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Algebraic Structures Formalizing the Logic of Quantum Mechanics Incorporating Time Dimension

Abstract. As Classical Propositional Logic finds its algebraic counterpart in Boolean algebras, the logic of Quantum Mechanics, as outlined within G. Birkhoff and J. von Neumann's approach to Quantum Theory (Birkhoff and von Neumann in Ann Math 37:823–843, 1936) [see also (Husimi in I Proc Phys-Math Soc Japan 19:766–789, 1937)] finds its algebraic alter ego in orthomodular lattices. However, this logic does not incorporate time dimension although it is apparent that the propositions occurring in the logic of Quantum Mechanics are depending on time. The aim of the present paper is to show that tense operators can be introduced in every logic based on a complete lattice, in particular in the logic of quantum mechanics based on a complete orthomodular lattice. If the time set is given together with a preference relation, we introduce tense operators in a purely algebraic way. We derive several important properties of such operators, in particular we show that they form dynamic pairs and, altogether, a dynamic algebra. We investigate connections of these operators with logical connectives conjunction and implication derived from Sasaki projections in an orthomodular lattice. Then we solve the converse problem, namely to find for given time set and given tense operators a time preference relation in order that the resulting time frame induces the given operators. We show that the given operators can be obtained as restrictions of operators induced by a suitable extended time frame.

Keywords: Complete orthomodular lattice, Event-state system, Logic of Quantum Mechanics, Tense operator, Time frame, Dynamic pair, Dynamic algebra.

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1. Introduction

It is well known that any physical theory determines an event-state system $(\mathcal{E}, \mathcal{S})$ where $\mathcal{E}$ contains the events that may occur with respect to the

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given system and $\mathcal{S}$ contains the states that such a physical system may assume. In Quantum Physics one usually identifies $\mathcal{E}$ with the set of projection operators of a Hilbert space $\mathbf{H}$. This set of operators is in bijective correspondence with the set $\mathcal{C}(\mathbf{H})$ of closed subspaces of $\mathbf{H}$. The set $\mathcal{C}(\mathbf{H})$ ordered by inclusion forms a complete orthomodular lattice. Such lattices were introduced in 1936 by G. Birkhoff and J. von Neumann [3] and independently in 1937 by K. Husimi [17] as a suitable algebraic tool for investigating the logical structure underlying physical theories that, like mentioned Quantum Mechanics, do not obey the laws of classical logic. For the theory of orthomodular lattices cf. the monographs [1] and [18].

However, the logic based on orthomodular lattices does not incorporate the dimension of time. Our goal is to find how the logic of Quantum Mechanics can be considered as a *tense logic* (or *time logic* in another terminology, see e.g. [7] and [21]). As mentioned above, we need not work directly with projections on a Hilbert space but with the corresponding orthomodular lattice. Hence, our task is to introduce certain tense operators P, F, H and G on a given orthomodular lattice. The meaning of these operators is as follows:

P ... "It has at some time been the case that",

F ... "It will at some time be the case that",

H ... "It has always been the case that",

G ... "It will always be the case that".

To obtain a suitable semantics for the above operators, a time scale is needed. For this reason a *time frame* is introduced. It is a pair (T, R) consisting of a non-empty set T of *time* and a non-empty binary relation R on T, the relation of *time preference*, i.e. for $s, t \in T$ we say that $s\ R\ t$ means s is *before* t or, equivalently, t is *after* s. For our purposes we will consider sometimes *serial relations* R (see [7]), i.e. binary relations R such that for each $s \in T$ there exist $t, u \in T$ with $t\ R\ s$ and $s\ R\ u$. Every reflexive binary relation is serial. In physical theories R is usually understood as an order or a quasiorder.

It is worth noticing that our tense operators are in fact special sorts of modal operators, see e.g. [10] and [19]. However, having a time frame (T, R), the tense operators H and G can be considered as universal quantifiers over the half sets of T and the tense operators P and F as existential operators over these segments. The theory of tense logic has its origin in works by

A. N. Prior (cf. [19] and [20]) and in the monographs and chapters [13–15] and [16]. For the classical propositional calculus, these operators were studied in [4], for MV-algebras in [8], for intuitionistic logic in [10], and for De Morgan algebras in [11]. Let us note that for other non-classical logics several papers on tense operators were published, see e.g. [9,12] and [22]. It should be noticed that tense operators were already introduced and investigated also in basic algebras, see [2], and that orthomodular lattices can be considered as a special kind of basic algebras. However, the difference between both is essential since in orthomodular lattices the unary operation is a complementation and, moreover, orthomodular lattices satisfy the rather strong orthomodular law. Due to this fact we will obtain stronger results.

2. Preliminaries

First we recall several concepts from lattice theory.

An *antitone involution* on a poset $(P, \leq)$ is a mapping $'$ from P to P satisfying the following conditions for all $x, y \in P$:

(i) $x \leq y$ implies $y' \leq x'$,

(ii) $x'' = x$.

