Set-Valued Means

Kazimierz Nikodem¹

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Abstract

The notion of set-valued means is introduced. Set-valued counterparts of the arithmetic, quasi-arithmetic and Lagrangian means are investigated and various properties of them are presented.

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1 Introduction

Let $I \subset \mathbb{R}$ be an interval and $n \geq 2$ be an integer. A function $M : I^n \rightarrow I$ is said to be a mean if

$$\min\{x_1, \dots, x_n\} \leq M(x_1, \dots, x_n) \leq \max\{x_1, \dots, x_n\}, \quad (1)$$

for all $x_1, \dots, x_n \in I$.

This formal definition was probably used first time by Cauchy [5] almost two hundred years ago, but the general idea of means is much older. The classical means, such as arithmetic, geometric or harmonic, were known already in the ancient time. At present the theory of means is an extensive mathematical theory having various applications in mathematics itself as well as in economics, information theory, engineering and natural science. There are books and numerous papers devoted to it (see e.g. [1, 3, 4, 6–10, 12] and the references given there).

The aim of this paper is to generalize the classical definition of mean (1) to the case of set-valued means in vector spaces. We extend the notion of arithmetic mean defining a family of set-valued means containing the arithmetic one and present various properties of them. Set-valued counterparts of the quasi-arithmetic and Lagrangian means are also investigated.

✉ Kazimierz Nikodem
knikodem@ath.bielsko.pl

¹ Department of Mathematics, University of Bielsko-Biała, ul. Willowa 2, 43-309, Bielsko-Biała, Poland

Let X be a real vector space and D be a convex nonempty subset of X . Denote by $S(D)$ the family of all nonempty subsets of D . We say that a function $M : D^n \rightarrow S(D)$ is a *set-valued mean* if

$$M(x_1, \dots, x_n) \subset \text{conv}\{x_1, \dots, x_n\}, \quad (2)$$

for all $x_1, \dots, x_n \in D$.

Clearly, if $X = \mathbb{R}$, $D = I$ and M is single-valued, then condition (2) reduces to (1). Note also that any $M : D^n \rightarrow S(D)$ defined by (2) is *reflexive*, that is $M(x, \dots, x) = \{x\}$ for every $x \in D$.

A set-valued mean $M : D^n \rightarrow S(D)$ is said to be *symmetric*, if

$$M(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = M(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in D$ and every permutation $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$.

M is *homogenous* if

$$M(sx_1, \dots, sx_n) = sM(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in D$ and every $s \in \mathbb{R}$ such that $sx_i \in D$, $i = 1, \dots, n$.

We say that M is *concave* if

$$M(\lambda x + (1 - \lambda)y) \subset \lambda M(x) + (1 - \lambda)M(y),$$

for all $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in D^n$ and $\lambda \in [0, 1]$. M is called *convex* if the opposite inclusions hold.

2 Set-valued extension of the arithmetic mean

Let us start with an example in which we define a family of set-valued means extending the classical arithmetic mean. For a given $x = (x_1, \dots, x_n) \in D^n$ we put $\bar{x} := \frac{1}{n}(x_1 + \dots + x_n)$.

Example 1 Let $M_0, M_1 : D^n \rightarrow S(D)$ be defined by

$$M_0(x_1, \dots, x_n) = \bar{x}, \quad x_1, \dots, x_n \in D;$$

$$M_1(x_1, \dots, x_n) = \text{conv}\{x_1, \dots, x_n\}, \quad x_1, \dots, x_n \in D.$$

More generally, for every $t \in [0, 1]$ define $M_t : D^n \rightarrow S(D)$ by

$$M_t(x_1, \dots, x_n) = t \text{ conv}\{x_1, \dots, x_n\} + (1 - t) \bar{x}, \quad x_1, \dots, x_n \in D. \quad (3)$$

It is clear that M_t are set-valued means and

$$M_0(x_1, \dots, x_n) \subset M_t(x_1, \dots, x_n) \subset M_1(x_1, \dots, x_n),$$

for every $t \in [0, 1]$.

