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Dynamics of Nonautomous Impulsive Multivalued Processes

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Abstract
In this paper we study the asymptotic behaviour of multivalued processes which are under the influence of impulsive action. We provide conditions to guarantee the existence of a pullback attractor and we illustrate the results with several examples.

Keywords Generalized process · Impulses · Pullback attractors · Multivalued process

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1 Introduction

The theory of attractors for nonautonomous dynamical systems has been well-studied in the last few years (see for example [1–7]). An attractor usually gives us information about the long-term behaviour of the solutions of the system. Here we will focus on the study of the multivalued situation, that is, when we may have more than one solution for a given initial data. These dynamical systems include, for example, some reaction-diffusion equations, other models of physics or differential inclusions. Some results can be found on [1, 4, 7–11].

Furthermore, solutions can experience discontinuities, that is, continuous trajectories could have perturbations or changes in their state when they reach a certain set in the phase space. These changes can be interpreted as jumps or some forced corrections in order to avoid undesirable situations. The analysis of systems with impulsive perturbations has been studied during the last 30 years, see for example the monographs [12, 13]. The theory of

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impulsive dynamical systems can be traced back to the 1970's, see [14, 15]. In [16, 17] it was studied the evolution of such systems and in [18, 19] some results were obtained related to the continuity of the impact time map (see (2) for the definition). More recently, there have been results on the existence of different types of attractors and some properties for different types of systems, most of them in the autonomous situation, see for example [20–24]. There are also several results regarding the stability and control theory of these systems, see for example [25–29]

In this paper we focus on the study of the dynamics of systems in the nonautonomous and impulsive multivalued situation. For example, in [23, 30, 31] they consider autonomous multivalued dynamical systems. We study the nonautonomous situation and we give conditions in order to guarantee the existence of the pullback attractor. In order to do that, we define the notion of impulsive generalized process, we state some properties, and later we prove the existence of the pullback attractor. In Section 2 we recall some facts about generalized processes and multivalued processes and we define the notion of impulsive generalized process (which consists of a generalized process $\mathscr{G}$, an impulsive family of sets $\hat{M}$ and a collection of multifunctions I). Then we study some properties of these processes. In Section 3 we define the notion of pullback attractor for these systems and we give conditions to guarantee its existence. In Section 4 we present some applications of the results. Finally, in an appendix we include the proofs of some results of Section 3.

2 Impulsive Generalized Processes

Let (X, d) be a complete metric space. The following definitions can be found, for example, in [6].

Definition 1 A generalized process $\mathscr{G} = \{\mathscr{G}(t)\}_{t\in\mathbb{R}}$ in X is a family of sets $\mathscr{G}(t)$ consisting of functions $\varphi : [t, +\infty) \longrightarrow X$ satisfying:

(G1) (Existence) For each $t \in \mathbb{R}$ and $x \in X$, there exists at least one $\varphi \in \mathscr{G}(t)$ such that $\varphi(t) = x$.

(G2) (Translation) If $\varphi \in \mathscr{G}(t)$ and $s \geq 0$, then the map $\varphi^{+s} \in \mathscr{G}(t+s)$, with $\varphi^{+s} = \varphi\mid_{[t+s,+\infty)}$.

(G3) (Upper semicontinuity with respect to initial data) If $\{\varphi_n\}_n \subset \mathscr{G}(s)$ and $\varphi_n(s) \longrightarrow x$, then there exist a subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}_n$ and $\varphi \in \mathscr{G}(s)$ with $\varphi(s) = x$ such that $\varphi_{n_k}(t) \longrightarrow \varphi(t)$ as $k \to \infty$ for each $t \geq s$.

In this work we will assume that:

(G4) (Continuity) Every map $\varphi : [\tau, +\infty) \longrightarrow X$ in $\mathscr{G}(\tau)$ is continuous.

Definition 2 We say that a generalized process $\mathscr{G} = \{\mathscr{G}(t)\}_{t\in\mathbb{R}}$ is exact (or strict) if it satisfies the following condition:

(G5) (Concatenation) If $\varphi \in \mathscr{G}(\tau)$, $\psi \in \mathscr{G}(r)$ and $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$, then $\theta \in \mathscr{G}(\tau)$, with θ defined as

$$\theta(t) := \begin{cases} \varphi(t), & t \in [\tau, s], \\ \psi(t), & t > s. \end{cases}$$

Definition 3 Let $\mathscr{G}$ be a generalized process. A multivalued process $\{U(t,s)\}_{t\geq s}$ is a family of multivalued operators $U(t,s) : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ defined as

$$U(t,s)D := \{\varphi(t) : \varphi \in \mathscr{G}(s),\ \varphi(s) \in D\}.$$

This multivalued process satisfies:

1. $U(t,t)x = x$ for all $t \in \mathbb{R}$ and $x \in X$
2. $U(t,s)x \subset U(t,\tau)\,(U(\tau,s)x)$ for all $s \leq \tau \leq t$ and $x \in X$.

Furthermore, if we have an exact generalized process, then on the second property we have an equality, that is,

$$U(t,s) = U(t,\tau)U(\tau,s).$$

We recall for completeness the following result. Its proof can be seen, for example, in [32, Theorem 2.2], for the autonomous case.

Proposition 1 *Let $\mathscr{G}$ be an exact generalized process, $s \in \mathbb{R}$ and $\{\varphi_n\}_n$, φ elements of $\mathscr{G}(s)$ such that $\varphi_n(t)$ converges to $\varphi(t)$ for all $t > s$. Then $\varphi_n(t)$ converges to $\varphi(t)$ uniformly for t in compact subsets of (s,∞). In particular, we have the following property:*

$$\begin{cases} \text{If } \{\varphi_n\}_n \subset \mathscr{G}(s) \text{ and } \varphi_n(s) \longrightarrow x, \text{ then there exist a subsequence} \\ \{\varphi_{n_k}\}_k \text{ and } \varphi \in \mathscr{G}(s) \text{ with } \varphi(s) = x \text{ and } \varphi_{n_k}(t) \longrightarrow \varphi(t) \\ \text{uniformly for } t \text{ in compact subsets of } (s,\infty). \end{cases}$$

This result, in general, is not valid for compact subsets of $[s,\infty)$. Examples can be found in [33] or in [32, Section 6.2].

Definition 4 Let $\mathscr{G}$ be a generalized process and $\hat{D} = \{D(t)\}_{t\in\mathbb{R}}$ a family of sets. We say that:

- $\hat{D}$ is positively invariant if $U(t,s)D(s) \subset D(t)$ for all $t \geq s$.
- $\hat{D}$ is negatively invariant if $D(t) \subset U(t,s)D(s)$ for all $t \geq s$.
- $\hat{D}$ is invariant if $\hat{D}$ is both positively and negatively invariant.

Definition 5 Let $\mathfrak{D}$ be a collection of non-empty families of sets. We say that $\mathfrak{D}$ is inclusion-closed if for any $\hat{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathfrak{D}$ and any $\hat{D}_1 = \{D_1(t)\}_{t\in\mathbb{R}}$ with $\emptyset \neq D_1(t) \subset D(t)$ for all $t \in \mathbb{R}$, then $\hat{D}_1 \in \mathfrak{D}$. Any collection of non-empty family of sets which is inclusion-closed is called a universe.

In applications, the two usual examples of universes are the "bounded universe" $\mathfrak{D}_B$, which consists of all the families $\{D(t)\}_{t\in\mathbb{R}}$ such that there exists B a bounded set with $D(t) \subset B$ for all $t \in \mathbb{R}$; and the "tempered universe", consisting of families $\{D(t)\}_{t\in\mathbb{R}}$ such that the map

$$t \longmapsto \sup\{\|x\| : x \in D(t)\}$$

grows subexponentially when $t \to -\infty$. See for example [3, 34].

From now on, $\mathfrak{D}$ will denote an arbitrary universe. We need to give some sense to the word "attraction". In order to do that, we consider the following definition of pullback attraction, see [3, Chapter 1] for more information.

Definition 6 Let $\hat{A}$ and $\hat{B}$ be two families of sets. We say that $\hat{A}$ pullback attracts $\hat{B}$ if

$$\lim_{s \to -\infty} \mathrm{d}_H(U(t,s)B(s), A(t)) = 0 \text{ for each } t \in \mathbb{R},$$

where d_H denotes the Hausdorff semidistance, which is given by

$$\mathrm{d}_H(C, D) := \sup_{c \in C} \inf_{d \in D} d(c, d).$$

We remark that $\mathrm{d}_H(C, D) = 0$ only implies that $\overline{C} \subset \overline{D}$. We recall here the definition of upper semicontinuity of multifunctions, as well as a known result in set-valued analysis which is useful in the study of the upper semicontinuity. The proof of this result can be found in [35]

Definition 7 Let X and Y be two metric spaces. A multifunction $F : X \longrightarrow \mathscr{P}(Y)$ is upper semicontinuous at $x \in X$ if for every open neighborhood V of $F(x)$ there exists an open neighborhood U of x such that $F(U) \subset V$.

