nonemptiness of the Greenberg-Pierskalla subdifferential of quasiconvex functions have been recently investigated in [29].

Definition 15 [9] The Greenberg-Pierskalla subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $x \in f^{-1}(\mathbb{R})$ is the set

$$\partial^{GP} f(x) := \{x^* \in \mathbb{R}^n : f(x) = \min\{f(y) : \left\langle x^*, y-x\right\rangle \geq 0\}\}.$$

If this set is nonempty, one says that f is GP-subdifferentiable at x.

The well known notion of stationary point is recalled next.

Definition 16 We say that $x \in \mathbb{R}^n$ is a stationary point of $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ if f is Gâteaux differentiable at x and '$\nabla f(x) = 0$ (equivalently, if f is tangentially subdifferentiable at x and $\partial_T f(x) = \{0\}$).

The last result will establish the close relationship linking quasiconvexity, quasimonotonicity of the tangential subdifferential, and the Greenberg-Pierskalla subdifferential. Other characterizations of quasiconvexity in terms of quasimonotonicity of various subdifferentials were obtained by A. Hassouni [11], D. T. Luc [19], and D. Aussel, J. N. Corvellec and M. Lassonde [2].

Proposition 17 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be continuous on $f^{-1}(\mathbb{R})$, the set $f^{-1}(\mathbb{R})$ be convex and open, and f be tangentially convex on $f^{-1}(\mathbb{R})$.*

If f is quasiconvex, then $\partial_T f$ is quasimonotone.

Conversely, if $\partial_T f$ is quasimonotone, then $\partial_T f \setminus \{0\} \subseteq \partial^{GP} f$ (pointwise); hence, if, moreover, f is GP-subdifferentiable at x, then f is quasiconvex.

Proof If f is quasiconvex, then, for every $x, y \in \mathbb{R}^n$, $x^* \in \partial_T f(x)$ and $y^* \in \partial_T f(y)$, using [19, Theorem 5.2] we obtain

$$\min\left\{\left\langle x^*, y-x\right\rangle, \left\langle y^*, x-y\right\rangle\right\} \leq \min\left\{f'\left(x, y-x, f'\left(y, x-y\right)\right)\right\} \leq 0,$$

which proves that $\partial_T f$ is quasimonotone.

Conversely, assume that $\partial_T f$ is quasimonotone, and let $x \in f^{-1}(\mathbb{R})$, $x^* \in \partial_T f(x) \setminus \{0\}$ and $y \in f^{-1}(\mathbb{R})$ be such that $\langle x^*, y-x\rangle \geq 0$. For sufficiently small $\epsilon > 0$, we have

$y + \epsilon x^* \in f^{-1}(\mathbb{R})$. Hence, by Theorem 6, there exist $t_0 \in]0, 1[$ and $z^* \in \partial_T f(x + t_0(y + \epsilon x^* - x))$ such that

$$f(y + \epsilon x^*) - f(x) = \left\langle z^*, y + \epsilon x^* - x \right\rangle. \tag{4}$$

From the quasimonotonicity of $\partial_T f$, it easily follows that

$$\min \left\{ \left\langle x^*, y + \epsilon x^* - x \right\rangle, \left\langle z^*, x - y - \epsilon x^* \right\rangle \right\} \leq 0;$$

hence, in view of the inequality $\langle x^*, y - x \rangle \geq 0$, we deduce that $\langle z^*, x - y - \epsilon x^* \rangle \leq 0$, which, by (4), implies that $f(y + \epsilon x^*) - f(x) \geq 0$. Setting $\epsilon \to 0^+$, since f is continuous we obtain $f(y) \geq f(x)$. This proves that $x^* \in \partial^{GP} f(x)$. Thus $\partial_T f(x) \setminus \{0\} \subseteq \partial^{GP} f(x)$ for every $x \in f^{-1}(\mathbb{R})$. If f is GP-subdifferentiable at x, this inclusion implies that f is GP-subdifferentiable on its domain, and therefore f is quasiconvex. □