A *complementation* on a bounded poset $(P, \leq, 0, 1)$ is a mapping $'$ from P to P satisfying $x \vee x' = 1$ and $x \wedge x' = 0$ for all $x \in P$. An *orthomodular lattice* is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice, $'$ is an antitone involution that is a complementation and the *orthomodular law* holds:

$$\text{If } x, y \in L \text{ and } x \leq y \text{ then } y = x \vee (y \wedge x').$$

Let us remark that according to the De Morgan's laws the orthomodular law is equivalent to the following condition:

$$\text{If } x, y \in L \text{ and } x \leq y \text{ then } x = y \wedge (x \vee y').$$

In the following we consider a non-trivial (i.e. not one-element) complete (possibly orthomodular) lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ ($\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$) and a given time frame (T, R). We can define the tense operators as quantifiers over the time frame as follows:

$$P(q)(s) := \bigvee\{q(t) \mid t\ R\ s\},$$

$$F(q)(s) := \bigvee\{q(t) \mid s\ R\ t\},$$

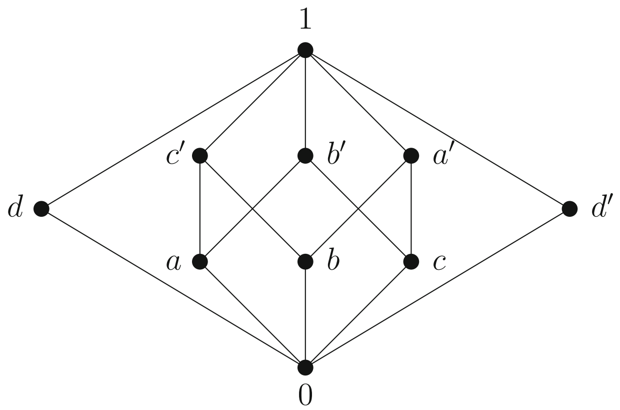

Figure 1. Orthomodular lattice

$$H(q)(s) := \bigwedge\{q(t) \mid t\ R\ s\},$$
$$G(q)(s) := \bigwedge\{q(t) \mid s\ R\ t\}$$

for every $q \in L^T$ and $s \in T$. In such a case we call the *tense operators* P, F, H and G to be *derived* from or *induced* by the time frame (T, R).

In complete orthomodular lattices there is a close connection between the tense operators P and H and the tense operators F and G. Namely, if these tense operators are induced by the time frame (T, R) then due to De Morgan's laws we have $H(q) = P(q')'$ and $G(q) = F(q')'$ for all $q \in L^T$. Here and in the following for every $q \in L^T$, q' denotes the mapping from T to L assigning to every $s \in T$ the element $\big(q(s)\big)'$ of L, and $P(q')'$ denotes the mapping from T to L assigning to every $s \in T$ the element $\big(P(q')(s)\big)'$ of L.

Example 2.1. Consider the orthomodular lattice $\mathbf{L}$ depicted in Figure 1:

Put $(T, R) := (\{1, 2, 3, 4, 5\}, \leq)$ and define time depending propositions $p, q \in L^T$ as follows:

t	1	2	3	4	5
$p(t)$	c'	b'	c'	a'	b'

t	1	2	3	4	5
$q(t)$	a	b'	d	a	a'

Then we have

t	1	2	3	4	5
$P(p)(t)$	c'	1	1	1	1
$F(p)(t)$	1	1	1	1	b'
$H(p)(t)$	c'	a	a	0	0
$G(p)(t)$	0	0	0	c	b'

t	1	2	3	4	5
$P(q)(t)$	a	b'	1	1	1
$F(q)(t)$	1	1	1	1	a'
$H(q)(t)$	a	a	0	0	0
$G(q)(t)$	0	0	0	0	a'

3. Dynamic Pairs

At first we prove that for tense operators as defined above the pairs (P, G) and (F, H) form *dynamic pairs*, thus $(\mathbf{L}, P, F, H, G)$ is a *dynamic algebra* (see [7] for details). For our reasons, the couple (X, Y) of operators forms a dynamic pair if they are monotone, $X(0) = Y(0) = 0$, $X(1) = Y(1) = 1$ and $XY(q) \leq YX(q)$ for every $q \in L$. If $\mathbf{L}$ is a complete lattice and (P, G) and (F, H) defined on L form dynamic pairs, then the quintuple $(\mathbf{L}, P, F, H, G)$ will be called a *dynamic algebra*. The term "dynamic" should express here the fact that the values of tense operators for a given proposition q may vary depending on time. This means that our system under consideration is dynamic.

THEOREM 3.1. *Let $(L, \vee, \wedge, 0, 1)$ be a complete lattice, (T, R) a time frame with serial relation R, P, F, H and G denote the tense operators induced by (T, R) and $p, q \in L^T$. Then the following holds:*

(i) $P(0) = F(0) = H(0) = G(0) = 0$ *and* $P(1) = F(1) = H(1) = G(1) = 1$,

(ii) $p \leq q$ *implies* $P(p) \leq P(q)$, $F(p) \leq F(q)$, $H(p) \leq H(q)$ *and* $G(p) \leq G(q)$,

(iii) $PG(q) \leq q \leq GP(q)$, $FH(q) \leq q \leq HF(q)$.

