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Paraconsistency in Non-Fregean Framework

Abstract. A non-Fregean framework aims to provide a formal tool for reasoning about semantic denotations of sentences and their interactions. Extending a logic to its non-Fregean version involves introducing a new connective $\equiv$ that allows to separate denotations of sentences from their logical values. Intuitively, $\equiv$ combines two sentences φ and ψ into a true one whenever φ and ψ have the same semantic correlates, describe the same situations, or have the same content or meaning. The paper aims to compare non-Fregean paraconsistent Grzegorczyk's logics (Logic of Descriptions LD, Logic of Descriptions with Suszko's Axioms LDS, Logic of Equimeaning LDE) with non-Fregean versions of certain well-known paraconsistent logics (Jaśkowski's Discussive Logic D_2, Logic of Paradox LP, Logics of Formal Inconsistency $\mathsf{LFI1}$ and $\mathsf{LFI2}$). We prove that Grzegorczyk's logics are either weaker than or incomparable to non-Fregean extensions of LP, $\mathsf{LFI1}$, $\mathsf{LFI2}$. Furthermore, we show that non-Fregean extensions of LP, $\mathsf{LFI1}$, $\mathsf{LFI2}$, and D_2 are more expressive than their original counterparts. Our results highlight that the non-Fregean connective $\equiv$ can serve as a tool for expressing various properties of the ontology underlying the logics under consideration.

Keywords: Non-Fregean logic, The identity connective, Grzegorczyk's logic of descriptions, Paraconsistent logic, 3-Valued logic, Jaśkowski's discussive logic.

1. Introduction

Non-Fregean logics and paraconsistent logics were introduced into the realm of logical systems and have typically been studied with relatively distinct motivations. The roots of non-Fregean logic can be traced back to the seminal paper [39] by Suszko. In this work, Suszko fervently advocated for the rejection of the *Fregean Principle* (Frege). Intuitively, this foundational principle states that all true (and, similarly, all false) sentences *describe* the same thing, thereby sharing a common semantic reference (or denotation). Indeed, in the classical paradigm, which has birthed many well-established logical systems including modal logic, the universes of structures

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that interpret a formal language do not contain semantic references of sentences: they merely attribute to each sentence one of two possible logical values. Thus, the Fregean approach treats terms and formulas of a language in a drastically distinct manner: terms have their ontological reference in models, while formulas are ascribed only certain abstract objects beyond the universes of models.

A pivotal moment in the development of non-Fregean logic was the construction of the *Sentential Calculus of Identity* SCI. The primary motivation behind the creation of SCI was not to deny the properties of classical connectives, but rather to provide a formal tool enabling reasoning about semantic denotations of sentences and their interactions. SCI rejects the principle (Frege), introducing a new *identity* connective $\equiv$ to the language of classical logic, with the following intended interpretation: a formula $\varphi \equiv \psi$ is true whenever φ and ψ describe the same situations, that is, they have the same semantic correlates. The semantics of SCI is based on structures with a nonempty universe, a distinguished subset of the universe with elements that can be interpreted as semantic correlates of true sentences, operations interpreting classical logical connectives, and the operation $\tilde{\equiv}$ corresponding to $\equiv$, assumed to be the identity. Consequently, SCI allows to express that equivalent formulas have different denotations in SCI. As a result, the formula

(FA) $(\varphi \equiv \psi) \leftrightarrow (\varphi \leftrightarrow \psi)$,

which can be seen as a formal representation of the Fregean principle, cannot be valid in SCI. The ontological commitments of SCI are relatively weak, assuming only the existence of at least two elements in the universes of structures that interpret a logical language. Indeed, as shown in [17], there exist uncountably many non-equivalent extensions of SCI. It is noteworthy that certain modal and many-valued logics can be interpreted as some SCI-theories. Further details on SCI and its extensions (propositional and of higher-order), see for instance [5,6,15,17,19,30,38,39,41].

Two properties of SCI (and any of its extensions) need to be highlighted. First, SCI is an extension of classical logic, and hence all classical laws remain valid in SCI. Secondly, the connective $\equiv$ represents the identity relation between denotations of sentences. However, proponents of the non-Fregean approach are not forced to accept classical logic. Furthermore, in a general non-Fregean approach $\equiv$ need not necessarily be interpreted as the identity. Consequently, depending on the purposes of our logic, we may impose weaker assumptions on the interpretation of $\equiv$ in non-Fregean structures. For instance, it can be a congruence relation or an equivalence relation that satisfies a form of the extensionality principle. Over the years of research on non-Fregean systems, it turned out that even minor changes in the axioms of SCI and/or its semantics can yield significantly distinct formalizations of the non-Fregean connective $\equiv$.

A deviant version of SCI is a logic introduced by Grzegorczyk in [22]. His main motivation for constructing a new system was to separate logical values of sentences from their meanings. Grzegorczyk argued that classical logic, with its paradoxes of implication, is too strong to serve as a suitable formal tool for expressing descriptions of reality. Unaware of Suszko's results, Grzegorczyk formulated his logic LD, called the *Logic of Descriptions*, from scratch. He was firmly convinced that the most primitive logical connectives, whose meaning and role in human language are readily comprehensible, are negation, conjunction, and disjunction. Their role in language is more fundamental than that of implication and equivalence connectives, which need to be discarded. Instead of implication and equivalence connectives, a new connective $\equiv$ of *descriptive equivalence* should be adopted. In Grzegorczyk's framework the connective $\equiv$ is non-Fregean in nature, aiming to express the assertion that two descriptions of reality describe the same, share the same meaning, refer to the same situations. Nonetheless, even though Grzegorczyk's logic LD and Suszko's logic SCI share philosophical motivation, they significantly differ in formalization. Further results presented in [18] and [20] have revealed that LD and SCI exhibit notably different semantics and properties. The most remarkable property of LD is paraconsistency.

Paraconsistent logics have a slightly longer history than non-Fregean logics. Their primary motivation is to allow inferences that tolerate a certain kind of inconsistency. Classical logic does not permit any contradiction, which is guaranteed by the *Principle of Explosion*, according to which from contradictory premises anything follows. In contrast, a paraconsistent paradigm challenges the *Principle of Explosion* and aims to formalize inferences with contradictions in a controlled and judicious way, avoiding trivial systems in which everything is true. Over the years, many different systems of paraconsistent logic have been developed, among which are the following well-known and extensively studied logics: Logic of Paradox, Logics of Formal Inconsistency, Jaśkowski's Discussive Logic. It is also known that many non-classical logics, such as relevant or many-valued logics, originally addressing other than paraconsistency aspects of inference, are also paraconsistent. For further reading on paraconsistent logics, see for instance [1,2,4,7,12,34,35].

In this paper, we introduce non-Fregean versions of some well-known paraconsistent logics. We focus on Jaśkowski's Discussive Logic D_2 and some 3-valued paraconsistent logics, including the Logic of Paradox LP. We also revisit Grzegorczyk's Logic of Descriptions LD and two of its variants, namely LDS (*Logic of Descriptions with Suszko's Axioms*) and LDE (*Logic of Equime aning*), which differ from LD on the formalization of the extensionality property. Our goal is to address the question posed in [21]:

> Is a non-Fregean Grzegorczyk's logic L, for L $\in$ {LD, LDS, LDE}, equivalent to a previously known paraconsistent logic? If not, how they are related?

The paper is organized as follows. In Section 2 we introduce definitions of basic concepts and the notation employed throughout the paper. We also revisit crucial ideas of the non-Fregean framework, providing its formal definition. In Section 3 we present Grzegorczyk's logics of descriptions, recapitulate some of their properties, and prove minor propositions that will be used in the subsequent sections. Section 4 considers non-Fregean versions of 3-valued logics LFI1, LFI2, and LP, along with their logical relationships to Grzegorczyk's logics. In particular, we show that logics LD, LDS, LDE are either weaker than or incomparable to non-Fregean extensions of LP, LFI1, LFI2. In Section 5, we study a non-Fregean version of Jaśkowski's logic D_2 and we show that it is incomparable with Grzegorczyk's logics. The paper concludes in Section 6 with a summary and prospects for further research. To avoid cluttering the main text, details of proofs that involve extensive calculations have been included in the "Appendix".

2. Preliminaries

Logic is a formal system usually determined by a formal language and a deduction (axiomatic) system or semantics. In this paper, we examine various logics through the lens of the non-Fregean methodology that presupposes the existence of a universe of semantic correlates of sentences. Consequently, we adopt a semantic (algebraic) approach.

Definition 2.1. (L-*formulas*) The set FOR_L of *formulas of a propositional logic* L is the smallest set that includes propositional variables from a countable infinite set Prop and is closed with respect to all the connectives from a finite set OP_L.

We assume that the set OP_L of each logic L studied in this paper includes a unary connective of negation ($\neg$) and binary connectives of conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), bi-implication ($\leftrightarrow$). The set OP_L may also include a binary connective of *descriptive equivalence* $\equiv$. We do not consider any other propositional connectives.

Definition 2.2. (L-*structure*) A structure $\mathcal{M} = (U, D, \mathcal{O})$ is said to be an L-*structure* whenever U and D are nonempty sets such that D is a proper subset of U, and $\mathcal{O}$ is a mapping which assigns to each n-ary propositional connective $\# \in \mathsf{OP_L}$, an operation $\tilde{\#} : U^n \to U$ in U.

It is worth to mention that the structures defined above are often referred to as *logical matrices*. Given an L-structure $\mathcal{M} = (U, D, \mathcal{O})$, the set U is referred to as a *domain* of $\mathcal{M}$, and D is the set of *distinguished elements of* $\mathcal{M}$. For simplicity of presentation, given a logic L, we will substitute $\mathcal{O}$ by the list of operations interpreting the connectives of L.

Definition 2.3. (*Logic* L) A *logic* L is a pair $(\mathsf{FOR_L}, \mathcal{K}_\mathsf{L})$, where $\mathsf{FOR_L}$ is the set of L-formulas and $\mathcal{K}_\mathsf{L}$ is a class of L-structures.

Definition 2.4. (*Valuation*) A *valuation* in an L-structure $\mathcal{M} = (U, D, \mathcal{O})$ is a mapping $v\colon \mathsf{FOR_L} \to U$ such that for all $p \in \mathsf{Prop}$, for all L-formulas $\varphi_1, \ldots, \varphi_n$, and for every n-ary connective $\# \in \mathsf{OP_L}$, the following holds:

(v1) $v(p) \in U$,

(v2) $v(\#(\varphi_1, \ldots, \varphi_n)) = \tilde{\#}(v(\varphi_1), \ldots, v(\varphi_n))$.

Note that a valuation in an L-structure $\mathcal{M} = (U, D, \mathcal{O})$ can be seen as a homomorphism from the algebra of formulas of L to $(U, \mathcal{O})$.

Definition 2.5. (*Satisfaction, truth, validity*) Let $\mathcal{M} = (U, D, \mathcal{O})$ be an L-structure and let v be a valuation in $\mathcal{M}$. An L-formula φ is said to be *satisfied* in $\mathcal{M}$ by v, $\mathcal{M}, v \models \varphi$ in short, if and only if $v(\varphi) \in D$. A formula φ is said to be *true* in $\mathcal{M}$, denoted by $\mathcal{M} \models \varphi$, whenever it is satisfied in $\mathcal{M}$ by all the valuations in $\mathcal{M}$. If $\mathsf{L} = (\mathsf{FOR_L}, \mathcal{K}_\mathsf{L})$, then structures in $\mathcal{K}_\mathsf{L}$ are referred to as L-*models*. We say that a formula φ is *valid in* L, denoted by $\models_\mathsf{L} \varphi$, if and only if φ is true in every $\mathcal{M} \in \mathcal{K}_\mathsf{L}$.

Definition 2.6. (*Semantic consequence*) Let $\mathsf{L} = (\mathsf{FOR_L}, \mathcal{K}_\mathsf{L})$ be a propositional logic. Let X and φ be a set of L-formulas and a single L-formula, respectively. Then, a formula φ is said to be a *semantic* L-*consequence* of X, denoted by $X \models_\mathsf{L} \varphi$, if and only if for every L-structure $\mathcal{M} \in \mathcal{K}_\mathsf{L}$ and for every valuation v in $\mathcal{M}$, if $\mathcal{M}, v \models \psi$ for all $\psi \in X$, then $\mathcal{M}, v \models \varphi$.

Definition 2.7. (*Non-Fregean logic*) A logic $\mathsf{L} = (\mathsf{FOR_L}, \mathcal{K}_\mathsf{L})$ is said to be *non-Fregean* if and only if there is a binary connective $\equiv\, \in \mathsf{OP_L}$ such that the following holds:

(nfl1) There exist $\mathcal{M} = (U, D, \mathcal{O})$ in $\mathcal{K}_\mathsf{L}$ and $a, b \in U$ such that $a \neq b$ and the following holds: $a \tilde{\equiv} b \notin D$ and either $a, b \in D$ or $a, b \notin D$.

(nfl2) For every L-model $\mathcal{M} = (U, D, \mathcal{O})$ in $\mathcal{K}_{\mathsf{L}}$ and for all $a, b, c \in U$, the following holds:

(i) $a \tilde{\equiv} a \in D$.

(ii) If $a \tilde{\equiv} b \in D$, then $b \tilde{\equiv} a \in D$.

(iii) If $a \tilde{\equiv} b \in D$ and $b \tilde{\equiv} c \in D$, then $a \tilde{\equiv} c \in D$.

(iv) If $a \in D$ and $a \tilde{\equiv} b \in D$, then $b \in D$.

A connective $\equiv \in \mathsf{OP}_{\mathsf{L}}$ that satisfies conditions (nfl1) and (nfl2) will be referred to as a *non-Fregean equivalence.*

Definition 2.8. (*Standard non-Fregean logic*) A non-Fregean logic $\mathsf{L} = (\mathsf{FOR}_{\mathsf{L}}, \mathcal{K}_{\mathsf{L}})$ is said to be *standard* if and only there exists a non-Fregean equivalence connective $\equiv \in \mathsf{OP}_{\mathsf{L}}$ such that for every L-model $\mathcal{M} = (U, D, \mathcal{O})$ in $\mathcal{K}_{\mathsf{L}}$ and for all $a, b \in U$, the following holds:

(snfl) $a \tilde{\equiv} b \in D$ iff $a = b$.

A non-Fregean connective $\equiv$ satisfying (snfl) will be referred to as *standard.*

In the light of non-Fregean methodology, elements of U are *denotations* (semantical correlates) of formulas. A denotation of a formula φ can be understood as a situation or a state of affairs described by φ, or in any other way, depending on the intended motivation of the logic (meaning of φ, a state of a machine after executing a program φ, etc.). Elements of D are *factual denotations* (real situations, actual states, etc.), and $U \setminus D$ consists of *nonfactual denotations.* Intuitively, the condition (nfl1) says that the connective of non-Fregean equivalence allows us to distinguish either at least two distinct factual denotations or at least two distinct nonfactual denotations. Thus, if a logic L is non-Fregean, then there must exist an L-structure with more than two elements. The condition (nfl2) implies that $\tilde{\equiv}$ represents an equivalence relation between denotations in U such that each of its equivalence classes is included either in D or in $U \setminus D$. In a standard non-Fregean logic, $\equiv$ is the *identity connective* that combines two sentences into a new true sentence whenever the semantic correlates of its arguments are the same.

Note that the classical propositional logic PC over the language with connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ is not non-Fregean. Indeed, the class $\mathcal{K}_{\mathsf{PC}}$ consists of a single PC-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$, where $U = \{0, 1\}$, $D = \{1\}$, and $(\{0, 1\}, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ forms a Boolean algebra. The only connective satisfying the condition (nfl2) is the material equivalence $\leftrightarrow$, which, however, does not satisfy (nfl1).

Clearly, if $\rightarrow$ and $\leftrightarrow$ of a logic L are classical connectives of implication and bi-implication, that is, for each L-model $\mathcal{M} = (U, D, \mathcal{O})$ and for all $a, b \in U$ the following conditions hold:

$a \tilde{\rightarrow} b \in D$ if and only if either $a \notin D$ or $b \in D$,

$a \tilde{\leftrightarrow} b \in D$ if and only if either $a, b \in D$ or $a, b \notin D$,

and, in addition, in L there exists a connective $\equiv$ satisfying conditions (nfl1) and (nfl2) as defined in Definition 2.7, then the formula $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \equiv \psi)$ is not L-valid.

The well-known example of a non-Fregean logic is SCI, *Sentential Calculus of Identity*, introduced by Suszko in [37]. Connectives of SCI are $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and the *identity* connective $\equiv$. SCI-models are structures of the form $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$, where U, D are any nonempty sets such that $D \subset U$, and for all $a, b \in U$ it holds that:

(SCI1) $\tilde{\neg} a \in D$ iff $a \notin D$,
(SCI2) $a \tilde{\wedge} b \in D$ iff $a \in D$ and $b \in D$,
(SCI3) $a \tilde{\vee} b \in D$ iff $a \in D$ or $b \in D$,
(SCI4) $a \tilde{\rightarrow} b \in D$ iff $a \notin D$ or $b \in D$,
(SCI5) $a \tilde{\leftrightarrow} b \in D$ iff $a \tilde{\rightarrow} b \in D$ and $b \tilde{\rightarrow} a \in D$,
(SCI6) $a \tilde{\equiv} b \in D$ iff $a = b$.

Definition 2.9. (*Paraconsistent logic*) A logic $\mathsf{L} = (\mathsf{FOR}_\mathsf{L}, \mathcal{K}_\mathsf{L})$ is said to be *paraconsistent* if and only if it has a unary connective of negation $\neg \in \mathsf{OP}_\mathsf{L}$ such that $\varphi, \neg\varphi \not\models_\mathsf{L} \psi$, for some L-formulas φ and ψ.