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References

1. Allevi, E., Martínez-Legaz, J.E., Riccardi, R.: Optimality conditions for convex problems on intersections of non necessarily convex sets. J. Glob. Optim. **77**, 143–155 (2020)
2. Aussel, D., Corvellec, J.-N., Lassonde, M.: Subdifferential characterization of quasiconvexity and convexity. J. Convex Anal. **1**, 195–201 (1994)
3. Avriel, M., Diewert, W.E., Schaible, S., Zang, I.: Generalized Concavity. Plenum Press, New York (1988)
4. Berge, C.: Espaces Topologiques. Fonctions Multivoques. Dunod, Paris (1966)
5. Clarke, F.H.: Necessary Conditions for Nonsmooth Problems in Optimal Control and the Calculus of Variations, Ph.D. thesis, Univ. of Washington (1973)
6. Correa, R., Jofré, A., Thibault, L.: Subdifferential monotonicity as characterization of convex functions. Numer. Funct. Anal. Optim. **15**, 531–535 (1994)
7. Correa, R., Jofré, A., Thibault, L.: Subdifferential characterization of convexity. In: Ding-Zhu, D. et al. (eds.) Recent Advances in Nonsmooth Optimization, pp. 18–23. World Scientific, Singapore (1995)
8. de Finetti, B.: Sulle stratificazioni convesse. Ann. Matematica **30**, 173–183 (1949)
9. Greenberg, H.J., Pierskalla, W.P.: Quasi-conjugate functions and surrogate duality. Cah. Cent. Étud. Rech. Opér. **15**, 437–448 (1973)
10. Hadjisavvas, N., Schaible, S.: Generalized monotone maps. In: Hadjisavvas, N. et al. (eds.) Handbook of Generalized Convexity and Generalized Monotonicity, pp. 389–420. Springer, New York (2005)
11. Hassouni, A.: Sous-différentiels des fonctions quasi-convexes, Thèse de 3ème Cycle, Université Paul Sabatier, Toulouse III (1983)
12. Hiriart-Urruty, J.-B.: Mean value theorems in nonsmooth analysis. Numer. Funct. Anal. Optim. **2**, 1–30 (1980)

13. Hiriart-Urruty, J.-B.: A note on the mean value theorem for convex functions. Boll. Unione Mat. Ital. V. Ser. B **17**, 765–775 (1980)
14. Hiriart-Urruty, J.-B., Ledyaev, Y.S.: A note on the characterization of the global maxima of a (tangentially) convex function over a convex set. J. Convex Anal. **3**, 55–61 (1996)
15. John, R.: Uses of generalized convexity and generalized monotonicity in economics. In: Hadjisavvas, N. et al. (eds.) Handbook of Generalized Convexity and Generalized Monotonicity, pp. 619–666. Springer, New York (2005)
16. Karamardian, S., Schaible, S.: Seven kinds of monotone maps. J. Optim. Theory Appl. **66**, 37–46 (1990)
17. Lebourg, G.: Valeur moyenne pour gradient généralise. C. R. Acad. Sci. Paris, Sér. A **281**, 795–797 (1975)
18. Lemaréchal, C.: An introduction to the theory of nonsmooth optimization. Optimization **17**, 827–858 (1986)
19. Luc, D.T.: Characterizations of quasiconvex functions. Bull. Austral. Math. Soc. **48**, 393–406 (1993)
20. Martínez-Legaz, J.E.: Generalized convex duality and its economic applications. In: Hadjisavvas, N. et al. (eds.) Handbook of Generalized Convexity and Generalized Monotonicity, pp. 237–292. Springer, New York (2005)
21. Martínez-Legaz, J.E.: Optimality conditions for pseudoconvex minimization over convex sets defined by tangentially convex constraints. Optim. Lett. **9**, 1017–1023 (2015)
22. Mashkoorzadeh, F., Movahedian, N., Nobakhtian, S.: Optimality conditions for nonconvex constrained optimization problems. Numer. Funct. Anal. Optim. **40**, 1918–1938 (2019)
23. Penot, J.P.: Mean-value theorem with small subdifferentials. J. Optim. Theory Appl. **94**, 209–221 (1997)
24. Pshenichnyi, B.N.: Necessary Conditions for an Extremum. Marcel Dekker Inc, New York (1971)
25. Schaible, S., Ziemba, W.T. (eds.): Generalized Concavity in Optimization and Economics. Academic Press, New York (1981)
26. Sisarat, N., Wangkeeree, R.: Characterizing the solution set of convex optimization problems without convexity of constraints. Optim. Lett. **14**, 1127–1144 (2020)
27. Sisarat, N., Wangkeeree, R., Lee, G.M.: KKT optimality conditions in non-smooth, non-convex optimization. J. Nonlinear Convex Anal. **19**, 1319–1329 (2018)
28. Uderzo, A.: Multidirectional mean value inequalities in quasidifferential calculus. Numer. Funct. Anal. Optim. **26**, 709–733 (2005)
29. Volle, M., Martínez-Legaz, J.E.: Some remarks on Greenberg-Pierskalla subdifferentiability of quasi-convex functions. Vietnam J. Math. **48**, 391–406 (2020)
30. Wegge, L.L.: Mean value theorem for convex functions. J. Math. Econ. **1**, 207–208 (1974)

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