Proposition 2 *Let X and Y be two metric spaces. A multifunction $F : X \longrightarrow \mathscr{P}(Y)$ is upper semicontinuous and compact valued at $x \in X$ if and only if for every sequence $x_n \longrightarrow x$ and every sequence $y_n \in F(x_n)$, there exist a subsequence $\{y_{n_k}\}_k$ and $y \in F(x)$ such that $\{y_{n_k}\}_k$ converges to y.*

Definition 8 A family of sets $\hat{D} = \{D(t)\}_{t \in \mathbb{R}}$ will be called collectively closed if for any $t_n \longrightarrow t$ and $x_n \in D(t_n)$ with $x_n \longrightarrow x$ we have $x \in D(t)$. It will be called collectively compact if for any $t_n \longrightarrow t$ and $x_n \in D(t_n)$, the sequence $\{x_n\}_n$ has a convergent subsequence with limit in $D(t)$.

After all these definitions, we are in position to define the notion of impulsive generalized process. The goal of this paper is to study these type of processes.

Definition 9 An impulsive generalized process $(\mathscr{G}, \hat{M}, I)$ consists of a generalized process $\mathscr{G}$, a collectively closed family of sets $\hat{M} = \{M(t)\}_{t \in \mathbb{R}}$ such that for every $s \in \mathbb{R}$, $x \in M(s)$ and $\varphi \in \mathscr{G}(s)$ with $\varphi(s) = x$,

$$\exists\, \varepsilon = \varepsilon(\varphi, s) > 0 \text{ such that } \bigcup_{r \in (0,\varepsilon)} \{\varphi(s+r)\} \cap M(s+r) = \emptyset, \tag{1}$$

and collection of collectively upper semicontinuous multifunctions which are compact-valued $I = \{I_t : M(t) \longrightarrow \mathscr{P}(X)\}_{t \in \mathbb{R}}$, that is:

$$\begin{aligned}&\text{for every sequences } t_n \longrightarrow t,\ x_n \longrightarrow x \text{ and } y_n \in I_{t_n}(x_n),\ \text{ there exists}\\ &\text{a convergent subsequence } \{y_n\}_n \text{ with limit in } I_t(x).\end{aligned}$$

Remark 1 Condition (1) is different from some previous papers (cf. [24, 30]), which also include a condition on φ "backwards" in time, that is, before "touching" the set $\hat{M}$. See also [36, Remark 2].

Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process. For each $s \in \mathbb{R}$ and $\varphi \in \mathscr{G}(s)$, we define the impact time map by

$$\phi(\varphi, s) := \inf\{t > 0 : \varphi(s+t) \in M(s+t)\}, \tag{2}$$

and we denote $\phi(\varphi, s) = \infty$ if $\varphi(s+t) \notin M(s+t)$ for all $t > 0$.

Proposition 3 *The map $\phi(\varphi, s) > 0$ for all $s \in \mathbb{R}$ and $\varphi \in \mathscr{G}(s)$.*

Proof Fix $s \in \mathbb{R}$ and $\varphi \in \mathscr{G}(s)$. If $\varphi(s) \in M(s)$, then $\phi(\varphi, s) \geq \varepsilon$, with ε given by (1). If $\varphi(s) \notin M(s)$ and $\phi(\varphi, s) = 0$, then there exists a sequence $\{r_n\}_n$ of positive numbers convergent to 0 such that $\varphi(s + r_n) \in M(s + r_n)$. As φ is continuous and $\hat{M}$ is collectively closed, then $\varphi(s) \in M(s)$, a contradiction. □

Remark 2 If $\phi(\varphi, s) \neq \infty$, then $\varphi(s + \phi(\varphi, s)) \in M(s + \phi(\varphi, s))$.

This positive number, if it exists, is the smallest number such that $\varphi(s+t) \in M(s+t)$, meaning that if $\varphi(r) \in M(r)$ for some $r > s$, then $s + \phi(\varphi, s) \leq r$. We note that this is a generalization of the impact time map in the single-valued case (see [24]), which was defined as a function from $X \times \mathbb{R}$. The properties of this map will help us understand better the evolution of the impulsive trajectories, which will be defined next. This definition of impulsive trajectories is a generalization from the single-valued case.

Definition 10 Given $s \in \mathbb{R}$, a map $\tilde{\varphi} : [s, \omega) \longrightarrow X$, with $\omega \in (s, +\infty)$, will be called an impulsive trajectory of $(\mathscr{G}, \hat{M}, I)$ if there exists a division of $[s, \omega)$ into a family of subintervals

$$[s, \omega) = [t_0, t_1) \cup [t_1, t_2) \cup \cdots$$

with $t_0 = s$, $t_k < t_{k+1}$ and the union could be finite or not finite. Furthermore, for each k, there exists $\varphi_k \in \mathscr{G}(t_k)$ satisfying:

(i) $\phi(\varphi_k, t_k) = \infty$ or $\phi(\varphi_k, t_k) = t_{k+1} - t_k$,
(ii) $\tilde{\varphi}(t) = \varphi_k(t)$ for $t \in [t_k, t_{k+1})$,
(iii) if $\phi(\varphi_k, t_k) \neq \infty$, then $\tilde{\varphi}(t_{k+1}) \in I_{t_{k+1}}(\varphi_k(t_{k+1}))$.

The times t_k will be called jump times of $\tilde{\varphi}$, the family of impulsive trajectories starting at s will be denoted by $\tilde{\mathscr{G}}(s)$, and we will also denote $\tilde{\mathscr{G}} = \{\tilde{\mathscr{G}}(s)\}_{s \in \mathbb{R}}$.

From the previous definition, our first result is existence of (local) impulsive impulsive trajectories. It follows from the existence property (G1) in the definition of generalized processes (see Definition 1).

Proposition 4 *For each $s \in \mathbb{R}$ and $x \in X$, there exists $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, defined on an interval $[s, \omega)$, with $\omega > s$, such that with $\tilde{\varphi}(s) = x$.*

Proof By definition of impulsive trajectory and (G1), there exists $\varphi_0 \in \mathscr{G}(s)$ with $\varphi_0(s) = x$. If $\phi(\varphi_0, s) = \infty$, then $\tilde{\varphi}(t) = \varphi_0(t)$ for all $t \geq s$. On the other hand, if $\phi(\varphi_0, s) \neq \infty$, then $\phi(\varphi_0, s) > 0$. Denote $t_1 := s + \phi(\varphi_0, s)$. We have that $\varphi_0(t_1) \in M(t_1)$. Take $x_1 \in I_{t_1}(\varphi_0(t_1))$ and $\varphi_1 \in \mathscr{G}(t_1)$ with $\varphi_1(t_1) = x_1$. If $\phi(\varphi_1, t_1) = \infty$, then we define

$$\tilde{\varphi}(t) = \begin{cases} \varphi_0(t), & s \leq t < t_1, \\ \varphi_1(t), & t_1 \leq t. \end{cases}$$

If $\phi(\varphi_1, t_1)$ is finite, we denote $t_2 = t_1 + \phi(\varphi_1, t_1)$, and then $\varphi_1(t_2) \in M(t_2)$. Take $x_2 \in I_{t_2}(\varphi_1(t_2))$ and $\varphi_2 \in \mathscr{G}(t_2)$ with $\varphi_2(t_2) = x_2$. We continue analogously. □

We introduce next a condition which will be used through the paper in order to prove the main results.

$$I_\tau(M(\tau)) \cap M(\tau) = \emptyset \quad \forall \tau \in \mathbb{R}. \tag{I}$$

With this condition, we are able to prove the following useful result.

Proposition 5 *Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process satisfying Condition (I). Then, for each $s \in \mathbb{R}$, $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ and $t \in (s, \omega)$, with the interval (s, ω) the domain of definition of the impulsive trajectory $\tilde{\varphi}$, we have $\tilde{\varphi}(t) \notin M(t)$.*

From now on we will assume:

$$\text{Every impulsive trajectory is defined on } [s, +\infty). \tag{3}$$

From the definition of impulsive trajectories we can define a new family of multivalued maps $\{\tilde{U}(t,s)\}_{t \geq s}$, given by $\tilde{U}(t,s) : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ and defined as

$$\tilde{U}(t,s)D := \{\tilde{\varphi}(t) : \tilde{\varphi} \in \tilde{\mathscr{G}}(s),\ \tilde{\varphi}(s) \in D\}.$$

Lemma 6 *Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process. Then*

1. *$\tilde{\mathscr{G}}$ satisfies (G2) and (G5),*
2. *$\tilde{U}(t,s) = \tilde{U}(t,\tau)\tilde{U}(\tau,s)$ for any $s \leq \tau \leq t$.*

The definitions of invariance and pullback attraction for $\tilde{U}$ are analogous, just replace U by $\tilde{U}$.