Definition 2.10. Let L and L' be propositional logics such that $\mathsf{OP}_\mathsf{L} \subseteq \mathsf{OP}_{\mathsf{L}'}$. We will say that L *is weaker than or equal to* L', denoted by $\mathsf{L} \preceq \mathsf{L}'$, if and only if each L-valid formula is L'-valid. A logic L (resp. L') is referred to as *weaker* (resp. *stronger*) than L' (resp. L), $\mathsf{L} \prec \mathsf{L}'$ in short, if and only if $\mathsf{L} \preceq \mathsf{L}'$ and $\mathsf{L}' \not\preceq \mathsf{L}$. Logics L and L' are said to be *incomparable*, denoted by $\mathsf{L} \prec\succ \mathsf{L}'$, whenever $\mathsf{L} \not\preceq \mathsf{L}'$ and $\mathsf{L}' \not\preceq \mathsf{L}$.

3. Grzegorczyk's Non-Fregean Logics

The first Grzegorczyk's logic of description, denoted by LD, was introduced in [22]. The logic LD was originally defined over the language with connectives $\neg, \wedge, \vee, \equiv$. Its Hilbert-style axiomatization consists of 17 axioms (of which only one does not involve the connective $\equiv$) and 4 rules of inference: substitution, introduction and elimination of conjunction, and modus ponens for ≡ (if $\varphi \equiv \psi$ and φ are theorems, then so is ψ). In LD, the connective ≡ represents a congruence relation, but its formalization in LD is essentially different than in SCI. In a comprehensive article [18], a sound and complete semantics for LD has been constructed, based on a new class of algebras named *Grzegorczyk algebras*. The paper [18] provides several unprovability results for LD, showing that the descriptive equivalence ≡ of LD is indeed different from the classical equivalence. Moreover, negation and disjunction of LD also behave in unexpected ways. In the paper [20], a weaker semantics for LD has been constructed. The new class of models was obtained by removing the consistency condition from the original LD-models. Consequently, it has been proved that the logic LD is paraconsistent. However, three axioms of LD, intended to express the extensionality of ≡, sparked heated controversy. In the paper [20], two new Grzegorczyk's logics have been introduced: LDE (*Logic of Equimeaning*) and LDS (*Logic of Descriptions with Suszko's Axioms*). Logics LDE and LDS are obtained from LD by some modification of the axioms reflecting the extensionality principle. Unexpectedly, it turned out that the three logics LD, LDS, and LDE are pairwise incomparable. For more detailed results on LD, LDS, and LDE, see the survey [21].

A reassessment of the philosophical legitimacy of LD-axioms resulted in the logic MGL, named *Minimal Grzegorczyk's logic* and presented in [16]. The discussion in [16] on the interpretations of LD, LDS, and LDE in natural language has led to the conclusion that axioms reflecting the extensionality principle lack philosophical intuitiveness, whereas the other axioms are overly strong. In logic MGL the extensionality property is not expressed by axioms, but by a rule of inference. Such an approach seems more warranted from both a philosophical and linguistic standpoint. In [16], a sound and complete semantics for MGL has been constructed. Moreover, it has been showed that MGL is paraconsistent, decidable, and can serve as a non-trivial generic logic that can be extended to some non-Fregean logics constructed in Grzegorczyk or Suszko style.

In this section we study the following Grzegorczyk's logics: WGL – *Weak Grzegorczyk's Logic*; MGL – *Minimal Grzegorczyk's Logic*; LD – *Logic of Descriptions*; LDS – *Logic of Descriptions with Suszko's Axioms*; LDE – *Logic of Equimeaning*. All these logics share their language, that is, they have the same set FOR_G of formulas, which are built with the use of the following connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\equiv$. Each logic $\mathsf{L} \in \{\mathsf{WGL}, \mathsf{MGL}, \mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$ is determined by a class of its models. Therefore, $\mathsf{L} = (\mathsf{FOR}_\mathsf{G}, \mathcal{K}_\mathsf{L})$, where $\mathcal{K}_\mathsf{L}$ consists of L-models which are defined as follows.

Definition 3.1. (WGL-*model*) A structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be a WGL-*model* whenever U, D are nonempty, $D \subset U$, and for all $a, b, c \in U$ the following conditions hold:

(WGL$_1$) $\tilde{\neg}(a \mathbin{\tilde{\wedge}} \tilde{\neg} a) \in D$,

(WGL$_2$) $a \mathbin{\tilde{\equiv}} a \in D$,

(WGL$_3$) $(a \mathbin{\tilde{\wedge}} b) \in D$ if and only if $a \in D$ and $b \in D$,

(WGL$_4$) If $a \in D$ and $(a \mathbin{\tilde{\equiv}} b) \in D$, then $b \in D$,

(WGL$_5$) If $a \mathbin{\tilde{\equiv}} b \in D$, then $b \mathbin{\tilde{\equiv}} a \in D$,

(WGL$_6$) If $a \mathbin{\tilde{\equiv}} b \in D$ and $b \mathbin{\tilde{\equiv}} c \in D$, then $a \mathbin{\tilde{\equiv}} c \in D$.

Definition 3.2. (MGL-*model*) An MGL-*model* is a WGL-structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ such that $\tilde{\equiv}$ satisfies the condition (snfl), that is, for all $a, b \in U$ the following holds:
(mgl) $(a \mathbin{\tilde{\equiv}} b) \in D$ if and only if $a = b$.

Models of LD, LDE, and LDS are based on G-structures.

Definition 3.3. (G-*structure*)
A WGL-model $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be a G-*structure* if and only if for all $a, b, c \in U$ and $\# \in \{\wedge, \vee\}$, the following conditions hold:

(G1) $a \mathbin{\tilde{\equiv}} (a \mathbin{\tilde{\#}} a) \in D$,

(G2) $(a \mathbin{\tilde{\#}} b) \mathbin{\tilde{\equiv}} (b \mathbin{\tilde{\#}} a) \in D$,

(G3) $(a \mathbin{\tilde{\#}} (b \mathbin{\tilde{\#}} c)) \mathbin{\tilde{\equiv}} ((a \mathbin{\tilde{\#}} b) \mathbin{\tilde{\#}} c) \in D$,

(G4) $(a \mathbin{\tilde{\wedge}} (b \mathbin{\tilde{\vee}} c)) \mathbin{\tilde{\equiv}} ((a \mathbin{\tilde{\wedge}} b) \mathbin{\tilde{\vee}} (a \mathbin{\tilde{\wedge}} c)) \in D$,

(G5) $(a \mathbin{\tilde{\vee}} (b \mathbin{\tilde{\wedge}} c)) \mathbin{\tilde{\equiv}} ((a \mathbin{\tilde{\vee}} b) \mathbin{\tilde{\wedge}} (a \mathbin{\tilde{\vee}} c)) \in D$,

(G6) $(\tilde{\neg}\tilde{\neg} a) \mathbin{\tilde{\equiv}} a \in D$,

(G7) $\tilde{\neg}(a \mathbin{\tilde{\wedge}} b) \mathbin{\tilde{\equiv}} (\tilde{\neg} a \mathbin{\tilde{\vee}} \tilde{\neg} b) \in D$,

(G8) $\tilde{\neg}(a \mathbin{\tilde{\vee}} b) \mathbin{\tilde{\equiv}} (\tilde{\neg} a \mathbin{\tilde{\wedge}} \tilde{\neg} b) \in D$,

(G9) $(a \mathbin{\tilde{\equiv}} b) \mathbin{\tilde{\equiv}} (b \mathbin{\tilde{\equiv}} a) \in D$,

(G10) $(a \mathbin{\tilde{\equiv}} b) \mathbin{\tilde{\equiv}} (\tilde{\neg} a \mathbin{\tilde{\equiv}} \tilde{\neg} b) \in D$.

Note that operations $\tilde{\rightarrow}$ and $\tilde{\leftrightarrow}$ in G-structures can be freely defined. In the original axiomatization of Grzegorczyk's logic LD in [22], none of its axioms involve connectives of implication $\rightarrow$ and bi-implication $\leftrightarrow$. Such an approach was motivated by Grzegorczyk's philosophical view on the need for a logic built from the ground up to reflect the fundamental interactions between descriptive equivalence $\equiv$ and the basic connectives of negation, disjunction and conjunction. As a result, LD introduces counterparts to many classical laws involving equivalence while omitting others. As shown in [18], even if the connectives $\rightarrow$ and $\leftrightarrow$ are defined in LD in terms of $\neg$ and $\wedge$ in a classical way, many of the classical laws of implication and bi-implication do not hold in LD. Thus, Grzegorczyk's approach can serve as a basis for introducing non-classical implication and bi-implication.

Definition 3.4. (*De Morgan bisemilattice*) *A de Morgan bisemilattice* is a structure $(U, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee})$ such that $\tilde{\neg}$ is a unary operation on U, $\tilde{\wedge}, \tilde{\vee}$ are binary operations on U that are commutative, associative, idempotent, and moreover, $\tilde{\wedge}$ distributes over $\tilde{\vee}$, $\tilde{\vee}$ distributes over $\tilde{\wedge}$, and for all $a, b \in U$, $\tilde{\neg}\tilde{\neg}a = a$ and $\tilde{\neg}(a \,\tilde{\vee}\, b) = \tilde{\neg}a \,\tilde{\wedge}\, \tilde{\neg}b$.

Definition 3.5. (*Standard* G-*structure*)
A WGL-model $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be a *standard* G-*structure* if and only if $(U, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee})$ is a de Morgan bisemilattice and for all $a, b, c \in U$, the following conditions hold:

$a \,\tilde{\equiv}\, b \in D$ iff $a = b$ (i.e., $\tilde{\equiv}$ satisfies the condition (snfl)),
$a \,\tilde{\equiv}\, b = b \,\tilde{\equiv}\, a$,
$a \,\tilde{\equiv}\, b = \tilde{\neg}a \,\tilde{\equiv}\, \tilde{\neg}b$.

Definition 3.6. (LD-*model*) A standard G-structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be an LD-*model* whenever for all $a, b, c \in U$, the following conditions hold:

(LD_1) $(a \,\tilde{\equiv}\, b) \,\tilde{\equiv}\, ((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, ((a \,\tilde{\equiv}\, c) \,\tilde{\equiv}\, (b \,\tilde{\equiv}\, c))) \in D$,
(LD_2) $(a \,\tilde{\equiv}\, b) \,\tilde{\equiv}\, ((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, ((a \,\tilde{\wedge}\, c) \,\tilde{\equiv}\, (b \,\tilde{\wedge}\, c))) \in D$,
(LD_3) $(a \,\tilde{\equiv}\, b) \,\tilde{\equiv}\, ((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, ((a \,\tilde{\vee}\, c) \,\tilde{\equiv}\, (b \,\tilde{\vee}\, c))) \in D$.

Definition 3.7. (LDS-*model*) A standard G-structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be an LDS-*model* whenever for all $a, b, c, d \in U$, the following conditions hold:

(LDS_1) $(a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, c \,\tilde{\equiv}\, d) \,\tilde{\equiv}\, ((a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, c \,\tilde{\equiv}\, d) \,\tilde{\wedge}\, ((a \,\tilde{\equiv}\, c) \,\tilde{\equiv}\, (b \,\tilde{\equiv}\, d))) \in D$,
(LDS_2) $(a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, c \,\tilde{\equiv}\, d) \,\tilde{\equiv}\, (a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, c \,\tilde{\equiv}\, d) \,\tilde{\wedge}\, ((a \,\tilde{\wedge}\, c) \,\tilde{\equiv}\, (b \,\tilde{\wedge}\, d))) \in D$,
(LDS_3) $(a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, c \,\tilde{\equiv}\, d) \,\tilde{\equiv}\, ((a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, c \,\tilde{\equiv}\, d) \,\tilde{\wedge}\, ((a \,\tilde{\vee}\, c) \,\tilde{\equiv}\, (b \,\tilde{\vee}\, d))) \in D$.

Thus, we have:

Theorem 3.8. *All models of* LD *and* LDS *are standard* G-*structures.*

Definition 3.9. (LDE-*model*) A G-structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be an LDE-*model* whenever for all $a, b, c \in U$, the following conditions hold:

(LDE$_1$) $(a \tilde{\equiv} b \mathbin{\tilde\wedge} a \tilde{\equiv} c) \tilde{\equiv} (a \tilde{\equiv} b \mathbin{\tilde\wedge} b \tilde{\equiv} c) \in D$,
(LDE$_2$) $(a \tilde{\equiv} b \mathbin{\tilde\wedge} (a \mathbin{\tilde\wedge} c)) \tilde{\equiv} (a \tilde{\equiv} b \mathbin{\tilde\wedge} (b \mathbin{\tilde\wedge} c)) \in D$,
(LDE$_3$) $(a \tilde{\equiv} b \mathbin{\tilde\wedge} (a \mathbin{\tilde\vee} c)) \tilde{\equiv} (a \tilde{\equiv} b \mathbin{\tilde\wedge} (b \mathbin{\tilde\vee} c)) \in D$.

The logic MGL is a simple extension of WGL to its standard version in which $\equiv$ represents the identity relation between elements of U. By Theorem 3.8, all models of LD and LDS are models of MGL, whereas all models of LDE are models of WGL. Therefore, we have the following:

Proposition 3.10. *Let* L $\in$ {MGL, LD, LDS, LDE}. *Every* L-*model is a* WGL-*model.*

A trivial example of a WGL-model is the two-element Boolean algebra with 1 as the only distinguished element. It is also easy to see that every SCI-model is an MGL-model. However, SCI is much stronger than MGL. In particular, in every SCI-model it holds that $\tilde{\neg} a \in D$ iff $a \notin D$, which does not hold in all MGL-models (see [16]). In consequence, we have:

Proposition 3.11. WGL $\preceq$ MGL $\prec$ SCI.

Below we present examples of non-trivial models of logics LD, LDS, LDE. Let $\mathcal{M}_1 = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be such that $U = \{0, 1, 2\}$, $D = \{2\}$, $\tilde{\rightarrow}$ and $\tilde{\leftrightarrow}$ are any binary operations on U, and the operations $\tilde{\neg}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\equiv}$ are defined as:

$$\tilde{\neg} a = \begin{cases} 2, & \text{if } a = 0 \\ 0, & \text{otherwise} \end{cases} \qquad a \tilde{\equiv} b = \begin{cases} 2, & \text{if } a = b \\ 0, & \text{otherwise} \end{cases}$$

$$a \mathbin{\tilde\wedge} b = \begin{cases} 2, & \text{if } a = b = 2 \\ 0, & \text{otherwise} \end{cases} \qquad a \mathbin{\tilde\vee} b = \begin{cases} 2, & \text{if } a = 2 \text{ or } b = 2 \\ 0, & \text{otherwise} \end{cases}$$

The structure $\mathcal{M}_1$ satisfies all the conditions imposed in MGL-models:

Proposition 3.12. *The structure* $\mathcal{M}_1$ *is a model of* WGL *and* MGL.

However, $\mathcal{M}_1$ is not a model of logics LD, LDS, LDE, as it does not satisfy the condition (G6) of G-structures. Indeed, $\tilde{\neg}\tilde{\neg} 1 = 2 \neq 1$. Note also that the operation $\tilde{\neg}$ does not behave in $\mathcal{M}_1$ as the classical negation: $1 \notin D$ and $\tilde{\neg} 1 = 0 \notin D$. Furthermore, $\mathcal{M}_1$ does not satisfy the semantic version of the excluded middle law, that is $a \mathbin{\tilde\vee} \tilde{\neg} a$ does not hold for all $a \in U$, as $(1 \mathbin{\tilde\vee} \tilde{\neg} 1) = (1 \mathbin{\tilde\vee} 0) = 0 \notin D$. Hence, the formula $p \vee \neg p$ is not valid in WGL and MGL. However, it should be noted that the formula $\neg(p \wedge \neg p)$ is valid in each logic L $\in$ {WGL, MGL, LD, LDS, LDE}. Indeed, its semantic version $\tilde{\neg}(a \mathbin{\tilde\wedge} \tilde{\neg} a) \in D$ holds in each L-model, for all $a \in U$.
Let $\mathcal{M}_2 = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be such that $U = \{0, 1, 2\}$, $D = \{1, 2\}$, $\tilde{\rightarrow}$ and $\tilde{\leftrightarrow}$ are any binary operations on U, and the operations $\tilde{\neg}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\equiv}$ are defined as:

$$\tilde{\neg} a = 2 - a \qquad a \,\tilde{\equiv}\, b = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{otherwise} \end{cases}$$
$$a \,\tilde{\wedge}\, b = \min(a, b) \qquad a \,\tilde{\vee}\, b = \max(a, b)$$

The following can be easily verified (cf. [20] and [21]):

Proposition 3.13. *The structure $\mathcal{M}_2$ is an* L*-model, for every* $\mathsf{L} \in \{\mathsf{WGL}, \mathsf{MGL}, \mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$.

The structure $\mathcal{M}_2$ enjoys the following property: $1 \in D$ and $\tilde{\neg} 1 \in D$. Thus, the operation $\tilde{\neg}$ in $\mathcal{M}_2$ is not classical. Indeed, if v is a valuation in $\mathcal{M}_2$ such that $v(p) = 1$ and $v(q) = 0$, then $\mathcal{M}_2, v \models p$ and $\mathcal{M}_2, v \models \neg p$, but $\mathcal{M}_2, v \not\models q$. Hence, $p, \neg p \not\models_{\mathsf{L}} q$, for every $\mathsf{L} \in \{\mathsf{WGL}, \mathsf{MGL}, \mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$.

