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Propositional Type Theory of Indeterminacy

Abstract. The aim of this paper is to define a partial Propositional Type Theory. Our system is partial in a double sense: the hierarchy of (propositional) types contains partial functions and some expressions of the language, including formulas, may be undefined. The specific interpretation we give to the undefined value is that of Kleene's strong logic of indeterminacy. We present a semantics for the new system and prove that every element of any domain of the hierarchy has a name in the object language. Finally, we provide a proof system and a (constructive) proof of completeness.

Keywords: Type theory, Partial logic, Three-valued logic, Kleene logic.

1. Introduction

The system of Propositional Type Theory (**PT**) was presented by Henkin [12]. It is a version of Church's Simple Type Theory where the set of truth-values ($\mathcal{D}_t = \{0, 1\}$) is the only basic type and any complex type (say, $\mathcal{D}_{ab}$) is the set of total functions from $\mathcal{D}_a$ to $\mathcal{D}_b$. Henkin gave a complete calculus for this logic taking nothing but the abstractor λ and equality as primitive symbols. In fact, the completeness proof is constructive, as it follows from the fact that every element of any domain $\mathcal{D}_a$ has a *name* in the object language [12, pp. 328-29]. As Andrews pointed out, "the decidability of Henkin's axiomatic system for propositional types follows directly from the results in his paper" [3, p. 68].

The idea of incorporating partiality into Church's Type Theory is not new in the literature. On the one hand, Farmer [8,9] has already defined a system in which partial functions are in the hierarchy of types. He distinguishes between kind e and kind t types. The former includes the type of individuals as well as the functions from elements of any type to elements of kind e, while the latter includes the type of truth values as well as the type of functions from elements of any type to elements of kind t. "Expressions of kind e may be non-denoting, but expressions of kind t must be denoting" [8, p. 1277].

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On the other hand, Lepage [16] and Lapierre [15] worked on a Type Theory where functions of any domain may be partial as a consequence of the introduction of a third truth-value. In particular, Lepage [17] presented a variation of Henkin's logic containing 0, 1 and the *undefined* as the basic type, so "the undefined becomes an object like the others and can thus be a value and an argument of a function" [17, p. 29]. In Lepage's system, connectives behave like those of Kleene's logic of indeterminacy [14, p. 153]: a disjunction, e.g., is true if one of its members is true, false if both are false and undefined otherwise.

However, neither a completeness proof nor a Nameability theorem is given in [17]. "The unavoidable problem linked to this approach is the impossibility of having a canonical name in the object language for every partial function" [17, p. 37]. For this reason, in this paper we provide an alternative to Lepage's Type Theory in which the Nameability theorem can be proved and completeness is obtained constructively. Since we keep Kleene's strong connectives, we call the new system Propositional Type Theory of Indeterminacy ($\mathbf{PT}_K$). The results we publish here are part of a broader research interest in combining partiality and Type Theory [4,19]. We show that, in a higher-order logic, the connectives borrowed from strong Kleene logics allow us to reason about partial functions and indeterminacy in a natural way.

Our approach differs from Lepage's in that we do not allow the undefined to become an object like the others (more precisely, for any domain $\mathcal{D}_a^K$, $* \notin \mathcal{D}_a^K$). Thus, the undefined value cannot be the argument of a function and $f(x) = *$ is used to signal the undefinedness of f at certain input (x in this case). The type of truth-values of $\mathbf{PT}_K$ only contains 0 and 1, but any complex type $\mathcal{D}_{ab}^K$ is the set of *partial* functions from $\mathcal{D}_a^K$ to $\mathcal{D}_b^K$. In order to axiomatize such a hierarchy, we must take additional primitive symbols beyond the language of $\mathbf{PT}$: one constant for each type always "denoting" $*$, infinitely many symbols $\simeq_{a\langle at\rangle}$ for weak equality (called "quasi-equality") and Kleene's strong disjunction ($\overset{*}{\vee}$). We take disjunction from strong Kleene logics rather than from weak ones, as we want to avoid *contamination* [7, pp. 73-4], also called *infectiousness* [11, p. 67], staying as close to classical logic as possible.

The paper is organized as follows. The syntax of $\mathbf{PT}_K$ is defined in Section 2.1 and the semantics is presented in Section 2.2. In Section 3, the Nameability theorem for this logic is stated and proved following Henkin's strategy. Section 4 provides a proof system for $\mathbf{PT}_K$, while some derived rules of inference and useful metatheorems are proved in Section 5. Finally, in Section 6, we give a constructive proof of completeness. We think, as Farmer

[10] does, that having partial functions and undefinedness at our disposal in Type Theory provides high benefits at low cost, since the main computational properties of **PT** are preserved in $\mathbf{PT}_K$.

2. Propositional Type Theory of Indeterminacy

2.1. Syntax

The syntax of $\mathbf{PT}_K$ is based on that of **PT**. Firstly, the set of type symbols is exactly the same, as we also get rid of the type of individuals:

Definition 1. (*Type Symbols*) We inductively define the set TYPES of type symbols as follows:

$$\mathsf{TYPES} := t \mid \langle ab \rangle,$$

with $a, b \in \mathsf{TYPES}$ and writing ab instead of $\langle ab \rangle$ when no confusion arises ($a, b, c, \ldots$ are syntactic variables ranging over type symbols).

Secondly, $\mathcal{L}_{\mathbf{PT}}$ contains parenthesis and the abstractor λ as improper symbols, a denumerably infinite set of variables of type a for each $a \in \mathsf{TYPES}$ ($f_a, g_a, h_a, x_a, y_a, z_a, \ldots$) and a logical constant $\mathsf{Q}_{a\langle at\rangle}$ for each $a \in \mathsf{TYPES}$. In $\mathbf{PT}_K$, we keep all these symbols as primitive and define an extension of $\mathcal{L}_{\mathbf{PT}}$.

Definition 2. (*Set of symbols of* $\mathbf{PT}_K$) The set of symbols of $\mathbf{PT}_K$ is defined as follows:

$$\mathcal{L}_{\mathbf{PT}_K} = \mathcal{L}_{\mathbf{PT}} \cup \{\overset{*}{\vee}, \overset{*}{\exists}\} \cup \bigcup \{U_a, \simeq_{a\langle at\rangle}\}_{a \in \mathsf{TYPES}}$$

Definition 2 can be simplified by Theorem 3 (see Corollary 4). We are now ready to define, for each $a \in \mathsf{TYPES}$, the set of meaningful expression of type a ($\alpha_a, \beta_a, \gamma_a, \ldots$ are syntactic variables ranging over expressions of type a):

Definition 3. (*Meaningful expressions of* $\mathbf{PT}_K$) The set of meaningful expressions, ME^K, is defined as follows:

$$x_a \in \mathsf{ME}^K_a \mid \mathsf{Q}_{a\langle at\rangle} \in \mathsf{ME}^K_{a\langle at\rangle} \mid U_a \in \mathsf{ME}^K_a \mid \gamma_{ab}\beta_a \in \mathsf{ME}^K_b \mid \lambda x_a \alpha_b \in \mathsf{ME}^K_{ab} \mid$$

$$\simeq_{a\langle at\rangle}(\alpha_a, \beta_a) \in \mathsf{ME}^K_t \mid \{\alpha_t \overset{*}{\vee} \beta_t, \overset{*}{\exists} x_a \alpha_t\} \in \mathsf{ME}^K_t$$

We easily see that the set of meaningful expressions of **PT** is a subset of ME^K. With regard to the specific expressions of $\mathbf{PT}_K$, U_a always “denote” the undefined value in our semantics (for any type a), $\simeq_{a\langle at\rangle}(\alpha_a, \beta_a)$ is true iff α_a and β_a have the same denotation or both are non-denoting and $\overset{*}{\vee}$ and $\overset{*}{\exists}$ are non-classical logical constants (see Definition 10).

