Set-Valued Var. Anal (2014) 22:533–555
DOI 10.1007/s11228-014-0276-5

About Semicontinuity of Set-valued Maps and Stability of Quasivariational Inclusions

Lam Quoc Anh · Phan Quoc Khanh · Dinh Ngoc Quy

Received: 16 April 2013 / Accepted: 24 February 2014 / Published online: 11 May 2014

Abstract We propose several additional kinds of semi-limits and corresponding notions of semicontinuity of a set-valued map. They can be used additionally to known basic concepts of semicontinuity to have a clearer insight of local behaviors of maps. Then, we investigate semicontinuity properties of solution maps to a general parametric quasivariational inclusion, which is shown to include most of optimization-related problems. Consequences are derived for several particular problems. Our results are new or generalize/improve recent existing ones in the literature.

Keywords Semi-limits · Semicontinuity · Solution maps · Quasivariational inclusions · Quasivariational relation problems · Quasivariational equilibrium problems

Mathematics Subject Classifications (2010) 90C31 · 49J53

1 Introduction

The aim of this paper is twofold. First, we propose several kinds of semicontinuity of a set-valued map, additionally to the fundamental notions (see [1–3]). We hope they can be somehow useful to give additional details of local behaviors of a set-valued map in some cases when the fundamental notions of semicontinuity are not enough. Next, we consider

L. Q. Anh · D. N. Quy (✉)
Department of Mathematics, Cantho University, Cantho, Vietnam
e-mail: dnquy@ctu.edu.vn

L. Q. Anh
e-mail: quocanh@ctu.edu.vn

P. Q. Khanh
Department of Mathematics, International University, Vietnam National University Hochiminh City, Hochiminh, Vietnam
e-mail: pqkhanh@hcmiu.edu.vn

various semicontinuity properties of solution maps of a quasivariational inclusion problem. We choose to study this model since, though simple and relatively little mentioned in the literature, it is equivalent to other frequently discussed models, which englobe most of optimization-related problems.

Semicontinuity properties are among the most important topics in analysis and optimization. Let X and Y be topological spaces. For $x \in X$, let $\mathcal{N}(x)$ stand for the set of neighborhoods of x. The basic semicontinuity concepts for $G : X \to 2^Y$ are the following (see [1–3]). G is called inner semicontinuous (isc in short) at $\bar{x}$ if $\text{liminf}_{x \to \bar{x}} G(x) \supset G(\bar{x})$, and outer semicontinuous (osc) at $\bar{x}$ if $\text{limsup}_{x \to \bar{x}} G(x) \subset G(\bar{x})$. Here liminf and limsup are the Painlevé-Kuratowski inferior and superior limits in terms of nets:

$$\text{liminf}_{x \to \bar{x}} G(x) := \{y \in Y : \forall x_\alpha \to \bar{x}, \exists y_\alpha \in G(x_\alpha), y_\alpha \to y\},$$

$$\text{limsup}_{x \to \bar{x}} G(x) := \{y \in Y : \exists x_\alpha \to \bar{x}, \exists y_\alpha \in G(x_\alpha), y_\alpha \to y\}.$$

Equivalently, G is isc at $\bar{x}$ if $\forall x_\alpha \to \bar{x}, \forall \bar{y} \in G(\bar{x}), \exists y_\alpha \in G(x_\alpha), y_\alpha \to \bar{y}$. If G is both outer and inner semicontinuous at $\bar{x}$, we say that G is Rockafellar-Wets continuous at this point. Close to outer and inner semicontinuity is the (Berge) upper and lower semicontinuity: G is called upper semicontinuous (usc) at $\bar{x}$ if for each open set $U \supset G(\bar{x})$, there is $N \in \mathcal{N}(\bar{x})$ such that $U \supset G(N)$; G is called lower semicontinuous (lsc) at $\bar{x}$ if for each open set U with $U \cap G(\bar{x}) \neq \emptyset$, there is $N \in \mathcal{N}(\bar{x})$ such that, for all $x' \in N$, $U \cap G(x') \neq \emptyset$. If G is usc and lsc at the same time, we say that G is Berge continuous. Lower semicontinuity agrees with inner semicontinuity, but upper semicontinuity differs from outer semicontinuity, though close to each other (see [2]). G is called closed at $\bar{x}$ if for each net $(x_\alpha, y_\alpha) \in \text{gr}G := \{(x, y) : z \in G(x)\}$ with $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y})$, $\bar{y}$ must belong to $G(\bar{x})$. We say that G satisfies a certain property in $A \subset X$ if G satisfies it at every point of A. If $A = X$ we omit "in X". Observe that G is closed if and only if its graph is closed.

In [4–6] several semicontinuity-related concepts were proposed. In [7] the inferior and superior open limits, respectively (resp, shortly), were proposed. Here, we use the following version of these definitions

$$\text{liminfo}_{x \to \bar{x}} G(x) := \{y \in Y : \exists U \in \mathcal{N}(\bar{x}), \exists V \in \mathcal{N}(y), \forall x \in U, V \subset G(x)\};$$

$$\text{limsupo}_{x \to \bar{x}} G(x) := \{y \in Y : \exists V \in \mathcal{N}(y), \exists x_\alpha \to \bar{x}, \forall \alpha, V \subset G(x_\alpha)\}.$$

Notice that, in [7], inferior and superior open limits were defined as follows (we add "st." and "w." in the notations to avoid confusions and write only st.limsup, by similarity):

$$\text{st.limsupo}_{x \to \bar{x}} G(x) := \{y \in Y : \exists V \in \mathcal{N}(y), \exists x_\alpha \to \bar{x} : x_\alpha \neq \bar{x}, \forall \alpha, V \subset G(x_\alpha)\}.$$

However, as more frequently met in the literature, we allow x_α to take the value $\bar{x}$ in this paper.

Remark 1 Observe that the following relations hold:

$$\text{limsupo}_{x \to \bar{x}} G(x) = \text{st.limsupo}_{x \to \bar{x}} G(x) \cup \text{int}G(\bar{x}),$$

$$\text{liminfo}_{x \to \bar{x}} G(x) = \text{w.liminfo}_{x \to \bar{x}} G(x) \cap \text{int}G(\bar{x}).$$

However, in the sequel, we will not use the semi-limits on the right-hand side of these relations. Here and later, int$A$, cl$A$, and bd$A$ stand for the interior, closure and boundary of A, resp.

A set-valued map G is called inner open (outer open) at $\bar{x} \in X$ (see [7]) if $\text{liminfo}_{x \to \bar{x}} G(x) \supset G(\bar{x})$ ($\text{limsupo}_{x \to \bar{x}} G(x) \subset G(\bar{x})$, resp). These concepts help to link semicontinuities of G with its complement G^c ($G^c(x) := Y \setminus G(x)$) and to characterize a map by its graph as follows.

Proposition 1 *([7]) The following assertions hold.*

(i) *G is outer open at λ_0 if and only if G^c is inner semicontinuous at λ_0.*
(ii) *G is outer semicontinuous at λ_0 if and only if G^c is inner open at λ_0.*
(iii) *G is outer semicontinuous and closed-valued (respectively, inner open and open-valued) on Λ if and only if its graph is closed (respectively, open).*
(iv) *If G is outer semicontinuous at λ_0, then it is outer open there.*
(v) *G is inner open at λ_0, then it is inner semicontinuous there.*

In Section 2, we go further in this direction by proposing other two kinds of semi-limits and corresponding semicontinuities to obtain a more detailed picture of local behaviors of a set-valued map. Sections 3 and 4 are devoted to discussing semicontinuity properties of solution maps of the following parametric quasivariational inclusion problem. Let X and Λ be Hausdorff topological spaces, Z a topological vector space. Let $K_1, K_2 : X \times \Lambda \to 2^X$ and $F : X \times X \times \Lambda \to 2^Z$. The problem under our investigation is of, for each $\lambda \in \Lambda$,

$$(\text{QVIP}_\lambda) : \text{finding } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for each } y \in K_2(\bar{x}, \lambda),\ 0 \in F(\bar{x}, y, \lambda).$$

To motivate our choice of this model, we state the following other two general settings. Let $P, Q : X \times X \times \Lambda \to 2^Z$. In [8–10] and [11], the following inclusion problem was investigated

$$(\text{QVIP}^1_\lambda) : \text{find } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for each } y \in K_2(\bar{x}, \lambda),\ P(\bar{x}, y, \lambda) \subset Q(\bar{x}, y, \lambda).$$

Notice, as seen in [8–10] and [11], that for the mentioned problems, but with other constraints or other types of the inclusions, analogous study methods can be applied.

Let $R(x, y, \lambda)$ be a relation linking $x, y \in X$ and $\lambda \in \Lambda$. Note that R can be identified as the subset $M := \{(x, y, \lambda) \in X \times X \times \Lambda : R(x, y, \lambda) \text{ holds}\}$ of the product space $X \times X \times \Lambda$. In [7, 12, 13] (with different constraints), the following quasivariational relation problem was studied

$$(\text{QVRP}_\lambda) : \text{find } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for each } y \in K_2(\bar{x}, \lambda),\ R(\bar{x}, y, \lambda) \text{ holds.}$$

As observed in the encountered references, (QVIP^1_λ) and (QVRP_λ) contain most of optimization-related problems as special cases. Now we show the equivalence of them and our model (QVIP_λ) when X and Λ are Hausdorff topological vector spaces. To convert (QVRP_λ) to a particular case of (QVIP_λ), we simply set $Z := X \times X \times \Lambda$ and $F(x, y, \lambda) := (x, y, \lambda) - M$. Then, $R(x, y, \lambda)$ holds if and only if $0 \in F(x, y, \lambda)$. Next, (QVIP_λ) is clearly a case of (QVIP^1_λ) with $F(x, y, \lambda) \equiv Q(x, y, \lambda)$ and $P(x, y, \lambda) \equiv \{0\}$. Finally, to see that (QVIP^1_λ) in turn is a case of (QVRP_λ), define that $R(x, y, \lambda)$ holds if and only if $P(x, y, \lambda) \subset Q(x, y, \lambda)$. In the sequel, let $(\text{QVIP})_{\lambda \in \Lambda}$ stand for the family of (QVIP_λ) for all $\lambda \in \Lambda$.

Section 5 is devoted to applying the results of the preceding sections to some special cases. Here, we consider only several quasiequilibrium problems as illustrative examples. In particular, in Subsection 5.3 we investigate a very specific scalar equilibrium problem to see that Ekeland's variational principle can be applied to get good semicontinuity results, which cannot be derived from our results for (QVIP$_\lambda$).

2 About Semicontinuity of Set-Valued Maps

Throughout this section, let X and Y be topological spaces and $G : X \to 2^Y$. We propose the following new definitions of semi-limits of set-valued maps

$$\operatorname{liminf}^*_{x\to\bar{x}} G(x) := \{y \in Y : \exists U \in \mathcal{N}(\bar{x}), \forall x \in U, y \in G(x)\},$$

$$\operatorname{limsup}^*_{x\to\bar{x}} G(x) := \{y \in Y : \exists x_\alpha \to \bar{x}, \forall \alpha, y \in G(x_\alpha)\}.$$

It is known that (Painlevé-Kuratowski) liminf and limsup of a map are always closed sets and that liminfo and limsupo of a map are always open. However, many examples in the remaining part of this section show that the above two new semi-limits may be neither open nor closed. The following relations ensure that the introduction of the two new semi-limits is helpful.

Proposition 2 *For $G : X \to 2^Y$, the following assertions hold*

(i) $\operatorname{limsupo}_{x\to\bar{x}} G(x) \subset \operatorname{limsup}^*_{x\to\bar{x}} G(x) \subset \operatorname{limsup}_{x\to\bar{x}} G(x)$;
(ii) $\operatorname{liminfo}_{x\to\bar{x}} G(x) \subset \operatorname{liminf}^*_{x\to\bar{x}} G(x) \subset \operatorname{liminf}_{x\to\bar{x}} G(x) \subset clG(\bar{x})$;
(iii) $\operatorname{liminf}^*_{x\to\bar{x}} G(x) = [\operatorname{limsup}^*_{x\to\bar{x}} G^c(x)]^c$;
(iv) $G(\bar{x}) \subset \operatorname{limsup}^*_{x\to\bar{x}} G(x)$ *and* $\operatorname{cl}G(\bar{x}) \subset \operatorname{limsup}_{x\to\bar{x}} G(x)$;
(v) $\operatorname{liminf}^*_{x\to\bar{x}} G(x) \subset G(\bar{x})$.