Theorem 3.14. *Let* $\mathsf{L} \in \{\mathsf{WGL}, \mathsf{MGL}, \mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. *The logic* L *is paraconsistent.*

Clearly, the model $\mathcal{M}_2$ satisfies the condition (nfl1). Furthermore, every logic $\mathsf{L} \in \{\mathsf{WGL}, \mathsf{MGL}, \mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$ satisfies the condition (nfl2) of Definition 2.7. Therefore, we obtain:

Theorem 3.15. *Let* $\mathsf{L} \in \{\mathsf{WGL}, \mathsf{MGL}, \mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. *The logic* L *is non-Fregean. Moreover,* MGL*,* LD*, and* LDS *are standard non-Fregean logics.*

MGL was originally defined in [16] by the Hilbert-style axiomatization which consists of axioms of the form $\neg(\varphi \wedge \neg\varphi)$ and $\varphi \equiv \varphi$, for all formulas φ, and the following four inference rules:

$$(\mathsf{MP}_{\equiv}) \quad \frac{\varphi, \varphi \equiv \psi}{\psi} \qquad\qquad (\mathsf{ext}) \quad \frac{\varphi \equiv \psi}{\alpha(\varphi) \equiv \alpha(\varphi/\psi)}$$

$$(\wedge_1) \quad \frac{\varphi, \psi}{\varphi \wedge \psi} \qquad\qquad (\wedge_2) \quad \frac{\varphi \wedge \psi}{\varphi, \psi}$$

The following properties of MGL are proved in [16]:

Theorem 3.16. (Strong Soundness and Completeness of MGL) *Let X be a set of* MGL*-formulas and φ be a single* MGL*-formula. Then, the following conditions are equivalent:*

1. $X \vdash_{\mathsf{MGL}} \varphi$
2. $X \models_{\mathsf{MGL}} \varphi$.

Theorem 3.17. (Decidability of MGL) *The logic* MGL *is decidable.*

In what follows, a rule $\frac{\varphi_1, \ldots, \varphi_n}{\psi_1, \ldots, \psi_k}$, $n, k \geq 1$, is said to be *strongly correct in a logic* L whenever for every L-structure $\mathcal{M}$ and for every valuation v in $\mathcal{M}$, if $\mathcal{M}, v \models \varphi_i$, for all $i \in \{1, \ldots, n\}$, then $\mathcal{M}, v \models \psi_j$, for all $j \in \{1, \ldots, k\}$.

In [16], it has been proved that all the MGL-rules are strongly correct in MGL. Now, it can be easily seen that WGL is a weaker version of MGL. Indeed, the connective $\equiv$ represents a congruence relation in MGL, but not in WGL. In particular, we have:

$\{p \equiv q\} \not\models_{\mathsf{WGL}} (p \wedge r) \equiv (q \wedge r)$.

Thus, the rule (ext) is not strongly correct in WGL, since there is a WGL-model $\mathcal{M}$ and a valuation v in $\mathcal{M}$ such that $\mathcal{M}, v \models p \equiv q$, but $\mathcal{M}, v \not\models (p \wedge r) \equiv (q \wedge r)$. The structure $\mathcal{M}_{\mathsf{LDE}}$ defined on page 15 is an example of such a WGL-model.

In fact the rule (ext) plays an important philosophical role. As argued in [16], if we read a formula $\varphi \equiv \psi$ as 'φ and ψ have equal meaning (have the same content or are the same descriptions)', then the rule (ext) can be seen as a formal representation of the following *Extensionality Property*:

(EXT) *Sentences that have equal meaning (have the same content or are the same descriptions) are interchangeable in all possible contexts.*

In [20] it has been proved that the rule (ext) holds in LD and LDS, but not in LDE. Consequently, logics WGL and LDE are beyond the expressive power of (ext). However, we may easily obtain WGL from MGL by replacing (ext) with the following two rules expressing symmetry and transitivity of $\equiv$:

$$(\mathsf{sym}) \quad \frac{\varphi \equiv \psi}{\psi \equiv \varphi} \qquad (\mathsf{tran}) \quad \frac{\varphi \equiv \psi, \psi \equiv \vartheta}{\varphi \equiv \vartheta}$$

Then, we obtain the soundness and completeness theorem for WGL, which can be proved essentially in the same way as Theorem 3.12 in [16], but without taking equivalence classes.

Hilbert-style axiomatizations of logics LD, LDS, and LDE can be found in [21] (cf. [18,20]). The inference rules of LD, LDS, and LDE are (MP), $(\wedge_1)$, and $(\wedge_2)$. All the logics LD, LDS, and LDE share most of their axioms that correspond to conditions imposed in G-structures. However, they differ on the last three axioms that are intended to express the extensionality of $\equiv$, which is also manifested in the semantics: conditions (LD_1)–(LD_3), (LDS_1)–(LDS_3), and (LDE_1)–(LDE_3) are semantic counterparts of the three specific axioms of the logics LD, LDS, and LDE, respectively, listed below. For simplicity of presentation, we admit the following shorthand notation for the connective of *the descriptive implication*:

$(\varphi \Rightarrow \psi) \stackrel{\mathrm{df}}{=} (\varphi \equiv (\varphi \wedge \psi))$.

The three specific axioms of LD, LDS, and LDE are:

(AxLD_1) $\quad (\varphi \equiv \psi) \Rightarrow ((\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta))$

(AxLD_2) $\quad (\varphi \equiv \psi) \Rightarrow ((\varphi \wedge \vartheta) \equiv (\psi \wedge \vartheta))$

(AxLD_3) $(\varphi \equiv \psi) \Rightarrow ((\varphi \vee \vartheta) \equiv (\psi \vee \vartheta))$
(AxLDS_1) $((\varphi \equiv \psi) \wedge (\vartheta \equiv \zeta)) \Rightarrow ((\varphi \equiv \vartheta) \equiv (\psi \equiv \zeta))$
(AxLDS_2) $((\varphi \equiv \psi) \wedge (\vartheta \equiv \zeta)) \Rightarrow ((\varphi \wedge \vartheta) \equiv (\psi \wedge \zeta))$
(AxLDS_3) $((\varphi \equiv \psi) \wedge (\vartheta \equiv \zeta)) \Rightarrow ((\varphi \vee \vartheta) \equiv (\psi \vee \zeta))$
(AxLDE_1) $((\varphi \equiv \psi) \wedge (\varphi \equiv \vartheta)) \equiv ((\varphi \equiv \psi) \wedge (\psi \equiv \vartheta))$
(AxLDE_2) $((\varphi \equiv \psi) \wedge (\varphi \wedge \vartheta)) \equiv ((\varphi \equiv \psi) \wedge (\psi \wedge \vartheta))$
(AxLDE_3) $((\varphi \equiv \psi) \wedge (\varphi \vee \vartheta)) \equiv ((\varphi \equiv \psi) \wedge (\psi \vee \vartheta))$

As proved in [20], logics LD, LDS, and LDE are pairwise incomparable (see [20, 21]), that is, their models generate essentially different classes of valid formulas. Therefore, the specific conditions of LD, LDS, and LDE formalize *extensionality* of $\equiv$ in an essentially different way.

Many non-trivial examples of models of logics LD, LDE, and LDS can be found in [18, 20, 21]. Below we present non-Fregean structures that are used to show incomparability of LD, LDE, and LDS: an LD-model $\mathcal{M}_{\mathsf{LD}}$ which is not a model of LDS nor LDE, an LDS-model $\mathcal{M}_{\mathsf{LDS}}$ which is not a model of LD nor LDE, and an LDE-model $\mathcal{M}_{\mathsf{LDE}}$ which is not a model of LD nor LDS. Let $A = \{0, 1\}$ and let $\mathcal{M}_{\mathsf{LD}} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be such that $U = \{\emptyset, \{0\}, \{1\}, A\}$, $D = \{A\}$, $\tilde{\rightarrow}$ and $\tilde{\leftrightarrow}$ are any binary operations on U, and the operations $\tilde{\neg}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\equiv}$ are defined as:

$$\tilde{\neg} a = A \setminus a \qquad a \,\tilde{\equiv}\, b = \begin{cases} A, & \text{if } a = b \\ \emptyset, & \text{if } a = A \setminus b \\ \{0\}, & \text{otherwise} \end{cases}$$

$$a \,\tilde{\wedge}\, b = a \cap b \qquad a \,\tilde{\vee}\, b = a \cup b$$

It is known that $\mathcal{M}_{\mathsf{LD}}$ is an LD-model (see [20]). However, the following can be proved (see the "Appendix" at the end of the paper):

Proposition 3.18. *The conditions* (LDS_1)–(LDS_3) *and* (LDE_1)–(LDE_3) *do not hold in* $\mathcal{M}_{\mathsf{LD}}$*. Consequently, formulas* (AxLDS_1)–(AxLDS_3) *and* (AxLDE_1)–(AxLDE_3) *are not valid in* LD*.*

Let $\mathcal{M}_{\mathsf{LDS}} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be such that $U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$, $\tilde{\rightarrow}$ and $\tilde{\leftrightarrow}$ are any binary operations on U, and the operations $\tilde{\neg}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\equiv}$ are defined as:

$$\tilde{\neg} a = 3 - a \qquad a \,\tilde{\equiv}\, b = \begin{cases} a \,\tilde{\vee}\, \tilde{\neg} a, & \text{if } a = b \\ 0, & \text{if } \{a, b\} = \{0, 3\} \\ 1, & \text{otherwise} \end{cases}$$

$$a \,\tilde{\wedge}\, b = \min(a, b) \qquad a \,\tilde{\vee}\, b = \max(a, b)$$

A structure $\mathcal{M}_{\mathsf{LDS}}$ is an LDS-model (for details, see [20]), but it does not satisfy conditions (LD_1)–(LD_3) and (LDE_1)–(LDE_3) (see the "Appendix").

PROPOSITION 3.19. *The model $\mathcal{M}_{\mathsf{LDS}}$ does not satisfy the conditions* (LD_1)–(LD_3) *and* (LDE_1)–(LDE_3)*. Hence, the formulas* (AxLD_1)–(AxLD_3) *and* (AxLDE_1)–(AxLDE_3) *are not valid in* LDS*.*

Now, we will study an LDE-model from [20].
Let $\mathcal{M}_{\mathsf{LDE}} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be such that $U = \{0,1,2\} \times \{0,1\}$, $D = \{0,1,2\} \times \{1\}$, $\tilde{\rightarrow}$ and $\tilde{\leftrightarrow}$ are any binary operations on U, and the operations $\tilde{\neg}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\equiv}$ are defined as:

$$\tilde{\neg}(a,b) = (2-a, 1-b)$$

$$(a,b)\,\tilde{\wedge}\,(c,d) = (\min(a,c), \min(b,d)) \qquad (a,b)\,\tilde{\vee}\,(c,d) = (\max(a,c), \max(b,d))$$

$$(a,b)\,\tilde{\equiv}\,(c,d) = \begin{cases} (0,1), & \text{if } a = b \\ & \text{or } \{(a,b),(c,d)\} = \{(0,0),(1,0)\} \\ & \text{or } \{(a,b),(c,d)\} = \{(1,1),(2,1)\} \\ (0,0), & \text{otherwise} \end{cases}$$

In the “Appendix”, we prove that $\mathcal{M}_{\mathsf{LDE}}$ is not an LD-model nor LDS-model.

PROPOSITION 3.20. *The model $\mathcal{M}_{\mathsf{LDE}}$ does not satisfy the conditions* (LD_2)–(LD_3) *and* (LDS_2)–(LDS_3)*. Consequently,* (AxLD_2)–(AxLD_3) *and* (AxLDS_2)–(AxLDS_3) *are not valid in* LDE*.*

Propositions 3.18, 3.19, and 3.20 imply:

PROPOSITION 3.21. *Logics* LD*,* LDS*, and* LDE *are pairwise incomparable.*

Observe that the original Suszko's logic SCI is beyond the expressive power of logics LD, LDS, LDE. Indeed, SCI does not impose any special conditions on the operations $\tilde{\neg}$, $\tilde{\wedge}$, $\tilde{\vee}$. In particular, $\neg\neg p \equiv p$ and $p \equiv (p \vee p)$ are not valid in SCI. On the other hand, every SCI-model satisfies the condition $\tilde{\neg} a \in D$ iff $a \notin D$, which does not hold for every model of $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. Therefore, we obtain:

PROPOSITION 3.22. *Let $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. Then,* L *and* SCI *are incomparable.*

Due to Propositions 3.11, 3.21, and 3.22, we obtain the following:

THEOREM 3.23.

1. $\mathsf{WGL} \prec \mathsf{L}$*, for each* $\mathsf{L} \in \{\mathsf{MGL}, \mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}, \mathsf{SCI}\}$*.*
2. $\mathsf{MGL} \prec \mathsf{L}$*, for each* $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{SCI}\}$*.*
3. *Logics* LD, LDS, LDE, SCI *are pairwise incomparable.*

The proofs of properties of Grzegorczyk's logics, listed in the next proposition, can be found in [18,20,21].

Proposition 3.24.

1. $(\varphi \equiv \psi) \Rightarrow (\psi \equiv \psi)$ *is valid in* LDE*, but not in* LD.
2. $(p \equiv q) \Rightarrow [(p \equiv (p \equiv (p \equiv p))) \equiv (q \equiv (q \equiv (q \equiv q)))]$ *is valid in* LDS*, but not in* LD.
3. *The following formulas are not valid in* LD*,* LDS*, and* LDE*:*

$$\neg(\varphi \equiv \neg\varphi) \qquad (\varphi \wedge \neg\varphi) \Rightarrow \psi$$
$$(\varphi \vee \neg\varphi) \equiv (\psi \vee \neg\psi) \qquad \psi \Rightarrow (\varphi \vee \neg\varphi)$$
$$(\varphi \equiv \varphi) \equiv (\psi \equiv \psi) \qquad (\varphi \equiv \psi \wedge \varphi) \equiv (\varphi \equiv \psi \wedge \psi)$$
$$(\varphi \wedge (\varphi \vee \psi)) \equiv \varphi \qquad (\varphi \equiv \psi \wedge \psi \wedge \vartheta) \Rightarrow (\varphi \equiv \vartheta).$$
$$(\varphi \vee (\varphi \wedge \psi)) \equiv \varphi$$

Below we present some further properties of Grzegorczyk's logics which will be used in the subsequent sections. For proofs, see the "Appendix".

Proposition 3.25. *Let* $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$*. Then,* $\varphi \vee \neg\varphi$ *is* L*-valid.*

Proposition 3.26. *Let* $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$*. If* $\varphi \equiv \psi$ *is valid in* L*, then* $\neg\varphi \vee \psi$ *and* $\neg\psi \vee \varphi$ *are* L*-valid.*

Proposition 3.27. *Let* $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$*. If* $\varphi \equiv \psi$ *is valid in* L*, then* $\neg(\varphi \wedge \neg\psi)$ *and* $\neg(\psi \wedge \neg\varphi)$ *are* L*-valid.*

4. Non-Fregean 3-Valued Paraconsistent Logics

In this section we consider non-Fregean extensions of 3-valued paraconsistent logics LP, LFI1, and LFI2, which belong to the family LFI of *Logics of Formal Inconsistency* as introduced in [9]. LFI-logics are paraconsistent logics that enable to express the notion of consistency within the object language in such a way that consistency may be logically independent of non-contradiction. For a detailed survey on LFI-logics, refer to [8]. The logic LP, named *Logic of Paradox* by Priest in [34], was first proposed by Asenjo in [3] and later popularized and developed by Priest and others (see [35,36]). As its name suggests, the main motivation of LP is to deal with logical paradoxes. The logic LFI1 is functionally equivalent to a 3-valued modal logic J3, whereas LFI2 provides a sound and complete semantics for the logic Ciore, as shown by Carnielli and others (see [7]).

We assume that the language of LFI1, LFI2, and LP contains the following connectives: negation $\neg$, conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$, and bi-implication $\leftrightarrow$. Models of LFI1, LFI2, and LP are defined as follows.

Definition 4.1. (LP-*model*) An LP-*model* is a structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ such that $U = \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, $D = \{\mathsf{n}, \mathsf{t}\}$, and the operations $\tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}$ are defined as:

$\tilde{\neg}$	
f	t
n	n
t	f

$\tilde{\wedge}$	f	n	t
f	f	f	f
n	f	n	n
t	f	n	t

$\tilde{\vee}$	f	n	t
f	f	n	t
n	n	n	t
t	t	t	t

$\tilde{\rightarrow}$	f	n	t
f	t	t	t
n	n	n	t
t	f	n	t

$\tilde{\leftrightarrow}$	f	n	t
f	t	n	f
n	n	n	n
t	f	n	t

Definition 4.2. (LFI1-*model*) An LFI1-*model* is a structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ such that $U = \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, $D = \{\mathsf{n}, \mathsf{t}\}$, and the operations $\tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}$ are defined as follows:

$\tilde{\neg}$	
f	t
n	n
t	f

$\tilde{\wedge}$	f	n	t
f	f	f	f
n	f	n	n
t	f	n	t

$\tilde{\vee}$	f	n	t
f	f	n	t
n	n	n	t
t	t	t	t

$\tilde{\rightarrow}$	f	n	t
f	t	t	t
n	f	n	t
t	f	n	t

$\tilde{\leftrightarrow}$	f	n	t
f	t	f	f
n	f	n	n
t	f	n	t

Definition 4.3. (LFI2-*model*) An LFI2-*model* is a structure $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ such that $U = \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, $D = \{\mathsf{n}, \mathsf{t}\}$, and the operations $\tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}$ are defined as:

$\tilde{\neg}$	
f	t
n	n
t	f

$\tilde{\wedge}$	f	n	t
f	f	f	f
n	f	n	t
t	f	t	t

$\tilde{\vee}$	f	n	t
f	f	t	t
n	t	n	t
t	t	t	t

$\tilde{\rightarrow}$	f	n	t
f	t	t	t
n	f	n	t
t	f	t	t

$\tilde{\leftrightarrow}$	f	n	t
f	t	f	f
n	f	n	t
t	f	t	t

Clearly, for every $\mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}, \mathsf{LP}\}$ there is exactly one model of L (up to isomorphism). We will denote the model of L by $\mathcal{M}_{\mathsf{L}}$. Note also that LP and LFI1 differ in their interpretations of $\rightarrow$ and $\leftrightarrow$, whereas LFI2 differs from both LP and LFI1 in the interpretations of the binary connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. The following is well known (see eg., [35]):

Proposition 4.4. *Let* $\mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}\}$*. Then, for every formula* φ*:*

1. φ *is valid in* LP *iff* φ *is valid in classical propositional logic.*
2. *If* φ *is valid in* L*, then* φ *is valid in classical propositional logic.*

The above easily implies:

Proposition 4.5. *Let* $\mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}\}$*. Then, every formula valid in* L *is valid in* LP*.*

The next proposition shows that LP is stronger than LFI1 and LFI2.