Meaningful expressions of type t are called *formulas* and those containing no free occurrence of a variable are called *closed expressions*. Closed formulas are *sentences*. We also introduce some abbreviations to improve readability (as Andrews does in [2, p. 212]):

- $\alpha_a \equiv \beta_a$ stands for $(\mathsf{Q}_{a\langle at\rangle}\alpha_a)\beta_a$.
- $\alpha_a \simeq \beta_a$ stands for $\simeq_{a\langle at\rangle} (\alpha_a, \beta_a)$.
- 1^{N} stands for $\lambda x_t x_t \equiv \lambda x_t x_t$ (Henkin's name for truth).
- 0^{N} stands for $\lambda x_t x_t \equiv \lambda x_t 1^{\mathsf{N}}$ (Henkin's name for falsity).
- $\neg\alpha_t$ stands for $(\lambda x_t(0^{\mathsf{N}} \equiv x_t))\alpha_t$.
- $\alpha_t \overset{*}{\wedge} \beta_t$ stands for $\neg(\neg\alpha_t \overset{*}{\vee} \neg\beta_t)$.
- $\alpha_t \overset{*}{\rightarrow} \beta_t$ stands for $\neg\alpha_t \overset{*}{\vee} \beta_t$.
- $\alpha_a \uparrow$ stands for $\alpha_a \simeq U_a$ and $\alpha_a \downarrow$ stands for $\neg(\alpha_a \uparrow)$.
- $\overset{*}{\forall} x_a\alpha_t$ stands for $\neg(\overset{*}{\exists} x_a\neg\alpha_t)$.
- $\overset{*}{\exists}! x_a\alpha_t$ stands for $\overset{*}{\exists} x_a\alpha_t \overset{*}{\wedge} \overset{*}{\forall} y_a(\mathsf{S}^{x_a}_{y_a}\alpha_t \overset{*}{\rightarrow} x_a \equiv y_a)$, where $\mathsf{S}^{x_a}_{y_a}\alpha_t$ is the result of replacing each free occurrence of x_a in α_t by y_a (y_a is the first variable of type a not occurring in α_t).

Before moving to the next section, let us consider a set of formulas which correspond to formulas of the ordinary propositional logic (called P-formulas). This set is useful for proving completeness (see Theorem 15) and was isolated, for the same purposes, in [12, p. 335].

Definition 4. (*P-formulas of* $\mathbf{PT}_K$) We recursively define the set of P-formulas as follows:

- $0^{\mathsf{N}}, 1^{\mathsf{N}} \in P$.
- For any $x_t \in \mathsf{VAR}_t$, $x_t \in P$.
- If $\varphi \in P$, then $\neg\varphi \in P$.
- If $\varphi, \psi \in P$, then $\varphi \overset{*}{\vee} \psi \in P$ and $\varphi \equiv \psi \in P$.

Clearly, no formula containing either U_t or $\simeq$ belongs to this set, because we want it to resemble the propositional fragment of classical logic. In fact, in the absence of the undefined value, $\overset{*}{\vee}$ behaves exactly as classical disjunction.

Table 1. The set of 9 partial functions in $\mathcal{D}_{tt}^K$

00	$0 \longrightarrow 0$ $1 \longrightarrow 0$	10	$0 \longrightarrow 1$ $1 \longrightarrow 0$	$*0$	$0 \longrightarrow *$ $1 \longrightarrow 0$
01	$0 \longrightarrow 0$ $1 \longrightarrow 1$	11	$0 \longrightarrow 1$ $1 \longrightarrow 1$	$*1$	$0 \longrightarrow *$ $1 \longrightarrow 1$
$0*$	$0 \longrightarrow 0$ $1 \longrightarrow *$	$1*$	$0 \longrightarrow 1$ $1 \longrightarrow *$	$**$	$0 \longrightarrow *$ $1 \longrightarrow *$

2.2. Semantics

Our partial semantics is based on a hierarchy of partial functions, so we first define the notion of *partial propositional type hierarchy* as a collection of non-empty domains satisfying the following conditions:

Definition 5. (*Partial propositional type hierarchy*) The partial propositional type hierarchy $\{\mathcal{D}_a^K\}_{a \in \mathsf{TYPES}}$ is defined by:

1. $\mathcal{D}_t^K = \mathcal{D}_t = \{0, 1\}$.
2. $\mathcal{D}_{ab}^K$ is the set of *partial functions* from $\mathcal{D}_a^K$ to $\mathcal{D}_b^K$.

For any $ab \in \mathsf{TYPES}$, if f is a partial function in $\mathcal{D}_{ab}^K$ not defined at x, we write $f(x) = *$ (but $* \notin \mathcal{D}_b^K$ for any $b \in \mathsf{TYPES}$). We say that $Def(f)$ is the set $Y \subseteq \mathcal{D}_a^K$ such that $y \in Y$ iff $f(y) \neq *$. This is usually called the domain of definition of f.

Notice that for any $a \in \mathsf{TYPES}$ $|\mathcal{D}_a| < |\mathcal{D}_a^K|$. The reason is that the number of partial functions from $\mathcal{D}_a^K$ to $\mathcal{D}_b^K$ is $(|\mathcal{D}_b^K| + 1)^{|\mathcal{D}_a^K|}$. Table 1 describes the domain $\mathcal{D}_{tt}^K$, making evident the difference with $\mathcal{D}_{tt}$ (00, 01, 10 and 11 are the only ones also in $\mathcal{D}_{tt}$).

It is important to remark that Kleene's strong disjunction is not a function in a domain of our partial hierarchy, and this is why we took $\overset{*}{\vee}$ as a primitive.[1] Table 2 shows that classical disjunction is in $\mathcal{D}_{t\langle tt\rangle}$ and also that Kleene's can be found in the corresponding domain of Lepage's partial Type Theory [17, p. 33]. Although classical disjunction is a function in $\mathcal{D}_{t\langle tt\rangle}^K$ (the one sending 0 to 01 and 1 to 11), it is pretty obvious that Kleene's cannot be in $\mathcal{D}_{t\langle tt\rangle}^K$. There is no $f \in \mathcal{D}_{tt}^K$ sending 0, 1 and the undefined value to 1, because in our approach $*$ cannot be the argument of f. In consequence, $\overset{*}{\vee}$ must be a *logical constant* that behaves like Kleene's strong disjunction.

[1] Of course, we could have taken Kleene's strong conjunction as primitive instead of disjunction, obtaining the same results.