Proof The relations (i), (ii), (iv) and (v) follow directly from definition. For (iii), let $y \in \operatorname{liminf}^*_{x\to\bar{x}} G(x)$. Suppose $y \in \operatorname{limsup}^*_{x\to\bar{x}} G^c(x)$. There is a net $\{x_\alpha\} \subset X$ converging to $\bar{x}$ such that $y \in G^c(x_\alpha)$ for all α. Since $y \in \operatorname{liminf}^*_{x\to\bar{x}} G(x)$, $\exists U \in \mathcal{N}(\bar{x}), \forall x \in U, y \in G(x)$. As $x_\alpha \to \bar{x}$, there exists α_0 such that $x_{\alpha_0} \in U$, which implies that $y \in G(x_{\alpha_0})$, contradicting the fact that $y \in G^c(x_\alpha)$ for all α. Hence, $\operatorname{liminf}^*_{x\to\bar{x}} G(x) \subset [\operatorname{limsup}^*_{x\to\bar{x}} G^c(x)]^c$. Conversely, suppose $y \in [\operatorname{limsup}^*_{x\to\bar{x}} G^c(x)]^c$ but $y \notin \operatorname{liminf}^*_{x\to\bar{x}} G(x)$. Then, $\forall U_\alpha \in \mathcal{N}(\bar{x})$, $\exists x_\alpha \in U_\alpha$, $y \notin G(x_\alpha)$. Therefore, there is a net $\{x_\alpha\} \subset X$ converging to $\bar{x}$ such that $y \in G^c(x_\alpha)$ for all α, which implies that $y \in \operatorname{limsup}^*_{x\to\bar{x}} G^c(x)$. This contradiction yields (iii), since $\operatorname{liminf}^*_{x\to\bar{x}} G(x) \supset [\operatorname{limsup}^*_{x\to\bar{x}} G^c(x)]^c$. □

Correspondingly, we propose the following new kinds of semicontinuity.

Definition 1 (i) G is termed star-outer semicontinuous (star-osc) at $\bar{x} \in X$ if $\operatorname{limsup}^*_{x\to\bar{x}} G(x) \subset G(\bar{x})$;
(ii) G is called star-inner semicontinuous (star-isc) at $\bar{x} \in X$ if $\operatorname{liminf}^*_{x\to\bar{x}} G(x) \supset G(\bar{x})$.

It is known that G is osc at $\bar{x} \in X$ if and only if $\operatorname{limsup}_{x\to\bar{x}} G(x) = G(\bar{x})$, and isc at $\bar{x} \in X$ if and only if $\operatorname{liminf}_{x\to\bar{x}} G(x) = \operatorname{cl}G(\bar{x})$. By Proposition 2(iv) and (v), we have the first similar but different thing for the above new semicontinuity notions:

- G is star-osc at $\bar{x} \in X$ if and only if $\operatorname{limsup}^*_{x\to\bar{x}} G(x) = G(\bar{x})$;
- G is star-isc at $\bar{x} \in X$ if and only if $\operatorname{liminf}^*_{x\to\bar{x}} G(x) = G(\bar{x})$.

Now we prove relations between the mentioned kinds of semicontinuity.

Proposition 3 *The following assertions hold.*

(i) *If G is outer semicontinuous at $\bar{x}$, then G is star-outer semicontinuous at $\bar{x}$.*
(ii) *If G is star-outer semicontinuity at $\bar{x}$, then G is outer open at $\bar{x}$.*
(iii) *If G is star-inner semicontinuous at $\bar{x}$, then G is inner semicontinuous at $\bar{x}$.*
(iv) *If G is inner open at $\bar{x}$, then G is star-inner semicontinuous at $\bar{x}$.*
(v) *If G is usc at $\bar{x}$, then G is star-outer semicontinuous at $\bar{x}$.*
(vi) *G is star-inner semicontinuous if and only if G^c is star-outer semicontinuous.*

Proof Assertions (i) and (ii) are derived from Proposition 2(i). Assertions (iii) and (iv) are consequences of Proposition 2(ii). Statements (vi) is obtained directly from Proposition 2(iii). For (v), suppose to the contrary the existence of $y \in \limsup^*_{x\to\bar{x}} G(x)$ and $\{x_\alpha\} \subset X$ converging to $\bar{x}$ such that $y \in G(x_\alpha)$ for all α, but $y \notin G(\bar{x})$. If U is a neighborhood of $G(\bar{x})$, then so is $U \setminus \{y\}$, as $y \notin G(\bar{x})$. Since G is usc at $\bar{x}$, there exists $V \in \mathcal{N}(\bar{x})$ such that $G(V) \subset U \setminus \{y\}$. There exists α_0 such that $x_{\alpha_0} \in V$. This implies that $G(x_{\alpha_0}) \subset U \setminus \{y\}$, contradicting the fact that $y \in G(x_\alpha)$ for all α. □

Remark 2 We discuss the considered definitions of semicontinuity for the special case of $g(.)$ being single-valued. All lower semicontinuity, upper semicontinuity, and continuity (in the sense of Berge) are equivalent and this is just the usual continuity of a single-valued map. But, continuity in the sense of Rockafellar-Wets is weaker. Simply think of the real function $y = x^{-1}$ if $x \neq 0$ and $y(0) = 0$, which is both inner and outer semicontinuous at zero, but it has an infinite discontinuity jump at zero. All these four definitions of semicontinuity have been proved to be fundamental for set-valued maps. However, in some cases they are still not convenient in use. We explain this in simple examples.

Example 1 (with non-closed images, a "good" set-valued map may be nether usc nor osc) Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $G(x) = (0, 2^x)$ for $x \in \mathbb{R}$. Then, at any point, G is neither usc nor osc, though its behavior is very good at all $x \in \mathbb{R}$. In this case, G is outer open at each point.

Example 2 (with unbounded non-closed images, a "good" set-valued map may be nether usc nor osc) Let $G : \mathbb{R} \to 2^{\mathbb{R}\times\mathbb{R}}$ be defined by $G(x) = \{(y, xy) \in \mathbb{R}^2 : y \in (0, +\infty)\}$ for $x \in \mathbb{R}$. Then, G is neither usc nor osc at any point. But, G is both outer open and star-osc at each point. Observe that if G is osc at $\bar{x}$, then $G(\bar{x})$ must be closed, which may be violated even when G has a constant open value (see also Example 5).

Unlike in these two examples, outer openness seems not to describe well a behavior in the following.

Example 3 (with images having empty interior, a "bad" map may be outer open) Let $G : \mathbb{R} \to 2^{\mathbb{R}\times\mathbb{R}}$ be defined by $G(x) = \{(y, 1) \in \mathbb{R}^2 : y \in \mathbb{R}\}$ for $x \neq 0$ and $G(0) = \{(y, 0) \in \mathbb{R}^2 : y \in \mathbb{R}\}$. Then, G is outer open at 0, but its behavior is "discontinuous" for our usual feeling. Observe that G is not star-osc at zero though this property is weaker than being osc.

To end this Remark 2, observe that from the definition and Proposition 3, any single-valued map is outer open and never inner open. The star-outer semicontinuity and star-inner semicontinuity notions are also not significant in this case, since the former is relatively too weak (weaker than the usual continuity) and the latter is too strong. Namely, a (single-valued) map, which is star-inner semicontinuous at a point, must be locally constant around it. Hence, these four notions are designed specially to insight local behaviors of set-valued maps. Observe further that a complete "symmetry" of liminf* and limsup* given in Proposition 2(iii) does not have counterparts neither for liminf and limsup, nor for liminfo and limsupo.

Now we show that all the non-mentioned reverse implications in the assertions (i)-(v) of Proposition 3 do not hold in general indeed.

Example 4 (for (i) and (iv), star-outer semicontinuity not outer semicontinuity, and star-inner semicontinuity not inner openness) Let $G(x) \equiv (-1, 0]$ for $x \in \mathbb{R}$. Then, G is star-outer semicontinuous at 0, since $\text{limsupo}^*_{x\to 0}G(x) = (-1, 0] = G(0)$. But, G is not outer semicontinuous at 0, as $\text{limsup}_{x\to 0}G(x) = [-1, 0]$. Furthermore, G is star-inner semicontinuous since $\text{liminf}^*_{x\to 0}G(x) = (-1, 0]$, but G is not inner open, because $\text{liminfo}_{x\to 0}G(x) = (-1, 0)$.

Example 5 (for (ii), outer openness not star-outer semicontinuity) Let $G(x) = (-1, |x|)$ for $x \in \mathbb{R}$. Then, $\text{limsupo}_{x\to 0}G(x) = (-1, 0) = G(0)$ and $\text{limsup}^*_{x\to 0}G(x) = (-1, 0]$. Hence, at 0, G is outer open but not star-outer semicontinuous.

Example 6 (for (iii) and (v), inner semicontinuity not star-inner semicontinuity, and star-outer semicontinuity not upper semicontinuity) Let $G(x) = \{(y, xy) \in \mathbb{R}^2 : y \in \mathbb{R}\}$ for all $x \in \mathbb{R}$. Then, G is inner semicontinuous at 0 as $\text{liminf}_{x\to 0}G(x) = \{(y, 0) : y \in \mathbb{R}\} = G(0)$. But, G is not star-inner semicontinuous at 0, since $\text{liminf}^*_{x\to 0}G(x) = \{(0, 0)\}$ does not contain $G(0)$. Furthermore, G is star-outer semicontinuous as $\text{limsup}^*_{x\to 0}G(x) = G(0)$. G is not usc, because for an arbitrary neighborhood U of $G(0)$, one cannot find a neighborhood N of zero such that $G(N) \subset U$.

Next, we propose notions which are closely related to star-inner semicontinuity and star-outer semicontinuity. In fact they are developments of Definition 2.1 of [14], Definition 2.2 of [4], and Definition 2.2 of [5] to more general settings. These notions will be used in the subsequent sections for studying semicontinuity properties of solution maps of our variational problems.

Definition 2 Let $G : X \to 2^Y$ and $\theta \in Y$.

(i) G is said to have the θ-inclusion property at $\bar{x}$ if, for any $x_\alpha \to \bar{x}$,

$$[\theta \in G(x_\alpha),\ \forall \alpha] \Longrightarrow [\theta \in G(\bar{x})].$$

(ii) G is said to have the θ-inclusion complement property at x_0 if, for any $x_\alpha \to \bar{x}$,

$$[\theta \in G(\bar{x})] \Longrightarrow [\exists \bar{\alpha}, \theta \in G(x_{\bar{\alpha}})].$$

To compare these properties with the corresponding definitions in [4] and [14], let Y be a topological vector space, $C, U \subset Y$ with nonempty interior, C being closed, and $H : X \to 2^Y$. Then, one can verify the following relations.

• For $G = H - (Y \setminus -\text{int}C)$, G^c has the 0-inclusion property (or G has the 0-inclusion complement property) at $\bar{x}$ if and only if H has the C-inclusion property at $\bar{x}$ (by Definition 2.1 of [14]). While, setting $G = H + \text{int}C$, G has the 0-inclusion property (or G^c has the

0-inclusion complement property) at $\bar{x}$ if and only if H has the strict C-inclusion property at $\bar{x}$ (by the mentioned definition).

• With $G = H - \text{int}U$, G^c has the 0-inclusion property (or G has the 0-inclusion complement property) at $\bar{x}$ if and only if H is U-lsc at $\bar{x}$ (defined in [4]). While, setting $G = H - (Y \setminus \text{int}U)$, G has the 0-inclusion property (or G^c has the 0-inclusion complement property) at $\bar{x}$ if and only if H is U-usc at $\bar{x}$ (defined in [4]).

About these inclusion properties, we have the following statement.

Proposition 4 (i) *G has the θ-inclusion property at $\bar{x}$ if and only if G^c has the θ-inclusion complement property at $\bar{x}$.*

(ii) *The set $\{x \in X : \theta \in G(x)\}$ is closed if and only if G has the θ-inclusion property.*

(iii) *The set $\{x \in X : \theta \notin G(x)\}$ is closed if and only if G has the θ-inclusion complement property.*

(iv) *G is star-outer semicontinuous at $\bar{x}$ if and only if G has the θ-inclusion property at $\bar{x}$ for every θ.*

(v) *G is star-inner semicontinuous at $\bar{x}$ if and only if G has the θ-inclusion complement property at $\bar{x}$ for every θ.*

Proof Assertions (i)-(iii) are obvious from definition. For (iv), let $\{x_\alpha\} \subset X$ converge to $\bar{x}$ such that $\theta \in G(x_\alpha)$ for all α. Then, $\theta \in \text{limsup}^*_{x\to\bar{x}} G(x)$. The star-outer semicontinuity at $\bar{x}$ implies that $\text{limsup}^*_{x\to\bar{x}} G(x) \subset G(\bar{x})$. Hence, $\theta \in G(\bar{x})$. Conversely, if $\theta \in \lim\sup^*_{x\to\bar{x}} G(x)$, there exists $\{x_\alpha\}$ converging to $\bar{x}$ such that $\theta \in G(x_\alpha)$ for all α. Since G has the θ-inclusion property at $\bar{x}$, $\theta \in G(\bar{x})$. Hence, $\text{limsup}^*_{x\to\bar{x}} G(x) \subset G(\bar{x})$, i.e., G is star-outer semicontinuous at $\bar{x}$. (v) is obvious from (vi) of Proposition 3, and (i),(iv), since one has the equivalent relations: G is star-inner semicontinuous at $\bar{x}$ $\iff$ G^c is star-outer semicontinuous at $\bar{x}$ $\iff$ G^c has the θ-inclusion property at $\bar{x}$ for every θ $\iff$ G has the θ-inclusion complement property at $\bar{x}$ for every θ. □

The rest of this section is devoted to calculus rules of semi-limits and semicontinuity for intersections and unions of maps.