Proposition 4.6.

1. *The formula* $(p \wedge \neg q) \rightarrow \neg(p \rightarrow q)$ *is valid in* LP *and* LFI1*, whereas it is not valid in* LFI2*.*
2. *The formula* $(\neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p)) \rightarrow (p \leftrightarrow q)$ *is valid in* LP *and* LFI2*, whereas it is not valid in* LFI1*.*

Proof. Let φ be the formula $(p \wedge \neg q) \rightarrow \neg(p \rightarrow q)$. Its possible values in the models $\mathcal{M}_{\mathsf{LP}}$, $\mathcal{M}_{\mathsf{LFI1}}$, $\mathcal{M}_{\mathsf{LFI2}}$ are presented in Example 6.1 ("Appendix"). Thus, it follows that $v(\varphi) \in D$ in the models $\mathcal{M}_{\mathsf{LP}}$ and $\mathcal{M}_{\mathsf{LFI1}}$, for every valuation v. However, for a valuation v such that $v(p) = \mathsf{t}$ and $v(q) = \mathsf{n}$, we have $v(\varphi) = \mathsf{f}$, which implies that φ is not LFI2-valid.
Let φ be the formula $(\neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p)) \rightarrow (p \leftrightarrow q)$. Its possible values are presented in Example 6.2 (see "Appendix"). Clearly, $v(\varphi) \in D$ in $\mathcal{M}_{\mathsf{LP}}$ and $\mathcal{M}_{\mathsf{LFI2}}$, for every valuation v. Nonetheless, if v is a valuation in $\mathcal{M}_{\mathsf{LFI1}}$ such that $v(p) = \mathsf{f}$ and $v(q) = \mathsf{n}$, then $v(\varphi) = \mathsf{f}$, which implies that φ is not LFI1-valid. ■

By Propositions 4.5 and 4.6, we obtain:

Theorem 4.7.

1. LFI1 $\prec$ LP *and* LFI2 $\prec$ LP*.*
2. LFI1 $\not\prec\!\succ$ LFI2*.*

In order to obtain a non-Fregean version of a logic L, we need to extend its language with a connective $\equiv$ and its structures with a properly defined operation $\tilde{\equiv}$ that satisfies Definition 2.7. In the case of a three valued logic with $U = \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$ and $D = \{\mathsf{n}, \mathsf{t}\}$, an operation $\tilde{\equiv}$ corresponding to the connective $\equiv$ must be defined on U in such a way that the following conditions hold: $\mathsf{n} \tilde{\equiv} \mathsf{t} \notin D$ (as required by condition (nfl1)) and both $\mathsf{f} \tilde{\equiv} \mathsf{n} \notin D$ and $\mathsf{f} \tilde{\equiv} \mathsf{t} \notin D$ (as required by condition (nfl2)). Hence, in every non-Fregean extension of a three valued logic L with $U = \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$ and $D = \{\mathsf{n}, \mathsf{t}\}$, the operation $\tilde{\equiv}$ must satisfy condition (snfl). Thus, such an extension of L becomes a standard non-Fregean logic. It can be easily verified that none of the binary connectives in logics LFI1, LFI2, LP satisfy (snfl). We will consider two natural possible ways of defining $\tilde{\equiv}$ on $U = \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$.

Definition 4.8. ($\mathsf{L}_{\equiv}$-*model*) Let $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. A structure $(\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}, \{\mathsf{n}, \mathsf{t}\}, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be an $\mathsf{L}_{\equiv}$-*model* whenever $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ is an L-model and $\tilde{\equiv}$ is defined on U as:

$\tilde{\equiv}$	f	n	t
f	t	f	f
n	f	n	f
t	f	f	t

DEFINITION 4.9. ($\mathsf{sL}_{\equiv}$-*model*) Let $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. A structure $(\{\mathsf{f}, \mathsf{n}, \mathsf{t}\}, \{\mathsf{n}, \mathsf{t}\}, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be an $\mathsf{sL}_{\equiv}$-*model* (*strong* $\mathsf{L}_{\equiv}$-*model*) whenever $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ is an L-model and $\tilde{\equiv}$ is defined on U as:

$\tilde{\equiv}$	f	n	t
f	t	f	f
n	f	t	f
t	f	f	t

Clearly, for every $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$, there exists exactly one $\mathsf{L}_{\equiv}$-model (up to isomorphism) and exactly one $\mathsf{sL}_{\equiv}$-model (up to isomorphism). By $\mathsf{L}_{\equiv}$ and $\mathsf{sL}_{\equiv}$ we denote the non-Fregean extensions of L with their respective $\mathsf{L}_{\equiv}$-model and $\mathsf{sL}_{\equiv}$-model.

PROPOSITION 4.10. *Let* $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. *Then,* $\mathsf{L}_{\equiv}$ *and* $\mathsf{sL}_{\equiv}$ *are standard non-Fregean logics.*

Curiously, for each $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$, sets of valid formulas of $\mathsf{L}_{\equiv}$ and $\mathsf{sL}_{\equiv}$ are essentially different. In the "Appendix", we prove the following:

PROPOSITION 4.11. *For every* $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$, *the following holds:*

1. *The formula* $(p \equiv \neg p) \equiv p \vee (p \equiv \neg p) \equiv \neg p$ *is valid in* $\mathsf{L}_{\equiv}$, *whereas it is not valid in* $\mathsf{sL}_{\equiv}$.
2. *The formula* $(p \wedge \neg p) \vee (p \rightarrow ((q \equiv q) \equiv p))$ *is valid in* $\mathsf{sL}_{\equiv}$, *whereas it is not valid in* $\mathsf{L}_{\equiv}$.

The language of logics LP, $\mathsf{LFI1}$, $\mathsf{LFI2}$ has significantly less expressive power than its extension with the connective $\equiv$ that satisfies the condition (snfl). In particular, the connective $\equiv$ allows to express certain properties of $\mathsf{LFI1}$ and $\mathsf{LFI2}$ which do not hold in LP.

PROPOSITION 4.12. *Let* $\mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}\}$. *The formula*

$$(p \wedge \neg p) \vee (p \rightarrow ((q \rightarrow (p \wedge \neg p)) \equiv p) \vee ((q \rightarrow (p \wedge \neg p)) \equiv \neg p)))$$

is valid in $\mathsf{L}_{\equiv}$ *and* $\mathsf{sL}_{\equiv}$, *whereas it is not valid in* $\mathsf{LP}_{\equiv}$ *and* $\mathsf{sLP}_{\equiv}$.

PROOF. See the "Appendix". ■

Due to Propositions 4.6, 4.11, and 4.12, we obtain:

THEOREM 4.13. *Let* $\mathsf{L}, \mathsf{L}' \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{LFI2}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$. *If* $\mathsf{L} \neq \mathsf{L}'$, *then it holds that* $\mathsf{L} \nprec\!\!\nsucc \mathsf{L}'$.

Let us compare non-Fregean 3-valued paraconsistent logics with Grzegorczyk's logics. Recall that the formula $\neg(p \wedge \neg p)$ is L-valid, for all $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. Hence, we obtain:

Proposition 4.14. *Let* $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$*. Then, models of* $\mathsf{L}_{\equiv}$ *and* $\mathsf{sL}_{\equiv}$ *are* MGL*-models.*

As we showed in the previous section, the formula $p \vee \neg p$ is not MGL-valid, whereas it is L-valid, for every $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. Therefore, we have:

Theorem 4.15. *Let* $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$*. Then,* $\mathsf{MGL} \prec \mathsf{L}_{\equiv}$ *and* $\mathsf{MGL} \prec \mathsf{sL}_{\equiv}$*.*

The above theorem is not particularly surprising, given the inherent weakness of the logic MGL. More intriguing results could emerge from a comparison between our non-Fregean extensions of 3-valued paraconsistent logics and the logics LD, LDS, LDE.

Proposition 4.16. *Let* $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{LFI2}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$ *and* $\mathsf{L}' \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$*. Then, there exists a formula valid in* L*, but not valid in* L'*.*

Proof. Let φ be the formula $\bigvee_{i,j\in\{0,\ldots,3\},i\neq j} p_i \equiv q_j$. It is easy to verify that φ is valid in $\mathsf{L}_{\equiv}$ and $\mathsf{sL}_{\equiv}$, for every $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. Indeed, suppose φ does not hold in the model of $\mathsf{L}_{\equiv}$ or $\mathsf{sL}_{\equiv}$. Then, there exists a valuation v such that for all $i, j \in \{0, \ldots, 3\}$, if $i \neq j$, then $(v(p_i) \tilde{\equiv} v(p_j)) = \mathsf{f}$. However, $(v(p_i) \tilde{\equiv} v(p_j)) = \mathsf{f}$ iff $v(p_i) \neq v(p_j)$. In consequence, we obtain that for all $i, j \in \{0, \ldots, 3\}$, if $i \neq j$, then $v(p_i) \neq v(p_j)$. It is known that the latter implies the existence of at least 4 elements, which cannot hold in any structure of $\mathsf{L}_{\equiv}$ and $\mathsf{sL}_{\equiv}$, for $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. On the contrary, it is well known that logics LD, LDS, and LDE have models with more than 3 elements in which the formula φ cannot hold (for specific examples see examples in [18]). ■

Recall that the connectives $\to$ and $\leftrightarrow$ can be interpreted in logics LD, LDS, and LDE in any way, whereas in logics LP, $\mathsf{LFI1}$, $\mathsf{LFI2}$ they have fixed meaning. Consequently, for a trivial reason, there are many formulas of LP, $\mathsf{LFI1}$, $\mathsf{LFI2}$, which are valid in these logics, but not valid in any $\mathsf{L}' \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$.

Proposition 4.17. *Let* $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$*. The formula* $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ *is valid in* L*, but not in* $\mathsf{LFI2}_{\equiv}$ *and* $\mathsf{sLFI2}_{\equiv}$*.*

Proof. Let $\mathcal{M}$ be the model of $\mathsf{LFI2}_{\equiv}$ or $\mathsf{sLFI2}_{\equiv}$ v. Let v be a valuation in $\mathcal{M}$ such that $v(p) = \mathsf{n}$ and $v(q) = \mathsf{t}$. Then, $v(\neg(p \wedge q)) = \mathsf{f}$ and $v(\neg p \vee \neg q) = \mathsf{t}$. Therefore, $v(\neg(p \wedge q)) \neq v(\neg p \vee \neg q)$, and so $\mathcal{M}$ does not satisfy $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$. On the other hand, the formula $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ must be valid in every logic $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$, as it corresponds to condition (G7) of G-structures. ■

Due to the semantics of $\neg$, $\wedge$, and $\vee$ in LP and $\mathsf{LFI1}$, we obtain:

PROPOSITION 4.18. *Let* $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}\}$*. Then, the model of* L *is a* G*-structure.*

PROOF. See the "Appendix". ■

PROPOSITION 4.19. *Let* $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}\}$ *and* $\mathsf{L}' \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$*. Then, the following holds:*

1. *The model of* $\mathsf{sL}_{\equiv}$ *is a model of* L'.
2. *The model of* $\mathsf{L}_{\equiv}$ *is a model of* LDS *and* LDE*, but it is not an* LD*-model.*

PROOF. Let $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}\}$ and $\mathsf{L}' \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. By Proposition 4.18, it holds that each $\mathsf{L}_{\equiv}$-model and each $\mathsf{sL}_{\equiv}$-model is a G-structure. Thus, it suffices to show that the specific conditions of L'-models hold in each $\mathsf{sL}_{\equiv}$-model, the specific conditions of LDS-models and LDE-models hold in each $\mathsf{L}_{\equiv}$-model, while at least one of the specific conditions of LD-models does not hold in $\mathsf{LP}_{\equiv}$ and $\mathsf{LFI1}_{\equiv}$. For a detailed proof see the "Appendix". ■

Finally, we obtain:

THEOREM 4.20.

1. $\mathsf{LD} \nprec\nsucc \mathsf{LP}_{\equiv}$ *and* $\mathsf{LD} \nprec\nsucc \mathsf{LFI1}_{\equiv}$.
2. $\mathsf{LD} \prec \mathsf{sLP}_{\equiv}$ *and* $\mathsf{LD} \prec \mathsf{sLFI1}_{\equiv}$.
3. $\mathsf{L} \prec \mathsf{L}'$*, for all* $\mathsf{L} \in \{\mathsf{LDS}, \mathsf{LDE}\}$*,* $\mathsf{L}' \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}\}$.
4. $\mathsf{L} \nprec\nsucc \mathsf{LFI2}_{\equiv}$ *and* $\mathsf{L} \nprec\nsucc \mathsf{sLFI2}_{\equiv}$*, for every* $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$.

PROOF. By Proposition 4.16, there is a formula valid in $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{LFI1}_{\equiv}\}$, which is not valid in LD. Clearly, (AxLD_2) is valid in LD. However, we know from the proof of Proposition 4.19 (see the "Appendix") that (AxLD_2) is not valid in $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{LFI1}_{\equiv}\}$. Thus, LD is incomparable with $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{LFI1}_{\equiv}\}$. Now, let φ be a formula valid in LD. Then, Proposition 4.19 implies that it is valid in $\mathsf{sLP}_{\equiv}$ and $\mathsf{sLFI1}_{\equiv}$. Consequently, due to Proposition 4.16, we have that $\mathsf{LD} \prec \mathsf{sLP}_{\equiv}$ and $\mathsf{LD} \prec \mathsf{sLFI1}_{\equiv}$. Furthermore, if φ is valid in $\mathsf{L} \in \{\mathsf{LDS}, \mathsf{LDE}\}$, then it is valid in the model of $\mathsf{L}' \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}\}$. Thus, by Proposition 4.16, the item 3. follows. Finally, Propositions 4.16 and 4.17 imply the item 4. ■

We conclude this section with a comparison between our non-Fregean paraconsistent 3-valued logics and Suszko's logic SCI.

THEOREM 4.21. *Let* $\mathsf{L} \in \{\mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{LFI2}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$*. Then,* $\mathsf{L} \nprec\nsucc \mathsf{SCI}$*, whereas* $\mathsf{SCI} \prec \mathsf{LP}_{\equiv}$.

Proof. First, note that the formula $\bigvee_{i,j\in\{0,\dots,3\},i\neq j} p_i \equiv q_j$, expressing the fact that there are at most three denotations of formulas, is valid in all non-Fregean 3-valued logics in question (see the proof of Proposition 4.16). However, it is not valid in SCI, as SCI has models of arbitrary cardinality. Thus, $\mathsf{L} \not\prec \mathsf{SCI}$, for any $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{LFI2}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$.

Clearly, SCI is not paraconsistent. In particular, models of SCI satisfy the condition: $\tilde{\neg}a \in D$ iff $a \notin D$, for all $a \in U$. Hence, the formula $\neg(p \equiv \neg p)$ is valid in SCI. Nonetheless, it is not valid in any $\mathsf{L} \in \{\mathsf{sLP}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$. Indeed, for a valuation v such that $v(p) = \mathsf{n}$, the formula $\neg(p \equiv \neg p)$ takes the value f. Now, let us note that the formula $(p \wedge \neg p) \to (q \wedge \neg q)$ is valid in SCI, while it is not true in the models of $\mathsf{LFI1}_{\equiv}$ and $\mathsf{LFI2}_{\equiv}$ for a valuation v such that $v(p) = \mathsf{n}$ and $v(q) \in \{\mathsf{f}, \mathsf{t}\}$. Hence, SCI and L are incomparable, for every $\mathsf{L} \in \{\mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{LFI2}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$. However, no formula valid in SCI is invalid in $\mathsf{LP}_{\equiv}$. All classical tautologies are valid in the logics SCI and $\mathsf{LP}_{\equiv}$. Moreover, the logic SCI does not impose specific conditions on the structure of its models, except for (snfl), which also holds in $\mathsf{LP}_{\equiv}$. Consequently, we obtain $\mathsf{SCI} \prec \mathsf{LP}_{\equiv}$. ■

5. Non-Fregean Jaśkowski's Discussive Logic

Discussive (discursive) logic D_2, introduced by Jaśkowski in [23] and [24], aims to formalize discussions involving contradictory opinions. According to Jaśkowski ([25]), "if a thesis $\mathfrak{T}$ is recorded in a discussive system, its intuitive meaning ought to be interpreted so as if it were preceded by the symbol *Pos*, that is, the sense: 'it is possible that $\mathfrak{T}$.' This is how an impartial arbiter might understand the theses of the various participants in the discussion". To obtain a non-trivial system capable of handling contradictory opinions, Jaśkowski interprets the connectives of conjunction $\wedge$ and implication $\to$ within the modal language with the possibility operator:

$$\tau(\varphi \wedge \psi) \stackrel{\text{df}}{=} \varphi \wedge \Diamond\psi \quad \text{and} \quad \tau(\varphi \to \psi) \stackrel{\text{df}}{=} \Diamond\varphi \to \psi.$$

Then, a formula φ of the new language is a theorem of D_2 whenever $\Diamond\tau(\varphi)$ is a theorem of the modal logic $\mathsf{S5}$, where τ is a translation of the classical language into the modal one. Jaśkowski and his continuators have shown that such an approach yields a paraconsistent system that can be applied to classically inconsistent propositions without leading to overfullness.