Table 2. A comparison between classical and Kleene's disjunction as functions of type $t\langle tt\rangle$

Disjunction in Henkin	Disjunction in Lepage
$0 \longrightarrow \begin{matrix} 0 \longrightarrow 0 \\ 1 \longrightarrow 1 \end{matrix}$	$0 \longrightarrow \begin{matrix} 0 \longrightarrow 0 \\ 1 \longrightarrow 1 \\ * \longrightarrow * \end{matrix}$
$1 \longrightarrow \begin{matrix} 0 \longrightarrow 1 \\ 1 \longrightarrow 1 \end{matrix}$	$1 \longrightarrow \begin{matrix} 0 \longrightarrow 1 \\ 1 \longrightarrow 1 \\ * \longrightarrow 1 \end{matrix}$
	$* \longrightarrow \begin{matrix} 0 \longrightarrow * \\ 1 \longrightarrow 1 \\ * \longrightarrow * \end{matrix}$

Table 2 also shows Lepage's characterization of partial functions. According to him, the undefined value has a status *inside* the hierarchy [16, p. 494] and, hence, functions from the set of truth-values are also defined for $*$. As a result, a function f such that $f(0) = 1$, $f(1) = 1$ and $f(*) = 1$ belongs to Lepage's system. However, in our opinion, his starting point was far from the intuitive understanding of partiality and undefinedness. A partial function from A to B is, simply stated, one that is defined for some arguments and not for others, so it may be identified with a mapping from A to $B \cup \{*\}$.

Now, we define the key semantic notions of $\mathbf{PT}_K$.

DEFINITION 6. (*Interpretation function*) The interpretation function $\mathcal{J}$ is the (total) mapping

$$\mathcal{J} : \bigcup_{a \in \mathsf{TYPES}} \{\mathsf{Q}_{a\langle at\rangle}, U_a\} \longrightarrow \bigcup_{a \in \mathsf{TYPES}} \mathcal{D}_a^K \cup \{*\}$$

such that

- $\mathcal{J}(\mathsf{Q}_{a\langle at\rangle})$ is the function $q \in \mathcal{D}^K_{a\langle at\rangle}$ such that, for any $x, y \in \mathcal{D}^K_a$, $q(x)(y) = 1$ iff $x = y$ and $q(x)(y) = 0$ otherwise. We say that q is the *identity*[2] of type $a\langle at\rangle$.
- $\mathcal{J}(U_a) = *$.

DEFINITION 7. ($\mathbf{PT}_K$ *model*) The *structure*, or *model*, for $\mathbf{PT}_K$ is the pair

$$\mathcal{M} = \langle \{\mathcal{D}_a^K\}_{a \in \mathsf{TYPES}}, \mathcal{J}\rangle,$$

[2]See Henkin [13] and [4,19].

where $\{\mathcal{D}_a^K\}_{a\in\mathsf{TYPES}}$ is the partial propositional type hierarchy and $\mathcal{J}$ the interpretation function.

DEFINITION 8. (*Assignment*) An assignment g is a (total) function

$$g : \bigcup_{a\in\mathsf{TYPES}} \mathsf{VAR}_a \longrightarrow \bigcup_{a\in\mathsf{TYPES}} \mathcal{D}_a^K$$

such that $g(x_a) \in \mathcal{D}_a^K$ for any $a \in \mathsf{TYPES}$.

An assignment g' is a x_a-variant of g if it coincides with g on all values except, perhaps, the value assigned to $x_a \in \mathsf{VAR}_a$. We will use $g_\theta^{x_a}$ to denote the x_a-variant assignment g whose value for x_a is θ.

DEFINITION 9. (*Interpretation*) An *interpretation* for $\mathbf{PT}_K$ is a pair $\langle \mathcal{M}, g\rangle$, where $\mathcal{M}$ is the *structure* for $\mathbf{PT}_K$ and g is an assignment.

DEFINITION 10. Let $\mathcal{M}$ be a structure such that $* \notin \mathcal{D}_a^K$ for any $a \in \mathsf{TYPES}$ and let g be any assignment on this structure. We recursively define, for each $\alpha_a \in \mathsf{ME}_a^K$, an interpretation $[[\alpha_a]]^{\mathcal{M},g}$ of α_a with respect to $\langle \mathcal{M}, g\rangle$ as follows:

1. $[[x_a]]^{\mathcal{M},g} = g(x_a)$.
2. $[[\mathsf{Q}_{a\langle at\rangle}]]^{\mathcal{M},g} = \mathcal{J}(\mathsf{Q}_{a\langle at\rangle})$.
3. $[[U_a]]^{\mathcal{M},g} = \mathcal{J}(U_a)$.
4. $[[\lambda x_a \alpha_b]]^{\mathcal{M},g} = f$, where f is the partial function in $\mathcal{D}_{ab}^K$ such that, for each $\theta \in \mathcal{D}_a^K$, $f(\theta) = [[\alpha_b]]^{\mathcal{M},g_\theta^{x_a}}$. Thus,
$$Def(f) = \{\theta \in \mathcal{D}_a^K \mid [[\alpha_b]]^{\mathcal{M},g_\theta^{x_a}} \neq *\}, \text{ which may be } \varnothing.$$
5. $$[[\gamma_{ab}\beta_a]]^{\mathcal{M},g} = \begin{cases} [[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}), & \text{if } [[\gamma_{ab}]]^{\mathcal{M},g} \neq *, [[\beta_a]]^{\mathcal{M},g} \neq * \text{ and} \\ & [[\gamma_{ab}]]^{\mathcal{M},g} \text{ is defined at } [[\beta_a]]^{\mathcal{M},g}; \\ *, & \text{otherwise.} \end{cases}$$
6. $$[[\alpha_a \simeq \beta_a]]^{\mathcal{M},g} = \begin{cases} 1, & \text{if } [[\alpha_a]]^{\mathcal{M},g} = [[\beta_a]]^{\mathcal{M},g} = * \text{ or both are } \neq * \\ & \text{and } [[\alpha_a]]^{\mathcal{M},g} = [[\beta_a]]^{\mathcal{M},g}; \\ 0, & \text{if } [[\alpha_a]]^{\mathcal{M},g} \neq [[\beta_a]]^{\mathcal{M},g}. \end{cases}$$

7.

$$[[\alpha_t \overset{*}{\vee} \beta_t]]^{\mathcal{M},g} = \begin{cases} 1, & \text{if } [[\alpha_t]]^{\mathcal{M},g} = 1 \text{ or } [[\beta_t]]^{\mathcal{M},g} = 1; \\ 0, & \text{if } [[\alpha_t]]^{\mathcal{M},g} = 0 \text{ and } [[\beta_t]]^{\mathcal{M},g} = 0; \\ *, & \text{otherwise.} \end{cases}$$

8.

$$[[\overset{*}{\exists} x_a \alpha_t]]^{\mathcal{M},g} = \begin{cases} 1, & \text{if there is some } x \in \mathcal{D}_a^K \text{ s.t. } [[\lambda x_a \alpha_t]]^{\mathcal{M},g}(x) = 1; \\ 0, & \text{if for all } x \in \mathcal{D}_a^K \text{ it holds that } [[\lambda x_a \alpha_t]]^{\mathcal{M},g}(x) = 0; \\ *, & \text{otherwise.} \end{cases}$$

Notice that from Definition 10 it follows that, for each $\alpha_a \in \mathsf{ME}_a^K$, $[[\alpha_a]]^{\mathcal{M},g} \in \mathcal{D}_a^K$ or $[[\alpha_a]]^{\mathcal{M},g} = *$.