Theorem 1 *Let D be a convex subset of a real vector space X and let $t \in [0, 1]$. The set-valued means $M_t : D^n \rightarrow S(D)$ defined by (3) are symmetric, homogenous and concave. Moreover, they have the following comparison property:*

$$0 \leq s \leq t \leq 1 \Rightarrow M_s(x_1, \dots, x_n) \subset M_t(x_1, \dots, x_n),$$

for all $x_1, \dots, x_n \in D$.

Proof The symmetry and homogeneity of M_t are clear. To prove the concavity, fix $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in D^n$, $\lambda \in [0, 1]$, and take a point

$$z \in M_t(\lambda x + (1 - \lambda)y) \\ = t \operatorname{conv}\{\lambda x_1 + (1 - \lambda)y_1, \dots, \lambda x_n + (1 - \lambda)y_n\} + (1 - t)\overline{(\lambda x + (1 - \lambda)y)}.$$

Then

$$z = tz' + (1 - t)\overline{(\lambda x + (1 - \lambda)y)}, \quad (4)$$

where $z' \in \operatorname{conv}\{\lambda x_1 + (1 - \lambda)y_1, \dots, \lambda x_n + (1 - \lambda)y_n\}$. By the definition of convex hull, there exist $\alpha_1, \dots, \alpha_n \geq 0$ with $\alpha_1 + \dots + \alpha_n = 1$ such that

$$\begin{aligned} z' &= \alpha_1(\lambda x_1 + (1 - \lambda)y_1) + \dots + \alpha_n(\lambda x_n + (1 - \lambda)y_n) \\ &= \lambda(\alpha_1 x_1 + \dots + \alpha_n x_n) + (1 - \lambda)(\alpha_1 y_1 + \dots + \alpha_n y_n) \\ &\in \lambda \operatorname{conv}\{x_1, \dots, x_n\} + (1 - \lambda) \operatorname{conv}\{y_1, \dots, y_n\}. \end{aligned} \quad (5)$$

We have also

$$\overline{(\lambda x + (1 - \lambda)y)} = \lambda \bar{x} + (1 - \lambda)\bar{y}. \quad (6)$$

By (4), (5) and (6), we obtain

$$\begin{aligned} z &\in t(\lambda \operatorname{conv}\{x_1, \dots, x_n\} + (1 - \lambda) \operatorname{conv}\{y_1, \dots, y_n\} + (1 - t)(\lambda \bar{x} + (1 - \lambda)\bar{y})) \\ &= \lambda(t \operatorname{conv}\{x_1, \dots, x_n\} + (1 - t)\bar{x}) + (1 - \lambda)(t \operatorname{conv}\{y_1, \dots, y_n\} + (1 - t)\bar{y}) \\ &= \lambda M_t(x) + (1 - \lambda)M_t(y), \end{aligned}$$

which proves that M_t is concave.

To prove the comparison property fix $0 \leq s < t \leq 1$ and take any $z \in M_s(x_1, \dots, x_n)$. There exist $\alpha_1, \dots, \alpha_n \geq 0$ with $\alpha_1 + \dots + \alpha_n = 1$ such that

$$z = \sum_{i=1}^n \alpha_i x_i + (1 - s)\bar{x} = t \sum_{i=1}^n \frac{s}{t} \alpha_i x_i + (1 - t)\bar{x} + (t - s)\bar{x}. \quad (7)$$

Since

$$t \sum_{i=1}^n \frac{s}{t} \alpha_i x_i + (t - s)\bar{x} = t \sum_{i=1}^n \left(\frac{s}{t} \alpha_i + \frac{t - s}{nt} \right) x_i \in t \operatorname{conv}\{x_1, \dots, x_n\},$$

by (7) we get $z \in M_t(x_1, \dots, x_n)$. This finishes the proof. $\square$

Remark 2 The mean M_0 is single-valued and it is an affine function from D^n to D . Since $M_0(x_1, \dots, x_n) \in M_t(x_1, \dots, x_n)$, $(x_1, \dots, x_n) \in D^n$, this shows that every set-valued mean M_t , $t \in (0, 1]$, has an affine selection.