PROOF. Let $s \in T$.

(i) Since R is serial we have $P(0)(s) = \bigvee\{0 \mid t\ R\ s\} = 0$ and $P(1)(s) = \bigvee\{1 \mid t\ R\ s\} = 1$. The situation for F, H and G is analogous.

(ii) Assume $p \leq q$. Then

$$p(t) \leq q(t) \leq \bigvee\{q(u) \mid u\ R\ s\} = P(q)(s)$$

for all $t \in T$ with $t\ R\ s$ and hence

$$P(p)(s) = \bigvee\{p(t) \mid t\ R\ s\} \leq P(q)(s).$$

This shows $P(p) \leq P(q)$. The inequality $F(p) \leq F(q)$ can be shown analogously. Moreover,

$$H(p)(s) = \bigwedge\{p(u) \mid u\ R\ s\} \leq p(t) \leq q(t)$$

for all $t \in T$ with $t\ R\ s$ and hence

$$H(p)(s) \leq \bigwedge\{q(t) \mid t\ R\ s\} = H(q)(s).$$

This shows $H(p) \leq H(q)$. The inequality $G(p) \leq G(q)$ can be shown analogously.

(iii) The following are equivalent:

$$\begin{aligned} PG(q)(s) &\leq q(s), \\ \bigvee\{G(q)(t) \mid t\ R\ s\} &\leq q(s), \\ G(q)(t) &\leq q(s) \text{ for all } t \in T \text{ with } t\ R\ s, \\ \bigwedge\{q(u) \mid t\ R\ u\} &\leq q(s) \text{ for all } t \in T \text{ with } t\ R\ s. \end{aligned}$$

Since the last statement is true, the same holds for the first statement. Analogously, one can prove $FH(q) \leq q$. Now the following are equivalent:

$$\begin{aligned} & q(s) \leq GP(q)(s), \\ & q(s) \leq \bigwedge\{P(q)(t) \mid s\ R\ t\}, \\ & q(s) \leq P(q)(t) \text{ for all } t \in T \text{ with } s\ R\ t, \\ & q(s) \leq \bigvee\{q(u) \mid u\ R\ t\} \text{ for all } t \in T \text{ with } s\ R\ t. \end{aligned}$$

Since the last statement is true, the same holds for the first statement. Analogously, one can prove $q \leq HF(q)$.

■

If the operators P, F, H and G on the complete lattice $\mathbf{L}$ satisfy (i), (ii) and (iii) of Theorem 3.1 then the quintuple $(\mathbf{L}, P, F, H, G)$ will be referred to as a *dynamic algebra*.

Example 3.2. For p and q of Example 2.1 we obtain

t	1	2	3	4	5
$p(t)$	c'	b'	c'	a'	b'
$PG(p)(t)$	0	0	0	c	b'
$GP(p)(t)$	c'	1	1	1	1

t	1	2	3	4	5
$q(t)$	a	b'	d	a	a'
$PG(q)(t)$	0	0	0	0	a'
$GP(q)(t)$	a	b'	1	1	1

showing that $PG(p) \leq p \leq GP(p)$ and $PG(q) \leq q \leq GP(q)$ and, moreover, that these inequalities are strict.

In the following we establish several properties of tense operators on complete lattices that are in accordance with the general approach presented in [7] and [21].

THEOREM 3.3. *Let* $(L, \vee, \wedge, 0, 1)$ *be a complete lattice,* (T, R) *a time frame with serial relation* R*,* P*,* F*,* H *and* G *denote the tense operators induced by* (T, R) *and* $q \in L^T$*. Then the following hold:*

(i) $H(q) \leq P(q)$ *and* $G(q) \leq F(q)$*,*

(ii) *if* R *is reflexive then* $H(q) \leq q \leq P(q)$ *and* $G(q) \leq q \leq F(q)$*.*

PROOF. Let $s \in T$.

(i) Since R is serial there exists some $u \in T$ with $u\ R\ s$ and we have

$$H(q)(s) = \bigwedge\{q(t) \mid t\ R\ s\} \leq q(u) \leq \bigvee\{q(t) \mid t\ R\ s\} = P(q)(s).$$

The proof for G and F is analogous.

(ii) We have

$$H(q)(s) = \bigwedge\{q(t) \mid t\ R\ s\} \leq q(s) \leq \bigvee\{q(t) \mid t\ R\ s\} = P(q)(s).$$

The proof for G and F is analogous.

■

We define $A \leq B$ for $A, B \in \{P, F, H, G\}$ by $A(q) \leq B(q)$ for all $q \in L^T$.

THEOREM 3.4. *Let* $(L, \vee, \wedge, 0, 1)$ *be a complete lattice,* (T, R) *a time frame with reflexive* R*,* P*,* F*,* H *and* G *denote the tense operators induced by* (T, R)*,* $A \in \{P, F, H, G\}$*,* $B \in \{P, F\}$ *and* $C \in \{H, G\}$*. Then the following hold:*

(i) $A \leq AB$ *and* $AC \leq A$*,*

(ii) *if* R *is, moreover, transitive then* $AA = A$*.*

PROOF.