3 Existence of the Pullback Attractor

Definition 11 Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process. We say that a family $\hat{A} \in \mathfrak{D}$ is a pullback $\mathfrak{D}$-semi attractor if:

(a) $A(t)$ is compact for all $t \in \mathbb{R}$,
(b) $\hat{A}$ pullback attracts each $\hat{D} \in \mathfrak{D}$.

When $\hat{A}$ satisfies

(c) the family $\hat{A} \setminus \hat{M} = \{A(t) \setminus M(t)\}_{t \in \mathbb{R}}$ is invariant

we will say that $\hat{A}$ is a pullback $\mathfrak{D}$-attractor.

We remark that a pullback $\mathfrak{D}$-semi attractor may not satisfy (c). An example on the autonomous case can be found in [20] or in [30].

The following result tells us that the pullback $\mathfrak{D}$-attractor is unique "up to $\hat{M}$".

Proposition 7 *Let $(\mathscr{G}, \hat{M}, I)$ be an impulsive generalized process. If $\hat{A}$ and $\hat{B}$ are two pullback $\mathfrak{D}$-attractors, then $\hat{A} \setminus \hat{M} = \hat{B} \setminus \hat{M}$.*

Proof Fix $t \in \mathbb{R}$. We know that $\hat{B} \in \mathfrak{D}$, which implies that $\hat{B} \setminus \hat{M}$ also belongs to $\mathfrak{D}$ because $\mathfrak{D}$ is a universe. Using the invariance of $\hat{B} \setminus \hat{M}$ we have that

$$\mathrm{d}_H(B(t) \setminus M(t), A(t)) = \mathrm{d}_H(\tilde{U}(t,s)(B(s) \setminus M(s)), A(t))$$

for any $s \leq t$. Using that $\hat{A}$ is a pullback $\mathfrak{D}$-attractor, taking $s \longrightarrow -\infty$ we have that

$$\lim_{s \to -\infty} \mathrm{d}_H(\tilde{U}(t,s)(B(s) \setminus M(s)), A(t)) = 0$$

so $\mathrm{d}_H(B(t) \setminus M(t), A(t)) = 0$, which implies $B(t) \setminus M(t) \subset A(t)$. Interchanging $\hat{A}$ and $\hat{B}$ we get the desired result. □

We present the definition of impulsive pullback ω-limit and the related concepts of pullback $\mathfrak{D}$-asymptotically compactness and pullback $\mathfrak{D}$-dissipativeness. These definitions will turn out to be very important on the construction of the pullback $\mathfrak{D}$-semi attractors and pullback $\mathfrak{D}$-attractors, and they are a little bit different that the continuous case.

Definition 12 Let $\hat{D}$ be a family of sets. The impulsive pullback ω-limit set of $\hat{D}$ at time $t \in \mathbb{R}$, denoted by $\tilde{\omega}(\hat{D}, t)$, is defined as the set of elements $x \in X$ such that there exist $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$ and $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$ with $\tilde{\varphi}_n(s_n) \in D(s_n)$ for each $n \in \mathbb{N}$ such that $\tilde{\varphi}_n(t+\varepsilon_n) \longrightarrow x$. The impulsive pullback ω-limit of $\hat{D}$ is the family $\tilde{\omega}(\hat{D}) = \{\tilde{\omega}(\hat{D}, t)\}_{t\in\mathbb{R}}$.

Definition 13 We say that $\tilde{\mathscr{G}}$ is pullback $\mathfrak{D}$-asymptotically compact if for each $D \in \mathfrak{D}$, $t \in \mathbb{R}$, $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$ and $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$ with $\tilde{\varphi}_n(s_n) \in D(s_n)$, then the sequence $\{\tilde{\varphi}_n(t+\varepsilon_n)\}_n$ has a convergent subsequence.

Definition 14 We say that $\tilde{\mathscr{G}}$ is pullback $\mathfrak{D}$-dissipative if there exists $\hat{B}_0 \in \mathfrak{D}$ collectively closed such that for all $\hat{D} \in \mathfrak{D}$, $t \in \mathbb{R}$, $s_n \longrightarrow -\infty$ and $\varepsilon_n \longrightarrow 0$, there exists $n_0 = n_0(\hat{D}, t) \in \mathbb{N}$ such that if $n \geq n_0$, $\tilde{\varphi} \in \tilde{\mathscr{G}}(s_n)$ and $\tilde{\varphi}(s_n) \in D(s_n)$, then $\tilde{\varphi}(t+\varepsilon_n) \in B_0(t+\varepsilon_n)$. The family $\hat{B}_0$ is called pullback $\mathfrak{D}$-absorbing family.

The main difference between these three definitions and the related ones in the continuous case is the presence of the sequence of $\{\varepsilon_n\}_n$.

3.1 Existence of the Pullback semi Attractor

We present some properties of the impulsive pullback ω-limit in combination with the previous definitions.

Proposition 8 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process, $\hat{D} \in \mathfrak{D}$ and $t \in \mathbb{R}$. Then the impulsive pullback ω-limit of $\hat{D}$, $\tilde{\omega}(\hat{D})$, is non-empty, collectively compact and pullback attracts $\hat{D}$.*

Proof The non-emptiness is trivial. First we prove the collective compactness. Take $t_n \longrightarrow t$ and $y_n \in \tilde{\omega}(\hat{D}, t_n)$. We want to prove that the sequence $\{y_n\}_n$ has a convergent subsequence with limit in $\tilde{\omega}(\hat{D}, t)$.

For each $n \in \mathbb{N}$, we have that $y_n \in \tilde{\omega}(\hat{D}, t_n)$. Then there exist $s_n \leq t - n$, $|\varepsilon_n| < 1/n$ and $\tilde{\varphi} \in \tilde{\mathscr{G}}(s_n)$ with $\tilde{\varphi}(s_n) \in D(s_n)$ such that $d(\tilde{\varphi}_n(t_n+\varepsilon_n), y_n) < 1/n$. As $\delta_n := t_n - t + \varepsilon_n \longrightarrow 0$, we have that $\{\tilde{\varphi}_n(t + (t_n - t + \varepsilon_n))\}_n$ has a convergent subsequence by pullback $\mathfrak{D}$-asymptotical compactness. Thus we may assume that $\tilde{\varphi}_n(t+\delta_n) = \tilde{\varphi}_n(t_n+\varepsilon_n) \longrightarrow y$ for some $y \in X$. But this implies that $y \in \tilde{\omega}(\hat{D}, t)$. Furthermore,

$$d(y_n, y) \leq d(y_n, \tilde{\varphi}_n(t_n+\varepsilon_n)) + d(\tilde{\varphi}(t_n+\varepsilon_n), y) \longrightarrow 0,$$

so we can say $y_n \longrightarrow y$.

Finally, we prove that $\tilde{\omega}(\hat{D})$ pullback attracts $\hat{D}$. If $\tilde{\omega}(\hat{D})$ does not pullback attract $\hat{D}$, there exist $t \in \mathbb{R}$, $\varepsilon > 0$, $s_n \longrightarrow -\infty$ and $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$ with $\tilde{\varphi}_n(s_n) \in D(s_n)$ for all $n \in \mathbb{N}$ such that $d(\tilde{\varphi}_n(t), \tilde{\omega}(\hat{D}, t)) \geq \varepsilon$. But $\{\tilde{\varphi}_n(t)\}_n$ has a convergent subsequence by the pullback $\mathfrak{D}$-asymptotical compactness, so we may assume that $\tilde{\varphi}_n(t) \longrightarrow x$ for some $x \in X$. But this implies that $x \in \tilde{\omega}(\hat{D}, t)$, a contradiction with $d(\tilde{\varphi}_n(t), \tilde{\omega}(\hat{D}, t)) \geq \varepsilon$. □

Proposition 9 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-dissipative impulsive generalized process with $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family. Then for any $\hat{D} \in \mathfrak{D}$ we have that $\tilde{\omega}(\hat{D}) \subset \hat{B}_0$.*

Proof Fix $t \in \mathbb{R}$ and $x \in \tilde{\omega}(\hat{D}, t)$. Then there exist $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$ and $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$ with $\tilde{\varphi}_n(s_n) \in D(s_n)$ such that $\tilde{\varphi}_n(t + \varepsilon_n) \longrightarrow x$. The definition of pullback $\mathfrak{D}$-dissipative implies that there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ we have that

$$\tilde{\varphi}_n(t + \varepsilon_n) \in B_0(t + \varepsilon_n).$$

As $\hat{B}_0$ is collectively closed, this implies that $x \in B_0(t)$. □

Theorem 10 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process and pullback $\mathfrak{D}$-dissipative. Then there exists a pullback $\mathfrak{D}$-semi attractor.*

Proof Take $\hat{A} = \tilde{\omega}(\hat{B}_0)$, with $\hat{B}_0$ a pullback $\mathfrak{D}$-absorbing family. The family $\hat{A}$ pullback $\mathfrak{D}$-attracts $\hat{B}_0$ and $\hat{A} \subset \hat{B}_0$, by Proposition 9. Then $\hat{A} \in \mathfrak{D}$ because $\hat{B}_0 \in \mathfrak{D}$ and $\mathfrak{D}$ is a universe. The family $\hat{A}$ is collectively compact by Proposition 8, so $A(t)$ is compact for all $t \in \mathbb{R}$. We have to prove that $\hat{A}$ pullback attracts every $\hat{D} \in \mathfrak{D}$.