One of the main problems in the study of D_2 has been its axiomatization within a propositional language not involving modality ([10,13,27,28,40])

A comprehensive survey on this issue is available in [33]. Various algebraic-style semantics for D_2 have been studied in [11, 14, 29]. Further (metalogical) properties and other issues related to Jaśkowski's logic have been presented in [26, 31, 32] Notably, it is known that the logic D_2 is not finite-valued ([29]). Hence, Jaśkowski's framework essentially differs from 3-valued approach to paraconsistency.

Subsequently, we assume that the set of propositional connectives of logic D_2 consists of negation $\neg$, conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$, and bi-implication $\leftrightarrow$, where $\wedge$, $\rightarrow$, and $\leftrightarrow$ are discussive connectives, in other works often denoted as $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$. The logic D_2 is defined semantically through the translation of its formulas into formulas of the propositional modal logic $\mathsf{S5}$.

Definition 5.1. (D_2) A D_2-formula φ is said to be D_2-*valid* if and only if $\Diamond\tau(\varphi)$ is $\mathsf{S5}$-valid, where $\tau : \mathsf{For}_{\mathsf{D}_2} \rightarrow \mathsf{For}_{\mathsf{S5}}$ is the translation defined as follows:

$$\begin{array}{ll}
\tau(p) = p, \text{ for } p \in \mathsf{Prop} & \tau(\neg\varphi) = \neg\tau(\varphi) \\
\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \Diamond\tau(\psi) & \tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi) \\
\tau(\varphi \rightarrow \psi) = \Diamond\tau(\varphi) \rightarrow \tau(\psi) & \tau(\varphi \leftrightarrow \psi) = (\Diamond\tau(\varphi) \rightarrow \tau(\psi)) \wedge \Diamond(\Diamond\tau(\psi) \rightarrow \tau(\varphi)).
\end{array}$$

Proposition 5.2. *The formulas* $\neg(p \wedge \neg p)$ *and* $\neg(p \vee \neg p) \rightarrow q$ *are valid in* D_2.

Proof. Let $\mathcal{M} = (U, R, m)$ be a Kripke $\mathsf{S5}$-model. If there exists $s \in U$ such that $s \notin m(p)$, then $\mathcal{M}, s \models \neg(p \wedge \Diamond\neg p)$, and so $\mathcal{M}, s \models \Diamond\neg(p \wedge \Diamond\neg p)$. If for all $s \in U$ it holds that $s \in m(p)$, then for all $t \in U$ we have that $\mathcal{M}, t \not\models \Diamond\neg p$, and so $\mathcal{M}, t \models \neg(p \wedge \Diamond\neg p)$. Thus, if $s \in m(p)$ for all $s \in U$, then there exists $t \in U$ such that $\mathcal{M}, t \models \neg(p \wedge \Diamond\neg p)$. Therefore, we have proved that for every Kripke $\mathsf{S5}$-model $\mathcal{M} = (U, R, m)$ and for every $s \in U$ it holds that $\mathcal{M}, s \models \Diamond\neg(p \wedge \Diamond\neg p)$. Since $\tau(\neg(p \wedge \neg p)) = \neg(p \wedge \Diamond\neg p)$, we obtain that $\Diamond\tau(\neg(p \wedge \neg p))$ is valid in $\mathsf{S5}$.

Now, we will show that the translation of $\neg(p \vee \neg p) \rightarrow q$, that is, the modal formula $\Diamond(\Diamond\neg(p \vee \neg p) \rightarrow q)$, is true in every Kripke $\mathsf{S5}$-model. Let $\mathcal{M} = (U, R, m)$ be a Kripke $\mathsf{S5}$-model. Then, $\mathcal{M}, s \not\models \neg(p \vee \neg p)$, for every $s \in U$, and so it also holds that $\mathcal{M}, s \not\models \Diamond\neg(p \vee \neg p)$. Thus, $\mathcal{M}, s \models \Diamond\neg(p \vee \neg p) \rightarrow q$, for every $s \in U$, which implies that $\mathcal{M} \models \Diamond(\Diamond\neg(p \vee \neg p) \rightarrow q)$. Hence, the translation of $\neg(p \vee \neg p) \rightarrow q$ is true in every $\mathsf{S5}$-model, and so it is valid in D_2. ■

Proposition 5.3. *The formulas* $p \rightarrow (\neg p \rightarrow \neg(p \vee \neg p))$ *and* $\neg((p \wedge q) \wedge \neg(q \wedge p))$ *are not valid in logic* D_2.

Proof. Denote the formulas $p \to (\neg p \to \neg(p \vee \neg p))$ and $\neg((p \wedge q) \wedge \neg(q \wedge p))$ by φ and ψ, respectively. The translations of φ and ψ are:

$\tau(\varphi) = \Diamond p \to (\Diamond\neg p \to \neg(p \vee \neg p))$,

$\tau(\psi) = \neg((p \wedge \Diamond q) \wedge \Diamond\neg(q \wedge \Diamond p))$.

We will show that $\Diamond\tau(\varphi)$ and $\Diamond\tau(\psi)$ are not valid in $\mathsf{S5}$, that is they are not true in some Kripke $\mathsf{S5}$-models.

Let $\mathcal{M} = (U, R, m)$ be such that $U = \{a, b\}$, $R = U^2$, and $m(p) = \{a\}$. Clearly, $\mathcal{M}$ is a model of $\mathsf{S5}$. First, note that $\mathcal{M}, s \not\models \neg(p \vee \neg p)$, for every $s \in \{a, b\}$. Since $(a, a), (b, a) \in R$ and $a \in m(p)$, we obtain that $\mathcal{M}, s \models \Diamond p$, for every $s \in U$. Similarly, $(a, b), (b, b) \in R$ and $b \notin m(p)$ imply that $\mathcal{M}, s \models \Diamond\neg p$, for every $s \in U$. Thus, each $s \in U$ satisfies $\Diamond p$ and $\Diamond\neg p$, while it does not satisfy $\neg(p \vee \neg p)$. Hence, for every $s \in U$ it holds that $\mathcal{M}, s \not\models \tau(\varphi)$, and so $\mathcal{M}, s \not\models \Diamond\tau(\varphi)$. Therefore, φ is not D_2-valid.

Now, let $\mathcal{M} = (U, R, m)$ be a Kripke $\mathsf{S5}$-model such that $U = \{a, b\}$, $R = U^2$, $m(p) = \{a, b\}$, and $m(q) = \{b\}$. Clearly, $\mathcal{M}, s \models p$, for every $s \in U$. Moreover, $\mathcal{M}, a \not\models q$, and so $\mathcal{M}, a \models \neg(q \wedge \Diamond p)$. Since R is the universal relation, we obtain that $\mathcal{M}, s \models \Diamond\neg(q \wedge \Diamond p)$ and $\mathcal{M}, s \models \Diamond q$, for every $s \in U$. Thus, $\mathcal{M}, s \models (p \wedge \Diamond q) \wedge \Diamond\neg(q \wedge \Diamond p)$, for every $s \in U$, which implies that $\mathcal{M}, s \not\models \neg((p \wedge \Diamond q) \wedge \Diamond\neg(q \wedge \Diamond p))$, for every $s \in U$. Therefore, there is no $s \in U$ such that $\mathcal{M}, s \models \tau(\psi)$, and hence $\mathcal{M}, s \not\models \Diamond\tau(\psi)$, for every $s \in U$. Consequently, $\Diamond\tau(\psi)$ is not $\mathsf{S5}$-valid, and so ψ is not valid in D_2. ■

Proposition 5.4. *Let* $\mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}\}$. *Then,* $\mathsf{L} \nprec\!\!\succ \mathsf{D}_2$ *and* $\mathsf{D}_2 \prec \mathsf{LP}$.

Proof. Let $\mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}\}$. By Proposition 5.2, the formula $\neg(p \vee \neg p) \to q$ is valid in D_2. However, it is not valid in L. Indeed, let v be a valuation in the model of L such that $v(p) = \mathsf{n}$ and $v(q) = \mathsf{f}$. Clearly, $v(\neg(p \vee \neg p)) = \mathsf{n}$, which implies that $v(\neg(p \vee \neg p) \to q) = (\mathsf{n} \to \mathsf{f}) = \mathsf{f}$. Furthermore, by Proposition 5.3, the formula $\varphi = (p \to (\neg p \to \neg(p \vee \neg p)))$ is not valid in D_2. Let us consider the possible values of φ in L. It can be easily checked that $v(\varphi) = \mathsf{t}$ whenever v is a valuation in the model of L such that $v(p) \in \{\mathsf{f}, \mathsf{t}\}$. So let v be a valuation in the model of L such that $v(p) = \mathsf{n}$. Then, each subformula of φ takes the value n, and so $v(\varphi) = \mathsf{n}$. Hence, $v(\varphi) \in \{\mathsf{n}, \mathsf{t}\}$, for all valuations v in the model of L. Consequently, we have proved that L and D_2 are incomparable.

The formula $(p \to (\neg p \to \neg(p \vee \neg p)))$ is a classical tautology, and so by Proposition 4.4, it is valid in LP. As it is not valid in D_2, we obtain that $\mathsf{LP} \not\preceq \mathsf{D}_2$. On the other hand, it is known (see Theorem 3 in [29]) that if φ is valid in D_2, then it is valid in classical propositional logic. Therefore, by Proposition 4.4, φ is valid in LP, from which it follows that $\mathsf{D}_2 \prec \mathsf{LP}$. ■

Various sound and complete algebraic semantics for D_2 have been presented in [29] and [11]. For the sake of simplicity of the presentation, we will use the semantics from [11].

Definition 5.5. (D_2-*model*) A D_2-model is a tuple $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ such that $1, 0 \in U$, $1 \neq 0$, $(U, \tilde{\neg}, \tilde{\vee}, 1, 0)$ is a Boolean algebra, $U \backslash D = \{0\}$, and the following conditions hold for all $a, b \in U$:

$$a \tilde{\wedge} b = \begin{cases} 0, & \text{if } b = 0 \\ a, & \text{otherwise} \end{cases} \quad a \tilde{\rightarrow} b = \begin{cases} 1, & \text{if } a = 0 \\ b, & \text{otherwise} \end{cases} \quad a \tilde{\leftrightarrow} b = \begin{cases} 1, & \text{if } a = b = 0 \\ 0, & \text{if } a = 0 \neq b \\ b, & \text{otherwise.} \end{cases}$$

The notions of satisfaction, truth, and validity are defined for D_2 as in Section 2. As stated in [11], the class of all D_2-models provides a sound and complete algebraic-style semantics for D_2.

Theorem 5.6. *For every formula φ, φ is D_2-valid if and only φ is true in all D_2-models.*

Observe that D_2 is not non-Fregean in the sense of Definition 2.7. Indeed, none of its binary connectives can serve as a connective $\equiv$ satisfying (nfl1) and (nfl2). In particular, none of the connectives $\wedge, \vee, \rightarrow$ fullfils the condition (nfl2): in any model of D_2, it holds that $0 \tilde{\wedge} 0 \notin D$, $0 \tilde{\vee} 0 \notin D$, $0 \tilde{\rightarrow} 1 \in D$, while $1 \tilde{\rightarrow} 0 \notin D$. Moreover, in any model of D_2, the connective $\tilde{\leftrightarrow}$ does not satisfy (nfl1), as $a \tilde{\leftrightarrow} b \in D$ for all $a, b \in D = U \setminus \{0\}$. In other words, the connective $\leftrightarrow$ of D_2 identifies all true formulas: if φ and ψ are true in a model of D_2, then $\varphi \leftrightarrow \psi$ is also true.

A non-Fregean version of D_2, denoted by $\mathsf{D}_2^{\equiv}$, is obtained by adding $\equiv$ to the language of D_2 and imposing the condition (snfl) on its interpretations.

Definition 5.7. ($\mathsf{D}_2^{\equiv}$-*model*) A structure $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ is said to be a $\mathsf{D}_2^{\equiv}$-*model* whenever $(U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow})$ is a D_2-model and $\tilde{\equiv}$ is a binary operation on U such that $a \tilde{\equiv} b \in D$ if and only if $a = b$, for all $a, b \in U$.

Theorem 5.8. $\mathsf{MGL} \prec \mathsf{D}_2^{\equiv}$.

Proof. Let $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be a $\mathsf{D}_2^{\equiv}$-model. By Proposition 5.2, $\tilde{\neg}(a \tilde{\wedge} \tilde{\neg} a) \in D$, for every $a \in D$. Hence, the condition (WGL_1) holds in $\mathcal{M}$. Since the operation $\tilde{\equiv}$ satisfies (snfl), the conditions (WGL_2) and (WGL_4)–(WGL_6) are true in $\mathcal{M}$. Now, observe that $a \tilde{\wedge} b = 0 \notin D$ iff $a = 0 \notin D$ or $b = 0 \notin D$. Thus, for all $a, b \in U$ it holds that $a \tilde{\wedge} b \in D$ iff $a \in D$ and $b \in D$. Consequently, the condition (WGL_3) holds in $\mathcal{M}$. Hence, every model of $\mathsf{D}_2^{\equiv}$ is an MGL-model. Therefore, if a formula is valid in MGL, then its is valid in $\mathsf{D}_2^{\equiv}$. On the other hand, the formula $\varphi \vee \neg\varphi$ is valid in $\mathsf{D}_2^{\equiv}$, while it is not valid in MGL. ■

It is easy to verify that $\mathsf{D}_2^{\equiv}$ satisfies the condition (nfl1) of Definition 2.7. Moreover, straightforwardly by the definition of D_2-models, we obtain that $p, \neg p \not\models q$, which by Proposition 5.8 yields the following:

THEOREM 5.9. *The logic* $\mathsf{D}_2^{\equiv}$ *is non-Fregean and paraconsistent.*

Let us compare $\mathsf{D}_2^{\equiv}$ with the logics LD, LDS, and LDE.

THEOREM 5.10. *Let* $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. *Then,* $\mathsf{L} \nleftrightarrow \mathsf{D}_2^{\equiv}$.

PROOF. Let $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. Due to the condition (G2) of G-structures, the formula $(p \wedge q) \equiv (q \wedge p)$ is valid in L. Thus, by Proposition 3.27, the formula $\neg((p \wedge q) \wedge \neg(q \wedge p))$ is also valid in L, for every $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$. However, by Proposition 5.3, the formula $\neg((p \wedge q) \wedge \neg(q \wedge p))$ is not valid in D_2, and consequently, it is not valid in $\mathsf{D}_2^{\equiv}$. On the other hand, models of $\mathsf{D}_2^{\equiv}$ assume some Boolean laws which do not hold in LD, LDS, LDE. In particular, $(U, \tilde{\neg}, \tilde{\vee}, 1, 0)$ is a Boolean algebra, and so the formula $(p \vee \neg(\neg p \vee q))) \equiv p$ expressing the absorption law must be true in all $\mathsf{D}_2^{\equiv}$-models. However, it is not valid in L. Indeed, for each $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$, it can be easily verified that the structure defined in Example 34 from [18] is an L-model in which there exist $a, b \in U$ such that $(a \,\tilde{\vee}\, \tilde{\neg}(\tilde{\neg} a \,\tilde{\vee}\, b))) \mathrel{\tilde{\equiv}} a \notin D$. Hence, for every $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$, there is a formula valid in L which is not valid in $\mathsf{D}_2^{\equiv}$, and there is a formula valid in $\mathsf{D}_2^{\equiv}$ which is not valid in L, which ends the proof. ∎

A comparison between $\mathsf{D}_2^{\equiv}$ and the non-Fregean 3-valued logics discussed in the previous section yields the following:

THEOREM 5.11. *Let* $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{LFI2}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$. *Then,* $\mathsf{L} \nleftrightarrow \mathsf{D}_2^{\equiv}$.

PROOF. By Proposition 5.4, D_2 is incomparable with LFI1 and LFI2. In consequence, $\mathsf{D}_2^{\equiv}$ is incomparable with each $\mathsf{L} \in \{\mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}, \mathsf{LFI2}_{\equiv}, \mathsf{sLFI2}_{\equiv}\}$. By Proposition 5.4, there exists a formula valid in LP, which is not valid in D_2. Hence, $\mathsf{LP}_{\equiv} \not\preceq \mathsf{D}_2^{\equiv}$ and $\mathsf{sLP}_{\equiv} \not\preceq \mathsf{D}_2^{\equiv}$. On the other hand, the formula $((p \vee \neg p) \wedge (q \vee \neg q)) \equiv (p \vee \neg p)$ is valid in $\mathsf{D}_2^{\equiv}$, since for every valuation v in a $\mathsf{D}_2^{\equiv}$-model, it holds that $v(p \vee \neg p) = v(q \vee \neg q) = 1$, which implies that $v((p \vee \neg p) \wedge (q \vee \neg q)) = v(p \vee \neg p)$. Nonetheless, the formula $((p \vee \neg p) \wedge (q \vee \neg q)) \equiv (p \vee \neg p)$ is not valid in $\mathsf{LP}_{\equiv}$ and $\mathsf{sLP}_{\equiv}$. Indeed, if v is a valuation such that $v(p) = \mathsf{t}$ and $v(q) = \mathsf{n}$, we have $v((p \vee \neg p) \wedge (q \vee \neg q)) = (\mathsf{t} \wedge \mathsf{n}) = \mathsf{n} \neq \mathsf{t} = v(p \vee \neg p)$. Therefore, $\mathsf{D}_2^{\equiv}$ is incomparable with $\mathsf{LP}_{\equiv}$ and $\mathsf{sLP}_{\equiv}$. ∎

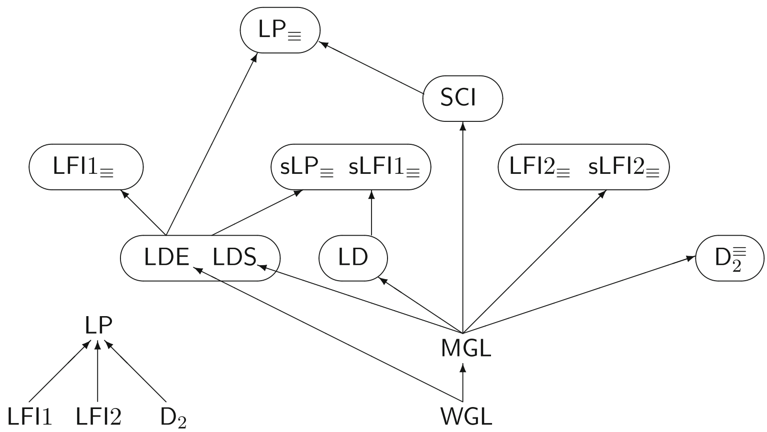

Figure 1. Dependencies among the considered non-Fregean logics

We conclude this section with the theorem showing incomparability of $\mathsf{D}_2^{\equiv}$ with the classical non-Fregean logic SCI.