(i) This follows from Theorems 3.3 and 3.1.

(ii) According to (i) we have $P \leq PP$. Let $q \in L^T$ and $s \in T$. Then the following are equivalent:

$$PP(q)(s) \leq P(q)(s),$$

$$\bigvee\{P(q)(t) \mid t\ R\ s\} \leq P(q)(s),$$
$$P(q)(t) \leq P(q)(s) \text{ for all } t \in T \text{ with } t\ R\ s,$$
$$\bigvee\{q(u) \mid u\ R\ t\} \leq P(q)(s) \text{ for all } t \in T \text{ with } t\ R\ s,$$
$$q(u) \leq \bigvee\{q(w) \mid w\ R\ s\} \text{ for all } t \in T \text{ with } t\ R\ s \text{ and all } u \in T \text{ with } u\ R\ t.$$

Since, due to transitivity of R, $t\ R\ s$ and $u\ R\ t$ together imply $u\ R\ s$, the last statement is true and hence the same holds for the first statement. Therefore $PP \leq P$, and together we obtain $PP = P$. The proof of $FF = F$ is analogous. According to (i) we have $HH \leq H$. Now the following are equivalent:

$$H(q)(s) \leq HH(q)(s),$$
$$H(q)(s) \leq \bigwedge\{H(q)(t) \mid t\ R\ s\},$$
$$H(q)(s) \leq H(q)(t) \text{ for all } t \in T \text{ with } t\ R\ s,$$
$$H(q)(s) \leq \bigwedge\{q(u) \mid u\ R\ t\} \text{ for all } t \in T \text{ with } t\ R\ s,$$
$$\bigwedge\{q(w) \mid w\ R\ s\} \leq q(u) \text{ for all } t \in T \text{ with } t\ R\ s \text{ and all } u \in T \text{ with } u\ R\ t.$$

Since, due to transitivity of R, $t\ R\ s$ and $u\ R\ t$ together imply $u\ R\ s$, the last statement is true and hence the same holds for the first statement. This shows $H \leq HH$, and together we obtain $HH = H$. The proof of $GG = G$ is analogous. ■

4. Connections with Logical Connectives

Considering the logic based on an orthomodular lattice $(L, \vee, \wedge, ', 0, 1)$ one can ask for logical connectives. One way how to introduce the *conjunction* $\odot$ and the implication $\rightarrow$ is based on the *Sasaki projections* (see [1]). This method was successfully used by the authors in [5] and [6] for investigating left adjointness. Let us recall the corresponding definitions:

$$\begin{aligned} x \odot y &:= (x \vee y') \wedge y, \\ x \rightarrow y &:= (y \wedge x) \vee x' \end{aligned} \tag{1}$$

for all $x, y \in L$. The Sasaki projection p_y on $[0, y]$ is given by $p_y(x) := (x \vee y') \wedge y$ for all $x \in L$. Hence we have $x \odot y = p_y(x)$ and $x \rightarrow y = \big(p_x(y')\big)'$ for all $x, y \in L$.

The following result was proved in [5] and [6].

Proposition 4.1. *Let* $(L, \vee, \wedge, ', 0, 1)$ *be an orthomodular lattice,* $\odot$ *and* $\rightarrow$ *defined by (1) and* $a, b, c \in L$*. Then the following holds:*

(i) $a \odot 1 = 1 \odot a = a$,

(ii) $a \odot b \leq c$ *if and only if* $a \leq b \rightarrow c$ *(left adjointness),*

(iii) $a' = a \rightarrow 0$.

The following lemma will be used in the next proof.

Lemma 4.2. *Let* $(L, \vee, \wedge, ', 0, 1)$ *be an orthomodular lattice,* $\odot$ *and* $\rightarrow$ *defined by (1) and* $a, b \in L$*. Then the following holds:*

(i) $(a \rightarrow b) \odot a = a \wedge b$,

(ii) $a \leq b \rightarrow (a \odot b)$.

Proof.

(i) Using the orthomodular law we obtain

$$(a \rightarrow b) \odot a = \big((b \wedge a) \vee a' \vee a'\big) \wedge a = a \wedge \big((a \wedge b) \vee a'\big) = a \wedge b.$$

(ii) Using again the orthomodular law we obtain

$$a \leq a \vee b' = b' \vee \big((a \vee b') \wedge b\big) = \big((a \vee b') \wedge b \wedge b\big) \vee b' = b \rightarrow (a \odot b).$$

■

Our next task is to show connections of tense operators with logical connectives $\odot$ and $\rightarrow$. Using Proposition 4.1 and Lemma 4.2 we can prove the following theorem.

Theorem 4.3. *Let* $(L, \vee, \wedge, ', 0, 1)$ *be a complete orthomodular lattice,* (T, R) *a time frame,* P*,* F*,* H *and* G *denote the tense operators induced by* (T, R) *and* $A \in \{P, F, H, G\}$*. Then the following two assertions are equivalent:*

(i) $A(x) \odot A(y) \leq A(x \odot y)$ *for all* $x, y \in L^T$,

(ii) $A(x \rightarrow y) \leq A(x) \rightarrow A(y)$ *for all* $x, y \in L^T$.