Proposition 5 *For $F, G : X \to 2^Y$, the following containments and inclusions hold for $\sharp$ being any of 'sup', 'sup*', 'supo', 'inf', 'inf*', 'info'.*

(i) $\lim\sharp_{x\to\bar{x}}(F \cap G)(x) \subset \lim\sharp_{x\to\bar{x}} F(x) \cap \lim\sharp_{x\to\bar{x}} G(x)$. *Moreover,*

$$\text{liminfo}_{x\to\bar{x}}(F \cap G)(x) = \text{liminfo}_{x\to\bar{x}} F(x) \cap \text{liminfo}_{x\to\bar{x}} G(x),$$

$$\text{liminf}^*_{x\to\bar{x}}(F \cap G)(x) = \text{liminf}^*_{x\to\bar{x}} F(x) \cap \text{liminf}^*_{x\to\bar{x}} G(x),$$

$$\text{liminf}_{x\to\bar{x}} F(x) \cap \text{liminfo}_{x\to\bar{x}} G(x) \subset \text{liminf}_{x\to\bar{x}}(F \cap G)(x).$$

(ii) $\lim\sharp_{x\to\bar{x}}(F \cup G)(x) \supset \lim\sharp_{x\to\bar{x}} F(x) \cup \lim\sharp_{x\to\bar{x}} G(x)$. *Moreover,*

$$\text{limsup}_{x\to\bar{x}}(F \cup G)(x) = \text{limsup}_{x\to\bar{x}} F(x) \cup \text{limsup}_{x\to\bar{x}} G(x),$$

$$\text{limsup}^*_{x\to\bar{x}}(F \cup G)(x) = \text{limsup}^*_{x\to\bar{x}} F(x) \cup \text{limsup}^*_{x\to\bar{x}} G(x),$$

$$\text{limsupo}_{x\to\bar{x}}(F \cup G)(x) \subset \text{limsupo}_{x\to\bar{x}} F(x) \cup \text{limsup}_{x\to\bar{x}} G(x).$$

Proof (i) The inclusion

$$\lim\sharp_{x\to\bar{x}}(F \cap G)(x) \subset \lim\sharp_{x\to\bar{x}} F(x) \cap \lim\sharp_{x\to\bar{x}} G(x)$$

for ♯ being 'sup', 'supo', 'inf', or 'info' and the equality for the inferior open limit are clear (cf. Lemma 2.4 [7]). The proof of the inclusion

$$\mathrm{limsup}^*_{x\to\bar{x}}(F\cap G)(x)\subset \mathrm{limsup}^*_{x\to\bar{x}}F(x)\cap \mathrm{limsup}^*_{x\to\bar{x}}G(x)$$

is direct by checking the definition. For showing the equality

$$\mathrm{liminf}^*_{x\to\bar{x}}(F\cap G)(x)=\mathrm{liminf}^*_{x\to\bar{x}}F(x)\cap \mathrm{liminf}^*_{x\to\bar{x}}G(x),$$

first let y belong to the left-hand side, i.e., there exists a neighborhood U of $\bar{x}$ such that $y\in(F\cap G)(x)=F(x)\cap G(x)$ for all $x\in U$. Thus, y belongs to the right-hand side. Let y now be in the right-hand side. There are two neighborhoods U_1 and U_2 of $\bar{x}$ such that $y\in F(x)$ for all $x\in U_1$ and $y\in G(x)$ for all $x\in U_2$. Then, $y\in F(x)\cap G(x)$ for all $x\in U:=U_1\cap U_2$. Thus, y belongs to the left-hand side.

Passing to the inclusion

$$\mathrm{liminf}_{x\to\bar{x}}F(x)\cap \mathrm{liminfo}_{x\to\bar{x}}G(x)\subset \mathrm{liminf}_{x\to\bar{x}}(F\cap G)(x),$$

let y be in the left-hand side. For any net $x_\alpha\to\bar{x}$, because $y\in \mathrm{liminf}_{x\to\bar{x}}F(x)$, there is $y_\alpha\in F(x_\alpha)$ such that $y_\alpha\to y$. Since $y\in \mathrm{liminfo}_{x\to\bar{x}}G(x)$, there are $U\in\mathcal{N}(\bar{x})$ and $V\in\mathcal{N}(y)$ such that $V\subset G(x)$ for all $x\in U$. Without loss of generality we may assume that $(x_\alpha,y_\alpha)\in U\times V$ for all α. This implies that $y_\alpha\in F(x_\alpha)\cap G(x_\alpha)$ and converging to y. Thus, y belongs to the right-hand side.

(ii) The containment

$$\mathrm{lim}\sharp_{x\to\bar{x}}(F\cup G)(x)\supset \mathrm{lim}\sharp_{x\to\bar{x}}F(x)\cup \mathrm{lim}\sharp_{x\to\bar{x}}G(x)$$

for ♯ being 'sup', 'supo', 'inf', or 'info', and the equality for the outer limit are easy to check (cf. Lemma 2.4 [7]). Let us prove the equality

$$\mathrm{limsup}^*_{x\to\bar{x}}(F\cup G)(x)=\mathrm{limsup}^*_{x\to\bar{x}}F(x)\cup \mathrm{limsup}^*_{x\to\bar{x}}G(x).$$

Let $y\in \mathrm{limsup}^*_{x\to\bar{x}}F(x)$, i.e., there exists a net $\{x_\alpha\}$ converging to $\bar{x}$ such that $y\in F(x_\alpha)$ for all α. Hence, $y\in(F\cup G)(x_\alpha)$ for all α. Thus, y belongs to the left-hand side. The case $y\in \mathrm{limsup}^*_{x\to\bar{x}}G(x)$ is similar. Let now $y\in \mathrm{limsup}^*_{x\to\bar{x}}(F\cup G)(x)$, i.e., there exists $\{x_\alpha\}$ converging to $\bar{x}$ such that $y\in F(x_\alpha)\cup G(x_\alpha)$ for all α. Therefore, there exists a subnet $\{x_{\alpha_\beta}\}$ such that $y\in F(x_{\alpha_\beta})$ for all β or $y\in G(x_{\alpha_\beta})$ for all β. Then, $y\in \mathrm{limsup}^*_{x\to\bar{x}}F(x)$ or $y\in \mathrm{limsup}^*_{x\to\bar{x}}G(x)$. Thus, y belongs to the right-hand side. The inclusion

$$\mathrm{liminf}^*_{x\to\bar{x}}(F\cup G)(x)\supset \mathrm{liminf}^*_{x\to\bar{x}}F(x)\cup \mathrm{liminf}^*_{x\to\bar{x}}G(x)$$

can also be verified by definition.

Finally, we check the inclusion

$$\mathrm{limsupo}_{x\to\bar{x}}(F\cup G)(x)\subset \mathrm{limsupo}_{x\to\bar{x}}F(x)\cup \mathrm{limsup}_{x\to\bar{x}}G(x).$$

If y lies in the left-hand side, there exist $V\in\mathcal{N}(y)$ and a net $\{x_\alpha\}$ converging to $\bar{x}$ such that $V\subset F(x_\alpha)\cup G(x_\alpha)$ for all α. If y belongs to $\mathrm{limsup}_{x\to\bar{x}}G(x)$, then we are done. If not, in view of Lemma 2.1(3) of [7], y belongs to $\mathrm{liminfo}_{x\to\bar{x}}G^c(x)$, which means that there are neighborhoods W of y and U of $\bar{x}$ such that $W\subset G^c(x)$ for all $x\in U$. Since $V\subset F(x_\alpha)\cup G(x_\alpha)$ and $W\subset G^c(x_\alpha)$ for all α, then $V\cap W\subset F(x_\alpha)$. Thus, $y\in \mathrm{limsupo}_{x\to\bar{x}}F(x)$. □

The following three examples explain the limitations of several inclusions/equalities in Proposition 5.

Example 7 (the equality in Proposition 5(i) fails for $\sharp$ being 'sup*'). Let $F, G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} (-1, x) & if \quad x \geq 0, \\ (0, 1) & if \quad x < 0, \end{cases} \qquad G(x) = \begin{cases} (-1, 0) & if \quad x \geq 0, \\ (x, 1) & if \quad x < 0. \end{cases}$$

Then, $\text{limsup}^*_{x\to 0} F(x) = \text{limsup}^*_{x\to 0} G(x) = (-1, 1)$ and $\text{limsup}^*_{x\to 0}(F \cap G)(x) = (-1, 0) \cup (0, 1)$. Hence,

$$\text{limsup}^*_{x\to 0}(F \cap G)(x) \not\subset \text{limsup}^*_{x\to 0} F(x) \cap \text{limsup}^*_{x\to 0} G(x).$$

Example 8 (the equality in Proposition 5(ii) fails for $\sharp$ being 'inf*'). Let $F, G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} [0, 2] & if \quad x \geq 0, \\ [1, 2] & if \quad x < 0, \end{cases} \qquad G(x) = \begin{cases} [1, 2] & if \quad x \geq 0, \\ [0, 2] & if \quad x < 0. \end{cases}$$

Then, $\text{liminf}^*_{x\to 0} F(x) = \text{liminf}^*_{x\to 0} G(x) = [1, 2]$ and $\text{liminf}^*_{x\to 0}(F \cup G)(x) = [0, 2]$. Hence,

$$\text{liminnf}^*_{x\to 0}(F \cup G)(x) \not\supset \text{liminf}^*_{x\to 0} F(x) \cup \text{liminf}^*_{x\to 0} G(x).$$

Example 9 Related to Proposition 5(i), we show a case where

$$\text{liminf}_{x\to \bar{x}} F(x) \cap \text{lim}\sharp_{x\to \bar{x}} G(x) \not\subset \text{liminf}_{x\to \bar{x}}(F \cap G)(x)$$

for $\sharp$ being 'inf*' or 'inf'. Let $F, G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $F(x) = (-\infty, -1] \cup [1 - 3^{-|x|}, +\infty)$ for $x \in \mathbb{R}$ and

$$G(x) = \begin{cases} (-\infty, 1 - 2^{-|x|}] \cup [1, +\infty) & if \quad x \neq 0, \\ (-\infty, 0.5] \cup [1, +\infty) & if \quad x = 0. \end{cases}$$

We have

$$(F \cap G)(x) = \begin{cases} (-\infty, -1] \cup [1, +\infty) & if \quad x \neq 0, \\ (-\infty, -1] \cup [0, 0.5] \cup [1, +\infty) & if \quad x = 0. \end{cases}$$

Then, $\text{liminf}_{x\to 0} F(x) = (-\infty, -1] \cup [0, +\infty)$ and $\text{liminf}_{x\to 0} G(x) = \text{liminf}^*_{x\to 0} G(x) = (-\infty, 0] \cup [1, +\infty)$. Hence,

$$\text{liminf}_{x\to 0} F(x) \cap \text{liminf}_{x\to 0} G(x) = (-\infty, -1] \cup \{0\} \cup [1, +\infty),$$

$$\text{liminf}_{x\to 0} F(x) \cap \text{liminf}^*_{x\to 0} G(x) = (-\infty, -1] \cup \{0\} \cup [1, +\infty).$$

Since $\text{liminf}_{x\to 0}(F \cap G)(x) = (-\infty, -1] \cup [1, +\infty)$, the mentioned inclusion does not holds for $\sharp$ being 'inf*' or 'inf' in this case.

Example 10 Related to Proposition 5(ii), we show a case where

$$\text{limsupo}_{x\to \bar{x}}(F \cup G)(x) \not\subset \text{limsupo}_{x\to \bar{x}} F(x) \cup \text{lim}\sharp_{x\to \bar{x}} G(x)$$

for $\sharp$ being 'limsupo' or 'limsup*'. Let $F, G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $F(x) \equiv (-1, 0]$ and $G(x) \equiv (0, 1)$ for $x \in \mathbb{R}$. We have $(F \cup G)(x) = (-1, 1)$ for all $x \in \mathbb{R}$. Then, $\text{limsupo}_{x\to 0} F(x) = (-1, 0)$, $\text{limsupo}_{x\to 0}(F \cup G)(x) = (-1, 1)$, and $\text{limsupo}_{x\to 0} G(x) = \text{limsup}^*_{x\to 0} G(x) = (0, 1)$. Hence,

$$\text{limsupo}_{x\to \bar{x}}(F \cup G)(x) \not\subset \text{limsupo}_{x\to \bar{x}} F(x) \cup \text{lim}\sharp_{x\to \bar{x}} G(x)$$

for $\sharp$ being 'limsupo' or 'limsup*'.

The following statement follows from Proposition 5(i).