For arbitrary $t \in (0, 1]$ the set-valued means M_t need not be convex. For example, assume that $D = [0, 1]$, $n = 2$ and take $x = (1, 0)$, $y = (0, 1)$. Then for the set-valued mean $M_t : [0, 1]^2 \rightarrow [0, 1]$ we have

$$M_t(x) = \left[\frac{1}{2} - \frac{t}{2}, \frac{1}{2} + \frac{t}{2} \right] = M_t(y) \text{ and } M_t\left(\frac{x + y}{2}\right) = \left\{ \frac{1}{2} \right\}.$$

Therefore, for each $t \in (0, 1]$,

$$\frac{1}{2}M_t(x) + \frac{1}{2}M_t(y) = \left[\frac{1}{2} - \frac{t}{2}, \frac{1}{2} + \frac{t}{2} \right] \not\subseteq M_t\left(\frac{x + y}{2}\right),$$

which shows that M_t is not convex.

Assume now that X, Y are topological vector spaces and $D \subset X$. A set-valued map $F : D \rightarrow S(Y)$ is said to be *continuous* at a point $x_0 \in D$ if for every neighbourhood W of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0) \subset F(x) + W \quad \text{and} \quad F(x) \subset F(x_0) + W$$

for every $x \in (x_0 + U) \cap D$. If only the first (the second) inclusions are fulfilled, F is called *lower semicontinuous* (*upper semicontinuous*) at x_0 . Lower (resp. upper) semicontinuity as we have defined is a stronger (resp. weaker) requirement than the usage of this terminology by Kuratowski.

The next result describes continuity properties of the set-valued means M_t .

Theorem 3 *Let X be a Hausdorff topological vector space and $D \subset X$ be a convex set. For every $t \in [0, 1]$ the set-valued mean $M_t : D^n \rightarrow S(D)$ is continuous on D^n . Moreover, for every fixed $(x_1, \dots, x_n) \in X^n$ the map $t \mapsto M_t(x_1, \dots, x_n)$ is continuous on $[0, 1]$.*

Proof Since the function $(x_1, \dots, x_n) \mapsto \frac{1}{n}(x_1 + \dots + x_n)$ is continuous, it suffices to show that the set-valued map $\text{conv} : (x_1, \dots, x_n) \mapsto \text{conv}\{x_1, \dots, x_n\}$ is continuous. To this aim fix $(x_1^0, \dots, x_n^0) \in D^n$ and take a neighbourhood W of zero in X . There exist a balanced neighbourhood V of zero in X such that the algebraic sum of n copies of V $V + \dots + V \subset W$. Let

$$(x_1, \dots, x_n) \in ((x_1^0, \dots, x_n^0) + V^n) \cap D^n, \quad (8)$$

that is $x_i \in (x_i^0 + V) \cap D$, $i = 1, \dots, n$. Then, for all $\alpha_1, \dots, \alpha_n \geq 0$ with $\alpha_1 + \dots + \alpha_n = 1$ we have

$$\sum_{i=1}^n \alpha_i x_i \in \sum_{i=1}^n \alpha_i (x_i + V) \subset \sum_{i=1}^n \alpha_i x_i^0 + (V + \dots + V) \subset \text{conv}\{x_1^0, \dots, x_n^0\} + W,$$

and hence

$$\text{conv}\{x_1, \dots, x_n\} \subset \text{conv}\{x_1^0, \dots, x_n^0\} + W.$$

This means that the map conv is upper semicontinuous at $(x_1^0, \dots, x_n^0)$.

Since V is balanced and $(x_1^0, \dots, x_n^0) \in D^n$, by (8) we have also

$$(x_1^0, \dots, x_n^0) \in ((x_1, \dots, x_n) + V^n) \cap D^n,$$

and hence, analogously as above,

$$\text{conv}\{x_1^0, \dots, x_n^0\} \subset \text{conv}\{x_1, \dots, x_n\} + W.$$

This shows that the map conv is lower semicontinuous at $(x_1^0, \dots, x_n^0)$ and finishes the proof of the continuity of M_t .