Observe that, according to the corresponding abbreviations and Definition 10(5), negation behaves as expected (classically for 0 and 1, yielding $*$ for an undefined argument). Taking Definition 10(7) and negation, we easily see that a conjunction is true iff both conjuncts are true, false whenever one of them is false and undefined in any other case. Finally, by Definition 10(8), we have:

$$[[\overset{*}{\exists}! x_a \alpha_t]]^{\mathcal{M},g} = \begin{cases} 1, & \text{if there is a unique } x \in \mathcal{D}_a^K \text{ s.t } [[\lambda x_a \alpha_t]]^{\mathcal{M},g}(x) = 1; \\ 0, & \text{if } [[\overset{*}{\exists} x_a \alpha_t]]^{\mathcal{M},g} = 0 \text{ or there are } x, y \in \mathcal{D}_a^K \text{ s.t} \\ & x \neq y, [[\lambda x_a \alpha_t]]^{\mathcal{M},g}(x) = 1 \text{ and } [[\lambda x_a \alpha_t]]^{\mathcal{M},g}(y) = 1; \\ *, & \text{otherwise.} \end{cases}$$

With regard to the concept of validity, Lepage [17, p. 34] introduced two notions of validity: being different from 0 for every assignment (weak validity) and being equal to 1 for every assignment. Since we do not want U_t to be valid, we restrict ourselves to the latter.

Definition 11. (*Validity*) For any $\alpha_t \in \mathsf{ME}_t^K$, α_t is valid iff, for every assignment g, $[[A_t]]^{\mathcal{M},g} = 1$, written $\models A_t$. If α_t is valid and $\alpha_t \in P$, we say that α_t is a *tautology*.

3. The Nameability Theorem

The Nameability theorem states the possibility of finding, for every element of any domain of the hierarchy, a closed expression in the object language whose interpretation is that particular element. To prove this result for $\mathbf{PT}_K$, we will follow Henkin's strategy in [12, 328–329]. The first step towards the Nameability theorem is to define an *election function* for each type, as follows (this function is marked in bold):

DEFINITION 12. For any $a \in \mathsf{TYPES}$, let $\mathbf{t}^a$ be a function in $\mathcal{D}^K_{\langle at\rangle a}$ such that, for any $f \in \mathcal{D}^K_{at}$, $\mathbf{t}^a(f)$ is the unique $x \in \mathcal{D}^K_a$ for which $f(x) = 1$ or $\mathbf{t}^a(f) = *$ if there is no such an x or if there are more than one.

Then, the next step is to show that it is possible to find a name (a closed expression of the corresponding type) for each of these election functions (see Lemma 1). After this, the desired result is obtained for any $a \in \mathsf{TYPES}$ and every element in $\mathcal{D}^K_a$ (Theorem 2), so let us start by proving the Lemma.

LEMMA 1. *For every* $a \in \mathsf{TYPES}$*, there exists a closed expression* $\iota_{\langle at\rangle a}$ *such that* $[[\iota_{\langle at\rangle a}]]^{\mathcal{M},g} = \mathbf{t}^a$.

PROOF. The proof is by induction.

1. *Base case*: $\mathcal{D}^K_t$. By definition of $\mathbf{t}^t$:
   - $\mathbf{t}^t(01) = \mathbf{t}^t(*1) = 1$,
   - $\mathbf{t}^t(10) = \mathbf{t}^t(1*) = 0$,
   - $\mathbf{t}^t(11) = \mathbf{t}^t(**) = \mathbf{t}^t(*0) = \mathbf{t}^t(0*) = \mathbf{t}^t(00) = *$.

   We want to prove that $[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g} = \mathbf{t}^t$, where $\iota_{\langle tt\rangle t} :=$

$$\lambda f_{tt}((U_t \overset{*}{\vee} (f_{tt} \equiv \lambda x_t x_t) \overset{*}{\vee} (f_{tt} \equiv \lambda x_t(x_t \overset{*}{\vee} U_t))) \overset{*}{\wedge} \neg(f_{tt} \equiv (\lambda x_t(0^{\mathsf{N}} \equiv x_t))) \overset{*}{\wedge} \neg(f_{tt} \equiv (\lambda x_t(\neg x_t \overset{*}{\vee} U_t)))).$$

   We see that $[[\lambda f_{tt}(f_{tt} \equiv \lambda x_t x_t)]]^{\mathcal{M},g}(01) = 1$ and hence $[[\lambda f_{tt}(U_t \overset{*}{\vee} (f_{tt} \equiv \lambda x_t x_t) \overset{*}{\vee} (f_{tt} \equiv \lambda x_t(x_t \overset{*}{\vee} U_t)))]]^{\mathcal{M},g}(01) = 1$. Since $[[\lambda f_{tt}\neg(f_{tt} \equiv (\lambda x_t(0^{\mathsf{N}} \equiv x_t)))]]^{\mathcal{M},g}(01) = 1$ and $[[\lambda f_{tt}\neg(f_{tt} \equiv (\lambda x_t(\neg x_t \overset{*}{\vee} U_t)))]]^{\mathcal{M},g}(01) = 1$, it follows that $[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(01) = 1$. The same argument works analogously for $*1$, as $[[\lambda f_{tt}(f_{tt} \equiv \lambda x_t(x_t \overset{*}{\vee} U_t))]]^{\mathcal{M},g}(*1) = 1$, so $[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(*1) = 1$. We can also check that $[[\lambda f_{tt}\neg(f_{tt} \equiv (\lambda x_t(0^{\mathsf{N}} \equiv x_t)))]]^{\mathcal{M},g}(10) = 0$ and $[[\lambda f_{tt}\neg(f_{tt} \equiv (\lambda x_t(\neg x_t \overset{*}{\vee} U_t)))]]^{\mathcal{M},g}(1*) = 0$, which is enough to conclude that $[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(10) = 0$ and $[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(1*) = 0$.

   Finally, $[[\lambda f_{tt}(U_t \overset{*}{\vee} (f_{tt} \equiv \lambda x_t x_t) \overset{*}{\vee} (f_{tt} \equiv \lambda x_t(x_t \overset{*}{\vee} U_t)))]]^{\mathcal{M},g}(11) = *$, $[[\lambda f_{tt}\neg(f_{tt} \equiv (\lambda x_t(0^{\mathsf{N}} \equiv x_t)))]]^{\mathcal{M},g}(11) = 1$ and $[[\lambda f_{tt}\neg(f_{tt} \equiv (\lambda x_t(\neg x_t \overset{*}{\vee} U_t)))]]^{\mathcal{M},g}(11) = 1$. Hence, $[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(11) = *$. The same argument works analogously for $**$, $*0$, $0*$ and 00, so:

$$[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(**) = [[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(*0) = [[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(0*) = [[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g}(00) = *.$$

Therefore, $[[\iota_{\langle tt\rangle t}]]^{\mathcal{M},g} = \mathbf{t}^t$.

2. *Inductive step*: $\mathcal{D}^K_{ab}$. By induction hypothesis, we assume that $\iota_{\langle bt\rangle b}$ and $\imath x_b\alpha_t$ have been defined and have the desired properties ($\imath x_b\alpha_t$ stands for $\iota_{\langle bt\rangle b}(\lambda x_b\alpha_t)$). Fix $\iota_{\langle\langle ab\rangle t\rangle ab} :=$

$$\lambda f_{\langle ab\rangle t}(\lambda x_a(\imath y_b(\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall} z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))))).$$

We must show that $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g} = \mathbf{t}^{ab}$. Let $h \in \mathcal{D}^K_{\langle ab\rangle t}$. There are five possibilities:

(a) h is a function such that there is exactly one $x \in \mathcal{D}^K_{ab}$, say d, such that $h(d) = 1$. Let g be an assignment such that $g(f_{\langle ab\rangle t}) = h$ and take

$$[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall} z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b)))]]^{\mathcal{M},g}.$$

We see immediately that $[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}))]]^{\mathcal{M},g} = 1$. Let $g(z_{ab}) \neq d$. Then, $[[f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow]]^{\mathcal{M},g} = 0$, because $[[f_{\langle ab\rangle t}z_{ab}]]^{\mathcal{M},g} = *$ (and hence $[[f_{\langle ab\rangle t}z_{ab}\downarrow]]^{\mathcal{M},g} = 0$) or $[[f_{\langle ab\rangle t}z_{ab}]]^{\mathcal{M},g} = 0$. Thus, $[[(f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))]]^{\mathcal{M},g} = 1$ in that case. Now, if $g(z_{ab}) = d$, $[[f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow]]^{\mathcal{M},g} = 1$. Since $[[(z_{ab}x_a \equiv y_b]]^{\mathcal{M},g}$ is 1 iff $g(y_b) = d(g(x_a))$, it holds that $[[(f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))]]^{\mathcal{M},g} = 1$ iff $g(y_b) = d(g(x_a))$, what also works for $[[\overset{*}{\forall} z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b)))]]^{\mathcal{M},g}$. Thus:

$$[[\imath y_b(\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall} z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab}\downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))))]]^{\mathcal{M},g} = d(g(x_a))$$

and consequently $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g}(h) = d$ for this h.