Proof. Let $p, q \in L^T$. First assume (i). According to (i) we have

$$A(p \rightarrow q) \odot A(p) \leq A\big((p \rightarrow q) \odot p\big)$$

Now

$$(p \rightarrow q) \odot p = p \wedge q$$

because of Lemma 4.2 and hence

$$A\big((p \to q) \odot p\big) = A(p \wedge q).$$

Applying Theorem 3.1 to $p \wedge q \le q$ yields

$$A(p \wedge q) \le A(q).$$

Altogether, we obtain

$$A(p \to q) \odot A(p) \le A(q).$$

Thus, by Proposition 4.1 we conclude

$$A(p \to q) \le A(p) \to A(q)$$

showing (ii). Conversely, assume (ii). According to Lemma 4.2 we have

$$p \le q \to (p \odot q).$$

Applying Theorem 3.1 we conclude

$$A(p) \le A\big(q \to (p \odot q)\big).$$

Using (ii) we obtain

$$A\big(q \to (p \odot q)\big) \le A(q) \to A(p \odot q).$$

Altogether, we have

$$A(p) \le A(q) \to A(p \odot q).$$

Thus, by Proposition 4.1 we conclude

$$A(p) \odot A(q) \le A(p \odot q)$$

showing (i). ■

However, we can prove also further interesting connections between these operators.

THEOREM 4.4. *Let $(L, \vee, \wedge, ', 0, 1)$ be a complete orthomodular lattice, (T, R) a time frame with reflexive R, P, F, H and G denote the tense operators induced by (T, R), $A, A_1, A_2 \in \{P, F\}$, $B, B_1, B_2 \in \{H, G\}$ and $p, q \in L^T$. Then the following holds:*

(i) $p \le q \to A_1\big(A_2(p) \odot q\big)$,

(ii) $B(p \odot q) \le A(p) \odot q$,

(iii) $B(p) \le q \to A(p \odot q)$,

(iv) $B_1\big(B_2(p) \odot q\big) \le p \odot q$,

(v) $p \to q \leq A_1\big(p \to A_2(q)\big)$,

(vi) $B(p \to q) \odot p \leq A(q)$,

(vii) $p \to B(q) \leq A(p \to q)$,

(viii) $B_1\big(p \to B_2(q)\big) \odot p \leq q$.

Proof. We use Theorems 3.1 and 3.3, (1) and Proposition 4.1.

(i) We have

$$\begin{aligned} p \odot q &\leq A_1(p \odot q) \text{ by Theorem 3.3},\\ p &\leq A_2(p) \text{ by Theorem 3.3},\\ p \odot q &\leq A_2(p) \odot q \text{ by (1)},\\ A_1(p \odot q) &\leq A_1\big(A_2(p) \odot q\big) \text{ by Theorem 3.1},\\ p \odot q &\leq A_1\big(A_2(p) \odot q\big) \text{ by transitivity},\\ p &\leq q \to A_1\big(A_2(p) \odot q\big) \text{ by Proposition 4.1}. \end{aligned}$$

The other statements follow in an analogous way.

(ii) follows from $B(p \odot q) \leq B\big(A(p) \odot q\big) \leq A(p) \odot q$,

(iii) follows from $B(p) \odot q \leq A\big(B(p) \odot q\big) \leq A(p \odot q)$ by applying Proposition 4.1,

(iv) follows from $B_1\big(B_2(p) \odot q\big) \leq B_1(p \odot q) \leq p \odot q$,

(v) follows from $p \to q \leq A_1(p \to q) \leq A_1\big(p \to A_2(q)\big)$,

(vi) follows from $B(p \to q) \leq B\big(p \to A(q)\big) \leq p \to A(q)$ by applying Proposition 4.1,

(vii) follows from $p \to B(q) \leq A\big(p \to B(q)\big) \leq A(p \to q)$,

(viii) follows from $B_1\big(p \to B_2(q)\big) \leq B_1(p \to q) \leq p \to q$ by applying Proposition 4.1. ∎

5. A Construction of the Time Frame

A tense logic is established if for a given logic a time frame (T, R) exists such that the lattice together with the tense operators forms a dynamic algebra and the logical connectives are related with tense operators in the way shown in Section 4. Hence, if such a logic incorporating time dimension is created, we can define tense operators P, F, H and G. The question is whether, conversely, for given time set T and given tense operators there exists a suitable time frame (T, R) such that the given tense operators are derived from it. In other words, we ask if for given tense operators P, F, H and G on a time set T one can find some time preference relation R such that these operators are induced by (T, R). To show that this is possible is the goal of Section 5.