THEOREM 5.12. $\mathsf{SCI} \not\prec\succ \mathsf{D}_2^{\equiv}$.

PROOF. We know from the proof of Theorem 5.10 that $(p \vee \neg(\neg p \vee q))) \equiv p$ is valid in $\mathsf{D}_2^{\equiv}$. Clearly, models of SCI do not impose any Boolean laws, and consequently $(p \vee \neg(\neg p \vee q))) \equiv p$ is not SCI-valid. Moreover, it is quite obvious that all formulas valid in classical propositional logic are valid in SCI. Therefore, the formulas from Proposition 5.3 are valid in SCI, whereas they are not valid in $\mathsf{D}_2^{\equiv}$. Hence, SCI and $\mathsf{D}_2^{\equiv}$ are incomparable. ■

6. Conclusions

We have considered several non-Fregean paraconsistent logics: Grzegorczyk's logics of descriptions, non-Fregean extensions of 3-valued logics LP, LFI1, LFI2, and the non-Fregean extension of Jaśkowski's discussive logic D_2. We have proved that Grzegorczyk's logics LD, LDS, LDE are either weaker than or incomparable to other non-Fregean paraconsistent logics. Due to Theorems 3.23, 4.7, 4.13, 4.15, 4.20, 4.21, 5.8, 5.10, 5.11, and 5.12, the logical dependencies among the non-Fregean logics under considerations are as depicted in Figure 1, where $\mathsf{L} \to \mathsf{L}'$ represents $\mathsf{L} \prec \mathsf{L}'$.

Hence, the results presented in the paper provide a partial negative answer to the question raised in [21]: Is L, for $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$, equivalent to a previously known paraconsistent logic?

It is worth emphasizing that the non-Fregean extensions of LP, LFI1, LFI2, and D_2 are more expressive than their original counterparts. Indeed, LP is the strongest logic among LP, LFI1, LFI2, D_2, and so we are not able to express any property of LFI1, LFI2, D_2 that does not hold in LP. The connective $\equiv$ significantly alters this landscape: all non-Fregean extensions of LP, LFI1, LFI2, D_2 considered in the paper become mutually incomparable. Hence, the connective $\equiv$ serves as a means to express diverse properties of the ontology underlying the logics under consideration.

Our results prompt further questions. Firstly, it is worth noting that the relation $\prec$ depends solely on valid formulas of logics under consideration. However, we can strengthen $\prec$ to $\prec^*$, defined as follows: $\mathsf{L} \prec^* \mathsf{L}'$ if and only if (i) for any set X of formulas and any formula φ, if $X \models_{\mathsf{L}} \varphi$, then $X \models_{\mathsf{L}'} \varphi$, and (ii) there is a set X of formulas and a formula φ such that $X \models_{\mathsf{L}'} \varphi$ and $X \not\models_{\mathsf{L}} \varphi$. Clearly, if $\mathsf{L} \prec \mathsf{L}'$, then $\mathsf{L}' \not\prec^* \mathsf{L}$. Certainly, $\prec^*$ allows to separate SCI and $\mathsf{LP}_{\equiv}$, as for instance $p, \neg p \models_{\mathsf{SCI}} q$, while $p, \neg p \not\models_{\mathsf{LP}_{\equiv}} q$. However, the general question of whether $\mathsf{L} \prec \mathsf{L}'$ implies $\mathsf{L} \prec^* \mathsf{L}'$ remains open.

Secondly, it would be interesting to investigate extensions of $\mathsf{D}_2^{\equiv}$ that introduce additional, yet philosophically natural, conditions on the interpretation of $\equiv$ in models of $\mathsf{D}_2^{\equiv}$ (for instance, conditions like $(a \mathrel{\tilde{\equiv}} b) = 1$ for all $a, b \in U$ such that $a = b$). Could we then express a property of the ontology of D_2 that is not expressible in $\mathsf{D}_2^{\equiv}$?

Finally, the potential for future research extends to a comparison between Grzegorczyk's logics LD, LDS, LDE and other paraconsistent logics not addressed in this paper, such as relevant or connexive logics. A comprehensive map of dependencies among Grzegorczyk's logics and other paraconsistent logics would offer a definite answer to Grzegorczyk's original question of whether LD is a new logic.

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Appendix

Proof of Proposition 3.18. (LDS_1) does not hold in $\mathcal{M}_{\mathsf{LD}}$ for $a = c = \emptyset$, $b = \{0\}$, and $d = \{1\}$:
$((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) = ((\emptyset \tilde{\equiv} \{0\}) \cap (\emptyset \tilde{\equiv} \{1\})) = (\{0\} \cap \{0\}) = \{0\}$, but $((a \tilde{\equiv} c) \tilde{\equiv} (b \tilde{\equiv} d)) = ((\emptyset \tilde{\equiv} \emptyset) \tilde{\equiv} (\{0\} \tilde{\equiv} \{1\})) = (A \tilde{\equiv} \emptyset) = \emptyset$, and so $(((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) \tilde{\wedge} ((a \tilde{\equiv} c) \tilde{\equiv} (b \tilde{\equiv} d))) = (\{0\} \cap \emptyset) = \emptyset$.
(LDS_2) does not hold in $\mathcal{M}_{\mathsf{LD}}$ for $a = d = A$, $b = \{1\}$, and $c = \{0\}$:
$((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) = ((A \tilde{\equiv} \{1\}) \cap (\{0\} \tilde{\equiv} A)) = (\{0\} \cap \{0\}) = \{0\}$, but $((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} d)) = ((A \cap \{0\}) \tilde{\equiv} (\{1\} \cap A)) = (\{0\} \tilde{\equiv} \{1\}) = \emptyset$, and so $(((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) \tilde{\wedge} ((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} d))) = (\{0\} \cap \emptyset) = \emptyset$.
(LDS_3) does not hold in $\mathcal{M}_{\mathsf{LD}}$ for $a = d = \emptyset$, $b = \{1\}$, and $c = \{0\}$:
$((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) = ((\emptyset \tilde{\equiv} \{1\}) \cap (\{0\} \tilde{\equiv} \emptyset)) = (\{0\} \cap \{0\}) = \{0\}$, but $((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} d)) = ((\emptyset \cup \{0\}) \tilde{\equiv} (\{1\} \cup \emptyset)) = (\{0\} \tilde{\equiv} \{1\}) = \emptyset$, and so $(((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) \tilde{\wedge} ((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} d))) = (\{0\} \cap \emptyset) = \emptyset$.
(LDE_1) does not hold in $\mathcal{M}_{\mathsf{LD}}$ for $a = \emptyset$, $b = \{1\}$, and $c = \{0\}$:
$((a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\equiv} c)) = ((\emptyset \tilde{\equiv} \{1\}) \tilde{\wedge} (\emptyset \tilde{\equiv} \{0\})) = (\{0\} \cap \{0\}) = \{0\}$, but $((a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\equiv} c)) = ((\emptyset \tilde{\equiv} \{1\}) \tilde{\wedge} (\{1\} \tilde{\equiv} \{0\})) = (\{0\} \cap \emptyset) = \emptyset$.
(LDE_2) does not hold in $\mathcal{M}_{\mathsf{LD}}$ for $a = \emptyset$ and $b = c = \{0\}$:
$((a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\wedge} c)) = ((\emptyset \tilde{\equiv} \{0\}) \tilde{\wedge} (\emptyset \tilde{\wedge} \{0\})) = (\{0\} \cap (\emptyset \cap \{0\})) = (\{0\} \cap \emptyset) = \emptyset$, but $((a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\wedge} c)) = ((\emptyset \tilde{\equiv} \{0\}) \tilde{\wedge} (\{0\} \tilde{\wedge} \{0\}) = (\{0\} \cap (\{0\} \cap \{0\})) = (\{0\} \cap \{0\}) = \{0\}$.
(LDE_3) does not hold in $\mathcal{M}_{\mathsf{LD}}$ for $a = c = \emptyset$ and $b = \{0\}$:
$((a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\vee} c)) = ((\emptyset \tilde{\equiv} \{0\}) \tilde{\wedge} (\emptyset \tilde{\vee} \emptyset)) = (\{0\} \cap (\emptyset \cup \emptyset)) = (\{0\} \cap \emptyset) = \emptyset$, but $((a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\vee} c)) = ((\emptyset \tilde{\equiv} \{0\}) \tilde{\wedge} (\{0\} \tilde{\vee} \emptyset)) = (\{0\} \cap (\{0\} \cup \emptyset)) = (\{0\} \cap \{0\}) = \{0\}$. ■

Proof of Proposition 3.19. (LD_1) does not hold in $\mathcal{M}_{\mathsf{LDS}}$ for $a = b = 0$ and $c = 1$:
$(a \tilde{\equiv} b) = (0 \tilde{\equiv} 0) = 3$, but $((a \tilde{\equiv} c) \tilde{\equiv} (b \tilde{\equiv} c)) = ((0 \tilde{\equiv} 1) \tilde{\equiv} (0 \tilde{\equiv} 1)) = (1 \tilde{\equiv} 1) = (1 \tilde{\vee} 2) = 2$, and so $((a \tilde{\equiv} b) \tilde{\wedge} ((a \tilde{\equiv} c) \tilde{\equiv} (b \tilde{\equiv} c))) = (3 \tilde{\wedge} 2) = 2$
(LD_2) does not hold in $\mathcal{M}_{\mathsf{LDS}}$ for $a = b = 3$ and $c = 1$:

$(a \tilde{\equiv} b) = (3 \tilde{\equiv} 3) = 3$, but $((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} c)) = ((3 \tilde{\wedge} 1) \tilde{\equiv} (3 \tilde{\wedge} 1)) = (1 \tilde{\equiv} 1) = (1 \tilde{\vee} 2) = 2$, and so $((a \tilde{\equiv} b) \tilde{\wedge} ((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} c))) = (3 \tilde{\wedge} 2) = 2$

(LD_3) does not hold in $\mathcal{M}_{\mathsf{LDS}}$ for $a = b = 0$ and $c = 1$:
$(a \tilde{\equiv} b) = (0 \tilde{\equiv} 0) = 3$, but $((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} c)) = ((0 \tilde{\vee} 1) \tilde{\equiv} (0 \tilde{\vee} 1)) = (1 \tilde{\equiv} 1) = (1 \tilde{\vee} 2) = 2$, and so $((a \tilde{\equiv} b) \tilde{\wedge} ((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} c))) = (3 \tilde{\wedge} 2) = 2$

(LDE_1) does not hold in $\mathcal{M}_{\mathsf{LDS}}$ for $a = 1$, $b = 0$, and $c = 3$:
$((a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\equiv} c)) = ((1 \tilde{\equiv} 0) \tilde{\wedge} (1 \tilde{\equiv} 3)) = (1 \tilde{\wedge} 1) = 1$, but $((a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\equiv} c)) = ((1 \tilde{\equiv} 0) \tilde{\wedge} (0 \tilde{\equiv} 3)) = (1 \tilde{\wedge} 0) = 0$.

(LDE_2) does not hold in $\mathcal{M}_{\mathsf{LDS}}$ for $a = 1$, $b = 0$, and $c = 3$:
$((a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\wedge} c)) = ((1 \tilde{\equiv} 0) \tilde{\wedge} (1 \tilde{\wedge} 3)) = (1 \tilde{\wedge} 1) = 1$, but $((a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\wedge} c)) = ((1 \tilde{\equiv} 0) \tilde{\wedge} (0 \tilde{\wedge} 3)) = (1 \tilde{\wedge} 0) = 0$.

(LDE_3) does not hold in $\mathcal{M}_{\mathsf{LDS}}$ for $a = 1$ and $b = c = 0$:
$((a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\vee} c)) = ((1 \tilde{\equiv} 0) \tilde{\wedge} (1 \tilde{\vee} 0)) = (1 \tilde{\wedge} 1) = 1$, but $((a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\vee} c)) = ((1 \tilde{\equiv} 0) \tilde{\wedge} (0 \tilde{\vee} 0)) = (1 \tilde{\wedge} 0) = 0$. ■

Proof of Proposition 3.20.
(LD_2) does not hold in $\mathcal{M}_{\mathsf{LDE}}$ for $a = (1,1)$, $b = (2,1)$, and $c = (2,0)$:
$(a \tilde{\equiv} b) = ((1,1) \tilde{\equiv} (2,1)) = (0,1)$, but $((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} c)) = (((1,1) \tilde{\wedge} (2,0)) \tilde{\equiv} ((2,1) \tilde{\wedge} (2,0))) = ((1,0) \tilde{\equiv} (2,0)) = (0,0)$, and so $((a \tilde{\equiv} b) \tilde{\wedge} ((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} c))) = ((0,1) \tilde{\wedge} (0,0)) = (0,0)$.

(LD_3) does not hold in $\mathcal{M}_{\mathsf{LDE}}$ for $a = (1,1)$, $b = (2,1)$, and $c = (0,2)$:
$(a \tilde{\equiv} b) = ((1,1) \tilde{\equiv} (2,1)) = (0,1)$, but $((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} c)) = (((1,1) \tilde{\vee} (0,2)) \tilde{\equiv} ((2,1) \tilde{\vee} (0,2))) = ((1,2) \tilde{\equiv} (2,2)) = (0,0)$, and so $((a \tilde{\equiv} b) \tilde{\wedge} ((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} c))) = ((0,1) \tilde{\wedge} (0,0)) = (0,0)$.

(LDS_2) does not hold in $\mathcal{M}_{\mathsf{LDE}}$ for $a = (1,1)$, $b = (2,1)$, and $c = d = (2,0)$:
$((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) = (((1,1) \tilde{\equiv} (2,1)) \tilde{\wedge} ((2,0) \tilde{\equiv} (2,0))) = ((0,1) \tilde{\wedge} (0,1)) = (0,1)$, but $((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} d)) = (((1,1) \tilde{\wedge} (2,0)) \tilde{\equiv} ((2,1) \tilde{\wedge} (2,0))) = ((1,0) \tilde{\equiv} (2,0)) = (0,0)$, and so $((a \tilde{\equiv} b) \tilde{\wedge} ((a \tilde{\wedge} c) \tilde{\equiv} (b \tilde{\wedge} d))) = ((0,1) \tilde{\wedge} (0,0)) = (0,0)$.

(LDS_3) does not hold in $\mathcal{M}_{\mathsf{LDE}}$ for $a = (1,1)$, $b = (2,1)$, and $c = d = (0,2)$:
$((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) = (((1,1) \tilde{\equiv} (2,1)) \tilde{\wedge} ((0,2) \tilde{\equiv} (0,2))) = ((0,1) \tilde{\wedge} (0,1)) = (0,1)$, but $((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} d)) = (((1,1) \tilde{\vee} (0,2)) \tilde{\equiv} ((2,1) \tilde{\vee} (0,2))) = ((1,2) \tilde{\equiv} (2,2)) = (0,0)$, and so $((a \tilde{\equiv} b) \tilde{\wedge} ((a \tilde{\vee} c) \tilde{\equiv} (b \tilde{\vee} c))) = ((0,1) \tilde{\wedge} (0,0)) = (0,0)$. ■

Proof of Proposition 3.25. Let $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$ and let $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be an L-model. Take any $a \in U$. Then, in all G-structures the following holds:

(1) $\tilde{\neg}(a \tilde{\wedge} \tilde{\neg} a) \in D$ — (WGL_1)
(2) $\tilde{\neg}(a \tilde{\wedge} \tilde{\neg} a) \tilde{\equiv} (\tilde{\neg} a \tilde{\vee} \tilde{\neg}\tilde{\neg} a) \in D$ — $(\mathsf{G7})$
(3) $(\tilde{\neg} a \tilde{\vee} \tilde{\neg}\tilde{\neg} a) \tilde{\equiv} (\tilde{\neg}\tilde{\neg} a \tilde{\vee} \tilde{\neg} a) \in D$ — $(\mathsf{G2})$
(4) $(\tilde{\neg} a \tilde{\vee} \tilde{\neg}\tilde{\neg} a) \in D$ — (WGL_4), (1), and (2)
(5) $(\tilde{\neg}\tilde{\neg} a \tilde{\vee} \tilde{\neg} a) \in D$ — (WGL_4), (3), and (4)

(6) $\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a \in D$ (G6)

Now, we have three cases.

Case: $\mathsf{L} = \mathsf{LD}$

(7_{LD}) $(\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\equiv}\, ((\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, ((\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a))) \in D$ (LD_3)

(8_{LD}) $((\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, ((\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a))) \in D$ (WGL_4), (6), (7_{LD})

(9_{LD}) $(\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_3) and (8_{LD})

(10_{LD}) $(a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_4), (5), and (9_{LD}).