Proposition 6 *The following assertions hold.*

(i) *If F and G are outer semicontinuous, star-outer semicontinuous, outer open, inner open, or star-inner semicontinuous at $\bar{x}$, then so is their intersection.*
(ii) *If F is inner semicontinuous and G is inner open at $\bar{x}$, then their intersection is inner semicontinuous at $\bar{x}$.*

Example 11 (Proposition 6(ii) is no longer true if the inner openness of G is replaced by star-inner semicontinuity or inner semicontinuity). Let $F, G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $F(x) = (-\infty, -1] \cup [1 - 2^{-|x|}, +\infty)$ and $G(x) = (-\infty, 0] \cup [1, +\infty)$ for all $x \in \mathbb{R}$. We have

$$(F \cap G)(x) = \begin{cases} (-\infty, -1] \cup [1, +\infty) & if \quad x \neq 0, \\ (-\infty, -1] \cup \{0\} \cup [1, +\infty) & if \quad x = 0. \end{cases}$$

F is inner semicontinuous at 0 but $F \cap G$ is not, since $\mathrm{liminf}_{x\to 0}(F \cap G)(x) = (-\infty, -1] \cup [1, +\infty) \not\supset (F \cap G)(0)$. The reason is that G is not inner open at 0 ($\mathrm{liminfo}_{x\to 0}G(x) = (-\infty, 0) \cup (1, +\infty) \not\supset G(0)$). Observe that G is both star-inner semicontinuous and inner continuous at 0 (since $\mathrm{liminf}^*_{x\to 0}G(x) = \mathrm{liminf}_{x\to 0}G(x) = G(0) = (-\infty, 0] \cup [1, +\infty)$).

From Proposition 5(ii), we easily obtain the following statement.

Proposition 7 *The following assertions hold.*

(i) *If F and G are outer semicontinuous, star-outer semicontinuous, inner open, inner semicontinuous, or star-inner semicontinuous at $\bar{x}$, then so is their union.*
(ii) *If F is outer open and G is outer semicontinuous at $\bar{x}$, then their union is outer open at $\bar{x}$.*

Example 12 (the outer openness in Proposition 7(ii) does not hold if the outer semicontinuity of G is replaced by star-outer semicontinuity or outer openness). Let $F, G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $G(x) = (0, 1)$ for $x \in \mathbb{R}$ and

$$F(x) = \begin{cases} (-1, 0] & \text{if} \quad x \neq 0, \\ (-1, 0) & \text{if} \quad x = 0. \end{cases}$$

Then,

$$(F \cup G)(x) = \begin{cases} (-1, 1) & \text{if} \quad x \neq 0, \\ (-1, 0) \cup (0, 1) & \text{if} \quad x = 0. \end{cases}$$

Clearly F is outer open at 0 but $F \cup G$ is not, since $\mathrm{limsupo}_{x\to 0}(F \cup G)(x) = \mathrm{limsup}^*_{x\to 0}(F \cup G)(x) = (-1, 1) \not\subset (F \cup G)(0)$. The cause is that G is not outer semicontinuous at 0 ($\mathrm{limsup}_{x\to 0}G(x) = [0, 1] \not\subset G(0)$). However, in this case, G is both star-outer semicontinuous and outer open at 0 (since $\mathrm{limsupo}_{x\to 0}G(x) = \mathrm{limsup}^*_{x\to 0}G(x) = G(0) = (0, 1)$).

Proposition 8 *The following assertions hold.*

(i) *If F is outer semicontinuous (resp, star-outer semicontinuous, outer open) at $\bar{x}$ and if* $\operatorname{limsup}_{x\to\bar{x}} G(x) \cap F(\bar{x}) \subset G(\bar{x})$ *(resp,* $\operatorname{limsup}^*_{x\to\bar{x}} G(x) \cap F(\bar{x}) \subset G(\bar{x})$, $\operatorname{limsupo}_{x\to\bar{x}} G(x) \cap F(\bar{x}) \subset G(\bar{x})$*), then $F \cap G$ is outer semicontinuous (resp, star-outer semicontinuous, outer open) at $\bar{x}$.*

(ii) *If F is star-inner semicontinuous (resp, inner open) at $\bar{x}$ and if* $\operatorname{liminf}^*_{x\to\bar{x}} G(x) \supset G(\bar{x}) \cap F(\bar{x})$ *(resp,* $\operatorname{liminfo}_{x\to\bar{x}} G(x) \supset G(\bar{x}) \cap F(\bar{x})$*), then $F \cap G$ is star-inner semicontinuous (resp, inner open) at $\bar{x}$.*

(iii) *If F is inner semicontinuous at $\bar{x}$ and if* $\operatorname{liminfo}_{x\to\bar{x}} G(x) \supset G(\bar{x}) \cap F(\bar{x})$, *then $F \cap G$ is inner semicontinuous at $\bar{x}$.*

Proof (i) By Proposition 5(i), we have

$$\operatorname{limsup}_{x\to\bar{x}} (F \cap G)(x) \subset \operatorname{limsup}_{x\to\bar{x}} F(x) \cap \operatorname{limsup}_{x\to\bar{x}} G(x)$$

$$\subset F(\bar{x}) \cap \operatorname{limsup}_{x\to\bar{x}} G(x) \subset F(\bar{x}) \cap G(\bar{x}),$$

where the second inclusion is due to the outer semicontinuity of F and the last one follows from the hypothesis on G. The proof for the star-outer semicontinuity and outer openness is similar.

(ii) Also from Proposition 5(i), we have

$$\operatorname{liminf}^*_{x\to\bar{x}} (F \cap G)(x) = \operatorname{liminf}^*_{x\to\bar{x}} F(x) \cap \operatorname{liminf}^*_{x\to\bar{x}} G(x)$$

$$\supset F(\bar{x}) \cap \operatorname{liminf}^*_{x\to\bar{x}} G(x) \supset F(\bar{x}) \cap G(\bar{x}),$$

where the second containment is obtained from the star-inner semicontinuity of F and the last one follows from the hypothesis on G. The proof for the inner openness is similar.

(iii) Proposition 5(i) implies also that

$$\operatorname{liminf}_{x\to\bar{x}} (F \cap G)(x) \supset \operatorname{liminf}_{x\to\bar{x}} F(x) \cap \operatorname{liminfo}_{x\to\bar{x}} G(x)$$

$$\supset F(\bar{x}) \cap \operatorname{liminfo}_{x\to\bar{x}} G(x) \supset F(\bar{x}) \cap G(\bar{x}),$$

where the second containment is obtained from the inner semicontinuity of F and the last one from the hypothesis on G.

□

Example 13 Proposition 6(iii) is no longer true if the inclusion $\operatorname{liminfo}_{x\to\bar{x}} G(x) \supset G(\bar{x}) \cap F(\bar{x})$ is replaced by $\lim\sharp_{x\to\bar{x}} G(x) \supset G(\bar{x}) \cap F(\bar{x})$ for $\sharp$ being 'inf*' or 'inf'. Indeed, let $F, G : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $F(x) = (-\infty, -1] \cup [1 - 2^{-|x|}, +\infty)$ and $G(x) = (-\infty, 0] \cup [1, +\infty)$ for $x \in \mathbb{R}$. We have

$$(F \cap G)(x) = \begin{cases} (-\infty, -1] \cup [1, +\infty) & \text{if} \quad x \neq 0, \\ (-\infty, -1] \cup \{0\} \cup [1, +\infty) & \text{if} \quad x = 0. \end{cases}$$

Then, it is easy to see that F is inner semicontinuous at 0 but $F \cap G$ is not, since $\operatorname{liminf}_{x\to 0} (F \cap G)(x) = (-\infty, -1] \cup [1, +\infty) \not\supset (F \cap G)(0)$. The cause is that $\operatorname{liminfo}_{x\to 0} G(x) = (-\infty, 0) \cup (1, +\infty) \not\supset G(0) \cap F(0)$. Although $\lim\sharp_{x\to 0} G(x) = (-\infty, 0] \cup [1, +\infty) \supset G(0) \cap F(0)$ for $\sharp$ being 'inf*' or 'inf'.

3 Upper Semicontinuity Properties of Solution Maps

For $\lambda \in \Lambda$ we denote the set of solutions of (QVIP$_\lambda$) by $S(\lambda)$. Let $E(\lambda) := \{x \in X : x \in K_1(x,\lambda)\}$. Throughout the paper assume that $S(\lambda) \neq \emptyset$ and $E(\lambda) \neq \emptyset$ for all mentioned λ in a neighborhood of $\bar{\lambda} \in \Lambda$. In this section, we investigate sufficient conditions for $S(\cdot)$ to satisfy various upper semicontinuity properties.

Theorem 1 *Impose for* (QVIP)$_{\lambda\in\Lambda}$ *that*

(i) *$K_2(x,.)$ is lsc at $\bar{\lambda}$ for all $x \in E(\bar{\lambda})$;*
(ii) *$F(x,.,.)$ has the 0-inclusion property in $K_2(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$ for all $x \in E(\bar{\lambda})$.*

If E is outer open or star-outer semicontinuous at $\bar{\lambda}$, then so is S.

Proof By the similarity, we consider only the case of star-outer semicontinuity. Let $x \in \limsup^*_{\lambda\to\bar{\lambda}} S(\lambda)$. There is $\{\lambda_\alpha\} \subset \Lambda$ converging to $\bar{\lambda}$ such that $x \in S(\lambda_\alpha)$ for all α. As $x \in E(\lambda_\alpha)$, the star-outer semicontinuity of E implies that $x \in E(\bar{\lambda})$. We claim that $x \in S(\bar{\lambda})$. Indeed, for $y \in K_2(x,\bar{\lambda})$, the lower semicontinuity of $K_2(x,.)$ at $\bar{\lambda}$ yields $y_\alpha \in K_2(x,\lambda_\alpha)$ such that $y_\alpha \to y$. Since $0 \in F(x,y_\alpha,\lambda_\alpha)$, it follows from (ii) that $0 \in F(x,y,\bar{\lambda})$. □

Theorem 2 *If E is outer semicontinuous at $\bar{\lambda}$, so is S, provided that*

(i) *K_2 is lsc in $E(\bar{\lambda}) \times \{\bar{\lambda}\}$;*
(ii) *F has the 0-inclusion property in $E(\bar{\lambda}) \times K_2(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$.*

Proof Let $x \in \limsup_{\lambda\to\bar{\lambda}} S(\lambda)$. There are nets $\{\lambda_\alpha\}$ converging to $\bar{\lambda}$ and $\{x_\alpha\}$ converging to x with $x_\alpha \in S(\lambda_\alpha)$. By the outer semicontinuity of E, $x \in E(\bar{\lambda})$. To see that $x \in S(\bar{\lambda})$, let $y \in K_2(x,\bar{\lambda})$. The lower semicontinuity of K_2 in $E(\bar{\lambda}) \times \{\bar{\lambda}\}$ implies the existence of $y_\alpha \in K_2(x_\alpha,\lambda_\alpha)$ with $y_\alpha \to y$. Because $0 \in F(x_\alpha,y_\alpha,\lambda_\alpha)$, (ii) implies that $x \in S(\bar{\lambda})$. □

Theorem 3 *The solution map S of* (QVIP)$_{\lambda\in\Lambda}$ *is both usc and closed at $\bar{\lambda}$, if*

(i) *K_2 is lsc in $E(\bar{\lambda}) \times \{\bar{\lambda}\}$;*
(ii) *F has the 0-inclusion property in $E(\bar{\lambda}) \times K_2(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$;*
(iii) *E is usc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is compact.*

Proof Suppose there is an open set $U \supset S(\bar{\lambda})$ such that $\forall\lambda_\alpha \to \bar{\lambda}$, $\exists x_\alpha \in S(\lambda_\alpha)$, $\forall\alpha$, $x_\alpha \notin U$. By the upper semicontinuity of E and the compactness of $E(\bar{\lambda})$, one can assume that $x_\alpha \to x \in E(\bar{\lambda})$. We claim that $x \in S(\bar{\lambda})$. Indeed, for $y \in K_2(x,\bar{\lambda})$, the lower semicontinuity of K_2 in $E(\bar{\lambda}) \times \{\bar{\lambda}\}$ yields $y_\alpha \in K_2(x_\alpha,\lambda_\alpha)$ with $y_\alpha \to y$. Since $0 \in F(x_\alpha,y_\alpha,\lambda_\alpha)$, (ii) gives that $x \in S(\bar{\lambda}) \subset U$, which is a contradiction, since $x_\alpha \notin U$, for all α. Now let $(\lambda_\alpha,x_\alpha) \to (\bar{\lambda},x)$ with $x_\alpha \in S(\lambda_\alpha)$. Arguing similarly as above, we see that $x \in S(\bar{\lambda})$. □

Remark 3 Assumption (iii) in Theorem 3 can be replaced by the condition (directly in terms of the problem data) that X is compact, K_1 is usc and closed-valued in $X \times \{\bar{\lambda}\}$. Indeed, let $x_\alpha \in E(\lambda_\alpha)$ and $\lambda_\alpha \to \bar{\lambda}$. We need in the proof of Theorem 3 that $x_\alpha \to x$ for some $x \in E(\lambda)$. Suppose $x_\alpha \to x \notin K_1(x,\bar{\lambda})$. Because $K_1(x,\bar{\lambda})$ is closed, there are neighborhoods N of x and V of $K_1(x,\bar{\lambda})$ such that $N \cap V = \emptyset$. Since K_1 is usc at $(x,\bar{\lambda})$, without loss of generality we may assume that $K_1(x_\alpha,\lambda_\alpha) \subset V$ for each α. Then, we have $x_\alpha \in K_1(x_\alpha, \lambda_\alpha) \subset V$ and hence $x_\alpha \notin N$ for each α, contradicting the convergence $x_\alpha \to x$.