(b) h is a total function in $\mathcal{D}^K_{\langle ab\rangle t}$ with constant value 0. Let g be an assignment such that $g(f_{\langle ab\rangle t}) = h$ and take

$$[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \\ \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b)))]]^{\mathcal{M},g}.$$

Clearly, $[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}))]]^{\mathcal{M},g} = 0$ and hence $[[\lambda y_b(\overset{*}{\exists}z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))]]^{\mathcal{M},g}$ is the function in $\mathcal{D}^K_{bt}$ with constant value 0. Therefore:

$$[[\imath y_b(\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \\ \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))))]]^{\mathcal{M},g} = *,$$

and consequently $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g}(h) = *$ for this h.

(c) h is a function in $\mathcal{D}^K_{\langle ab\rangle t}$ such that $h(u) = 1$ and $h(s) = 1$, with $u, s \in \mathcal{D}^K_{ab}$ and $u \neq s$. Let $g(f_{\langle ab\rangle t}) = h$ and take

$$[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \\ \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b)))]]^{\mathcal{M},g}.$$

We see again that $[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}))]]^{\mathcal{M},g} = 0$, so the argument in (b) works here. Thus:

$$[[\imath y_b(\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \\ \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))))]]^{\mathcal{M},g} = *,$$

and consequently $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g}(h) = *$ for this h.

(d) h is the function in $\mathcal{D}^K_{\langle ab\rangle t}$ such that $Def(h) = \varnothing$ (empty function). Let $g(f_{\langle ab\rangle t}) = h$ and take

$$[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \\ \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b)))]]^{\mathcal{M},g}.$$

In this case, $[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}))]]^{\mathcal{M},g} = *$. On the other hand, we know that $[[f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow]]^{\mathcal{M},g} = 0$, because $[[f_{\langle ab\rangle t}z_{ab} \downarrow]]^{\mathcal{M},g} = 0$. Thus, $[[\overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv y_b))]]^{\mathcal{M},g} = 1$, and hence $[[\lambda y_b(\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab}) \overset{*}{\wedge} \overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab} \overset{*}{\wedge} f_{\langle ab\rangle t}z_{ab} \downarrow) \overset{*}{\rightarrow} (z_{ab}x_a \equiv$

$y_b)))]]^{\mathcal{M},g}$ is the empty function in $\mathcal{D}^K_{bt}$. Therefore:

$$[[\imath y_b(\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab})\overset{*}{\wedge}\overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab}\overset{*}{\wedge}f_{\langle ab\rangle t}z_{ab}\downarrow)\\ \overset{*}{\rightarrow}(z_{ab}x_a\equiv y_b))))]]^{\mathcal{M},g}=*,$$

and consequently $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g}(h)=*$ for this h.

(e) h is the proper partial function from $\mathcal{D}^K_{ab}$ to $\mathcal{D}^K_t$ such that, for every $x\in\mathcal{D}^K_{ab}$, if $x\in Def(h)$, then $h(x)=0$. Let $g(f_{\langle ab\rangle t})=h$ and take

$$[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab})\overset{*}{\wedge}\overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab}\overset{*}{\wedge}f_{\langle ab\rangle t}z_{ab}\downarrow)\\ \overset{*}{\rightarrow}(z_{ab}x_a\equiv y_b)))]]^{\mathcal{M},g}.$$

Obviously, $[[\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab})]]^{\mathcal{M},g}=*$. Now, notice that $[[(f_{\langle ab\rangle t}z_{ab}\overset{*}{\wedge}f_{\langle ab\rangle t}z_{ab}\downarrow)]]^{\mathcal{M},g}$ is 0 in case $[[f_{\langle ab\rangle t}z_{ab}]]^{\mathcal{M},g}=0$, as well as in case $[[f_{\langle ab\rangle t}z_{ab}]]^{\mathcal{M},g}=*$. Thus, $[[\overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab}\overset{*}{\wedge}f_{\langle ab\rangle t}z_{ab}\downarrow)\overset{*}{\rightarrow}(z_{ab}x_a\equiv y_b))]]^{\mathcal{M},g}=1$, so the argument in (d) also works here. Hence:

$$[[\imath y_b(\overset{*}{\exists}!z_{ab}(f_{\langle ab\rangle t}z_{ab})\overset{*}{\wedge}\overset{*}{\forall}z_{ab}((f_{\langle ab\rangle t}z_{ab}\overset{*}{\wedge}f_{\langle ab\rangle t}z_{ab}\downarrow)\\ \overset{*}{\rightarrow}(z_{ab}x_a\equiv y_b))))]]^{\mathcal{M},g}=*,$$

and consequently $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g}(h)=*$ for this h.

Thus, we showed that $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g}(h)=\mathbf{t}^{ab}(h)$ for every $h\in\mathcal{D}^K_{\langle ab\rangle t}$, so $[[\iota_{\langle\langle ab\rangle t\rangle ab}]]^{\mathcal{M},g}=\mathbf{t}^{ab}$, as claimed.

■

Now, before generalizing this result to each element of every domain of our partial propositional type hierarchy, let us introduce the following convention concerning our primitive constants for the undefined (see also Definition 5):

Definition 13. For any $ab\in\mathsf{TYPES}$, if $f\in\mathcal{D}^K_{ab}$, $x\in\mathcal{D}^K_a$ and $x\notin Def(f)$, then we define $(f(x))^{\mathsf{N}}$ as U_b.

Thus, we are now ready to prove the theorem for $\mathbf{PT}_K$.

Theorem 2. (*Nameability Theorem*) *For any* $a\in\mathsf{TYPES}$ *and each* $x\in\mathcal{D}^K_a$, *there exists a closed formula* x^{N} *of type* a *such that* $[[x^{\mathsf{N}}]]^{\mathcal{M},g}=x$.

Proof. The proof is by induction.