Case: $\mathsf{L} = \mathsf{LDS}$

(7_{LDS}) $((\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (\tilde{\neg}a \,\tilde{\equiv}\, \tilde{\neg}a)) \tilde{\equiv}$
$(((\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (\tilde{\neg}a \,\tilde{\equiv}\, \tilde{\neg}a)) \,\tilde{\wedge}\, ((\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a))) \in D$ (LDS_3)

(8_{LDS}) $(\tilde{\neg}a \,\tilde{\equiv}\, \tilde{\neg}a) \in D$ (WGL_2)

(9_{LDS}) $(\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (\tilde{\neg}a \,\tilde{\equiv}\, \tilde{\neg}a)) \in D$ (WGL_3), (6), and (8_{LDS})

(10_{LDS}) $((\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (\tilde{\neg}a \,\tilde{\equiv}\, \tilde{\neg}a)) \,\tilde{\wedge}\, ((\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a)) \in D$
(WGL_4), (7_{LDS}), and (9_{LDS})

(11_{LDS}) $(\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_3) and (10_{LDS})

(12_{LDS}) $(a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_4), (5), and (11_{LDS}).

Case: $\mathsf{L} = \mathsf{LDE}$

(7_{LDE}) $((\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a)) \,\tilde{\equiv}\, ((\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (a \,\tilde{\vee}\, \tilde{\neg}a)) \in D$ (LDE_3)

(8_{LDE}) $(\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (\tilde{\neg}\tilde{\neg}a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_3), (6), and (5)

(9_{LDE}) $(\tilde{\neg}\tilde{\neg}a \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, (a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_4), (7_{LDE}), (8_{LDE})

(10_{LDE}) $(a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_3) and (9_{LDE}).

Hence, we have proved that for every $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$, for every L-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$, and for every $a \in U$ it holds that $a \vee \tilde{\neg}a \in D$, which means that a formula $\varphi \vee \neg\varphi$ is valid in L. ■

Proof of Proposition 3.26. Let $\mathsf{L} \in \{\mathsf{LD}, \mathsf{LDS}, \mathsf{LDE}\}$ and let $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be a model of L. We will show that for all $a, b \in U$ the following holds:

(*) If $(a \,\tilde{\equiv}\, b) \in D$, then $(\tilde{\neg}a \,\tilde{\vee}\, b) \in D$.

Let $a, b \in U$ be such that $a \,\tilde{\equiv}\, b \in D$. Then, we have:

(1) $a \,\tilde{\equiv}\, b \in D$ assumption

(2) $b \,\tilde{\equiv}\, a \in D$ (WGL_5) and (1)

(3) $a \,\tilde{\vee}\, \tilde{\neg}a \in D$ Proposition 3.25

(4) $(b \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (\tilde{\neg}a \,\tilde{\vee}\, b) \in D$ (G2)

Now, three cases are possible:

Case: $\mathsf{L} = \mathsf{LD}$

(5_{LD}) $(b \,\tilde{\equiv}\, a) \,\tilde{\equiv}\, ((b \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, ((b \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a))) \in D$ (LD_3)

(6_{LD}) $(b \,\tilde{\equiv}\, a) \,\tilde{\wedge}\, ((b \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a)) \in D$ (WGL_4), (2), (5_{LD})

(7_{LD}) $(b \,\tilde{\vee}\, \tilde{\neg}a) \,\tilde{\equiv}\, (a \,\tilde{\vee}\, \tilde{\neg}a) \in D$ (WGL_3) and (6_{LD})

(8_{LD}) $(a \tilde{\vee} \tilde{\neg} a) \tilde{\equiv} (b \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_5) and (7_{LD})
(9_{LD}) $(b \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_4), (3), and (8_{LD})
(10_{LD}) $(\tilde{\neg} a \tilde{\vee} b) \in D$ (WGL_4), (4), and (9_{LD})
Case: $\mathsf{L} = \mathsf{LDS}$
(5_{LDS}) $((b \tilde{\equiv} a) \tilde{\wedge} (\tilde{\neg} a \tilde{\equiv} \tilde{\neg} a))\tilde{\equiv}$
$(((b \tilde{\equiv} a) \tilde{\wedge} (\tilde{\neg} a \tilde{\equiv} \tilde{\neg} a)) \tilde{\wedge} ((b \tilde{\vee} \tilde{\neg} a) \tilde{\equiv} (a \tilde{\vee} \tilde{\neg} a))) \in D$ (LDS_3)
(6_{LDS}) $\tilde{\neg} a \tilde{\equiv} \tilde{\neg} a \in D$ (WGL_2)
(7_{LDS}) $(b \tilde{\equiv} a) \tilde{\wedge} (\tilde{\neg} a \tilde{\equiv} \tilde{\neg} a) \in D$ (WGL_3), (2), and (6_{LDS})
(8_{LDS}) $(((b \tilde{\equiv} a) \tilde{\wedge} (\tilde{\neg} a \tilde{\equiv} \tilde{\neg} a)) \tilde{\wedge} ((b \tilde{\vee} \tilde{\neg} a) \tilde{\equiv} (a \tilde{\vee} \tilde{\neg} a))) \in D$
(WGL_4), (5_{LDS}), (7_{LDS})
(9_{LDS}) $(b \tilde{\vee} \tilde{\neg} a) \tilde{\equiv} (a \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_3) and (8_{LDS})
(10_{LDS}) $(a \tilde{\vee} \tilde{\neg} a) \tilde{\equiv} (b \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_5) and (9_{LDS})
(11_{LDS}) $(b \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_4), (3), and (10_{LDS})
(12_{LDS}) $(\tilde{\neg} a \tilde{\vee} b) \in D$ (WGL_4), (4), and (11_{LDS})
Case: $\mathsf{L} = \mathsf{LDE}$
(5_{LDE}) $((a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\vee} \tilde{\neg} a)) \tilde{\equiv} ((a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\vee} \tilde{\neg} a))$ (LDE_3)
(6_{LDE}) $(a \tilde{\equiv} b) \tilde{\wedge} (a \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_3), (1), and (3)
(7_{LDE}) $(a \tilde{\equiv} b) \tilde{\wedge} (b \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_4), (5_{LDE}), (6_{LDE})
(8_{LDS}) $(b \tilde{\vee} \tilde{\neg} a) \in D$ (WGL_3) and (7_{LDE})
(9_{LDS}) $(\tilde{\neg} a \tilde{\vee} b) \in D$ (WGL_4), (4), and (8_{LDE})

Suppose $\neg\varphi \vee \psi$ is not L-valid. Then, there is an L-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ and a valuation v in $\mathcal{M}$ such that $(\tilde{\neg} v(\varphi) \tilde{\vee} v(\psi)) \notin D$. Thus, by the property (*), we obtain that $v(\varphi) \tilde{\equiv} v(\psi) \notin D$, which means that $\varphi \equiv \psi$ is not L-valid. Suppose $\neg\psi \vee \varphi$ is not L-valid. Then, as we already have proved, $\psi \equiv \varphi$ is not L-valid. However, if $\psi \equiv \varphi$ is not L-valid, then by the condition (WGL_5), $\varphi \equiv \psi$ is also not L-valid. ■

Proof of Proposition 3.27. By way of example, we will prove the proposition for LD, as proofs for LDS and LDE are similar. Let $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ be a model of LD. We will show that for all $a, b \in U$ the following holds:

(*) If $(a \tilde{\equiv} b) \in D$, then $\tilde{\neg}(a \tilde{\wedge} \tilde{\neg} b) \in D$.

Let $a, b \in U$ be such that $a \tilde{\equiv} b \in D$. Then, we have:
(1) $a \tilde{\equiv} b \in D$ assumption
(2) $\tilde{\neg} a \tilde{\vee} b \in D$ Proposition 3.26
(3) $b \tilde{\vee} \tilde{\neg} a \in D$ (G2)
(4) $\tilde{\neg}\tilde{\neg} b \tilde{\equiv} b \in D$ (G6)
(5) $b \tilde{\equiv} \tilde{\neg}\tilde{\neg} b \in D$ (WGL_5) and (4)
(6) $\tilde{\neg}(a \tilde{\wedge} \tilde{\neg} b) \tilde{\equiv} (\tilde{\neg} a \tilde{\vee} \tilde{\neg}\tilde{\neg} b) \in D$ (G7)
(7) $(\tilde{\neg} a \tilde{\vee} \tilde{\neg}\tilde{\neg} b) \tilde{\equiv} \tilde{\neg}(a \tilde{\wedge} \tilde{\neg} b) \in D$ (WGL_5) and (6)

(8) $(b \mathrel{\tilde{\equiv}} \tilde{\neg}\tilde{\neg}b) \mathrel{\tilde{\equiv}} ((b \mathrel{\tilde{\equiv}} \tilde{\neg}\tilde{\neg}b) \mathbin{\tilde{\wedge}} ((b \mathbin{\tilde{\vee}} \tilde{\neg}a) \mathrel{\tilde{\equiv}} (\tilde{\neg}\tilde{\neg}b \mathbin{\tilde{\vee}} \tilde{\neg}a))) \in D$ (LD_3)

(9) $(b \mathrel{\tilde{\equiv}} \tilde{\neg}\tilde{\neg}b) \mathbin{\tilde{\wedge}} ((b \mathbin{\tilde{\vee}} \tilde{\neg}a) \mathrel{\tilde{\equiv}} (\tilde{\neg}\tilde{\neg}b \mathbin{\tilde{\vee}} \tilde{\neg}a)) \in D$ (WGL_4), (5), (8)

(10) $(b \mathbin{\tilde{\vee}} \tilde{\neg}a) \mathrel{\tilde{\equiv}} (\tilde{\neg}\tilde{\neg}b \mathbin{\tilde{\vee}} \tilde{\neg}a) \in D$ (WGL_3) and (9)

(11) $(\tilde{\neg}\tilde{\neg}b \mathbin{\tilde{\vee}} \tilde{\neg}a) \in D$ (WGL_4), (3), and (10)

(12) $\tilde{\neg}a \mathbin{\tilde{\vee}} \tilde{\neg}\tilde{\neg}b \in D$ (WGL_5) and (11)

(13) $\tilde{\neg}(a \mathbin{\tilde{\wedge}} \tilde{\neg}b) \in D$ (WGL_4), (7), and (12)

Now, assume that $\varphi \equiv \psi$ is LD-valid, while $\neg(\varphi \wedge \neg\psi)$ is not LD-valid. Then, there is an LD-model $\mathcal{M} = (U, D, \tilde{\neg}, \tilde{\wedge}, \tilde{\vee}, \tilde{\rightarrow}, \tilde{\leftrightarrow}, \tilde{\equiv})$ and a valuation v in $\mathcal{M}$ such that $\tilde{\neg}(v(\varphi) \mathbin{\tilde{\wedge}} \tilde{\neg}v(\psi)) \notin D$. Thus, by the property (*), $v(\varphi) \mathrel{\tilde{\equiv}} v(\psi) \notin D$, which implies that $\varphi \equiv \psi$ is not L-valid, a contradiction. Note that if $\varphi \equiv \psi$ is LD-valid, then the formula $\psi \equiv \varphi$ is also LD-valid. Thus, LD-validity of $\neg(\psi \wedge \neg\varphi)$ easily follows from the property we have already proved. ■

Tables with Possible Valuations

In Examples 6.1 and 6.2, we present possible values of given formulas in the models $\mathcal{M}_{\mathsf{LP}}$, $\mathcal{M}_{\mathsf{LFI1}}$, $\mathcal{M}_{\mathsf{LFI2}}$. Each row of a table indicates the value of the initial formula and all its subformulas under a valuation that assigns to each propositional variable of the initial formula the value entered in a column named with that variable. The value of a complex formula is entered under the main connective of that formula. If a given valuation does not yield the same value of φ in all three logics, the possible values of φ are marked under its main connective as xyz, where x, y, z are the values of φ in LP, LFI1, LFI2, respectively.

Example 6.1. The formula $(p \wedge \neg q) \rightarrow \neg(p \rightarrow q)$ takes the following values in logics LP, LFI1, and LFI2:

$(p$	$\wedge$	$\neg$	$q)$	$\rightarrow$	$\neg$	$(p$	$\rightarrow$	$q)$
(f	f	t	f)	t	f	(f	t	f)
(f	f	n	n)	t	f	(f	t	n)
(f	f	f	t)	t	f	(f	t	t)
(n	nnt	t	f)	ntt	ntt	(n	nff	f)
(n	n	n	n)	n	n	(n	n	n)
(n	f	f	t)	t	f	(n	t	t)
(t	t	t	f)	t	t	(t	f	f)
(t	nnt	n	n)	nnf	nnf	(t	nnt	n)
(t	f	f	t)	t	f	(t	t	t)

Example 6.2. The formula $(\neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p)) \rightarrow (p \leftrightarrow q)$ take the following values in logics LP, LFI1, and LFI2:

(¬	(p	∧	¬	q)	∧	¬	(q	∧	¬	p))	→	(p	↔	q)
t	(f	f	t	f)	t	t	(f	f	t	f)	t	(f	t	f)
t	(f	f	n	n)	nnf	nnf	(n	nnt	t	f)	n$\underline{\underline{\mathsf{f}}}$t	(f	nff	n)
t	(f	f	f	t)	f	f	(t	t	t	f)	t	(f	f	t)
nnf	(n	nnt	t	f)	nnf	t	(f	f	n	n)	n$\underline{\underline{\mathsf{f}}}$t	(n	nff	f)
n	(n	n	n	n)	n	n	(n	n	n	n)	n	(n	n	n)
t	(n	f	f	t)	nnf	nnf	(t	nnt	n	n)	nnt	(n	nnt	t)
f	(t	t	t	f)	f	t	(f	f	f	t)	t	(t	f	f)
nnf	(t	nnt	n	n)	nnf	t	(n	f	f	t)	nnt	(t	nnt	n)
t	(t	f	f	t)	t	t	(t	f	f	t)	t	(t	t	t)

Proof of Proposition 4.11. Let $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$. Then, in every model $\mathcal{M}_{\mathsf{L}_{\equiv}}$:

$((\mathsf{n} \tilde{\equiv} \tilde{\neg}\mathsf{n}) \tilde{\equiv} \mathsf{n}) \tilde{\vee} ((\mathsf{n} \tilde{\equiv} \tilde{\neg}\mathsf{n}) \tilde{\equiv} \tilde{\neg}\mathsf{n}) = (\mathsf{n} \tilde{\equiv} \mathsf{n}) \tilde{\vee} (\mathsf{n} \tilde{\equiv} \mathsf{n}) = (\mathsf{n} \tilde{\vee} \mathsf{n}) = \mathsf{n}$.

For $\mathsf{x} \in \{\mathsf{f}, \mathsf{t}\}$, it holds that $(\mathsf{x} \tilde{\equiv} \tilde{\neg}\mathsf{x}) = \mathsf{f}$. Thus, we have:

$$((\mathsf{x} \tilde{\equiv} \tilde{\neg}\mathsf{x}) \tilde{\equiv} \mathsf{x}) = (\mathsf{f} \tilde{\equiv} \mathsf{x}) = \begin{cases} \mathsf{t}, \text{ if } \mathsf{x} = \mathsf{f} \\ \mathsf{f}, \text{ if } \mathsf{x} = \mathsf{t} \end{cases}$$

$$((\mathsf{x} \tilde{\equiv} \tilde{\neg}\mathsf{x}) \tilde{\equiv} \tilde{\neg}\mathsf{x}) = (\mathsf{f} \tilde{\equiv} \tilde{\neg}\mathsf{x}) = \begin{cases} \mathsf{t}, \text{ if } \mathsf{x} = \mathsf{t} \\ \mathsf{f}, \text{ if } \mathsf{x} = \mathsf{f} \end{cases}$$

Consequently, for $\mathsf{x} \in \{\mathsf{f}, \mathsf{t}\}$, we have $(((\mathsf{x} \tilde{\equiv} \tilde{\neg}\mathsf{x}) \tilde{\equiv} \mathsf{x}) \tilde{\vee} ((\mathsf{x} \tilde{\equiv} \tilde{\neg}\mathsf{x}) \tilde{\equiv} \tilde{\neg}\mathsf{x})) = \mathsf{t}$. Thus, $v((p \equiv \neg p) \equiv p \vee (p \equiv \neg p) \equiv \neg p) \in \{\mathsf{n}, \mathsf{t}\}$, for every valuation v in $\mathcal{M}_{\mathsf{L}_{\equiv}}$, which implies that the formula is valid in $\mathsf{L}_{\equiv}$.

Let $\mathcal{M}_{\mathsf{sL}_{\equiv}}$ be the model of $\mathsf{sL}_{\equiv}$. Recall that in $\mathsf{sL}_{\equiv}$, the operation $\tilde{\equiv}$ takes either the value f or t. Let v be a valuation in $\mathcal{M}_{\mathsf{sL}_{\equiv}}$ such that $v(p) = \mathsf{n}$. Then, we have:

$((\mathsf{n} \tilde{\equiv} \tilde{\neg}\mathsf{n}) \tilde{\equiv} \mathsf{n}) \tilde{\vee} ((\mathsf{n} \tilde{\equiv} \tilde{\neg}\mathsf{n}) \tilde{\equiv} \tilde{\neg}\mathsf{n}) = (\mathsf{t} \tilde{\equiv} \mathsf{n}) \tilde{\vee} (\mathsf{t} \tilde{\equiv} \mathsf{n}) = (\mathsf{f} \tilde{\vee} \mathsf{f}) = \mathsf{f}$.

Hence, $v((p \equiv \neg p) \equiv p \vee (p \equiv \neg p) \equiv \neg p) = \mathsf{f}$, and so the formula is not valid in $\mathsf{sL}_{\equiv}$, which ends the proof of the item 1. of Proposition 4.11.