The outer openness (resp, start-outer semicontinuity, outer semicontinuity) assumption of the mapping E in Theorems 1 and 2 can be replaced by the condition that the mapping K_1 is outer open (resp, start-outer semicontinuous, outer semicontinuous) in $X \times \{\bar{\lambda}\}$. By the similarity, we check only the outer openness. Indeed, let $\bar{x} \in \text{limsupo}_{\lambda \to \bar{\lambda}} E(\lambda)$. Then, $\bar{x} \in \text{limsupo}_{(x,\lambda)\to(\bar{x},\bar{\lambda})} K_1(x, \lambda)$. Since the outer openness of K_1 implies that $\bar{x} \in K_1(\bar{x}, \bar{\lambda})$, i.e., $\bar{x} \in E(\bar{\lambda})$.

The following example indicates that the assumptions on outer semicontinuity of the mapping E in Theorems 4 and 5may be satisfied even when neither outer continuity nor another upper semicontinuity of the mapping K_1 is fulfilled.

Example 14 Let $X = Z = \mathbb{R}$, $\Lambda = [0, 1]$, $K_2(x, \lambda) \equiv [0, 1]$, $\bar{\lambda} = 0$,

$$K_1(x, \lambda) = \begin{cases} \{x\} & \text{if } \lambda = 0, \\ [x - 1, x + 1] & \text{otherwise,} \end{cases}$$

and

$$F(x, y, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ [-1, 1] & \text{otherwise.} \end{cases}$$

Direct computations yield $E(\lambda) = (-\infty, +\infty)$ for all $\lambda \in [0, 1]$, and hence E is outer open, star-outer semicontinuous, outer semicontinuous, usc and closed at 0. Then, the assumptions of Theorems 4 and 5 are satisfied and, according to them, S is outer open, star-outer semicontinuous, outer semicontinuous, usc and closed at 0 (in fact $S(\lambda) = (-\infty, +\infty)$ for all $\lambda \in [0, 1]$). Checking directly, we see that K_1 is neither outer open, nor star-outer semicontinuous, nor outer semicontinuous, and nor usc in $X \times \{0\}$, and F is neither outer open, nor star-outer open, nor star-outer semicontinuous, nor outer semicontinuous, nor usc at $(0, 0, 0)$.

The following example shows that assumption (ii) in Theorems 1–3 may be satisfied even when neither outer continuity nor other upper semicontinuity of F is fulfilled.

Example 15 Let $X = Z = \mathbb{R}$, $\Lambda = [0, 1]$, $K_1(x, \lambda) \equiv K_2(x, \lambda) \equiv [0, 1]$, $\bar{\lambda} = 0$, and

$$F(x, y, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ [-1, 1] & \text{otherwise.} \end{cases}$$

Then, it is not hard to see that all the assumptions of Theorems 1–3 are satisfied and, accordingly, S is outer open, star-outer semicontinuous, outer semicontinuous, usc and closed at 0 (in fact $S(\lambda)$=[0,1] for all $\lambda \in [0, 1]$). One easily checks that F is neither outer open, nor star-outer open, nor star-outer semicontinuous, nor outer semicontinuous, nor usc at $(0, 0, 0)$.

The following three examples illustrate Theorems 1 and 3.

Example 16 Let $X = Z = \mathbb{R}$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = (-1, \lambda)$, $K_2(x, \lambda) \equiv [0, 1]$, $\bar{\lambda} = 0$, and

$$F(x, y, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ [-1, 1] & \text{otherwise.} \end{cases}$$

We have $E(\lambda) = (-1, \lambda)$ for $\lambda \in [0, 1]$. Hence, E is outer-open (but neither star-outer semicontinuous, nor outer semicontinuous, nor usc) at 0. It is not hard to see that all the assumptions in Theorem 1 are satisfied and, according to it, S is outer open at 0 (in fact $S(\lambda) = (-1, \lambda)$ for all $\lambda \in [0, 1]$). Evidently in this case, S is neither star-outer semicontinuous, nor outer semicontinuous, nor usc at 0.

Example 17 Let $X = \mathbb{R}^2$, $Z = \mathbb{R}$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = \{(t, \lambda t) : t > 0\}$, $K_2(x, \lambda) \equiv [0, 1] \times [0, 1]$, $\bar{\lambda} = 0$, and

$$F(x, y, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ [-1, 1] & \text{otherwise.} \end{cases}$$

Since $E(\lambda) = \{(t, \lambda t) : t > 0\}$ for $\lambda \in [0, 1]$, E is star-outer semicontinuous (but neither outer semicontinuous nor usc) at 0. It is not hard to see that all the assumptions in Theorem 1 are satisfied and, according to this statement, S is star-outer semicontinuous at 0 (in fact $S(\lambda) = \{(t, \lambda t) : t > 0\}$ for $\lambda \in [0, 1]$). Evidently in this case, S is neither outer semicontinuous nor usc at 0.

Example 18 Let $X = \mathbb{R}^2$, $Z = \mathbb{R}$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = \{(t, \lambda t) : t \in \mathbb{R}\}$, $K_2(x, \lambda) \equiv [0, 1] \times [0, 1]$, $\bar{\lambda} = 0$, and

$$F(x, y, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ [-1, 1] & \text{otherwise.} \end{cases}$$

Then, E is outer semicontinuous (but not usc) at 0, since $E(\lambda) = \{(t, \lambda t) : t \in \mathbb{R}\}$ for $\lambda \in [0, 1]$. It is not hard to see that all the assumptions in Theorem 2 are satisfied and, accordingly, S is outer semicontinuous at 0 (in fact $S(\lambda) = \{(t, \lambda t) : t \in \mathbb{R}\}$ for $\lambda \in [0, 1]$). Evidently in this case, S is not usc at 0.

4 Lower Semicontinuity Properties of Solution Maps

Theorem 4 *For* $(\mathrm{QVIP})_{\lambda \in \Lambda}$, *if E is inner open or lsc at $\bar{\lambda}$, then so is S, provided that*

(i) *K_2 is usc and compact-valued in $E(\bar{\lambda}) \times \{\bar{\lambda}\}$;*
(ii) *F has the 0-inclusion complement property in $E(\bar{\lambda}) \times K_2(E(\bar{\lambda}), \bar{\lambda}) \times \{\bar{\lambda}\}$.*

Proof By the similarity, we check only the inner openness. Suppose to the contrary there exists $x \in S(\bar{\lambda})$ such that $x \notin \mathrm{liminfo}_{\lambda \to \bar{\lambda}} S(\lambda)$. As $x \in E(\bar{\lambda})$, by the inner openness of E, one has $x \in \mathrm{liminfo}_{\lambda \to \bar{\lambda}} E(\lambda)$. Then, $\exists U \in \mathcal{N}(\bar{\lambda})$, $\exists V \in \mathcal{N}(x)$, $\forall \lambda \in U$, $V \subset E(\lambda)$. Since $\mathrm{liminfo}_{\lambda \to \bar{\lambda}} S(\lambda) = [\limsup_{\lambda \to \bar{\lambda}} S^c(\lambda)]^c$, $x \in \limsup_{\lambda \to \bar{\lambda}} S^c(\lambda)$. Therefore, there exist a net λ_α converging to $\bar{\lambda}$ and a net $x_\alpha \in S^c(\lambda_\alpha)$ converging x. We can assume that $(\lambda_\alpha, x_\alpha) \in U \times V$ for all α, and hence $x_\alpha \in E(\lambda_\alpha)$. Then, there is $y_\alpha \in K_2(x_\alpha, \lambda_\alpha)$ such that $0 \notin F(x_\alpha, y_\alpha, \lambda_\alpha)$. As K_2 is usc at $(x, \bar{\lambda})$ and $K_2(x, \bar{\lambda})$ is compact, one finds $y \in K_2(x, \bar{\lambda})$ such that $y_\alpha \to y$ (taking a subnet). As $x \in S(\bar{\lambda})$, we have $0 \in F(x, y, \bar{\lambda})$. Assumption (ii) implies the existence of $\bar{\alpha}$ such that $0 \in F(x_{\bar{\alpha}}, y_{\bar{\alpha}}, \lambda_{\bar{\alpha}})$, a contradiction. □

Theorem 5 *The star-inner semicontinuity of E at $\bar{\lambda}$ implies the same property for S, if*

(i) *$K_2(x, .)$ is usc at $\bar{\lambda}$ and $K_2(x, \bar{\lambda})$ is compact for all $x \in E(\bar{\lambda})$;*

(ii) $F(x,.,.)$ *has the 0-inclusion complement property in* $K_2(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$ *for all* $x \in E(\bar{\lambda})$.

Proof Suppose to the contrary the existence of $x \in S(\bar{\lambda})$ such that $x \notin \liminf^*_{\lambda\to\bar{\lambda}} S(\lambda)$. Then, there exists λ_α converging $\bar{\lambda}$ such that $x \notin S(\lambda_\alpha)$ for all α. The star-inner semicontinuity of E implies that $x \in \liminf^*_{\lambda\to\bar{\lambda}} E(\lambda)$. Hence, there exists a neighborhood U of $\bar{\lambda}$ such that $x \in K_1(x,\lambda)$ for all $\lambda \in U$. Assuming that $\lambda_\alpha \in U$ for all α, one has $x \in K_1(x,\lambda_\alpha)$ and $x \notin S(\lambda_\alpha)$ for all α. Therefore, there exists $y_\alpha \in K_2(x,\lambda_\alpha)$ with $0 \notin F(x,y_\alpha,\lambda_\alpha)$. Since $K_2(x,.)$ is usc at $\bar{\lambda}$ and $K_2(x,\bar{\lambda})$ is compact, one has $y \in K_2(x,\bar{\lambda})$ such that $y_\alpha \to y$ (taking a subnet if necessary). As $x \in S(\bar{\lambda})$, we have $0 \in F(x,y,\bar{\lambda})$. Assumption (ii) yields some $\bar{\alpha}$ such that $0 \in F(x,y_{\bar{\alpha}},\lambda_{\bar{\alpha}})$, a contradiction. □

The following example indicates that assumption (ii) and the assumption on inner openness and inner semicontinuity of the mapping E in Theorems 4 and 5 may be satisfied even when many properties related to inner semicontinuity of K_1 and F are not fulfilled.

Example 19 Let $X = Z = \mathbb{R}$, $\Lambda = [0,1]$, $K_2(x,\lambda) \equiv [0,1]$, $\bar{\lambda} = 0$,

$$K_1(x,\lambda) = \begin{cases} [x-1, x+1] & \text{if } \lambda = 0, \\ \{x\} & \text{otherwise,} \end{cases}$$

and

$$F(x,y,\lambda) = \begin{cases} [-1,1] & \text{if } \lambda = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

Direct computations yield $E(\lambda) = (-\infty,+\infty)$ for all $\lambda \in [0,1]$, and hence E is inner open, star-inner semicontinuous and lsc at 0. Then, the assumptions of Theorems 4 and 5 are satisfied and, according to them, S is inner open, star-inner semicontinuous and lsc at 0 (in fact $S(\lambda) = (-\infty,+\infty)$ for all $\lambda \in [0,1]$). Checking directly we see that K_1 is neither inner open, nor star-inner semicontinuous, nor lsc in $X \times \{0\}$ and F is neither inner open, nor star-inner semicontinuous, nor lsc at $(0,0,0)$.

The following example ensures us that the inner semicontinuity assumption on the mapping E in Theorems 4 and 5 is essential.

Example 20 Let $X = Z = \mathbb{R}$, $\Lambda = [0,1]$, $K_2(x,\lambda) \equiv [0,1]$, $\bar{\lambda} = 0$,

$$K_1(x,\lambda) = \{(\lambda+1)x\},$$

and

$$F(x,y,\lambda) = \begin{cases} [-1,1] & \text{if } \lambda = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

Then, it is easy to verify that (i), (ii) of Theorems 4 and 5 are satisfied. But, $S(0) = (-\infty,+\infty)$ and $S(\lambda) = \{0\}$ for all $\lambda \in (0,1]$, and thus S is neither inner open, nor star-inner semicontinuous, nor lsc at 0. The cause is that the assumed inner semicontinuity of the mapping E is violated (in fact $E(0) = (-\infty,+\infty)$ and $E(\lambda) = \{0\}$ for all $\lambda \in (0,1]$). By direct checking, we see that K_1 is star-inner continuous and lsc in in $X \times \Lambda$.

To develop other conditions for lower semicontinuity of S, which are more suitable than the above results in some cases, we need the following definition. $G : X \times X \to 2^Z$ is called generalized 0-convex in a convex set $A \subset X$ if, for all $x, y_1, y_2 \in A$, from $0 \in G(x, y_1)$ and $0 \in \mathrm{int}G(x, y_2)$, it follows that $0 \in \mathrm{int}G(x, (1-t)y_1 + ty_2)$ for all $t \in (0, 1)$.