Fix $t \in \mathbb{R}$, $\hat{D} \in \mathfrak{D}$ and $\varepsilon > 0$. We want to prove that there exists $r \leq t$ such that if $s \leq r$ and $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in D(s)$, then $d(\tilde{\varphi}(t), A(t)) < \varepsilon$.

We know that $\hat{A}$ pullback attracts $\hat{B}_0$, so there exists $s_0 \leq t$ such that if $s \leq s_0$ and $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in B_0(s)$, then $d(\tilde{\varphi}(t), A(t)) < \varepsilon$.

By pullback $\mathfrak{D}$-dissipativity, there exists $s_1 \leq s_0$ such that if $s \leq s_1$ and $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in D(s)$, then $\tilde{\varphi}(s) \in B_0(s)$.

Finally, take $r := s_1$. If $s \leq r$ and $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in D(s)$, then we know that $\tilde{\varphi}(s_0) \in B_0(s_0)$, so $\tilde{\varphi}_{|[s_0,\infty)} \in \tilde{\mathscr{G}}(s_0)$ and $\tilde{\varphi}(s_0) \in B_0(s_0)$. This implies that $d(\tilde{\varphi}(t), A(t)) < \varepsilon$ because $\hat{A}$ pullback attracts $\hat{B}_0$. □

3.2 Invariance

In this subsection we find conditions to obtain the invariance of the impulsive pullback ω-limits. In particular we look for conditions to guarantee the invariance of $\hat{A} \setminus \hat{M}$ when $\hat{A}$ is a pullback $\mathfrak{D}$-semi attractor.

First, we need a condition closely related to (G3) in Definition 1 and to Proposition 1, but a little stronger.

(G3') If $\tau_n \longrightarrow \tau$, $\varphi_n \in \mathscr{G}(\tau_n)$ and $\varphi_n(\tau_n) \longrightarrow x$, then there is a subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}_n$ and $\varphi \in \mathscr{G}(\tau)$ with $\varphi(\tau) = x$ and satisfying the following condition:

$$\text{For every } \{t_k\}_k \text{ with } t_k \geq \tau_{n_k} \text{ and } t_k \longrightarrow t, \text{ we have } \varphi_{n_k}(t_k) \longrightarrow \varphi(t).$$

Furthermore, we need to add some conditions in order to get the invariance. The first condition asks about the behavior of the trajectories near the impulsive family $\hat{M}$.

$$\begin{cases} \text{Fix } s \in \mathbb{R}, x \in X \setminus M(s), \{\varphi_n\}_n \text{ a sequence in } \mathscr{G}(s) \\ \text{and } \varphi \in \mathscr{G}(s) \text{ such that } \varphi(s) = x \text{ and } \varphi_n(t) \longrightarrow \varphi(t) \\ \text{for each } t \geq s. \text{ Then } \liminf_{n\to\infty} \phi(\varphi_n, s) \leq \phi(\varphi, s). \end{cases} \tag{NT}$$

The second condition implies some restrictions on the jump times.

$$\begin{cases} \text{There exists } \xi > 0 \text{ such that } \phi(\varphi, s) \geqslant 2\xi \text{ for all } s \in \mathbb{R} \\ \text{and } \varphi \in \tilde{\mathscr{G}}(s) \text{ with } \varphi(s) \in I_s(M(s)). \end{cases} \tag{H}$$

Remark 3 Condition (NT) generalizes other conditions in the literature, for example the tubes conditions in [20–22] or Condition (T) in [30].

This means that if an impulsive generalized process satisfies the tubes conditions or Condition (T) in [30], then it satisfies Condition (NT).

Remark 4 Condition (H) implies (3), that is, all impulsive trajectories are defined until $+\infty$. It also implies that if $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ and $t_1 < t_2$ are two different jump times of $\tilde{\varphi}$, then $t_2 - t_1 \geq 2\xi$.

Theorem 11 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions (G3'), (H), (I) and (NT). Then $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is negatively invariant for any $\hat{D} \in \mathfrak{D}$.*

The proof of this result is shown in the appendix. The following result tells us that, in the particular case that $\tilde{\omega}(\hat{D})$ is a pullback $\mathfrak{D}$-semi attractor, then negative invariance implies positive invariance.

Theorem 12 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process and pullback $\mathfrak{D}$-dissipative, $\hat{A}$ a pullback $\mathfrak{D}$-semi attractor such that $\hat{A} \setminus \hat{M}$ is negatively invariant and $\tilde{\mathscr{G}}$ satisfies Condition (I). Then $\hat{A} \setminus \hat{M}$ is also positively invariant.*

Proof Let $t > s$. The negative invariance of $\hat{A} \setminus \hat{M}$ implies that

$$B(s) \subset \tilde{U}(s, s-n)B(s-n)$$

for any $n \in \mathbb{N}$, with $B(r) = A(r) \setminus M(r)$. This implies that $\tilde{U}(t, s)B(s) \subset \tilde{U}(t, s-n)B(s-n)$, so we can say

$$\mathrm{d}_H\left(\tilde{U}(t, s)B(s), A(t)\right) \leq \mathrm{d}_H\left(\tilde{U}(t, s-n)B(s-n), A(t)\right).$$

We have that $\hat{A} \setminus \hat{M} \in \mathfrak{D}$, which implies that

$$\lim_{n\to\infty} \mathrm{d}_H\left(\tilde{U}(t, s-n)B(s-n), A(t)\right) = 0.$$

As a consequence we can say that

$$\mathrm{d}_H\left(\tilde{U}(t, s)\left(A(s) \setminus M(s)\right), A(t)\right) = 0,$$

that is, $\tilde{U}(t,s)\,(A(s)\setminus M(s)) \subset A(t)$. Finally, Proposition 5 implies the positive invariance. □

Corollary 13 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process and pullback $\mathfrak{D}$-dissipative, $\hat{A}$ a pullback $\mathfrak{D}$-semi attractor such that $\hat{A}\setminus\hat{M}$ is negatively invariant and $\tilde{\mathscr{G}}$ satisfies Condition* (I). *Then $\hat{A}$ is a pullback $\mathfrak{D}$-attractor.*

Positive invariance can also be proved for impulsive pullback ω-limits different from the pullback $\mathfrak{D}$-semi attractor, but it requires more work. However, the idea is similar as the negative invariance case. The proof of the next result will be given in the Appendix.

Theorem 14 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions (G3'),* (H), (I) *and* (NT). *Then $\tilde{\omega}(\hat{D})\setminus\hat{M}$ is positively invariant for any $\hat{D}\in\mathfrak{D}$.*

Corollary 15 *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions (G3'),* (H), (I) *and* (NT). *Then $\tilde{\omega}(\hat{D})\setminus\hat{M}$ is invariant for any $\hat{D}\in\mathfrak{D}$.*

4 Examples

In this section we provide some examples and applications of the previous results.

Example 1 Let $F:\mathbb{R}\times\mathbb{R}\longrightarrow\mathscr{P}(\mathbb{R})$ be given by

$$F(t,x)=\begin{cases}-|\cos(t)|x, & |x|>1,\\ x\,(|\cos(t)|-1/2)-1/2, & -1<x<0,\\ x\,(|\cos(t)|-1/2)+1/2, & 0<x<1,\\ [-|\cos(t)|,|\cos(t)|], & x\in\{-1,1\},\\ [-1/2,1/2], & x=0;\end{cases}$$

and consider the ordinary differential inclusion

$$x'(t)\in F(t,x(t))$$

The solutions of the differential inclusion are absolutely continuous functions. Given an initial data (τ,x_τ), we will say that $x:[\tau,+\infty)\longrightarrow\mathbb{R}$ is a solution with initial data (τ,x_τ) if x is absolutely continuous, it satisfies the inclusion for almost every $t\geq\tau$ and $x(\tau)=x_\tau$.