1. *Base case*: $\mathcal{D}_t^K$. Clearly, $[[\lambda x_t x_t \equiv \lambda x_t x_t]]^{\mathcal{M},g} = 1$, so $1^{\mathsf{N}} := \lambda x_t x_t \equiv \lambda x_t x_t$. In addition to this, $[[\lambda x_t x_t \equiv \lambda x_t 1^{\mathsf{N}}]]^{\mathcal{M},g} = 0$, so $0^{\mathsf{N}} := \lambda x_t x_t \equiv \lambda x_t 1^{\mathsf{N}}$.
2. *Inductive step*: $\mathcal{D}_{ab}^K$. Suppose that $y_1, \ldots, y_q$ are distinct and are all the elements of $\mathcal{D}_a^K$. By induction hypothesis, we assume that to every x of $\mathcal{D}_a^K$ and of $\mathcal{D}_b^K$ we have already assigned a name. Let f be any function in $\mathcal{D}_{ab}^K$. Take $f^{\mathsf{N}} :=$

$$\lambda x_a(\imath z_b((x_a \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_1))^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((x_a \equiv y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_q))^{\mathsf{N}})))$$

and consider an assignment g such that $g(x_a) = y_i$. Since $[[x_a \equiv y_j^{\mathsf{N}}]]^{\mathcal{M},g} \neq *$ for every $j \in \{1, \ldots, q\}$, $[[x_a \equiv y_j^{\mathsf{N}}]]^{\mathcal{M},g}$ will be 1 or 0 according as $\{i = j\}$ or $i \neq j$. It follows that, for any $i \neq j$, $[[(x_a \equiv y_j^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_j))^{\mathsf{N}})]]^{\mathcal{M},g} = 0$. Now, there are two possibilities:

(a) f is defined at y_i. In this case,

$$[[((x_a \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_1))^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((x_a \equiv y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_q))^{\mathsf{N}}))]]^{\mathcal{M},g}$$

$= 1$ iff $g(z_b) = f(y_i)$ and consequently

$$[[\imath z_b((x_a \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_1))^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((x_a \equiv y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_q))^{\mathsf{N}})))]]^{\mathcal{M},g}$$

$= f(y_i)$.

(b) $f(y_i) = *$. In this case,

$$[[((x_a \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_1))^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((x_a \equiv y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_q))^{\mathsf{N}}))]]^{\mathcal{M},g} = *,$$

for $[[x_a \equiv y_i^{\mathsf{N}}]]^{\mathcal{M},g} = 1$ and $[[z_b \equiv (f(y_i))^{\mathsf{N}}]]^{\mathcal{M},g} = *$ (independently of $g(z_b)$), so $[[(x_a \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_1))^{\mathsf{N}})]]^{\mathcal{M},g} = *$. Thus, we know that $[[(\lambda z_b((x_a \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_1))^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((x_a \equiv y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_q))^{\mathsf{N}})))]]^{\mathcal{M},g}$ is the empty function in $\mathcal{D}_{bt}^K$. Hence:

$$[[\imath z_b((x_a{\equiv}y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b{\equiv}(f(y_1))^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((x_a{\equiv}y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b{\equiv}(f(y_q))^{\mathsf{N}})))]]^{\mathcal{M},g} = *.$$

Therefore, $[[f^{\mathsf{N}}]]^{\mathcal{M},g}(y_i) = f(y_i)$ iff f is defined at y_i and $[[f^{\mathsf{N}}]]^{\mathcal{M},g}(y_i) = *$ otherwise. Since this fact holds for each i=$\{1, \ldots, q\}$, we get $[[f^{\mathsf{N}}]]^{\mathcal{M},g} = f$, as claimed.

■

Table 3. The names of all the functions in $\mathcal{D}_{tt}^{K}$

00 $\lambda x_t 0^{\mathsf{N}}$	10 $\lambda x_t(0^{\mathsf{N}} \equiv x_t)$	$*0$ $\lambda x_t(\neg x_t \overset{*}{\wedge} U_t)$
01 $\lambda x_t x_t$	11 $\lambda x_t 1^{\mathsf{N}}$	$*1$ $\lambda x_t(x_t \overset{*}{\vee} U_t)$
$0*$ $\lambda x_t(x_t \overset{*}{\wedge} U_t)$	$1*$ $\lambda x_t(\neg x_t \overset{*}{\vee} U_t)$	$**$ $\lambda x_t U_t$

Table 3 illustrates the simplest closed expressions of type tt corresponding to each function in $\mathcal{D}_{tt}^{K}$.[3]

Once the Nameability theorem has been stated, we explore its consequences to show how the set of primitive symbols of $\mathbf{PT}_K$ can be simplified (Definition 2). In the present (finitary) context, existential statements can be equivalently re-written as disjunctions, i.e. $\overset{*}{\exists}$ can be defined in terms of $\overset{*}{\vee}$. This is stated in the following theorem:

THEOREM 3. *Let $y_1, \ldots, y_q$ be a list of all the distinct elements of $\mathcal{D}_a^K$ and consider $\alpha_t \in \mathsf{ME}_t^K$. Let $\mathsf{S}^{x_a}_{(y_i)^{\mathsf{N}}}\alpha_t$ be the result of replacing each free occurrence of x_a in α_t by $(y_i)^{\mathsf{N}}$. Then, $[[\overset{*}{\exists} x_a \alpha_t]]^{\mathcal{M},g} = [[\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}}\alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}}\alpha_t]]^{\mathcal{M},g}$.*

PROOF.

1. Suppose that $[[\overset{*}{\exists} x_a \alpha_t]]^{\mathcal{M},g} = 1$. We have to show that $[[\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}}\alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}}\alpha_t]]^{\mathcal{M},g} = 1$. By Definition 10(8), it follows that there is a $j \in \{1, ..., q\}$ such that $[[\lambda x_a \alpha_t]]^{\mathcal{M},g}(y_j) = 1$. Thus, $[[\mathsf{S}^{x_a}_{(y_j)^{\mathsf{N}}}\alpha_t]] = 1$, what is enough to conclude that $[[\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}}\alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}}\alpha_t]]^{\mathcal{M},g} = 1$.

2. Suppose that $[[\overset{*}{\exists} x_a \alpha_t]]^{\mathcal{M},g} = 0$. We have to show that $[[\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}}\alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}}\alpha_t]]^{\mathcal{M},g} = 0$. By Definition 10(8), it follows that, for every $k \in \{1, ..., q\}$, $[[\lambda x_a \alpha_t]]^{\mathcal{M},g}(y_k) = 0$ and hence $[[\mathsf{S}^{x_a}_{(y_k)^{\mathsf{N}}}\alpha_t]] = 0$. Thus, $[[\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}}\alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}}\alpha_t]]^{\mathcal{M},g} = 0$.

[3] Although Lepage [16] refuses add a primitive constant "denoting" the undefined value, we believe that it is not possible to find a name for a function f such that $f(0) = 1, f(1) = 1$ and $f(*) = *$ (not even following the strategy of Table 3).

3. Suppose that $[[\overset{*}{\exists} x_a \alpha_t]]^{\mathcal{M},g} = *$. We have to show that $[[\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}}\alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}}\alpha_t]]^{\mathcal{M},g} = *$. By Definition 10(8), it follows that, for every $k \in \{1,\ldots,q\}$, $[[\lambda x_a \alpha_t]]^{\mathcal{M},g}(y_k) \neq 1$ and hence $[[\mathsf{S}^{x_a}_{(y_k)^{\mathsf{N}}}\alpha_t]] \neq 1$. We also know that there exists at least a $j \in \{1,\ldots,q\}$ such that $[[\lambda x_a \alpha_t]]^{\mathcal{M},g}(y_j) = *$, so $[[\mathsf{S}^{x_a}_{(y_j)^{\mathsf{N}}}\alpha_t]] = *$. Therefore, $[[\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}}\alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}}\alpha_t]]^{\mathcal{M},g} = *$, as required.