Now, let us note that the model of $\mathsf{sL}_{\equiv}$ satisfies the following conditions: $(\mathsf{x} \tilde{\wedge} \tilde{\neg}\mathsf{x}) \in \{\mathsf{f}, \mathsf{n}\}$ and $(\mathsf{y} \tilde{\equiv} \mathsf{y}) = \mathsf{t}$, for all $\mathsf{x}, \mathsf{y} \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$. Let $\mathsf{x}, \mathsf{y} \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$. Then, we have:

$$(\mathsf{x} \tilde{\rightarrow} ((\mathsf{y} \tilde{\equiv} \mathsf{y}) \tilde{\equiv} \mathsf{x})) = (\mathsf{x} \tilde{\rightarrow} (\mathsf{t} \tilde{\equiv} \mathsf{x})) = \begin{cases} \mathsf{t}, \text{ if } \mathsf{x} \in \{\mathsf{f}, \mathsf{t}\} \text{ and } \mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\} \\ \mathsf{n}, \text{ if } \mathsf{x} = \mathsf{n} \text{ and } \mathsf{L} = \mathsf{LP} \\ \mathsf{f}, \text{ if } \mathsf{x} = \mathsf{n} \text{ and } \mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}\} \end{cases}$$

Therefore, for all $\mathsf{x}, \mathsf{y} \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, it holds that:

$$(\mathsf{x} \tilde{\wedge} \tilde{\neg}\mathsf{x}) \tilde{\vee} (\mathsf{x} \tilde{\rightarrow} ((\mathsf{y} \tilde{\equiv} \mathsf{y}) \tilde{\equiv} \mathsf{x})) = \begin{cases} \mathsf{t}, \text{ if } \mathsf{x} \in \{\mathsf{f}, \mathsf{t}\} \text{ and } \mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\} \\ \mathsf{t}, \text{ if } \mathsf{x} = \mathsf{n} \text{ and } \mathsf{L} = \mathsf{LFI2} \\ \mathsf{n}, \text{ if } \mathsf{x} = \mathsf{n} \text{ and } \mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}\} \end{cases}$$

Hence, $v((p \wedge \neg p) \vee (p \rightarrow ((q \equiv q) \equiv p))) \in \{\mathsf{n}, \mathsf{t}\}$, for every valuation v in $\mathcal{M}_{\mathsf{sL}_{\equiv}}$, and so the formula is valid in $\mathsf{sL}_{\equiv}$, for every $\mathsf{L} \in \{\mathsf{LP}, \mathsf{LFI1}, \mathsf{LFI2}\}$.

On other hand, if v is a valuation in the model of $\mathsf{L}_{\equiv}$ such that $v(p) = \mathsf{t}$ and $v(q) = \mathsf{n}$, then we have:

$v((p \wedge \neg p) \vee (p \rightarrow ((q \equiv q) \equiv p))) = (\mathsf{f} \tilde{\vee} (\mathsf{t} \tilde{\rightarrow} (\mathsf{n} \tilde{\equiv} \mathsf{t}))) = (\mathsf{f} \tilde{\vee} (\mathsf{t} \tilde{\rightarrow} \mathsf{f})) = (\mathsf{f} \tilde{\vee} \mathsf{f}) = \mathsf{f}$.

Hence, the formula $(p \wedge \neg p) \vee (p \rightarrow ((q \equiv q) \equiv p))$ is not valid in $\mathsf{L}_{\equiv}$. ■

Proof of Proposition 4.12. Let $\mathsf{L} \in \{\mathsf{LFI1}, \mathsf{LFI2}\}$. Let φ be the formula $(p \wedge \neg p) \vee \psi$, where ψ has the following form:

$p \to ((q \to (p \wedge \neg p)) \equiv p) \vee ((q \to (p \wedge \neg p)) \equiv \neg p))$.

If v is a valuation in the model of $\mathsf{L}_{\equiv}$ or $\mathsf{sL}_{\equiv}$ such that $v(p) = \mathsf{f}$, then $v(\psi) = \mathsf{t}$, and so $v(\varphi) = \mathsf{t}$. If $v(p) = \mathsf{n}$, then $v(p \wedge \neg p) = \mathsf{n}$, which implies that $v(\varphi) \in \{\mathsf{n}, \mathsf{t}\}$. Thus, let v be a valuation such that $v(p) = \mathsf{t}$. Then, possible values of $v(\psi)$ are:

p	$\to$	$(((q$	$\to$	$(p \wedge \neg p))$	$\equiv$	$p)$	$\vee$	$((q$	$\to$	$(p \wedge \neg p))$	$\equiv$	$\neg p))$
t	t	((f	t	f)	t	t	t	(f	t	f)	f	f)
t	t	((n	f	f)	f	t	t	(n	f	f)	t	f)
t	t	((t	f	f)	f	t	t	(t	f	f)	t	f)

Therefore, if $v(p) = \mathsf{t}$, then $v(\psi) = \mathsf{t}$, which yields $v(\varphi) = \mathsf{t}$. Hence, we have proved that $v(\varphi \in \{\mathsf{n}, \mathsf{t}\})$ in each of the following logics: $\mathsf{LFI1}_{\equiv}$, $\mathsf{sLFI1}_{\equiv}$, $\mathsf{LFI2}_{\equiv}$, $\mathsf{sLFI2}_{\equiv}$. In contrary, it does not hold in $\mathsf{LP}_{\equiv}$ and $\mathsf{sLP}_{\equiv}$. Indeed, let v be a valuation such that $v(p) = \mathsf{t}$ and $v(q) = \mathsf{n}$. Then, the following holds in both logics $\mathsf{LP}_{\equiv}$ and $\mathsf{sLP}_{\equiv}$:

$v(\varphi) = (\mathsf{f} \mathbin{\tilde{\vee}} (\mathsf{t} \mathbin{\tilde{\to}} (((\mathsf{n} \mathbin{\tilde{\to}} \mathsf{f}) \mathbin{\tilde{\equiv}} \mathsf{f}) \mathbin{\tilde{\vee}} ((\mathsf{n} \mathbin{\tilde{\to}} \mathsf{f}) \mathbin{\tilde{\equiv}} \mathsf{t})))) =$

$(\mathsf{f} \mathbin{\tilde{\vee}} (\mathsf{t} \mathbin{\tilde{\to}} ((\mathsf{n} \mathbin{\tilde{\equiv}} \mathsf{f}) \mathbin{\tilde{\vee}} (\mathsf{n} \mathbin{\tilde{\equiv}} \mathsf{t})))) = (\mathsf{f} \mathbin{\tilde{\vee}} (\mathsf{t} \mathbin{\tilde{\to}} (\mathsf{f} \mathbin{\tilde{\vee}} \mathsf{f}))) = (\mathsf{f} \mathbin{\tilde{\vee}} (\mathsf{t} \mathbin{\tilde{\to}} \mathsf{f})) = \mathsf{f}$.

Hence, if v is a valuation such that $v(p) = \mathsf{t}$ and $v(q) = \mathsf{n}$, then $v(\varphi) = \mathsf{f}$ in the model of $\mathsf{LP}_{\equiv}$ and in the model of $\mathsf{sLP}_{\equiv}$. Consequently, φ is not valid in $\mathsf{LP}_{\equiv}$ and $\mathsf{sLP}_{\equiv}$. ∎

Proof of Proposition 4.18. Let $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{LP}^*_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{LFI1}^*_{\equiv}\}$. Let $\mathcal{M}_{\mathsf{L}}$ be the model of L. By Proposition 4.14, $\mathcal{M}_{\mathsf{L}}$ is an MGL-model. The operations $\tilde{\wedge}$ and $\tilde{\vee}$ are defined in $\mathsf{LFI1}$ and LP in the same way and can be interpreted as max and min operations on $\{0, \frac{1}{2}, 1\}$. Thus, they are associative, commutative, idempotent, and distributive over each other, which implies that conditions (G1)–(G5) of G-structures hold in $\mathcal{M}_{\mathsf{L}}$. Moreover, it can be easily verified that all $a, b \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$ satisfy the following conditions: $\tilde{\neg}\tilde{\neg} a = a$, $\tilde{\neg}(a \mathbin{\tilde{\wedge}} b) = (\tilde{\neg} a \mathbin{\tilde{\vee}} \tilde{\neg} b)$, and $\tilde{\neg}(a \mathbin{\tilde{\vee}} b) = (\tilde{\neg} a \mathbin{\tilde{\wedge}} \tilde{\neg} b)$. Consequently, the conditions (G6)–(G8) hold in $\mathcal{M}_{\mathsf{L}}$. Finally, since $\tilde{\equiv}$ is symmetric and $\tilde{\neg}$ is involution, conditions (G9) and (G10) also hold in $\mathcal{M}_{\mathsf{L}}$. ∎

Proof of Proposition 4.19. In every logic $\mathsf{L} \in \{\mathsf{LP}_{\equiv}, \mathsf{sLP}_{\equiv}, \mathsf{LFI1}_{\equiv}, \mathsf{sLFI1}_{\equiv}\}$, the operation $\tilde{\equiv}$ satisfies the condition: $a \mathbin{\tilde{\equiv}} b \in D$ iff $a = b$. Thus, we can simplify the specific conditions of LD, LDS, LDE as follows, for $\tilde{\#} \in \{\tilde{\equiv}, \tilde{\wedge}, \tilde{\vee}\}$:

($\mathsf{LD}_{\tilde{\#}}$) $\quad (a \mathbin{\tilde{\equiv}} b) = ((a \mathbin{\tilde{\equiv}} b) \mathbin{\tilde{\wedge}} ((a \mathbin{\tilde{\#}} c) \mathbin{\tilde{\equiv}} (b \mathbin{\tilde{\#}} c)))$,

($\mathsf{LDS}_{\tilde{\#}}$) $\quad ((a \mathbin{\tilde{\equiv}} b) \mathbin{\tilde{\wedge}} (c \mathbin{\tilde{\equiv}} d)) = (((a \mathbin{\tilde{\equiv}} b) \mathbin{\tilde{\wedge}} (c \mathbin{\tilde{\equiv}} d)) \mathbin{\tilde{\wedge}} ((a \mathbin{\tilde{\#}} c) \mathbin{\tilde{\equiv}} (b \mathbin{\tilde{\#}} d)))$,

($\mathsf{LDE}_{\tilde{\#}}$) $\quad (a \mathbin{\tilde{\equiv}} b \mathbin{\tilde{\wedge}} (a \mathbin{\tilde{\#}} c)) = (a \mathbin{\tilde{\equiv}} b \mathbin{\tilde{\wedge}} (b \mathbin{\tilde{\#}} c))$.

Clearly, for every $\mathsf{L} \in \{\mathsf{LP}_\equiv, \mathsf{sLP}_\equiv, \mathsf{LFI1}_\equiv, \mathsf{sLFI1}_\equiv\}$, it holds that $a \,\tilde{\equiv}\, b = \mathsf{f}$ iff $a \neq b$. Therefore, if $a \neq b$, then $((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, c) = \mathsf{f}$, for all $c \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$. Hence, the conditions $(\mathsf{LD}_{\tilde{\#}})$ and $(\mathsf{LDE}_{\tilde{\#}})$ hold in the model of L, for all $a, b \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$ such that $a \neq b$, and the conditions $(\mathsf{LDS}_{\tilde{\#}})$ hold in the model of L, for all $a, b, c, d \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$ such that either $a \neq b$ or $c \neq d$ (as the left side and the right side of each equality equals f).

Recall that for all $a, b \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, if $a = b$, then it holds that $(a \,\tilde{\equiv}\, b) = \mathsf{t}$ whenever $\mathsf{L} \in \{\mathsf{sLP}_\equiv, \mathsf{sLFI1}_\equiv\}$, and $(a \,\tilde{\equiv}\, b) \in \{\mathsf{n}, \mathsf{t}\}$, if $\mathsf{L} \in \{\mathsf{LP}_\equiv, \mathsf{LFI1}_\equiv\}$. Moreover, if $a = b$, then $(a \,\tilde{\#}\, c) = (b \,\tilde{\#}\, c)$, for all $c \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$. Similarly, if $a = b$ and $c = d$, then $(a \,\tilde{\#}\, c) = (b \,\tilde{\#}\, d)$.

Case: $\mathsf{L} \in \{\mathsf{sLP}_\equiv, \mathsf{sLFI1}_\equiv\}$

Let $(a \,\tilde{\#}\, c) = (b \,\tilde{\#}\, c) = \mathsf{x}$. Then, the following holds:

$\mathsf{t} = (a \,\tilde{\equiv}\, b) = ((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, ((a \,\tilde{\#}\, c) \,\tilde{\equiv}\, (b \,\tilde{\#}\, c))) = (\mathsf{t} \,\tilde{\wedge}\, (\mathsf{x} \,\tilde{\equiv}\, \mathsf{x})) = (\mathsf{t} \,\tilde{\wedge}\, \mathsf{t}) = \mathsf{t}.$

Thus, the conditions $(\mathsf{LD}_{\tilde{\#}})$ hold in the model of L. Next, we have:

$(\mathsf{t} \,\tilde{\wedge}\, \mathsf{x}) = (a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, (a \,\tilde{\#}\, c)) = (a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, (b \,\tilde{\#}\, c)) = (\mathsf{t} \,\tilde{\wedge}\, \mathsf{x}).$

The above shows that the conditions $(\mathsf{LDE}_{\tilde{\#}})$ are satisfied in the model of L. Now, let $(a \,\tilde{\#}\, c) = (b \,\tilde{\#}\, d) = \mathsf{x}$. Then, the conditions $(\mathsf{LDS}_{\tilde{\#}})$ hold in L:

$\mathsf{t} = ((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) = (((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) \tilde{\wedge} ((a \tilde{\#} c) \tilde{\equiv} (b \tilde{\#} d))) = (\mathsf{t} \tilde{\wedge} (\mathsf{x} \tilde{\equiv} \mathsf{x})) = (\mathsf{t} \,\tilde{\wedge}\, \mathsf{t}) = \mathsf{t}.$

Therefore, we have proved that the conditions $(\mathsf{LD}_{\tilde{\#}})$, $(\mathsf{LDS}_{\tilde{\#}})$, $(\mathsf{LDE}_{\tilde{\#}})$ hold in the model of $\mathsf{L} \in \{\mathsf{sLP}_\equiv, \mathsf{sLFI1}_\equiv\}$.

Case: $\mathsf{L} \in \{\mathsf{LP}_\equiv, \mathsf{LFI1}_\equiv\}$

Assume $(a \,\tilde{\equiv}\, b) = \mathsf{x} \in \{\mathsf{n}, \mathsf{t}\}$. Since $a = b$, it holds that $(a \,\tilde{\#}\, c) = (b \,\tilde{\#}\, c) = \mathsf{y} \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, for all $c \in \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$. Then, the conditions $(\mathsf{LDE}_{\tilde{\#}})$ hold in the model of L, which can be easily verified:

$(\mathsf{x} \,\tilde{\wedge}\, \mathsf{y}) = (a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, (a \,\tilde{\#}\, c)) = (a \,\tilde{\equiv}\, b \,\tilde{\wedge}\, (b \,\tilde{\#}\, c)) = (\mathsf{x} \,\tilde{\wedge}\, \mathsf{y}).$

Now, assume that $(a \,\tilde{\equiv}\, b), (c \,\tilde{\equiv}\, d) \in \{\mathsf{n}, \mathsf{t}\}$. Since $a = b$ and $c = d$, we obtain that $(a \tilde{\#} c) = (b \tilde{\#} d)$. Therefore, $(a \tilde{\#} c) \tilde{\equiv} (b \tilde{\#} d) \in \{\mathsf{n}, \mathsf{t}\}$. If $((a \tilde{\equiv} b) \tilde{\wedge} (c \tilde{\equiv} d)) = \mathsf{t}$, then $(a, b), (c, d) \in \{(\mathsf{f}, \mathsf{f}), (\mathsf{t}, \mathsf{t})\}$, and in consequence $(a \,\tilde{\#}\, c) \,\tilde{\equiv}\, (b \,\tilde{\#}\, d) = \mathsf{t}$, which means that the conditions $(\mathsf{LDS}_{\tilde{\#}})$ are satisfied in this case. Now, suppose that $((a \tilde{\equiv} b) \,\tilde{\wedge}\, (c \tilde{\equiv} d)) = \mathsf{n}$. Then, since $(a \tilde{\#} c) \tilde{\equiv} (b \tilde{\#} d) = \mathsf{x} \in \{\mathsf{n}, \mathsf{t}\}$, the following holds:

$(((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, (c \,\tilde{\equiv}\, d)) \,\tilde{\wedge}\, ((a \,\tilde{\#}\, c) \,\tilde{\equiv}\, (b \,\tilde{\#}\, d))) = (\mathsf{n} \,\tilde{\wedge}\, \mathsf{x}) = \mathsf{n} = ((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, (c \,\tilde{\equiv}\, d)).$

Hence, the conditions $(\mathsf{LDS}_{\tilde{\#}})$ hold in the model of $\mathsf{L} \in \{\mathsf{LP}_\equiv, \mathsf{LFI1}_\equiv\}$.

Nonetheless, the model of $\mathsf{L} \in \{\mathsf{LP}_\equiv, \mathsf{LFI1}_\equiv\}$ does not satisfy the condition $(\mathsf{LD}_{\tilde{\wedge}})$. Indeed, if $a = b = \mathsf{t}$ and $c = \mathsf{n}$, then:

$\mathsf{t} = (a \,\tilde{\equiv}\, b) \neq ((a \,\tilde{\equiv}\, b) \,\tilde{\wedge}\, ((a \,\tilde{\wedge}\, c) \,\tilde{\equiv}\, (b \,\tilde{\wedge}\, c))) = (\mathsf{t} \,\tilde{\wedge}\, (\mathsf{n} \,\tilde{\equiv}\, \mathsf{n})) = (\mathsf{t} \,\tilde{\wedge}\, \mathsf{n}) = \mathsf{n}.$

Therefore, the model of $\mathsf{L} \in \{\mathsf{LP}_\equiv, \mathsf{LFI1}_\equiv\}$ is not an LD-model. ■

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