We have uniqueness of solution if $x_\tau\neq 0$. If $x_\tau\notin\{-1,0,1\}$, then the solution is given by the solution of the differential equation until it reaches 1 or -1 (depending on the sign of the initial condition). When it reaches 1 or -1, the solution of the differential inclusion stays at that point. If $|x_\tau|=1$, then the unique solution is the constant function $x(t)=x_\tau$ for all $t\geq\tau$.

If $x_\tau = 0$ then we have infinite many solutions. For any $T > \tau$ we have the following solutions:

$$\begin{cases} x(t) = 0 \text{ for all } t \geq \tau \\ x(t) = \begin{cases} 0, & \tau \leq t \leq T, \\ \alpha(t), & T \leq t \leq T^*, \\ 1, & T^* \leq t. \end{cases} \\ x(t) = \begin{cases} 0, & \tau \leq t \leq T, \\ \beta(t), & T \leq t \leq T^*, \\ 1, & T^* \leq t. \end{cases} \end{cases}$$

where α denotes the solution of $x' = x(|\cos(t)| - 1/2) + 1/2$ with initial data $x(T) = 0$ and T^* is the time that α reaches 1 (respectively for β and the equation defined for values between -1 and 0 and the time it reaches -1).

We have an exact generalized process. In order to consider an impulsive generalized process, we can define, for example, for any $t \in \mathbb{R}$,

$$M_t = \left\{ \frac{6 + \arctan(t)}{4} \right\}, I_t(x) = \{5 + \sin(t), 3\},$$

and we consider the family of sets $\{M(t)\}_{t\in\mathbb{R}}$ and the collection of multifunctions $I = \{I_t\}_{t\in\mathbb{R}}$. We take $\mathfrak{D}$ the universe of all time-dependent families $\hat{D}$ such that there exists a bounded set D with $D(t) \subset D$ for all $t \in \mathbb{R}$. It is easy to see that all conditions of an impulsive generalized process are satisfied, it is pullback $\mathfrak{D}$-asymptotically compact and pullback $\mathfrak{D}$-dissipative. Furthermore, Conditions (H), (I) and (NT) are also fulfilled, so we there exists a pullback $\mathfrak{D}$-attractor. It is not hard to see that the pullback $\mathfrak{D}$-attractor is given by

$$A(t) = [-1, 1] \cup \left[\frac{6 + \arctan(t)}{4}, 5 + \sin(t) \right].$$

Example 2 The following example is a multivalued generalization of [24, Section 3.2]. Let Ω be an open and bounded subset of $\mathbb{R}^N$ with smooth boundary, and consider the problem

$$\begin{cases} u_t - \Delta u = 0, & (x, t) \in \Omega \times (s, +\infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (s, +\infty), \\ u(x, s) = u_s(x), & x \in \Omega. \end{cases} \tag{4}$$

If $u_s \in L^2(\Omega)$ we have uniqueness of solution in $L^2(\Omega)$ which is defined for all $t \geq s$. Take $\{\lambda_n\}_n$ the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and $\{e_n\}_n$ the eigenfunctions, which are orthonormal. A classical result says that the eigenvalues are different, positive, $\lambda_n < \lambda_{n+1}$ and $\lambda_n \longrightarrow +\infty$. Furthermore, any $u_s \in L^2(\Omega)$ is

$$u_s = \sum_{n=1}^{\infty} \alpha_n e_n,$$

and the unique solution of (4) is given by

$$U(t, s)u_s = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n (t-s)} e_n.$$

For each $t \in \mathbb{R}$ we define

$$M(t) = \left\{ v \in L^2(\Omega) : \|v\| = \frac{\arctan(t)}{\pi} + 1 \right\}$$

and for any $v \in M(t)$ with $v = \sum_{n=1}^{\infty} \beta_n e_n$,

$$I_t(v) = (\beta_1 + 10 + [-\sin(t), \sin(t)])e_1 + \sum_{n=2}^{\infty} \beta_n e_n$$

We take $\mathfrak{D}$ as the universe consisting of families $\hat{D}$ such that the union over all $t \in \mathbb{R}$ of all $D(t)$ is bounded. It can be checked that we have an impulsive generalized process, Conditions (H), (I) and (NT) are satisfied and we also have pullback $\mathfrak{D}$-asymptotically compactness and pullback $\mathfrak{D}$-dissipativeness. See [24, Section 3.2] for the proof of a similar case. Then we have a pullback $\mathfrak{D}$-attractor $\hat{A}$.

Example 3 Consider the nonautonomous differential inclusion

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b(t)H_0(u) + \omega(t)u, \text{ on } (\tau, +\infty) \times (0, 1), \\ u(0, t) = 0 = u(1, t), \\ u(\tau, x) = u_\tau(x), \end{cases} \tag{5}$$

where $b : \mathbb{R} \longrightarrow \mathbb{R}^+$, $\omega : \mathbb{R} \longrightarrow \mathbb{R}^+$ are continuous functions with

$$0 < b_0 \leq b(t) \leq b_1, 0 \leq \omega_0 \leq \omega(t) \leq \omega_1$$

and H_0 is the Heaviside function, that is,

$$H_0(u) = \begin{cases} -1, & u < 0, \\ [-1, 1], & u = 0, \\ 1, & u > 0. \end{cases}$$

This problem and similar ones have been studied, for example, in [10, 11, 37], where some results on the structure of the attractor were obtained.

We say that a continuous map $u : [\tau, +\infty) \longrightarrow L^2(0, 1)$ is a strong solution of (5) if

1. $u(\tau) = u_\tau$
2. For any $\delta > 0$ and $T > t + \delta$, u is absolutely continuous on $[t + \delta, T]$ and $u(t) \in H^2(0, 1) \cap H_0^1(0, 1)$ for almost all $t \in (\tau, T)$
3. there exists $r : [\tau, +\infty) \longrightarrow L^2(0, 1)$ such that:
   - $r(t) \in L^2(0, 1)$,
   - $r(t)(x) \in b(t)H_0(u(t, x)) + \omega(t)u(t, x)$ for almost all $x \in (0, 1)$,
   - $r \in L^2(\tau, T; L^2(0, 1))$ for any $T > \tau$
   - $\dfrac{du}{dt} - \Delta u = r(t)$, for almost all $t \in (\tau, +\infty)$

Theorem 16 (Theorem 1 in [10]) *For any $u_\tau \in L^2(0, 1)$, Problem* (5) *has at least one strong solution.*

It is also proved that we have a continuous and exact generalized process $\mathscr{G} = \{\mathscr{G}(t)\}_{t \in \mathbb{R}}$.

Let $\hat{M} = \{M(t)\}_{t\in\mathbb{R}}$ a collectively closed family of sets with $M(t) \subset L^2(0,1)$ for all $t \in \mathbb{R}$ and $I = \{I_t : M(t) \longrightarrow \mathscr{P}(X)\}_{t\in\mathbb{R}}$ a collection of collectively upper semicontinuous multifunctions such that $(\mathscr{G}, \hat{M}, I)$ is an impulsive generalized process. We will assume that $\hat{M}$ and I satisfy Conditions (H), (I) and (NT) and that the impulsive generalized process is pullback $\mathfrak{D}$-dissipative and pullback $\mathfrak{D}$-asymptotically compact and it satisfies (G3'). Then we can say that there exists a pullback $\mathfrak{D}$-attractor $\hat{A}$. For example, if we assume that $\|I_t(u)\|^2 \leq C$ for some $C > 0$ and for all $t \in \mathbb{R}$ and $u \in M(t)$ we would have the pullback $\mathfrak{D}$-dissipativeness and the pullback $\mathfrak{D}$-asymptotically compactness.

Appendix: Proofs of Theorems 11 and 14

Theorem *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions (G3'), (H), (I) and (NT). Then $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is negatively invariant for any $\hat{D} \in \mathfrak{D}$.*

Proof We restrict ourselves to the case where $t > s$ and $t - s \in (0, \xi]$. By recursion the general case follows easily. Take $x \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$. We want to prove that there exists $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s)$, $\tilde{\varphi}(s) \notin M(s)$ and $\tilde{\varphi}(t) = x$.

As $x \in \tilde{\omega}(\hat{D}, t)$, there exist $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$ and $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$ with $\tilde{\varphi}_n(s_n) \in D(s_n)$ such that $\tilde{\varphi}_n(t + \varepsilon_n) \longrightarrow x$. Each impulsive trajectory $\tilde{\varphi}_n$, which is defined on $[s_n, +\infty)$, has $N_n \geq 0$ jump times. We consider τ_n the last jump time on the interval $[s_n, t + \xi/4]$. If there are no jump times in that interval, we take $\tau_n = s_n$. We will split the proof into three different cases.

CASE 1. Up to a subsequence (denoted the same), there exists $\varepsilon \in (0, \xi/2)$ such that $\tau_n < s - \varepsilon$.