■

Corollary 4. *The set of symbols of* $\mathbf{PT}_K$ *is simplified to*

$$\mathcal{L}_{\mathbf{PT}_K} = \mathcal{L}_{\mathbf{PT}} \cup \{\overset{*}{\vee}\} \cup \bigcup \{U_a, \simeq_{a\langle at\rangle}\}_{a\in\mathsf{TYPES}}$$

Finally, let us introduce two more Definitions which depend essentially on Theorem 2 and which play a very important role in proving both Theorem 17 and Lemma 20:

Definition 14. For any $a \in \mathsf{TYPES}$, if $\alpha_a \in \mathsf{ME}^K_a$ and $[[\alpha_a]]^{\mathcal{M},g} = *$, then we define $([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}}$ as U_t.

Next we define a uniform way of replacing free variables in α_c by the name of their denotations (in a way analogous to Henkin) without changing the meaning of function abstractions (where variables may occur bound).

Definition 15. Let $\alpha_c \in \mathsf{ME}^K_c$ and let g be an assignment. Take $V \subset \bigcup_{a\in\mathsf{TYPES}} \mathsf{VAR}_a$.

We define $\alpha_c^{(g_V)}$ as follows:

- $$x_a^{(g_V)} = \begin{cases} x_a, & \text{if } x_a \in V; \\ (g(x_a))^{\mathsf{N}}, & \text{if } x_a \notin V. \end{cases}$$
- $\mathsf{Q}^{(g_V)}_{a\langle at\rangle} = \mathsf{Q}_{a\langle at\rangle}$
- $U_a^{(g_V)} = U_a$
- $(\gamma_{ab}\beta_a)^{(g_V)} = \gamma_{ab}^{(g_V)}\beta_a^{(g_V)}$
- $(\lambda x_a \alpha_b)^{(g_V)} = \lambda x_a (\alpha_b)^{(g_V \cup \{x_a\})}$
- $(\alpha_a \simeq \beta_a)^{(g_V)} = \alpha_a^{(g_V)} \simeq \beta_a^{(g_V)}$
- $(\alpha_a \overset{*}{\vee} \beta_a)^{(g_V)} = \alpha_a^{(g_V)} \overset{*}{\vee} \beta_a^{(g_V)}$

We will use $\alpha_c^{(g)}$ to denote $\alpha_c^{(g_\varnothing)}$.

4. Proof System

In this section, we present the proof system of $\mathbf{PT}_K$, which finds its inspiration in Henkin [12], Farmer [9], Blackburn et al. [6] and Manzano et al . [19]. Let $y_1, \ldots, y_q$ be a list of all the distinct elements of $\mathcal{D}_a^K$. The axioms and axiom schemes of $\mathbf{PT}_K$ are the following:

1. *Partial propositional types*:
   a. $\vdash (\alpha_t \equiv 1^{\mathsf{N}}) \simeq \alpha_t$.
   b. $\vdash (g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}}) \simeq \overset{*}{\forall} x_t(g_{tt}x_t)$.
   c. $\vdash (f_{ab} \equiv g_{ab}) \simeq \overset{*}{\forall} x_a(f_{ab}x_a \simeq g_{ab}x_a)$.
   d. $\vdash \beta_a \downarrow \overset{*}{\rightarrow} ((\lambda x_a \alpha_b)\beta_a \simeq (\mathsf{S}^{x_a}_{\beta_a}\alpha_b))$, provided β_a is free for x_a in α_b.
2. *Quasi-equality*:
   a. $\vdash \alpha_a \simeq \alpha_a$.
   b. $\vdash (\alpha_a \simeq \beta_a) \simeq (\beta_a \simeq \alpha_a)$.
   c. $\vdash ((\alpha_a \simeq \beta_a) \simeq (\beta_a \simeq \gamma_a)) \simeq (\alpha_a \simeq \gamma_a)$.
3. *Truth-table of* $\simeq$:
   a. $\vdash (1^{\mathsf{N}} \simeq 0^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$.
   b. $\vdash (1^{\mathsf{N}} \simeq U_t) \simeq 0^{\mathsf{N}}$.
   c. $\vdash (0^{\mathsf{N}} \simeq U_t) \simeq 0^{\mathsf{N}}$.
4. *Equality and quasi-equality*:
   a. $\vdash \alpha_a \downarrow \overset{*}{\rightarrow} (\beta_a \downarrow \overset{*}{\rightarrow} ((\alpha_a \simeq \beta_a) \simeq (\alpha_a \equiv \beta_a)))$.
5. *Negation*:
   a. $\vdash (\alpha_t \simeq 1^{\mathsf{N}}) \simeq (\neg \alpha_t \simeq 0^{\mathsf{N}})$.
   b. $\vdash (\alpha_t \simeq 0^{\mathsf{N}}) \simeq (\neg \alpha_t \simeq 1^{\mathsf{N}})$.
   c. $\vdash (\alpha_t \simeq U_t) \simeq (\neg \alpha_t \simeq U_t)$.
   d. $\vdash \neg\neg \alpha_t \simeq \alpha_t$.
6. *Commutative property*:
   a. $\vdash (\alpha_a \overset{*}{\vee} \beta_a) \simeq (\beta_a \overset{*}{\vee} \alpha_a)$.
   b. $\vdash (\alpha_a \overset{*}{\wedge} \beta_a) \simeq (\beta_a \overset{*}{\wedge} \alpha_a)$.
7. *Truth-table of* $\overset{*}{\vee}$:
   a. $\vdash (\alpha_t \overset{*}{\vee} 1^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$.
   b. $\vdash (0^{\mathsf{N}} \overset{*}{\vee} 0^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$.
   c. $\vdash (U_t \overset{*}{\vee} 0^{\mathsf{N}}) \simeq U_t$.

d. $\vdash (U_t \overset{*}{\vee} U_t) \simeq U_t$.
e. $\vdash (\alpha \overset{*}{\wedge} \beta) \simeq \neg(\neg\alpha \overset{*}{\vee} \neg\beta)$

8. *Definedness*:

a. $\vdash c_a \downarrow$, for any primitive constant $c_a \neq U_a$.
b. $\vdash \lambda x_a \alpha_b \downarrow$.
c. $\vdash \alpha_t \downarrow$, for any $\alpha_t \in P$ (see Definition 4).

9. *Quantification*:

a. $\vdash \overset{*}{\exists} x_a \alpha_t \simeq (\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}} \alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}} \alpha_t)$.
b. $\vdash \overset{*}{\forall} x_a \alpha_t \simeq \neg \overset{*}{\exists} x_a \neg \alpha_t$.
c. $\vdash (\overset{*}{\forall} x_a \alpha_t \simeq 1^{\mathsf{N}}) \overset{*}{\to} (\lambda y_a 1^{\mathsf{N}} \equiv \lambda x_a \alpha_t)$.

10. *Definite descriptions and definedness*:

a. $\vdash \beta_a \downarrow \overset{*}{\to} (\imath x_a (x_a \equiv \beta_a) \downarrow)$.
b. $\vdash (\imath x_a \alpha_t \downarrow) \overset{*}{\to} (\lambda x_a \alpha_t)(\imath x_a \alpha_t) \equiv 1^{\mathsf{N}}$.
c. $\vdash \beta_a \uparrow \overset{*}{\to} (\imath x_a (x_a \equiv \beta_a) \uparrow)$.