The second statement follows from the facts that for every fixed $(x_1, \dots, x_n) \in X^n$ the set $\text{conv}\{x_1, \dots, x_n\}$ is bounded and for any bounded set $B \subset X$ the map $t \mapsto tB$ is continuous on $\mathbb{R}$ (see e.g. [11, Lemma 3.2]). $\square$

3 Quasi-arithmetic-type set-valued means

Let $I, J \subset \mathbb{R}$ be intervals. If a function $f : I \rightarrow J$ is continuous and strictly monotonic, then $A_f : I^n \rightarrow I$ defined by

$$A_f(x_1, \dots, x_n) = f^{-1}\left(\frac{f(x_1) + \dots + f(x_n)}{n}\right)$$

is the quasi-arithmetic mean generated by f (see e.g. [8] and the references therein). In a similar way we can define its set-valued counterpart putting, for a given set-valued map $F : I \rightarrow S(J)$,

$$A_F(x_1, \dots, x_n) = F^+\left(\frac{F(x_1) + \dots + F(x_n)}{n}\right),$$

where $F^+(B) := \{x \in I : F(x) \subset B\}$. However, without any additional assumptions, A_F need not be a set-valued mean.

Example 2 Let $n = 2$ and $F : [0, 1] \rightarrow S([0, 1])$ be defined by $F(x) = [x^2, \sqrt{x}]$, $x \in [0, 1]$. Then

$$A_F(0, 1) = F^+\left(\frac{F(0) + F(1)}{2}\right) = F^+\left(\left\{\frac{1}{2}\right\}\right) = \emptyset.$$

Since, by the definition, values of set-valued means are nonempty, this shows that A_F is not a mean.

Example 3 Let $n = 2$ and $F : \mathbb{R} \rightarrow S(\mathbb{R})$ be defined by $F(x) = [x, +\infty)$, $x \in \mathbb{R}$. Then for any $x, y \in \mathbb{R}$

$$A_F(x, y) = F^+\left(\frac{F(x) + F(y)}{2}\right) = \left[\frac{x+y}{2}, +\infty\right) \not\subset \text{conv}\{x, y\}.$$

Thus A_F is not a mean.

The next theorem gives some conditions under which A_F is a set-valued mean.

Theorem 4 Let $f, g : I \rightarrow J$ be strictly increasing functions such that f is concave and g is convex. Assume that $f \leq g$ on I and $F(x) = [f(x), g(x)]$ for all $x \in I$. Then, for every $n \geq 2$, the map $A_F : I^n \rightarrow S(I)$ given by

$$A_F(x_1, \dots, x_n) = F^+\left(\frac{1}{n} \sum_{i=1}^n F(x_i)\right), \quad x_1, \dots, x_n \in I \quad (9)$$

is a set-valued mean.

Proof Fix $n \geq 2$ and take $x_1, \dots, x_n \in I$. We first prove that $A_F(x_1, \dots, x_n) \neq \emptyset$. By the concavity of f and convexity of g , we have

$$f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \geq \frac{1}{n} \sum_{i=1}^n f(x_i) \quad \text{and} \quad g\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \frac{1}{n} \sum_{i=1}^n g(x_i),$$

and hence

$$\begin{aligned} F\left(\frac{1}{n} \sum_{i=1}^n x_i\right) &= \left[f\left(\frac{1}{n} \sum_{i=1}^n x_i\right), g\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right] \\ &\subset \left[\frac{1}{n} \sum_{i=1}^n f(x_i), \frac{1}{n} \sum_{i=1}^n g(x_i) \right] = \frac{1}{n} \sum_{i=1}^n F(x_i). \end{aligned}$$

Consequently,

$$\frac{1}{n} \sum_{i=1}^n x_i \in F^+\left(\frac{1}{n} \sum_{i=1}^n F(x_i)\right) = A_F(x_1, \dots, x_n),$$

which proves that $A_F(x_1, \dots, x_n) \neq \emptyset$.