We have that there exist $\psi_n \in \mathscr{G}(s - \varepsilon/2)$ such that $\tilde{\varphi}_n(r) = \psi_n(r)$ for $r \in [s - \varepsilon/2, t + \xi/4]$, because $\tau_n < s - \varepsilon$ and τ_n was the last jump time of the trajectory $\tilde{\varphi}_n$ in $[s_n, t + \xi/4]$.

The sequence $\{\tilde{\varphi}_n(s - \varepsilon/2)\}_n$ has a convergent subsequence by pullback $\mathfrak{D}$-asymptotical compactness, so we can assume $\tilde{\varphi}_n(s - \varepsilon/2) \longrightarrow y$, or equivalently $\psi_n(s - \varepsilon/2) \longrightarrow y$. By the definition of generalized process, in particular (G3), there exist a subsequence (still denoted the same) and $\psi \in \mathscr{G}(s - \varepsilon/2)$ such that $\psi(s - \varepsilon/2) = y$ and $\psi_n(r) \longrightarrow \psi(r)$ for each $r \geq s - \varepsilon/2$. We claim that $\psi(r) \notin M(r)$ for $r \in [s - \varepsilon/2, t]$. Suppose that $\psi(r) \in M(r)$ for some $r \in [s - \varepsilon/2, t]$. This implies that $\phi(\psi, s - \varepsilon/2) \leq t - (s - \varepsilon/2)$. But $\tilde{\varphi}_n$ has no jump times on $[s - \varepsilon/2, t + \xi/4]$, so $(t + \xi/4) - (s - \varepsilon/2) \leq \phi(\psi_n, s - \varepsilon/2)$. Then Condition (NT) would imply that

$$(t + \xi/4) - (s - \varepsilon/2) \leq \phi(\psi, s - \varepsilon/2) \leq t - (s - \varepsilon/2),$$

which is a contradiction.

We have that $\tilde{\varphi}_n(t + \varepsilon_n) = \psi_n(t + \varepsilon_n)$ for n sufficiently large, so $x = \psi(t)$. Consider $\tilde{\theta} \in \tilde{\mathscr{G}}(t)$ such that $\tilde{\theta}(t) = x$ and define $\tilde{\psi} \in \tilde{\mathscr{G}}(s)$ as

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \leq r \leq t, \\ \tilde{\theta}(r), & t \leq r. \end{cases}$$

We have that $\tilde{\varphi}(s) = \psi(s) \notin M(s)$ and $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s)$ because $\tilde{\varphi}_n(s) = \psi_n(s)$ and $\psi_n(s) \longrightarrow \psi(s)$. This implies that $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$ and $\tilde{\varphi}(t) = x$.

CASE 2. Up to a subsequence (denoted the same), there exists $\varepsilon \in (0, \xi/2)$ such that $s + \varepsilon < \tau_n$.

We have that $\tau_n \in (s+\varepsilon, t+\xi/4]$, so we may assume that $\tau_n \longrightarrow \bar{\tau} \in [s+\varepsilon, t+\xi/4]$. The sequence $\{\tilde{\varphi}_n(t-3\xi/2)\}_n$ has a convergent subsequence by pullback $\mathfrak{D}$-asymptotical compactness, so we may assume $\tilde{\varphi}_n(t-3\xi/2) \longrightarrow v$. Note that $t-3\xi/2 < s$. We have that τ_n is the only jump time in $[t-3\xi/2, t+\xi/4]$, because $|(t+\xi/4)-(t-3\xi/2)| = 7\xi/4 < 2\xi$ and Condition (H) applies (see Remark 4). This implies that there exist $\psi_n \in \mathscr{G}(t-3\xi/2)$ and $\theta_n \in \mathscr{G}(\tau_n)$ such that

$$\tilde{\varphi}_n(r) = \begin{cases} \psi_n(r), & t-3\xi/2 \le r < \tau_n, \\ \theta_n(r), & \tau_n \le r \le t+\xi/4. \end{cases}$$

By definition of generalized process, there exist a subsequence (still denoted the same) and $\psi \in \mathscr{G}(t-3\xi/2)$ such that $\psi(t-3\xi/2) = v$ and $\psi_n(r) \longrightarrow \psi(r)$ for $r \ge t - 3\xi/2$. This implies that $\psi_n(\tau_n) \longrightarrow \psi(\bar{\tau})$ by Proposition 1. The fact that $\hat{M}$ is collectively closed implies that $\psi(\bar{\tau}) \in M(\bar{\tau})$, because $\psi_n(\tau_n) \in M(\tau_n)$. We also have that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n) \in I_{\tau_n}(\psi_n(\tau_n))$. By the collective upper semicontinuity of I, there exist a subsequence of $\{\tilde{\varphi}_n(\tau_n)\}_n$ (denoted the same) and $y \in I_{\bar{\tau}}(\psi(\bar{\tau}))$ such that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n)$ converges to y. In particular this implies that $y \in \tilde{\omega}(\hat{D}, \bar{\tau})$, because $\tilde{\varphi}_n(\tau_n) = \tilde{\varphi}_n(\bar{\tau} + (\tau_n - \bar{\tau}))$.

Using Condition (G3'), there exist a subsequence (denoted the same) and $\theta \in \mathscr{G}(\bar{\tau})$ such that $\theta_n(r_n)$ converges to $\theta(r)$ for any sequence $\{r_n\}_n$ with $r_n \ge \tau_n$ and r_n converging to r. In particular, $\theta_n(\tau_n)$ converges to $\theta(\bar{\tau})$, so $\theta(\bar{\tau}) = y$.

We claim that $\psi(r) \notin M(r)$ for $r \in (t-3\xi/2, \bar{\tau})$. If there exists $r \in [t-3\xi/2, \bar{\tau})$ such that $\psi(r) \in M(r)$, then

$$\phi(\psi, t-3\xi/2) \le r - (t-3\xi/2) < \bar{\tau} - (t-3\xi/2).$$

But τ_n was the only jump time of $\tilde{\varphi}_n$ on $[t-3\xi/2, t+\xi/4]$, so $\phi(\psi_n, t-3\xi/2) = \tau_n - (t-3\xi/2)$. Finally Condition (NT) implies

$$\bar{\tau} - (t-3\xi/2) \le \phi(\psi, t-3\xi/2) < \bar{\tau} - (t-3\xi/2),$$

a contradiction.

First, take $z = \psi(s) \in \tilde{\omega}(\hat{D}, s)$ because $\tilde{\varphi}_n(s) = \psi_n(s)$ and $\psi_n(s)$ converges to $\psi(s)$. Then $z \notin M(s)$ by the previous claim. Consider $\tilde{\alpha} \in \tilde{\mathscr{G}}(t+\xi/4)$ with $\tilde{\alpha}(t+\xi/4) = \theta(t+\xi/4)$ and define

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \le r < \bar{\tau}, \\ \theta(r), & \bar{\tau} \le r \le t+\xi/4, \\ \tilde{\alpha}(r), & t+\xi/4 \le r. \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. We want to prove that $\tilde{\varphi}(t) = x$.

Subcase 1. For a subsequence (denoted the same), $\tau_n \le t + \varepsilon_n$.

This implies that $\bar{\tau} \le t$. Then $\tilde{\varphi}_n(t+\varepsilon_n) = \theta_n(t+\varepsilon_n) \longrightarrow \theta(t)$, so $x = \theta(t) = \tilde{\varphi}(t)$.

Subcase 2. For a subsequence (denoted the same), $t + \varepsilon_n < \tau_n$.

This implies that $t \le \bar{\tau}$. On the one hand, if $t = \bar{\tau}$, then $\tilde{\varphi}_n(t+\varepsilon_n) = \psi_n(t+\varepsilon_n)$, which converges to $\psi(t) = \psi(\bar{\tau}) \in M(\bar{\tau})$ by Proposition 1, so $x = \psi(\bar{\tau}) \in M(t)$, a contradiction. This implies that t cannot be equal to $\bar{\tau}$ in this Subcase. On the other hand, if $t < \bar{\tau}$, then $\tilde{\varphi}_n(t+\varepsilon_n) = \psi_n(t+\varepsilon_n)$, which converges to $\psi(t) = \tilde{\varphi}(t)$, so $x = \tilde{\varphi}(t)$.

CASE 3. τ_n converges to s.