Note that this is a modification of the generalized Δ-concavity defined in Definition 2.1 of [15]. Indeed, let $g : X \times X \to Z$ be a single-valued map, $\Delta : X \to 2^Z$, and $A \subset X$. Set $H(x, y) := g(x, y) - \Delta(x)$. Then, H is generalized 0-convex in A if and only if g is generalized Δ-concave in A. We use the term "convex" instead of "concave" to suit the following known definition. $G : X \to 2^Z$ is said to be convex (concave) in $A \subset X$ if, for each $x, y \in A$ and $t \in [0, 1]$, $(1-t)G(x) + tG(y) \subset G((1-t)x + ty)$ $(G((1-t)x + ty) \subset (1-t)G(x) + tG(y)$, resp).

We consider also the following problem $(\widetilde{QVIP}_\lambda)$ as auxiliary to (QVIP_λ)

$$(\widetilde{\mathrm{QVIP}}_\lambda): \text{ find } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for each } y \in K_2(\bar{x}, \lambda),\ 0 \in \mathrm{int}\, F(\bar{x}, y, \lambda).$$

Let $\widetilde{S}(\lambda)$ be the solution set of $(\widetilde{\mathrm{QVIP}}_\lambda)$. Clearly $\widetilde{S}(\lambda) \subset S(\lambda)$.

Theorem 6 *Assume for problem* $(\widetilde{\mathrm{QVIP}}_\lambda)$ *that* $\widetilde{S}(\lambda) \neq \emptyset$ *in a neighborhood of* $\bar{\lambda} \in \Lambda$ *and*

(i) *K_2 is usc and has the compact values in $E(\bar{\lambda}) \times \{\bar{\lambda}\}$; $K_2(., \bar{\lambda})$ is concave in $E(\bar{\lambda})$;*
(ii) *int$F$ has the 0-inclusion complement property in $E(\bar{\lambda}) \times K_2(E(\bar{\lambda}), \bar{\lambda}) \times \{\bar{\lambda}\}$;*
(iii) *E is lsc at $\bar{\lambda}$ and $E(\bar{\lambda})$ is convex;*
(iv) *$F(., ., \bar{\lambda})$ is generalized 0-convex in $E(\bar{\lambda}) \times K_2(E(\bar{\lambda}), \bar{\lambda})$.*

Then, S is lsc at $\bar{\lambda}$.

Proof First, we prove that $\widetilde{S}$ is lsc at $\bar{\lambda}$. Suppose to the contrary that $\exists x \in \widetilde{S}(\bar{\lambda})$, $\exists \lambda_\alpha \to \bar{\lambda}$, $\forall x_\alpha \in \widetilde{S}(\lambda_\alpha)$, $x_\alpha \not\to x$. Since E is lsc at $\bar{\lambda}$, there is a net $\bar{x}_\alpha \in K_1(\bar{x}_\alpha, \lambda_\alpha)$, $\bar{x}_\alpha \to x$. By the above contradiction assumption, there must be a subnet $\bar{x}_\beta$ such that $\bar{x}_\beta \notin \widetilde{S}(\lambda_\beta)$ for all β, i.e., for some $y_\beta \in K_2(\bar{x}_\beta, \lambda_\beta)$,

$$0 \notin \mathrm{int}F(\bar{x}_\beta, y_\beta, \lambda_\beta)$$

As K_2 is usc at $(x, \bar{\lambda})$ and $K_2(x, \bar{\lambda})$ is compact, one has $y \in K_2(x, \bar{\lambda})$ such that $y_\beta \to y$ (taking a subnet if necessary). As $x \in \widetilde{S}(\bar{\lambda})$, we have $0 \in \mathrm{int}F(x, y, \bar{\lambda})$. Since $(\bar{x}_\beta, y_\beta, \lambda_\beta) \to (x, y, \bar{\lambda})$, assumption (ii) implies the existence of an index $\bar{\beta}$ such that

$$0 \in \mathrm{int}F(\bar{x}_{\bar{\beta}}, y_{\bar{\beta}}, \lambda_{\bar{\beta}}),$$

which is a contradiction.

Let $\bar{x} \in S(\bar{\lambda})$, $\bar{x}^1 \in \widetilde{S}(\bar{\lambda})$ and $x_t = (1-t)\bar{x} + t\bar{x}^1$ with $t \in (0, 1)$. By the convexity of $E(\bar{\lambda})$, $x_t \in E(\bar{\lambda})$. Since $K_2(., \bar{\lambda})$ is concave, for all $y_t \in K_2(x_t, \bar{\lambda})$, there exist $\bar{y} \in K_2(\bar{x}, \bar{\lambda})$, $\bar{y}_1 \in K_2(\bar{x}^1, \bar{\lambda})$ such that $y_t = (1-t)\bar{y} + t\bar{y}_1$. Since $F(., ., \bar{\lambda})$ is generalized 0-convex, $0 \in \mathrm{int}F(x_t, y_t, \bar{\lambda})$, i.e., $x_t \in \widetilde{S}(\bar{\lambda})$. Hence, $S(\bar{\lambda}) \subset \mathrm{cl}\widetilde{S}(\bar{\lambda})$. By the lower semicontinuity of $\widetilde{S}$ at $\bar{\lambda}$, S has the same property, since

$$S(\bar{\lambda}) \subset \mathrm{cl}\widetilde{S}(\bar{\lambda}) \subset \mathrm{liminf}\widetilde{S}(\lambda_\alpha) \subset S(\lambda_\alpha) \subset \mathrm{liminf}S(\lambda_\alpha).$$

□

The following example ensures us that the new assumption (iv) is essential.

Example 21 Let $X = Z = \mathbb{R}$, $\Lambda = [0, 1]$, $K_1(x, \lambda) = K_2(x, \lambda) = [\lambda, \lambda + 3]$, $\bar{\lambda} = 0$, and $F(x, y, \lambda) = (-\infty, x - \lambda - 1] \cup [x, +\infty)$. Then, it is easy to verify that (i), (ii) and (iii) of Theorem 6 are satisfied. But, $S(0) = \{0\} \cup [1, 3]$ and $S(\lambda) = [\lambda + 1, \lambda + 3]$ for all $\lambda \in (0, 1]$, and thus S is not lsc at 0. The cause is that (iv) is violated. Indeed, let $x_1 = 0$ and

$x_2 = 2$. Then, for all $y \in K_2(X, 0) = [0, 3]$, we have $F(0, y, 0) = (-\infty, -1] \cup [0, +\infty)$, $F(2, y, 0) = (-\infty, 1] \cup [2, +\infty)$, and $F(\frac{1}{2}x_1 + \frac{1}{2}x_2, y, 0) = (-\infty, 0] \cup [1, +\infty)$. Hence, $0 \in F(x_1, y, 0)$ and $0 \in \text{int}F(x_2, y, 0)$, but $0 \notin \text{int}F(\frac{1}{2}x_1 + \frac{1}{2}x_2, y, 0)$.

Theorem 6 is useful while Theorem 4 is inapplicable in the following.

Example 22 Let $X = Z = \mathbb{R}$, $\Lambda = [0, 1]$, $K_1(x, \lambda) \equiv K_2(x, \lambda) \equiv [-2, 2]$, $\bar{\lambda} = 0$, and $F(x, y, \lambda) = (-\infty, x - \lambda]$. Then, $K_2(x, \lambda)$ satisfies assumption (i) of Theorem 6. The set $\{(x, y, \lambda) : 0 \notin \text{int}F(x, y, \lambda)\} = \{(x, y, \lambda) : x - \lambda \leq 0\}$ is closed. Therefore, int F has the 0-inclusion complement property in $E(\bar{\lambda}) \times K_2(E(\bar{\lambda}), \bar{\lambda}) \times \{\bar{\lambda}\}$. Furthermore, $E(\lambda) \equiv [-2, 2]$ fulfils (iii). To check the generalized 0-convexity in $E(0) \times K_2(E(0), 0)$ of $F(., ., 0)$ in (iv), let $0 \in F(x_1, y_1, 0)$ and $0 \in \text{int}F(x_2, y_2, 0)$, i.e., $0 \in (-\infty, x_1]$ and $0 \in (-\infty, x_2)$. If $x_1 \geq 0, x_2 > 0$, for all $t \in (0, 1)$, we have $0 \in (-\infty, (1 - t)x_1 + tx_2)$, i.e., $0 \in \text{int}F[(1 - t)(x_1, y_1, 0) + t(x_2, y_2, 0)]$. According to Theorem 6, S is lsc at 0 (in fact $S(\lambda) = [\lambda, 2]$ for all $\lambda \in [0, 1]$). However, F does not have the 0-inclusion complement property in $E(0) \times K_2(E(0), 0) \times \{0\}$. Indeed, let $(-\frac{1}{n}, 0, 0) \to (0, 0, 0)$. As $F(0, 0, 0) = (-\infty, 0]$ and $F(-\frac{1}{n}, 0, 0) = (-\infty, -\frac{1}{n}]$, $0 \in F(0, 0, 0)$ but $0 \notin F(-\frac{1}{n}, 0, 0)$. Therefore, we cannot apply Theorem 4.

5 Particular Cases

Since our quasivariational inclusion problem contains many problems as special cases, including equilibrium problems, variational inequalities, optimization problems, fixed-point and coincidence-point problems, complementarity problems, Nash equilibrium problems, etc, from the results of Sections 3 and 4 we can derive consequences for such particular cases. In this section, we discuss only several corollaries for quasiequilibrium problems in connection with Ekeland's variational principle as examples.

5.1 Quasiequilibrium problems of type 1

Let X, Λ be Hausdorff topological spaces, Z a topological vector space, $C \subset Z$ closed with $\text{int}C \neq \emptyset$. Let $K : X \times \Lambda \longrightarrow 2^X$ and $G : X \times X \times \Lambda \to 2^Z$. We consider the following vector quasiequilibrium problems, for each $\lambda \in \Lambda$,

(QEP$^1_\lambda$) : find $\bar{x} \in \text{cl}K(\bar{x}, \lambda)$ such that, for each $y \in K(\bar{x}, \lambda)$, $G(\bar{x}, y, \lambda) \cap (Z \setminus -\text{int}C) \neq \emptyset$;

(SQEP$^1_\lambda$) : find $\bar{x} \in \text{cl}K(\bar{x}, \lambda)$ such that, for each $y \in K(\bar{x}, \lambda)$, $G(\bar{x}, y, \lambda) \subset Z \setminus -\text{int}C$.

Denote the set of the solutions of (QEP$^1_\lambda$) by $S^1(\lambda)$ and that of (SQEP$^1_\lambda$) by $\hat{S}^1(\lambda)$. Let $E(\lambda) := \{x \in X : x \in \text{cl}K(x, \lambda)\}$. We assume that $S^1(\lambda)$ and $\hat{S}^1(\lambda)$ are nonempty for all mentioned λ in a neighborhood of $\bar{\lambda} \in \Lambda$. To convert (QEP$^1_\lambda$) ((SQEP$^1_\lambda$), resp) to a special case of (QVIP$_\lambda$), simply set $K_1(x, \lambda) := \text{cl}K(x, \lambda)$, $K_2(x, \lambda) := K(x, \lambda)$, and $F(x, y, \lambda) := G(x, y, \lambda) - (Z \setminus -\text{int}C)$ ($F(x, y, \lambda) := Z \setminus (G(x, y, \lambda) + \text{int}C)$, resp).

To derive semicontinuity results for (QEP$^1_\lambda$) and (SQEP$^1_\lambda$) from those obtained in Sections 3 and 4, we recall here some notions defined in [14], which are particular cases of the θ-inclusion property (see the comparisons after Definition 2). $H : X \to 2^Z$ is said to have the C-inclusion property (strict C-inclusion property, resp) at x if, for any $x_\alpha \to x$,

$$[H(x) \cap (Z \setminus -\text{int}C) \neq \emptyset] \Rightarrow [\exists \bar{\alpha}, H(x_{\bar{\alpha}}) \cap (Z \setminus -\text{int}C) \neq \emptyset]$$

$$([H(x) \subset Z \setminus -\text{int}C] \Rightarrow [\exists \bar{\alpha}, H(x_{\bar{\alpha}}) \subset Z \setminus -\text{int}C], \text{resp}).$$

The following first result is a consequence of Theorem 3.

Corollary 1 *(Theorems 3.2 and 3.4 of* [14]*) Consider* $(\mathrm{QEP}^1)_{\lambda\in\Lambda}$ *(*$(\mathrm{SQEP}^1)_{\lambda\in\Lambda}$*, resp). Assume that*

(i) *K is lsc in $E(\bar\lambda)\times\{\bar\lambda\}$;*
(ii) *G is usc (lsc, resp) in $E(\bar\lambda)\times K(E(\bar\lambda),\bar\lambda)\times\{\bar\lambda\}$;*
(iii) *E is usc at $\bar\lambda$ with $E(\bar\lambda)$ being compact.*

Then, S^1 ($\hat S^1$, resp) is both usc and closed at $\bar\lambda$.