Now, fix an $x \in A_F(x_1, \dots, x_n)$. Then

$$F(x) \subset \frac{1}{n} \sum_{i=1}^n F(x_i) = \left[\frac{1}{n} \sum_{i=1}^n f(x_i), \frac{1}{n} \sum_{i=1}^n g(x_i) \right],$$

which implies that

$$f(x) \geq \frac{1}{n} \sum_{i=1}^n f(x_i) \quad \text{and} \quad g(x) \leq \frac{1}{n} \sum_{i=1}^n g(x_i). \quad (10)$$

Since f and g are strictly increasing, from (10) we get

$$x \geq \min\{x_1, \dots, x_n\} \quad \text{and} \quad x \leq \max\{x_1, \dots, x_n\}.$$

Hence

$$A_F(x_1, \dots, x_n) \subset [\min\{x_1, \dots, x_n\}, \max\{x_1, \dots, x_n\}] = \text{conv}\{x_1, \dots, x_n\},$$

which finishes the proof. $\square$

Remark 5 The assumptions that f is concave, g is convex and $f \leq g$ imply that the set-valued map $F(x) = [f(x), g(x)]$ is concave. If we have any concave set-valued map $F' : I \rightarrow S(I)$, then the set-valued mean $A_{F'}$ generated by F' has nonempty values (because $\frac{1}{n}(x_1 + \dots + x_n) \in A_{F'}(x_1, \dots, x_n)$), but $A_{F'}(x_1, \dots, x_n) \not\subseteq \text{conv}\{x_1, \dots, x_n\}$, in general. Consider, for example, the set-valued map $F' : \mathbb{R} \rightarrow S(\mathbb{R})$ defined by $F'(x) = [0, |x|]$, $x \in \mathbb{R}$. Then, for $n = 2$, we have

$$A_{F'}(-1, 2) = F'^+\left(\frac{[0, 1] + [0, 3]}{2}\right) = F'^+([0, 2]) = [-2, 2] \not\subseteq \text{conv}\{-1, 2\}.$$

Assume now that $M, N : I^n \rightarrow I$ are two means such that $M \leq N$ on I . Then the map $MN : I^n \rightarrow S(I)$ defined by

$$MN(x_1, \dots, x_n) = [M(x_1, \dots, x_n), N(x_1, \dots, x_n)], \quad x_1, \dots, x_n \in I,$$

is a set-valued mean. Using this observation we can construct various set-valued means from single-valued ones. For instance, consider the classical harmonic, geometric, arithmetic and quadratic means defined for $x_1, \dots, x_n > 0$ by

$$\begin{aligned} H(x_1, \dots, x_n) &= \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}, \quad G(x_1, \dots, x_n) = \sqrt[n]{x_1 \cdots x_n}, \\ A(x_1, \dots, x_n) &= \frac{x_1 + \dots + x_n}{n}, \quad K(x_1, \dots, x_n) = \sqrt{\frac{x_1^2 + \dots + x_n^2}{n}}. \end{aligned}$$

All these means are quasi-arithmetic means generated by the functions $f(x) = -1/x$, $f(x) = \ln x$, $f(x) = x$ and $f(x) = x^2$, respectively. It is known that

$$H(x_1, \dots, x_n) \leq G(x_1, \dots, x_n) \leq A(x_1, \dots, x_n) \leq K(x_1, \dots, x_n), \quad (11)$$

for all $x_1, \dots, x_n > 0$. Using these inequalities we can define the following set valued means:

$$HG(x) := [H(x), G(x)], \quad HA(x) := [H(x), A(x)], \quad HK(x) := [H(x), K(x)],$$

$$GA(x) := [G(x), A(x)], \quad GK(x) := [G(x), K(x)], \quad AK(x) := [A(x), K(x)],$$

where $x = (x_1, \dots, x_n) \in (0, \infty)^n$.

Observe that all but the first of the above set-valued means are examples of the means (9) obtained for appropriate generators F .