We have that τ_n is the only jump time of $\tilde{\varphi}_n$ in $(t-3\xi/2, t+\xi/4)$ for n sufficiently large. Then there exist $\psi_n \in \mathscr{G}(t-3\xi/2)$ and $\theta_n \in \mathscr{G}(\tau_n)$ such that

$$\tilde{\varphi}_n(r) = \begin{cases} \psi_n(r), & t-3\xi/2 \leq r < \tau_n, \\ \theta_n(r), & \tau_n \leq r \leq t+\xi/4. \end{cases}$$

The sequence $\{\tilde{\varphi}_n(t-3\xi/2)\}_n$ has a convergent subsequence by pullback $\mathfrak{D}$-asymptotical compactness, so we may assume $\tilde{\varphi}_n(t-3\xi/2) \longrightarrow v$, with $v \in X$. By definition of generalized process, there exist a subsequence (still denoted the same) and $\psi \in \mathscr{G}(t-3\xi/2)$ such that $\psi(t-3\xi/2) = v$ and $\psi_n(r) \longrightarrow \psi(r)$ for $r \geq t-3\xi/2$. This implies that $\psi(s) \in M(s)$, because $\psi_n(\tau_n) \in M(\tau_n)$, $\hat{M}$ is collectively closed and Proposition 1 applies. Furthermore, $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n) \in I_{\tau_n}(\psi_n(\tau_n))$. By the collective upper semicontinuity of I, there exist a subsequence of $\{\tilde{\varphi}_n(\tau_n)\}_n$ (denoted the same) and $y \in I_s(\psi(s))$ such that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n)$ converges to y, so $y \in \tilde{\omega}(\hat{D}, s)$.

By Condition (G3'), there exist a subsequence (denoted the same) and $\theta \in \mathscr{G}(s)$ such that $\theta_n(r_n)$ converges to $\theta(r)$ for any sequence $\{r_n\}_n$ with $r_n \geq \tau_n$ and r_n converging to r. This implies that $\theta_n(\tau_n)$ converges to $\theta(s) = y \in I_s(\psi(s))$, so $y \notin M(s)$ by Condition (I). Furthermore $\tilde{\varphi}_n(\tau_n) = \tilde{\varphi}_n(s + (\tau_n - s))$ converges to y, so $y \in \tilde{\omega}(\hat{D}, s)$.

We have that $\theta(r) \notin M(r)$ for $r \in [s, t+\xi/4]$ because $\theta(s) = y \in I_s(\psi(s))$ and Condition (H) applies. Take $\tilde{\alpha} \in \tilde{\mathscr{G}}(t+\xi/4)$ with $\tilde{\alpha}(t+\xi/4) = \theta(t+\xi/4)$ and define

$$\tilde{\varphi}(r) = \begin{cases} \theta(r), & s \leq r \leq t+\xi/4, \\ \tilde{\alpha}(r), & t+\xi/4 \leq r \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ and $\tilde{\varphi}(s) = y \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. Finally,

$$\tilde{\varphi}_n(t+\varepsilon_n) = \theta_n(t+\varepsilon_n) \longrightarrow \theta(t) = \tilde{\varphi}(t),$$

which implies that $x = \tilde{\varphi}(t)$. □

Theorem *Let $\tilde{\mathscr{G}}$ be a pullback $\mathfrak{D}$-asymptotically compact impulsive generalized process satisfying Conditions (G3'),* (H), (I) *and* (NT)*. Then $\tilde{\omega}(\hat{D}) \setminus \hat{M}$ is positively invariant for any $\hat{D} \in \mathfrak{D}$.*

Proof We restrict ourselves to the case where $t > s$ and $t - s \in (0, \xi]$. By recursion the general case follows easily. Let $x \in \tilde{\omega}(\hat{D}, s) \setminus M(s)$. We want to prove that there exists $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$ with $\tilde{\varphi}(s) = x$ such that $\tilde{\varphi}(t) \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$.

As $x \in \tilde{\omega}(\hat{D}, s)$, there exist $s_n \longrightarrow -\infty$, $\varepsilon_n \longrightarrow 0$ and $\tilde{\varphi}_n \in \tilde{\mathscr{G}}(s_n)$ such that $\tilde{\varphi}_n(s_n) \in D(s_n)$ and $\tilde{\varphi}_n(s+\varepsilon_n) \longrightarrow x$. We may assume, without loss of generality, that $s + \varepsilon_n < t$ for all $n \in \mathbb{N}$.

Each impulsive trajectory $\tilde{\varphi}_n$, which is defined on $[s_n, +\infty)$, has $N_n \geq 0$ jump times. We consider τ_n the last jump time of $\tilde{\varphi}_n$ on the interval $[s_n, s+5\xi/4]$. If there are no jump times we take $\tau_n = s_n$. We will split the proof into three different cases.

CASE 1. Up to a subsequence (denoted the same), there exists $\varepsilon \in (0, \xi/2)$ such that $\tau_n < s - \varepsilon$.

We know that there exist $\psi_n \in \mathscr{G}(s-\varepsilon/2)$ such that $\tilde{\varphi}_n(r) = \psi_n(r)$ for $r \in [s-\varepsilon/2, s+5\xi/4]$. By pullback $\mathfrak{D}$-asymptotical compactness, we may assume $\tilde{\varphi}_n(s-\varepsilon/2) \longrightarrow y$. We may assume that there exists $\psi \in \mathscr{G}(s-\varepsilon/2)$ with $\psi_n(r) \longrightarrow \psi(r)$ for $r \geq s-\varepsilon/2$, by definition of generalized process. First, $\tilde{\varphi}_n(s+\varepsilon_n) = \psi_n(s+\varepsilon_n)$, which converges to $\psi(s)$, so $x = \psi(s)$. This implies that $\tilde{\varphi}_n(s)$ converges to x.

We claim that $\psi(r) \notin M(r)$ for $r \in [s-\varepsilon/2, s+\xi]$. If $\psi(r) \in M(r)$ for some $r \in [s-\varepsilon/2, s+\xi]$, then $\phi(\psi, s-\varepsilon/2) \le r-(s-\varepsilon/2)$. But $\tilde{\varphi}_n$ has no jump times on $[s-\varepsilon/2, s+5\xi/4]$, so $\phi(\psi_n, s-\varepsilon/2) \ge (s+5\xi/4)-(s-\varepsilon/2)$. Then Condition (NT) would imply that

$$(s+5\xi/4)-(s-\varepsilon/2) \le \phi(\psi, s-\varepsilon/2) \le r-(s-\varepsilon/2) < (s+\xi)-(s-\varepsilon/2),$$

a contradiction.

We take $\tilde{\alpha} \in \tilde{\mathscr{G}}(s+\xi)$ with $\tilde{\alpha}(s+\xi) = \psi(s+\xi)$ and define

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \le r \le s+\xi, \\ \tilde{\alpha}(r), & s+\xi \le r. \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, $\tilde{\varphi}(s) = \psi(s) = x$ and finally $\tilde{\varphi}_n(t) = \psi_n(t) \longrightarrow \psi(t) = \tilde{\varphi}(t)$, so $\tilde{\varphi}(t) \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$.

CASE 2. Up to a subsequence (denoted the same), there exists $\varepsilon \in (0, \xi/2)$ such that $\tau_n > s+\varepsilon$.

We know that τ_n is the only jump time of $\tilde{\varphi}_n$ in $[s-\xi/2, s+5\xi/4]$ because of Condition (H), and we can assume that $\tau_n > s+\varepsilon$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ there exist $\psi_n \in \mathscr{G}(s-\xi/2)$ and $\theta_n \in \mathscr{G}(\tau_n)$ such that

$$\tilde{\varphi}_n(r) = \begin{cases} \psi_n(r), & s-\xi/2 \le r < \tau_n, \\ \theta_n(r), & \tau_n \le r \le s+5\xi/4. \end{cases}$$

We may assume that $\tilde{\varphi}_n(s-\xi/2) \longrightarrow y$ by pullback $\mathfrak{D}$-asymptotical compactness, and by definition of generalized process for ψ_n we may assume that there exist a subsequence (denoted the same) and $\psi \in \mathscr{G}(s-\xi/2)$ such that $\psi_n(r) \longrightarrow \psi(r)$ for $r \ge s-\xi/2$. Furthermore we have that $\tilde{\varphi}_n(s+\varepsilon_n) = \psi_n(s+\varepsilon_n)$, which converges to $\psi(s)$ by Proposition 1, so $x = \psi(s)$. This implies that $\tilde{\varphi}_n(s)$ also converges to x.

Subcase 1. Up to a subsequence (denoted the same), there exists $\delta > 0$ such that $\tau_n > t+\delta$.

We claim that $\psi(r) \notin M(u)$ for $r \in [s-\xi/2, t+\delta)$. If $\psi(r) \in M(r)$ for some $r \in [s-\xi/2, t+\delta)$, then $\phi(\psi, s-\xi/2) \le r-(s-\xi/2)$. But we know that $\tau_n > t+\delta$, so $\phi(\psi_n, s-\xi/2) \ge (t+\delta)-(s-\xi/2)$ and Condition (NT) implies that

$$(t+\delta)-(s-\xi/2) \le \phi(\psi, s-\xi/2) \le r-(s-\xi/2) < (t+\delta)-(s-\xi/2),$$

a contradiction.