If P, F, H and G are tense operators on a complete lattice $(L, \vee, \wedge, 0, 1)$ with time set T then the relations

$$R_1 := \{(s,t) \in T^2 \mid q(s) \leq P(q)(t) \text{ and } q(t) \leq F(q)(s) \text{ for all } q \in L^T\},$$
$$R_2 := \{(s,t) \in T^2 \mid H(q)(t) \leq q(s) \text{ and } G(q)(s) \leq q(t) \text{ for all } q \in L^T\},$$
$$R_3 := R_1 \cap R_2$$

are called *the relation induced by P and F*, *the relation induced by H and G* and *the relation induced by P, F, H and G*, respectively.

Observe that whenever tense operators P, F, H and G on a complete lattice $\mathbf{L}$ are induced by an arbitrarily given time frame then $(\mathbf{L}, P, F, H, G)$ forms a dynamic algebra, and if, moreover, $\mathbf{L}$ is a complete orthomodular lattice then Theorems 4.3 and 4.4 hold for these operators.

At first we show the relationship between given tense operators P and F and the corresponding operators P^* and F^* induced by the time frame (T, R) where R is induced by P and F.

THEOREM 5.1. *Let P and F be tense operators on a complete lattice $(L, \vee, \wedge, 0, 1)$ with time set T, R denote the relation induced by these operators and P^* and F^* denote the tense operators induced by the time frame (T, R). Then $P^* \leq P$ and $F^* \leq F$.*

PROOF. If $q \in L^T$ and $s \in T$ then

$$P^*(q)(s) = \bigvee\{q(t) \mid t\, R\, s\} \leq P(q)(s),$$
$$F^*(q)(s) = \bigvee\{q(t) \mid s\, R\, t\} \leq F(q)(s).$$

■

Analogously, one can prove

THEOREM 5.2. *Let H and G be tense operators on a complete lattice $(L, \vee, \wedge, 0, 1)$ with time set T, R denote the relation induced by these operators and H^* and G^* denote the tense operators induced by the time frame (T, R). Then $H \leq H^*$ and $G \leq G^*$.*

From Theorems 5.1 and 5.2 we obtain

COROLLARY 5.3. *Let P, F, H and G be tense operators on a complete lattice $(L, \vee, \wedge, 0, 1)$ with time set T, R denote the relation induced by these* *operators and P^*, F^*, H^* and G^* denote the tense operators induced by the time frame (T, R). Then $P^* \leq P$, $F^* \leq F$, $H \leq H^*$ and $G \leq G^*$.*

EXAMPLE 5.4. Consider the lattice $\mathbf{L}$ and the time set $T = \{1, 2, 3, 4, 5\}$ from Example 2.1. Define new tense operators P, F, H and G as follows:

$$P(q)(t) := \begin{cases} q(t) & \text{if } t = 2, \\ 1 & \text{otherwise} \end{cases} \qquad F(q)(t) := \begin{cases} q(t) & \text{if } t = 1, \\ 1 & \text{otherwise} \end{cases}$$

$$H(q)(t) := \begin{cases} q(t) & \text{if } t = 1, \\ 0 & \text{otherwise} \end{cases} \qquad G(q)(t) := \begin{cases} q(t) & \text{if } t = 2, \\ 0 & \text{otherwise} \end{cases}$$

for all $q \in L^T$ and all $t \in T$. Note that these operators satisfy the conditions

$$H(q) \leq q \leq P(q) \text{ and } G(q) \leq q \leq F(q)$$

for all $q \in L^T$ which were considered in Theorem 3.3. Let R denote the relation induced by P, F, H and G. Then $R = \{1\}^2 \cup \{2\}^2 \cup \{3, 4, 5\}^2$. This can be seen as follows: Obviously, $\{1\}^2 \cup \{2\}^2 \cup \{3, 4, 5\}^2 \subseteq R$. Now let $(s, t) \in R$.
$s = 1 \neq t$ would imply $q(t) \leq F(q)(1) = q(1)$ for all $q \in L^T$, a contradiction.
$s = 2 \neq t$ would imply $q(2) = G(q)(2) \leq q(t)$ for all $q \in L^T$, a contradiction.
$s \neq 1 = t$ would imply $q(1) = H(q)(1) \leq q(s)$ for all $q \in L^T$, a contradiction.
$s \neq 2 = t$ would imply $q(s) \leq P(q)(2) = q(2)$ for all $q \in L^T$, a contradiction.
This shows $R = \{1\}^2 \cup \{2\}^2 \cup \{3, 4, 5\}^2$. For p from Example 2.1 we have

t	1	2	3	4	5
$p(t)$	c'	b'	c'	a'	b'
$P(p)(t)$	1	b'	1	1	1
$F(p)(t)$	c'	1	1	1	1
$P^*(p)(t)$	c'	b'	1	1	1
$F^*(p)(t)$	c'	b'	1	1	1

showing $P^* \leq P$ and $F^* \leq F$ in accordance with Corollary 5.3, but $P^* \neq P$ and $F^* \neq F$, thus this inequality is strict.