Example 4 Let $F_1(x) = [\ln x, x^2]$, $x > 0$. Then F_1 satisfies the assumptions of Theorem 4 and therefore A_{F_1} is a set-valued mean (in particular, it has nonempty values). Assume, for simplicity, that $n = 2$. Since

$$A_{F_1}(x, y) = F_1^+\left(\frac{1}{2}([\ln x, x^2] + [\ln y, y^2])\right) = F_1^+\left(\left[\ln \sqrt{xy}, \frac{x^2 + y^2}{2}\right]\right),$$

we obtain

$$z \in A_{F_1}(x, y) \Leftrightarrow [\ln z, z^2] \subset \left[\ln \sqrt{xy}, \frac{x^2 + y^2}{2}\right] \Leftrightarrow z \in \left[\sqrt{xy}, \sqrt{\frac{x^2 + y^2}{2}}\right].$$

This proves that $A_{F_1}(x, y) = GK(x, y)$ for all $x, y > 0$.

Now, let $F_2(x) = \left[-\frac{1}{x}, x\right]$, $x > 0$. Then

$$A_{F_2}(x, y) = F_2^+\left(\frac{1}{2}\left[-\frac{1}{x}, x\right] + \frac{1}{2}\left[-\frac{1}{y}, y\right]\right) = F_2^+\left(\left[\frac{x+y}{-2xy}, \frac{x+y}{2}\right]\right).$$

Hence

$$z \in A_{F_2}(x, y) \Leftrightarrow \left[-\frac{1}{z}, z\right] \subset \left[\frac{x+y}{-2xy}, \frac{x+y}{2}\right] \Leftrightarrow z \in \left[\frac{2xy}{x+y}, \frac{x+y}{2}\right],$$

which shows that $A_{F_2}(x, y) = HA(x, y)$ for all $x, y > 0$.

Similarly, taking one by one $F_3(x) = [\ln x, x]$, $F_4(x) = [x, x^2 + 1]$ and $F_5(x) = \left[-\frac{1}{x}, x^2\right]$ we can show that $A_{F_3}(x, y) = GA(x, y)$, $A_{F_4}(x, y) = AK(x, y)$ and $A_{F_5}(x, y) = HK(x, y)$, for all $x, y > 0$.

Observe also that some of the inequalities (11) between the classical means can be obtained as corollaries to Theorem 4. Let us formulate explicitly a more general result.

Theorem 6 Let $f, g : I \rightarrow J$ be strictly increasing functions such that f is concave and g is convex. Then

$$A_f(x_1, \dots, x_n) \leq A(x_1, \dots, x_n) \leq A_g(x_1, \dots, x_n) \quad (12)$$

for all $x_1, \dots, x_n \in I$.

Proof Fix $x_1, \dots, x_n \in I$. By the concavity of f and convexity of g we have

$$f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \geq \frac{1}{n} \sum_{i=1}^n f(x_i) \quad \text{and} \quad g\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \frac{1}{n} \sum_{i=1}^n g(x_i). \quad (13)$$

Since f and g are strictly increasing, also f^{-1} and g^{-1} are strictly increasing. Therefore, from (13) we obtain

$$f^{-1}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)\right) \leq f^{-1}\left(f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right) = \frac{1}{n}\sum_{i=1}^n x_i = g^{-1}\left(g\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right) \leq g^{-1}\left(\frac{1}{n}\sum_{i=1}^n g(x_i)\right), \quad (14)$$

which gives (12) and finishes the proof. $\square$

Example 5 Let us take $f(x) = \ln x$ and $g(x) = x^2$, $x > 0$. Clearly, f and g satisfy the assumptions of Theorem 6. Since f generates the geometric mean and g generates the quadratic mean, from (12) we get

$$G(x_1, \dots, x_n) \leq A(x_1, \dots, x_n) \leq K(x_1, \dots, x_n),$$

for all $x_1, \dots, x_n > 0$.

Now, take $f(x) = -\frac{1}{x}$ and $g(x) = x^2$, $x > 0$. Since f generates the harmonic mean, by (12) we obtain

$$H(x_1, \dots, x_n) \leq A(x_1, \dots, x_n) \leq K(x_1, \dots, x_n),$$

for all $x_1, \dots, x_n > 0$.