We take $\tilde{\alpha} \in \tilde{\mathscr{G}}(t+\delta/2)$ with $\tilde{\alpha}(t+\delta/2) = \psi(t+\delta/2)$ and define

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \le r \le t+\delta/2, \\ \tilde{\alpha}(r), & t+\delta/2 \le r. \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, $\tilde{\varphi}(s) = \psi(s) = x$ and $\tilde{\varphi}_n(t) = \psi_n(t)$, which converges to $\psi(t) = \tilde{\varphi}(t)$, so $\tilde{\varphi}(t) \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$.

Subcase 2. Up to a subsequence (denoted the same), there exists $\delta > 0$ such that $\tau_n < t-\delta$.

As $\tau_n \in (s+\varepsilon, t-\delta)$, we may assume that τ_n converges to $\bar{\tau} \in [s+\varepsilon, t-\delta]$. We have $\psi_n(\tau_n) \in M(\tau_n)$, so $\psi(\bar{\tau}) \in M(\bar{\tau})$ by Proposition 1. We also have that $\theta_n(\tau_n) = \tilde{\varphi}(\tau_n) \in I_{\tau_n}(\psi_n(\tau_n))$. By the collective upper semicontinuity of I, there exist a subsequence of $\{\tilde{\varphi}_n(\tau_n)\}_n$, still denoted the same, and $z \in I_{\bar{\tau}}(\psi(\bar{\tau}))$ such that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n)$ converges to z, so $z \in \tilde{\omega}(\hat{D}, \bar{\tau})$.

Once again, we claim that $\psi(r) \notin M(r)$ for $r \in (s - \xi/2, \bar{\tau})$. If $\psi(r) \in M(r)$ for some $r \in (s - \xi/2, \bar{\tau})$, then $\phi(\psi, s - \xi/2) \leq r - (s - \xi/2)$. But we know that $\phi(\varphi_n, s - \xi/2) = \tau_n - (s - \xi/2)$ and Condition (NT) implies that

$$\bar{\tau} - (s - \xi/2) \leq \phi(\psi, s - \xi/2) \leq r - (s - \xi/2) < \bar{\tau} - (s - \xi/2),$$

a contradiction.

We can assume by Condition (G3') that there exists $\theta \in \mathscr{G}(\bar{\tau})$ such that $\theta_n(u_n)$ converges to $\theta(u)$ for any sequence $\{u_n\}_n$ with $u_n \geq \tau_n$ and u_n converging to u. Furthermore, $\theta(r) \notin M(r)$ for $r \in [\tau_n, s + 5\xi/4]$ because $\theta(\bar{\tau}) \in I_{\bar{\tau}}(M(\bar{\tau}))$ and Condition (H) applies.

We take $\tilde{\alpha} \in \tilde{\mathscr{G}}(s + \xi)$ with $\tilde{\alpha}(s + \xi) = \theta(s + \xi)$ and define

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \leq r < \bar{\tau}, \\ \theta(r), & \bar{\tau} \leq r < s + \xi, \\ \tilde{\alpha}(r), & s + \xi \leq r. \end{cases}$$

We have that $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, $\tilde{\varphi}(s) = x$ and $\tilde{\varphi}_n(t) = \theta_n(t)$, which converges to $\theta(t) = \tilde{\varphi}(t)$, so $\tilde{\varphi}(t) \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$.

Subcase 3. τ_n converges to t.

We have that $\psi_n(\tau_n) \in M(\tau_n)$, so $\psi(t) \in M(t)$ using Proposition 1 and that $\hat{M}$ is collectively closed. We also have that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n) \in I_{\tau_n}(\psi_n(\tau_n))$. By the collective upper semicontinuity of I, there exist a subsequence of $\{\tilde{\varphi}_n(\tau_n)\}_n$, still denoted the same, and $z \in I_t(\psi(t))$ such that $\theta_n(\tau_n) = \tilde{\varphi}_n(\tau_n)$ converges to z, so $z \in \tilde{\omega}(\hat{D}, t)$.

We claim that $\psi(r) \notin M(r)$ for $r \in [s - \xi/2, t)$. If $\psi(r) \in M(r)$ for some $r \in [s - \xi/2, t)$, then $\phi(\psi, s - \xi/2) \leq r - (s - \xi/2)$. But we know that τ_n is the only jump time of $\tilde{\varphi}_n$ in $[s - \xi/2, s + 5\xi/4]$, so $\phi(\psi_n, s - \xi/2) = \tau_n - (s - \xi/2)$, and Condition (NT) implies that

$$t - (s - \xi/2) \leq \phi(\psi, s - \xi/2) \leq r - (s - \xi/2) < t - (s - \xi/2),$$

a contradiction. We take $\tilde{\alpha} \in \tilde{\mathscr{G}}(t)$ with $\tilde{\alpha}(t) = z$ and we define

$$\tilde{\varphi}(r) = \begin{cases} \psi(r), & s \leq r < t, \\ \tilde{\alpha}(r), & t \leq r. \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, $\tilde{\varphi}(s) = x$ and $\tilde{\varphi}_n(\tau_n) = \theta_n(\tau_n)$ converges to $z = \tilde{\alpha}(t) = \tilde{\varphi}(t)$, so $\tilde{\varphi}(t) \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$.

CASE 3. τ_n converges to s

In this case τ_n is the only jump time of $\tilde{\varphi}_n$ in $[s - \xi/2, s + 5\xi/4]$. Then there exist $\psi_n \in \mathscr{G}(s - \xi/2)$ and $\theta_n \in \mathscr{G}(\tau_n)$ such that

$$\tilde{\varphi}_n(r) = \begin{cases} \psi_n(r), & s - \xi/2 \leq r < \tau_n, \\ \theta_n(r), & \tau_n \leq r \leq s + 5\xi/4. \end{cases}$$

By pullback $\mathfrak{D}$-asymptotical compactness we may assume $\tilde{\varphi}_n(s - \xi/2)$ converges to y, and by definition of generalized process we may assume that there exists $\psi \in \mathscr{G}(s - \xi/2)$ such that $\psi_n(r) \longrightarrow \psi(r)$ for $r \geq s - \xi/2$.

Subcase 1. Up to a subsequence, still denoted the same, $s + \varepsilon_n < \tau_n$

We have that $\tilde{\varphi}_n(s+\varepsilon_n) = \psi_n(s+\varepsilon_n)$, which converges to $\psi(s)$ by Proposition 1, so $x = \psi(s)$. Furthermore, $\psi_n(\tau_n) \in M(\tau_n)$, so $\psi(s) \in M(s)$ by Proposition 1 and the collective closedness of $\hat{M}$. This implies that $x \in M(s)$, a contradiction. As a consequence, this case cannot happen.

Subcase 2. Up to a subsequence, still denoted the same, $\tau_n \leq s + \varepsilon_n$.

We have $\psi_n(\tau_n) \in M(\tau_n)$, so $\psi(s) \in M(s)$ by Proposition 1 and the fact that $\hat{M}$ is collectively closed. Furthermore, $\tilde{\varphi}_n(\tau_n) = \theta_n(\tau_n) \in I_{\tau_n}(\psi_n(\tau_n))$. Using the collective upper semicontinuity of I, there exist a subsequence $\{\tilde{\varphi}_n(\tau_n)\}_n$, still denoted the same, and $z \in I_s(\psi(s))$ such that $\theta_n(\tau_n) = \tilde{\varphi}(\tau_n)$ converges to z, so $z \in \tilde{\omega}(\hat{D}, s)$.

By Condition (G3') we assume that there exists $\theta \in \mathscr{G}(\bar{\tau})$ such that $\theta_n(r_n)$ converges to $\theta(r)$ for any sequence $\{r_n\}_n$ with $r_n \geq \tau_n$ and r_n converging to r. We have that $\theta(u) \notin M(u)$ for $u \in [s, s+\xi]$ because of Condition (H). We take $\tilde{\alpha} \in \tilde{\mathscr{G}}(s+\xi)$ with $\tilde{\alpha}(s+\xi) = \theta(s+\xi)$ and define

$$\tilde{\varphi}(r) = \begin{cases} \theta(r), & s \leq r \leq s+\xi, \\ \tilde{\alpha}(r), & s+\xi \leq r. \end{cases}$$

Then $\tilde{\varphi} \in \tilde{\mathscr{G}}(s)$, $\tilde{\varphi}_n(s+\varepsilon_n) = \theta_n(s+\varepsilon_n)$ converges to $\theta(s) = z$, so $x = z = \tilde{\varphi}(s)$. Finally $\tilde{\varphi}_n(t) = \theta_n(t)$ converges to $\theta(t) = \tilde{\varphi}(t)$, so $\tilde{\varphi}(t) \in \tilde{\omega}(\hat{D}, t) \setminus M(t)$. □

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