REMARK 5.5. Although the new tense operators P^*, F^*, H^* and G^* constructed as shown in Corollary 5.3 satisfy only the inequalities $P^* \leq P$, $F^* \leq F$, $H \leq H^*$ and $G \leq G^*$ and, by Example 5.4, these inequalities may be strict, it is almost evident from the construction of these operators that $(\mathbf{L}, P^*, F^*, H^*, G^*)$ forms a dynamic algebra and that these operators are connected with the logical connectives $\odot$ and $\to$ in the way shown in Theorems 4.3 and 4.4 provided $\mathbf{L}$ is a complete orthomodular lattice.

Conversely, if a time frame (T, R) on a complete lattice is given and we consider the tense operators P, F, H and G induced by (T, R) then the relation induced by these operators coincides with R, see the following result.

Theorem 5.6. *Let $(L, \vee, \wedge, 0, 1)$ be a complete lattice, (T, R) a time frame, P, F, H and G denote the tense operators induced by (T, R) and R^* denote the relation induced by these operators. Then $R = R^*$ and hence the tense operators induced by the time frame (T, R^*) coincide with those induced by the time frame (T, R).*

Proof. If $s\ R\ t$ then

$$H(q)(t) = \bigwedge\{q(u) \mid u\ R\ t\} \leq q(s) \leq \bigvee\{q(u) \mid u\ R\ t\} = P(q)(t),$$
$$G(q)(s) = \bigwedge\{q(u) \mid s\ R\ u\} \leq q(t) \leq \bigvee\{q(u) \mid s\ R\ u\} = F(q)(s)$$

for all $q \in L^T$ and hence $s\ R^*\ t$. This shows $R \subseteq R^*$. Now assume $R \neq R^*$. Then there exists some $(s, t) \in R^* \setminus R$. For every $u \in T$ let q_u denote the following element of L^T:

$$q_u(t) := \begin{cases} 1 \text{ if } t = u, \\ 0 \text{ otherwise} \end{cases}$$

$(t \in T)$. Now we would obtain

$$1 = q_s(s) \leq P(q_s)(t) = \bigvee\{q_s(u) \mid u\ R\ t\} = \bigvee\{0 \mid u\ R\ t\} = 0,$$

a contradiction. This shows $R = R^*$. ∎

Remark 5.7. Assume that tense operators P, F, H and G on a complete lattice with time set T are given. We want to know if such operators are induced by a suitable time frame with possibly unknown time preference relation. We construct the relation R on T induced by these operators and then we construct the tense operators P^*, F^*, H^* and G^* induced by the time frame (T, R). Now two cases can happen: Either $P^* = P$, $F^* = F$, $H^* = H$ and $G^* = G$ or at least one of these equalities is violated, it means it is a proper inequality. In the first case the given operators P, F, H and G are induced by the time frame (T, R) whereas in the second case P, F, H and G are not induced by any time frame because of Theorems 5.6.

In Theorem 5.6 we showed that if a complete lattice $(L, \vee, \wedge, 0, 1)$ and a time frame (T, R) are given and P, F, H and G denote the tense operators induced by this time frame then the relation R^* induced by these operators coincides with R. If, conversely, the tense operators P and F are given on a complete lattice with a given time set T, we can ask whether we can construct a relation inducing these operators. In Theorem 5.1 we showed that if R is induced by given tense operators P and F on a given time set T in the complete lattice $\mathbf{L}$ then the operators P^* and F^* induced by the time frame (T, R) need not coincide with P and F, respectively, they satisfy only the inequalities $P^* \leq P$ and $F^* \leq F$. However, such tense operators P^* and F^* are still related with the logical connectives $\odot$ and $\rightarrow$ as shown in Theorems 4.3 and 4.4 provided the complete lattice $\mathbf{L}$ is orthomodular. We are going to show that the given time set T can be extended to some set $\bar{T}$ and R can be extended to some binary relation $\bar{R}$ on $\bar{T}$ such that the tense operators induced by the time frame $(\bar{T}, \bar{R})$ can be considered in some sense as extensions of the given tense operators P and F, respectively. Put

$$\bar{T} := T_1 \cup T \cup T_2 \text{ where } T_1 := T \times \{1\} \text{ and } T_2 := T \times \{2\}. \tag{2}$$

We extend our "world" L^T by adding two of its copies, "parallel worlds", namely the "past" L^{T_1} and the "future" L^{T_2}. In this way we obtain our "new world" $L^{\bar{T}}$ over the extended time set $\bar{T}$. We also extend our time depending propositions $q \in L^T$ to $\bar{q} \in L^{\bar{T}}$ by defining

$$\begin{aligned}&\bar{q}\big((s,1)\big) := P(q)(s),\\ &\bar{q}(s) := q(s),\\ &\bar{q}\big((s,2)\big) := F(q)(s)\\ &\text{for all } s \in T.\end{aligned} \tag{3}$$

Now we show that the given operators P and F can be considered in some sense as restrictions of the operators $\bar{P}$ and $\bar{F}$ induced by the time frame $(\bar{T}, \bar{R})$, respectively.