4 Lagrangian-type set-valued means

Now, assume that $f : I \rightarrow \mathbb{R}$ is a continuous function. By the classical integral mean value theorem, for each $x_1, x_2 \in I$, $x_1 < x_2$, there exists $\xi \in (x_1, x_2)$ such that

$$f(\xi) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x) dx.$$

If, moreover, f is strictly monotonic, then $f^{-1}(f(\xi)) = \xi$, and hence the function $L_f : I^2 \rightarrow I$ defined by

$$L_f(x_1, x_2) = \begin{cases} f^{-1}\left(\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x) dx\right) & \text{if } x_1 \neq x_2, \\ x_1 & \text{if } x_1 = x_2 \end{cases}$$

is a mean. It is called the Lagrangian mean.

For a given set-valued map $F : I \rightarrow S(\mathbb{R})$ the integral of F is understood in the sense of Aumann, i.e. it is the set of integrals of all Lebesgue integrable selections of this map (cf. [2]):

$$\int_a^b F(x) dx = \left\{ \int_a^b f(x) dx : f : [a, b] \rightarrow \mathbb{R} \text{ is integrable and } f(x) \in F(x), x \in [a, b] \right\}.$$

Analogously as above we can define a potential set-valued Lagrangian-type mean for a map $F : I \rightarrow S(\mathbb{R})$ putting for $x_1, x_2 \in I$

$$L_F(x_1, x_2) = \begin{cases} F^+\left(\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} F(x) dx\right) & \text{if } x_1 \neq x_2, \\ \{x_1\} & \text{if } x_1 = x_2 \end{cases} \quad (15)$$

The next theorem gives conditions under which the map L_F is a set-valued mean.

Theorem 7 Let $f, g : I \rightarrow \mathbb{R}$ be strictly increasing functions such that f is concave and g is convex. Assume that $f \leq g$ on I and $F(x) = [f(x), g(x)]$ for all $x \in I$. Then the map $L_F : I^2 \rightarrow S(I)$ given by (15) is a set-valued mean.

Proof Note first that if $F(x) = [f(x), g(x)]$ with integrable $f, g : I \rightarrow \mathbb{R}$, then for each $x_1, x_2 \in I, x_1 < x_2$

$$\int_{x_1}^{x_2} F(x)dx = \left[\int_{x_1}^{x_2} f(x)dx, \int_{x_1}^{x_2} g(x)dx \right],$$

because f, g are selections of F and the Aumann integral is a convex set. Assume that $x_1, x_2 \in I, x_1 < x_2$ and fix $x \in L_F(x_1, x_2)$. Then

$$F(x) \subset \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} F(x)dx = \frac{1}{x_2 - x_1} \left[\int_{x_1}^{x_2} f(x)dx, \int_{x_1}^{x_2} g(x)dx \right],$$

and hence

$$f(x) \geq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)dx \quad \text{and} \quad g(x) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} g(x)dx. \quad (16)$$

By the integral mean value theorem there exist $\xi_1, \xi_2 \in (x_1, x_2)$ such that

$$f(\xi_1) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)dx \quad \text{and} \quad g(\xi_2) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} g(x)dx. \quad (17)$$

Since f and g are strictly increasing, by (16) and (17) we get

$$x \geq \xi_1 > x_1 \quad \text{and} \quad x \leq \xi_2 < x_2,$$

which proves that $L_F(x_1, x_2) \subset \text{conv}\{x_1, x_2\}$.

Now, we will prove that $L_F(x_1, x_2)$ is nonempty. Since f is concave and g is convex, by the Hermite-Hadamard theorem we have

$$f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)dx \quad \text{and} \quad g\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} g(x)dx.$$

From here

$$F\left(\frac{x_1 + x_2}{2}\right) = \left[f\left(\frac{x_1 + x_2}{2}\right), g\left(\frac{x_1 + x_2}{2}\right) \right] \subset \frac{1}{x_2 - x_1} \left[\int_{x_1}^{x_2} f(x)dx, \int_{x_1}^{x_2} g(x)dx \right].$$

Consequently,

$$\frac{x_1 + x_2}{2} \in F^+\left(\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} F(x)dx\right) = L_F(x_1, x_2).$$

This shows that $L_F(x_1, x_2) \neq \emptyset$ and finishes the proof. $\square$

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