Proof Because of the similarity we consider only S^1. We need to check only that F, defined by $F(x,y,\lambda) := (G(x,y,\lambda) - (Z\setminus -\mathrm{int}C)$, has the 0-inclusion property in $E(\bar\lambda)\times K(E(\bar\lambda),\bar\lambda)\times\{\bar\lambda\}$. Assume that a net $\{(x_\alpha,y_\alpha,\lambda_\alpha)\}$ converges to $(\bar x,\bar y,\bar\lambda)$ in $E(\bar\lambda)\times K(E(\bar\lambda),\bar\lambda)\times\{\bar\lambda\}$, with $0\in F(x_\alpha,y_\alpha,\lambda_\alpha)$. Suppose to the contrary that $0\notin F(\bar x,\bar y,\bar\lambda)$, or what is the same, $G(\bar x,\bar y,\bar\lambda)\subset -\mathrm{int}C$. By the upper semicontinuity of G at $(\bar x,\bar y,\bar\lambda)$, there is α such that $G(x_\alpha,y_\alpha,\lambda_\alpha)\subset -\mathrm{int}C$, which implies $0\notin F(x_\alpha,y_\alpha,\lambda_\alpha)$, a contradiction. □

By the same arguments, from Theorems 1 and 1 we have

Corollary 2 *Assume for* $(\mathrm{QEP}^1)_{\lambda\in\Lambda}$ *(*$(\mathrm{SQEP}^1)_{\lambda\in\Lambda}$*, resp) that*

(i) *K is lsc in $E(\bar\lambda)\times\{\bar\lambda\}$;*
(ii) *G is usc (lsc, resp) in $E(\bar\lambda)\times K(E(\bar\lambda),\bar\lambda)\times\{\bar\lambda\}$.*

If E is outer open, star-outer semicontinuous or outer semicontinuous at $\bar\lambda$, then so is S^1 ($\hat S^1$, resp).

Analogously, from Theorems 4 and 5 we obtain

Corollary 3 *Assume for problem* $(\mathrm{QEP}^1)_{\lambda\in\Lambda}$ *(*$(\mathrm{SQEP}^1)_{\lambda\in\Lambda}$*, resp) that*

(i) *K is usc and has compact values in $E(\bar\lambda)\times\{\bar\lambda\}$;*
(ii) *G has the C-inclusion (strict C-inclusion, resp) property in $E(\bar\lambda)\times K(E(\bar\lambda),\bar\lambda)\times\{\bar\lambda\}$.*

If E is inner open, star-inner semicontinuous or lsc at $\bar\lambda$, then so is S^1 ($\hat S^1$, resp).

The case where E is lsc at $\bar\lambda$ of Corollary 3 coincides with Theorems 2.2 and 2.4 of [14]. To end this subsection, notice that by similar arguments we can consider quasiequilibrium problems with other types of constraints, e.g., with those studied in [16] and [17]. Of course, then stability results are derived as consequences of properties of quasivariational inclusion problems with the corresponding constraints.

5.2 Quasiequilibrium Problems of Type 2

Let X, Z and Λ be Hausdorff topological vector spaces, $A\subset X$ nonempty, $K : A\times\Lambda\to 2^A$, $\Gamma : A\times\Lambda\to 2^Z$, and $f : A\times A\times\Lambda\to Z$. Assume that the values of Γ are closed with nonempty interior, different from Z. For $\lambda\in\Lambda$ consider

$$(\mathrm{QEP}^2_\lambda):\ \text{find } \bar x\in K(\bar x,\lambda) \text{ such that, for all } y\in K(\bar x,\lambda),\ f(\bar x,y,\lambda)\in\Gamma(\bar x,\lambda).$$

Denote the set of solutions of (QEP^2_λ) by $S^2(\lambda)$ and $E(\lambda) := \{x \in A : x \in K(x,\lambda)\}$. Assume that $S^2(\lambda) \neq \emptyset$ in a neighborhood of $\bar{\lambda}$. (QEP^2_λ) is seen to be a special case of (QVIP_λ) by setting $K_1(x,\lambda) \equiv K_2(x,\lambda) := K(x,\lambda)$ and $F(x,y,\lambda) := f(x,y,\lambda) - \Gamma(\bar{x},\lambda)$. For X, Y, Γ and f as in (QEP^2_λ) and $\theta \in Z$, we use the following level-type sets

$$\mathrm{lev}_{\theta.\Gamma} f := \{(x,y,\lambda) : f(x,y,\lambda) \in \theta + \Gamma(x,\lambda)\},$$

$$\mathrm{lev}_{\theta.\Gamma(.,\bar{\lambda})} f := \{(x,y) : f(x,y,\bar{\lambda}) \in \theta + \Gamma(x,\bar{\lambda})\}.$$

Corollary 4 *(Theorem 2.1 of* [15]*) S^2 is both usc and closed at $\bar{\lambda}$, provided that*

(i) *K is lsc in $E(\bar{\lambda}) \times \{\bar{\lambda}\}$;*
(ii) $\mathrm{lev}_{0.\Gamma(.,\bar{\lambda})} f(.,.,\bar{\lambda})$ *is closed in $K(A,\Lambda) \times K(A,\Lambda)$;*
(iii) *for all $x, y \in K(A,\Lambda)$, $f(x,y,.)$ is $Z \setminus \Gamma(x,.)$-usc at $\bar{\lambda}$, uniformly with respect to $x, y \in X$ in the sense that, if $f(x,y,\bar{\lambda}) \in Z \setminus \Gamma(x,\bar{\lambda})$, there is a neighborhood N of $\bar{\lambda}$ not depending on x, y, such that, for every $\lambda \in N$, $f(x,y,\lambda) \in Z \setminus \Gamma(x,\lambda)$;*
(iv) *E is usc at $\bar{\lambda}$ with $E(\bar{\lambda})$ being compact.*

Proof Set $F(x,y,\lambda) := f(x,y,\lambda) - \Gamma(x,\lambda)$. To apply Theorem 3, we need to prove that $F(.,.,.)$ has the 0-inclusion property in $E(\bar{\lambda}) \times K(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$. Let $(x_\alpha, y_\alpha, \lambda_\alpha) \to (\bar{x},\bar{y},\bar{\lambda})$ in $E(\bar{\lambda}) \times K(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$, with $0 \in F(x_\alpha, y_\alpha, \lambda_\alpha)$. Suppose $0 \notin F(\bar{x},\bar{y},\bar{\lambda})$. Condition (ii) allows one to assume that $f(x_\alpha, y_\alpha, \bar{\lambda}) \in Z \setminus \Gamma(x_\alpha, \bar{\lambda})$ for all α. Since $f(x,y,.)$ is $Z \setminus \Gamma(x,.)$-usc at $\bar{\lambda}$, there is $N \in \mathcal{N}(\bar{\lambda})$ such that, for every $\lambda \in N$, $f(x_\alpha, y_\alpha, \lambda) \in Z \setminus \Gamma(x_\alpha, \lambda)$, which is impossible as $f(x_\alpha, y_\alpha, \lambda_{\alpha)} \in \Gamma(x_\alpha, \lambda_\alpha)$ for all α. □

For the special case where $K(x,\lambda) \equiv K$ and $\Gamma(x,\lambda) \equiv \Gamma$, [15] shows that Corollary 4 improves Theorem 3.1 of [18] and Theorem 2.1 of [19], since here the assumptions are required only for x, y in K (not globally in A like there) and the semicontinuity assumption in (iii) is weaker than the corresponding one in these theorems.

Corollary 5 *(Theorem 2.2 of* [15]*) Corollary 4 is still valid if we replace assumptions* (ii) *and* (iii) *by*

(ii') $\mathrm{lev}_{0.\Gamma} f$ *is closed in $K(A,\Lambda) \times K(A,\Lambda) \times \{\bar{\lambda}\}$.*

Proof Set $F(x,y,\lambda) := f(x,y,\lambda) - \Gamma(x,\lambda)$. To apply Theorem 3, we prove that $F(.,.,.)$ has the 0-inclusion property in $E(\bar{\lambda}) \times K(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$. Indeed, let $(x_\alpha, y_\alpha, \lambda_\alpha) \to (\bar{x},\bar{y},\bar{\lambda})$ in $E(\bar{\lambda}) \times K(E(\bar{\lambda}),\bar{\lambda}) \times \{\bar{\lambda}\}$, with $0 \in F(x_\alpha, y_\alpha, \lambda_\alpha)$. Then, we have $f(x_\alpha, y_\alpha, \lambda_{\alpha)} \in \Gamma(x_\alpha, \lambda_\alpha)$ for all α. By (ii'), $f(\bar{x},\bar{y},\bar{\lambda}) \in \Gamma(\bar{x},\bar{\lambda})$ and then $0 \in F(\bar{x},\bar{y},\bar{\lambda})$. □

As indicated in [15], when $\Gamma(x,\lambda) = Z \setminus -\mathrm{int}C(x,\lambda)$, $C(x,\lambda)$ being a convex cone, Corollary 5 corrects and improves Theorem 4.1 of [20]. Furthermore, setting $F(x,y,\lambda) := f(x,y,\lambda) - \Gamma(x,\lambda)$ and applying Theorem 6, we easily obtain Theorems 3.1 of [15] on lower semicontinuity of solutions maps of $(\mathrm{QEP}^2)_{\lambda\in\Lambda}$.

5.3 A Scalar Problem and Ekeland's Variational Principle

Now we investigate a particular scalar case of (QEP$^1_\lambda$) and (SQEP$^1_\lambda$), defined in Subsection 5.1, in connection with an application of versions of Ekeland's variational principle considered in [21] and [22]. Let (X, d) be a complete metric space, Λ a metric space and $f : X \times X \times \Lambda \to \mathbb{R}$. For $\lambda \in \Lambda$, we are concerned with the following scalar equilibrium problem

$$(\text{EP}_\lambda) \text{ find } \bar{x} \in X \text{ such that, for all } y \in X,\ f(\bar{x}, y, \lambda) + d(\bar{x}, y) \geq 0.$$

Assume that its solution set $\Sigma(\lambda)$ is nonempty for λ in a neighborhood of $\bar{\lambda}$.

Corollary 6 (i) *If, for all $x, y \in X$ and $(y_n, \lambda_n) \to (y, \bar{\lambda})$, from $f(x, y_n, \lambda_n) + d(x, y_n) \geq 0$ it follows that $f(x, y, \bar{\lambda}) + d(x, y) \geq 0$, then Σ is star-outer semicontinuous at $\bar{\lambda}$.*

(ii) *If, for all $x, y \in X$ and $(x_n, y_n, \lambda_n) \to (x, y, \bar{\lambda})$, $f(x_n, y_n, \lambda_n) + d(x_n, y_n) \geq 0$ implies $f(x, y, \bar{\lambda}) + d(x, y) \geq 0$, then Σ is outer semicontinuous at $\bar{\lambda}$. Moreover, if X is compact, then Σ is both usc and closed at $\bar{\lambda}$.*

(iii) *If X is compact and from $(x_n, y_n, \lambda_n) \to (x, y, \bar{\lambda})$ with $x \in \Sigma(\bar{\lambda})$ one has an index n_0 such that $f(x_{n_0}, y_{n_0}, \lambda_{n_0}) + d(x_{n_0}, y_{n_0}) \geq 0$, then Σ is inner open at $\bar{\lambda}$.*

(iv) *If X is compact, $x \in \Sigma(\bar{\lambda})$, $y \in X$ and from $(y_n, \lambda_n) \to (y, \bar{\lambda})$ we have an index n_0 such that $f(x, y_{n_0}, \lambda_{n_0}) + d(x, y_{n_0}) \geq 0$, then Σ is star-inner semicontinuous at $\bar{\lambda}$.*

Proof Notice that $E(\lambda) \equiv X$ and hence $E(.)$ is continuous in any sense. Hence, to apply Theorems 1, 2, 4 and 4 simply observe that, from the assumptions in (i)-(iv), by setting $F(x, y, \lambda) := f(x, y, \lambda) + d(x, y) - \mathbb{R}_+$ it follows the 0-inclusion or 0-inclusion complement property required in these theorems. □

Observe that, by the implications (see Proposition 3): inner openness ⇒ star-inner semicontinuity ⇒ lower semicontinuity, the lower semicontinuity of Σ has been obtained in Corollary 6 as consequences of stronger properties. However, the assumptions to guarantee stronger properties may be too restrictive (see Example 22). To seek for other sufficient conditions, we use the auxiliary problem

$$(\widetilde{\text{EP}}_\lambda) : \text{ find } \bar{x} \in X \text{ such that, for each } y \in X,\ f(\bar{x}, y, \lambda) + d(\bar{x}, y) > 0.$$

(This is problem $(\widetilde{QVIP}_\lambda)$ for this situation.) Let $\widetilde{\Sigma}(\lambda)$ be the solution set of $(\widetilde{\text{EP}}_\lambda)$.

Corollary 7 *Assume for problem $(\widetilde{\text{EP}}_\lambda)$ that X is compact and $\widetilde{\Sigma}(\lambda) \neq \emptyset$ in a neighborhood of $\bar{\lambda}$ and that*

(i) *for $x \in \widetilde{\Sigma}(\bar{\lambda})$, $y \in X$, and $(y_n, \lambda_n) \to (y, \bar{\lambda})$, there exists an index n_0 such that $f(x, y_{n_0}, \lambda_{n_0}) + d(x, y_{n_0}) > 0$;*

(ii) $\Sigma(\bar{\lambda}) \subset cl\widetilde{\Sigma}(\bar{\lambda})$.