Theorem 5.8. *Let P and F be tense operators on a complete lattice $(L, \vee, \wedge, 0, 1)$ with time set T and R denote the relation induced by these operators. Define $\bar{T}$ by (2), put*

$$\bar{R} := \{\big((s,1), s\big) \mid s \in T\} \cup R \cup \{\big(s, (s,2)\big) \mid s \in T\}.$$

and let $\bar{P}$ and $\bar{F}$ denote the tense operators induced by the time frame $(\bar{T}, \bar{R})$. Moreover, for every $q \in L^T$ let $\bar{q} \in L^{\bar{T}}$ denote the extension of q defined by (3). Then $\bar{R}|T = R$ and

$$\big(\bar{P}(\bar{q})\big)|T = P(q) \text{ and } \big(\bar{F}(\bar{q})\big)|T = F(q)$$

for all $q \in L^T$.

Proof. We have $\bar{R}|T = \bar{R} \cap T^2 = R$. If $q \in L^T$ and $s \in T$ then $q(t) \leq P(q)(s)$ for all $t \in T$ with $t\ R\ s$ and hence $\bigvee\{q(t) \mid t\ R\ s\} \leq P(q)(s)$ which implies

$$\bar{P}(\bar{q})(s) = \bigvee\{\bar{q}(\bar{t}) \mid \bar{t}\bar{R}s\} = \bar{q}\big((s,1)\big) \vee \bigvee\{\bar{q}(t) \mid t\ R\ s\}$$

$$= P(q)(s) \vee \bigvee\{q(t) \mid t\ R\ s\} == P(q)(s)$$

showing $\big(\bar{P}(\bar{q})\big)|T = P(q)$. Analogously, one can prove $\big(\bar{F}(\bar{q})\big)|T = F(q)$. ■

An analogous result holds for H and G instead of P and F, respectively, but the extensions of $q \in L^T$ to $\bar{q} \in L^{\bar{T}}$ must be slightly modified.

Theorem 5.9. *Let H and G be tense operators on a complete lattice* $(L, \vee, \wedge, 0, 1)$ *with time set T and R denote the relation induced by these operators. Define $\bar{T}$ by (2), put*

$$\bar{R} := \{\big((s,1),s\big) \mid s \in T\} \cup R \cup \{\big(s,(s,2)\big) \mid s \in T\}.$$

and let $\bar{H}$ and $\bar{G}$ denote the tense operators induced by the time frame $(\bar{T}, \bar{R})$. Moreover, for every $q \in L^T$ let $\bar{q} \in L^{\bar{T}}$ denote the extension of q defined by

$$\begin{aligned} \bar{q}\big((s,1)\big) &:= H(q)(s), \\ \bar{q}(s) &:= q(s), \\ \bar{q}\big((s,2)\big) &:= G(q)(s) \end{aligned}$$

for all $s \in T$. Then $\bar{R}|T = R$ and

$$\big(\bar{H}(\bar{q})\big)|T = H(q) \text{ and } \big(\bar{G}(\bar{q})\big)|T = G(q)$$

for all $q \in L^T$.

Example 5.10. Consider the time set T, the proposition p and the tense operators P and F from Example 2.1 and write ti instead of (t,i) for $t \in T$ and $i = 1,2$. Let R denote the relation induced by P and F. Then

$$\begin{aligned} R &= \{(s,t) \in \{1,2,3,4,5\}^2 \mid s \leq t\}, \\ \bar{R} &= \{(s1,s) \mid s \in T\} \cup R \cup \{(s,s2) \mid s \in T\}. \end{aligned}$$

Let $\bar{P}$ and $\bar{F}$ denote the tense operators induced by the time frame $(\bar{T}, \bar{R})$. Then we have

$\bar{t}$	11	21	31	41	51	1	2	3	4	5	12	22	32	42	52
$p(t)$						c'	b'	c'	a'	b'					
$P(p)(t)$						c'	1	1	1	1					
$F(p)(t)$						1	1	1	1	b'					
$\bar{p}(\bar{t})$	c'	1	1	1	1	c'	b'	c'	a'	b'	1	1	1	1	b'
$\bar{P}(\bar{p})(\bar{t})$	0	0	0	0	0	c'	1	1	1	1	c'	b'	c'	a'	b'
$\bar{F}(\bar{p})(\bar{t})$	c'	b'	c'	a'	b'	1	1	1	1	b'	0	0	0	0	0

where

$$T_1 = \{11, 21, 31, 41, 51\}, T = \{1, 2, 3, 4, 5\} \text{ and } T_2 = \{12, 22, 32, 42, 52\}.$$

Evidently, $\bar{R}|T = R$, $\big(\bar{P}(\bar{p})\big)|T = P(p)$ and $\big(\bar{F}(\bar{p})\big)|T = F(p)$ in accordance with Theorem 5.8.

6. Concluding Remarks

It is well-known that the logic of Quantum Mechanics based on orthomodular lattices forms an algebraizable logic. We have shown that also the tense operators introduced on a complete orthomodular lattice can be formalized in a purely algebraic way. However, the study of algebraizable tense logic based on orthomodular lattices would be beyond the scope of this paper and hence it is postponed to a subsequent paper. We encourage the readers to go on in this direction.

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