Then, Σ is lsc at $\bar{\lambda}$.

Proof Set $K_1(x, \lambda) \equiv K_2(x, \lambda) \equiv X$ and $F(x, y, \lambda) := f(x, y, \lambda) + d(x, y) - \mathbb{R}_+$, which implies that $E(\lambda) = X$. Note that $x \in \widetilde{\Sigma}(\lambda)$ if and only if $0 \in \text{int}F(x, y, \lambda)$ for all $y \in X$. By (i), $\text{int}F(x, ., .)$ has the 0-inclusion complement property in $X \times \{\bar{\lambda}\}$ for all $x \in \widetilde{\Sigma}(\bar{\lambda})$. According to Theorem 5, $\widetilde{\Sigma}$ is star-inner semicontinuous at $\bar{\lambda}$. Then, Proposition 3(iii) implies that $\widetilde{\Sigma}$ is lsc at $\bar{\lambda}$.

By the lower semicontinuity of $\widetilde{\Sigma}$ at $\bar{\lambda}$ and (ii), Σ is lsc at $\bar{\lambda}$ since

$$\Sigma(\bar{\lambda}) \subset \mathrm{cl}\widetilde{\Sigma}(\bar{\lambda}) \subset \mathrm{liminf}_{\lambda\to\bar{\lambda}}\widetilde{\Sigma}(\lambda) \subset \mathrm{liminf}_{\lambda\to\bar{\lambda}}\Sigma(\bar{\lambda}).$$

□

Remark 4 Assumption (i) in Corollary 7 can be replaced by the lower semicontinuity of $f(x,.,.)$ in $X \times \{\bar{\lambda}\}$ for all $x \in \widetilde{\Sigma}(\bar{\lambda})$. Indeed, let $y \in X$ and $(y_n, \lambda_n) \to (y, \bar{\lambda})$. Since $x \in \widetilde{\Sigma}(\bar{\lambda})$, then $f(x, y, \bar{\lambda}) + d(x, y) > 0$. By the lower semicontinuity of $f(x,.,.)$, we have

$$0 < f(x, y, \bar{\lambda}) + d(x, y) \leq \lim_{(y_n,\lambda_n)\to(y,\bar{\lambda})} [f(x, y_n, \lambda_n) + d(x, y_n)].$$

Then, there exists an index n_0 such that $f(x, y_{n_0}, \lambda_{n_0}) + d(x, y_{n_0}) > 0$.

To explain the need of developing still another sufficient condition for lower semicontinuity, let us consider the following example.

Example 23 Let $X = [0, \frac{5}{2}]$, $\Lambda = (0, +\infty)$, $f(x, y, \lambda) = \frac{1}{\lambda}(g(y) - g(x))$, and

$$g(x) = \begin{cases} \dfrac{x^2}{2} & \text{if } x \in [0, 2], \\ 2 & \text{if } x \in (2, \dfrac{5}{2}]. \end{cases}$$

Corollary 7 cannot be in use since $\widetilde{\Sigma}(\lambda) = \emptyset$ for all $\lambda \in (0, +\infty)$. Moreover, for any $\bar{\lambda} \in (0, 2)$, Corollary 6s(iii) and (iv) give us nothing, since the assumptions are not satisfied. Direct computations yield $\Sigma(\lambda) = [0, \lambda] \cup [\frac{\lambda^2+4}{2\lambda}, \frac{5}{2}]$, and hence Σ is lsc in $(0, 1) \cup (1, \infty)$.

Now we try to employ the following auxiliary problem called a parametric Ekeland's variational problem, for $\lambda \in \Lambda$,

$$(\mathrm{EVP}_\lambda) \quad \text{find } \bar{x} \in X \text{ such that}, \ \forall y \in X \setminus \{\bar{x}\},\ f(\bar{x}, y, \lambda) + d(\bar{x}, y) > 0.$$

Let $\hat{\Sigma}(\lambda)$ stand for its solution set. Note that, if $f(x, x, \lambda) = 0$ for all $x \in X$, then $\hat{\Sigma}(\lambda) \subset \Sigma(\lambda)$. The name of this problem is justified as follows. Set $f(x, y, \lambda) := g(y, \lambda) - g(x, \lambda)$. Then, $\bar{x} \in \hat{\Sigma}(\lambda)$ means that, for all $y \in X \setminus \{\bar{x}\}$,

$$g(y, \lambda) + d(\bar{x}, y) > g(\bar{x}, \lambda).$$

Thus, the assertion of the existence of a solution $\bar{x}$ is just an existence conclusion for (parametric) Ekeland's variational principle. But, here we are not concerned with stability for this principle. Instead, we will apply Proposition 9 below to obtain a stability result for $(\mathrm{EP})_{\lambda\in\Lambda}$ in Theorem 7. Observe that contributions to parametric Ekeland's variational principle usually deal with continuity properties (with respect to parameters) of the points given by the principle, see, e.g., [23].

The following existence result is an immediate consequence of Theorem 2.1 of [21], and Lemma 3.8(iii), Theorem 4.1 of [22].

Proposition 9 *Assume for problem* (EVP_λ)*, for all* λ *and* $x, y, z \in X$,

(i) $f(x, y, \lambda) + f(y, z, \lambda) \geq f(x, z, \lambda)$ *and* $f(x, x, \lambda) = 0$;
(ii) $f(x, ., \lambda)$ *is bounded from below;*
(iii) $f(x, ., \lambda)$ *is lsc.*

Then, $\hat{\Sigma}(\lambda) \neq \emptyset$. *Moreover, for each* $x \in X$, *there exists* $\bar{x} \in \hat{\Sigma}(\lambda)$ *such that*

$$f(x, \bar{x}, \lambda) + d(x, \bar{x}) \leq 0.$$

This proposition implies the following result for the lower semicontinuity of Σ.

Theorem 7 *For each* λ *in a neighborhood of* $\bar{\lambda} \in \Lambda$, *impose the assumptions of Proposition 9 and assume further that* X *is compact and*

(a) $f(x, ., .)$ *is lsc for all* $x \in \hat{\Sigma}(\bar{\lambda})$;
(b) $\Sigma(\bar{\lambda}) \subset cl\hat{\Sigma}(\bar{\lambda})$.

Then, Σ *is lsc at* $\bar{\lambda}$.

Proof First, we claim that $\hat{\Sigma}$ is lsc at $\bar{\lambda}$. Indeed, suppose to the contrary that there are $x \in \hat{\Sigma}(\bar{\lambda})$ and $\lambda_n \to \bar{\lambda}$ such that, for any $x_n \in \hat{\Sigma}(\lambda_n)$, $x_n \not\to x$. Without loss of generality, we may assume that $x \notin \hat{\Sigma}(\lambda_n)$ for all n, i.e., for some $y_n \neq x$, $f(x, y_n, \lambda_n) + d(x, y_n) \leq 0$. For each y_n and λ_n, Proposition 9 yields $x_n \in \hat{\Sigma}(\lambda_n)$ such that $f(y_n, x_n, \lambda_n) + d(y_n, x_n) \leq 0$. The above two inequalities together with (i) of Proposition 9 imply that

$$\begin{aligned} f(x, x_n, \lambda_n) + d(x, x_n) &\leq (f(x, y_n, \lambda_n) + f(y_n, x_n, \lambda_n)) + (d(x, y_n) + d(y_n, x_n)) \\ &= (f(x, y_n, \lambda_n) + d(x, y_n)) + (f(y_n, x_n, \lambda_n) + d(y_n, x_n)) \leq 0. \end{aligned}$$

As X is compact, one has $x_n \to \bar{x}$ (taking a subsequence if necessary). By (a), the last inequality implies that $f(x, \bar{x}, \bar{\lambda}) + d(x, \bar{x}) \leq 0$. By the contradiction assumption, we have $\bar{x} \neq x$. Hence, as $x \in \hat{\Sigma}(\bar{\lambda})$, $f(x, \bar{x}, \bar{\lambda}) + d(x, \bar{x}) > 0$. This contradiction shows that $\hat{\Sigma}$ is lsc at $\bar{\lambda}$. Since $f(x, x, \lambda) = 0$ for all $x \in X$, then $\hat{\Sigma}(\bar{\lambda}) \subset \Sigma(\bar{\lambda})$. By the lower semicontinuity of $\hat{\Sigma}$ and (b), Σ is lsc at $\bar{\lambda}$ since

$$\Sigma(\bar{\lambda}) \subset \mathrm{cl}\hat{\Sigma}(\bar{\lambda}) \subset \mathrm{liminf}_{\lambda \to \bar{\lambda}} \hat{\Sigma}(\lambda) \subset \mathrm{liminf}_{\lambda \to \bar{\lambda}} \Sigma(\lambda).$$

□

Now we apply Theorem 7 to consider Example 23. We can check that, for all $\bar{\lambda} \in (0, 1) \cup (1, \infty)$, the assumptions of Theorem 7 are fulfilled. Consequently, Σ is lower semicontinuous in this set. (Only at $\bar{\lambda} = 1$, Theorem 7 says nothing, since $\hat{\Sigma}(1) = [0, 1]$ does not contain $\Sigma(1) = [0, 1] \cup \left\{\frac{5}{2}\right\}$).

This work was supported by the Vietnam National University Hochiminh City (VNU-HCM) under the grant number B2013-28-01. The authors would like to thank the anonymous referee for the valuable remarks and suggestions, which have helped them to improve the paper.

References

1. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
2. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis, 3rd edn. Springer, Berlin (2009)
3. Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings. A View from Variational Analysis. Springer, Berlin (2009)

4. Anh, L.Q., Khanh, P.Q.: Various kinds of semicontinuity and the solution sets of parametric multivalued symmetric vector quasiequilibrium problems. J. Global Optim. **41**, 539–558 (2008)
5. Anh, L.Q., Khanh, P.Q.: Semicontinuity of solution sets to parametric quasivariational inclusions with applications to traffic networks I: Upper semicontinuities. Set-Valued Anal. **16**, 267–279 (2008)
6. Anh, L.Q., Khanh, P.Q.: Semicontinuity of solution sets to parametric quasivariational inclusions with applications to traffic networks II: Lower semicontinuities. Set-Valued Anal. **16**, 943–960 (2008)
7. Khanh, P.Q., Luc, D.T.: Stability of solutions in parametric variational relation problems. Set-Valued Anal. **16**, 1015–1035 (2008)
8. Hai, N.X., Khanh, P.Q.: The solution existence of general variational inclusion problems. J. Math. Anal. Appl. **328**, 1268–1277 (2007)
9. Sach, P.H., Lin, L.J., Tuan, L.A.: Generalized vector quasivariational inclusion problems with moving cones. J. Optim. Theory Appl. **147**, 607–620 (2010)
10. Hai, N.X., Khanh, P.Q., Quan, N.H.: Some existence theorems in nonlinear analysis for mappings on GFC-spaces and applications. Nonlinear Anal. **71**, 6170–6181 (2009)
11. Khanh, P.Q., Quan, N.H.: The solution existence of general inclusions using generalized KKM theorems with applications to minimax problems. J. Optim. Theory Appl. **146**, 640–653 (2010)
12. Luc, D.T.: An abstract problem in variational analysis. J. Optim. Theory Appl. **138**, 65–76 (2008)
13. Luc, D.T., Sarabi, E.: Existence of solutions in variational relation problems without convexity. J. Math. Anal. Appl. **364**, 544–555 (2010)
14. Anh, L.Q., Khanh, P.Q.: Semicontinuity of the solution sets of parametric multivalued vector quasiequilibrium problems. J. Math. Anal. Appl. **294**, 699–711 (2004)
15. Anh, L.Q., Khanh, P.Q.: Continuity of solution maps of parametric quasiequilibrium problems. J. Global Optim. **46**, 247–259 (2010)
16. Anh, L.Q., Khanh, P.Q.: On the stability of the solution sets of general multivalued vector quasiequilibrium problems. J. Optim. Theory Appl. **135**, 271–284 (2007)
17. Hai, N.X., Khanh, P.Q.: Existence of solutions to general quasi-equilibrium problems and applications. J. Optim. Theory Appl. **133**, 317–327 (2007)
18. Bianchi, M., Pini, R.: A note on stability for parametric equilibrium problems. Oper. Res. Lett. **31**, 445–450 (2003)
19. Bianchi, M., Pini, R.: Sensitivity for parametric vector equilibria. Optim. **55**, 221–230 (2006)
20. Kimura, K.: Sensitivity analysis of solution mappings of parametric vector quasiequilibrium problems. J. Global Optim. **41**, 187–202 (2008)
21. Bianchi, M., Kassay, G., Pini, R.: Existence of equilibria via Ekeland's principle. J. Math. Anal. Appl. **305**, 502–512 (2005)
22. Khanh, P.Q., Quy, D.N.: A generalized distance and enhanced Ekeland's variational principle for vector functions. Nonlinear Anal. **73**, 2245–2259 (2010)
23. Georgiev, P.G.: Parametric Ekelands variational principle. Appl. Math. Lett. **14**, 691–696 (2001)