The rules of inference of $\mathbf{PT}_K$ are quite standard. The Rule of Replacement differs from that of Henkin [12, p. 330] and was taken from Farmer [9] (he called it "Quasi-Equality Substitution"):

1. *Rule of Replacement*: If $\vdash \alpha_a \simeq \beta_a$ and $\vdash \gamma_t$, then $\vdash \delta_t$, where δ_t is the result of replacing one occurrence of α_a in γ_t by an occurrence of β_a, provided that the occurrence of α_a in γ_t is not immediately preceded by λ or in a meaningful part $\lambda x_b \varepsilon_c$ of γ_t where $x_b \in FreeVar(\alpha_a \simeq \beta_a)$.
2. *Modus Ponens*: If $\vdash \alpha_t$ and $\vdash \alpha_t \overset{*}{\to} \beta_t$, then $\vdash \beta_t$.
3. $\forall$-*Generalization*: If $\vdash \alpha_t$, then $\vdash \overset{*}{\forall} x_a \alpha_t$.
4. $\exists$-*Generalization*: If $\vdash \mathsf{S}^{x_a}_{\beta_a} \alpha_t$ and $\vdash \beta_a \downarrow$, then $\vdash \overset{*}{\exists} x_a \alpha_t$.
5. $\downarrow$-*Generalization*: If $\vdash \alpha_t$, then $\vdash \alpha_t \downarrow$.

Definition 16. (*Proof*) For any $\alpha_t \in \mathsf{ME}^K_t$, a proof of α_t in $\mathbf{PT}_K$ is a finite sequence of formulas, ending with α_t such that each member in the sequence is an axiom or an instance of an axiom schema of $\mathbf{PT}_K$ or is inferred from preceding formulas in the sequence by a rule of inference of $\mathbf{PT}_K$. A theorem of $\mathbf{PT}_K$ is a formula for which there is a proof in $\mathbf{PT}_K$, written $\vdash \alpha_t$.

5. Some Metatheorems

5.1. Derived Rules of Inference

Now, we introduce some derived rules of inference that can be easily obtained from our proof system. These rules are used to state the results of Sects. 5.2 and 6 and the proofs are based on those of [1,2,9,12]. Propositions 5, 6, 7 and 8 are called Rules 5, 6, 7 and 8, respectively.

PROPOSITION 5. *If* $\vdash \alpha_t$ *and* $\vdash \alpha_t \simeq \beta_t$*, then* $\vdash \beta_t$*.*

PROOF. Suppose that $\vdash \alpha_t$ and $\vdash \alpha_t \simeq \beta_t$. Immediate by Rule 1. ■

PROPOSITION 6. $\vdash \alpha_t$ *iff* $\vdash \alpha_t \equiv 1^{\mathsf{N}}$.

PROOF. We prove both sides of the implication.

($\Rightarrow$) Suppose that $\vdash \alpha_t$. We know that $\vdash (\alpha_t \equiv 1^{\mathsf{N}}) \simeq \alpha_t$ (Axiom 1a). By Rule 5, we obtain $\vdash \alpha_t \equiv 1^{\mathsf{N}}$.

($\Leftarrow$) Suppose that $\vdash \alpha_t \equiv 1^{\mathsf{N}}$. We know that $\vdash (\alpha_t \equiv 1^{\mathsf{N}}) \simeq \alpha_t$ (Axiom 1a). By Rule 5, we get $\vdash \alpha_t$.

■

PROPOSITION 7. $\vdash \alpha_t$ *iff* $\vdash \alpha_t \simeq 1^{\mathsf{N}}$.

PROOF. We prove both sides of the implication.

($\Rightarrow$) Suppose that $\vdash \alpha_t$. By applying Rule 6, $\vdash \alpha_t \equiv 1^{\mathsf{N}}$, as well as $\vdash \alpha_t \downarrow$ by Rule V and $\vdash 1^{\mathsf{N}} \downarrow$ by 8c. Thus, we get $\vdash \alpha_t \simeq 1^{\mathsf{N}}$ by Axioms 4a, 2b and Rules II and 5.

($\Leftarrow$) Suppose that $\vdash \alpha_t \simeq 1^{\mathsf{N}}$. We know that $\vdash 1^{\mathsf{N}} \simeq 1^{\mathsf{N}}$ by Axioms 2a. Since $\vdash 1^{\mathsf{N}} \downarrow$ is an instance of Axiom 8c, we get $\vdash 1^{\mathsf{N}} \equiv 1^{\mathsf{N}}$ by Axiom 4a and II. Thus, $\vdash 1^{\mathsf{N}}$ by Rule 6 and hence $\vdash \alpha_t$ by the assumption, Axiom 2b and Rule 5.

■

PROPOSITION 8. *If* $\vdash \neg\beta_t$ *and* $\vdash \alpha_t \simeq \beta_t$*, then* $\vdash \neg\alpha_t$*.*

PROOF. Suppose that $\vdash \neg\beta_t$ and $\vdash \alpha_t \simeq \beta_t$. Then, $\vdash \neg\beta_t \simeq 1^{\mathsf{N}}$ by Rule 7, so $\vdash \beta_t \simeq 0^{\mathsf{N}}$ by Rules 5 and 1 and Axioms 5a and 5d. It follows that $\vdash \alpha_t \simeq 0^{\mathsf{N}}$ by Rule 1 again and hence $\vdash \neg\alpha_t \simeq 1^{\mathsf{N}}$ by Rule 5 and Axiom 5b. In consequence, $\vdash \neg\alpha_t$ by Rule 7. ■

PROPOSITION 9. *The following formulas are theorems of* $\mathbf{PT}_K$*:*

1. $\vdash U_t \uparrow$.
2. $\vdash U_t \simeq \neg U_t$.
3. $\vdash (\alpha_t \overset{*}{\wedge} 0^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$.
4. $\vdash (1^{\mathsf{N}} \overset{*}{\wedge} 1^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$.
5. $\vdash (U_t \overset{*}{\wedge} 1^{\mathsf{N}}) \simeq U_t$.
6. $\vdash (U_t \overset{*}{\wedge} U_t) \simeq U_t$.
7. $\vdash \overset{*}{\forall} x_a \alpha_t \simeq (\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}} \alpha_t \overset{*}{\wedge} \ldots \overset{*}{\wedge} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}} \alpha_t)$.

PROOF.

1. $\vdash U_t \simeq U_t$ is an instance of Axiom 2a. By the definition of $\uparrow$, it follows that $\vdash U_t \uparrow$.
2. Again, $\vdash U_t \simeq U_t$ is an instance of Axiom 2a. By Rule 5 and Axiom 5c, we get $\vdash \neg U_t \simeq U_t$, so $\vdash U_t \simeq \neg U_t$ by Axiom 2b and Rule 5 again.
3. Take $\vdash (\neg \alpha_t \overset{*}{\vee} 1^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$, which is an instance of Axiom 7a. Then, $\vdash \neg(\neg \alpha_t \overset{*}{\vee} 1^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$ by Rule 5 and Axiom 5a. Finally, by Axiom 7e and Rule 1, we obtain $\vdash (\alpha_t \overset{*}{\wedge} 0^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$.
4. Immediate by Axioms 7b, 5b and 7e and Rules 5 and 1.
5. Immediate by Axioms 7c, 5c and 7e and Rules 5 and 1.
6. Immediate by Axioms 7d, 5c and 7e, Proposition 9(2) and Rules 5 and 1.
7. Consider $\vdash \overset{*}{\forall} x_a \alpha_t \simeq \neg \overset{*}{\exists} x_a \neg \alpha_t$ (Axiom 9b). Then, by Axiom 9a and Rule 1, it follows that $\vdash \overset{*}{\forall} x_a \alpha_t \simeq \neg(\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}} \neg \alpha_t \overset{*}{\vee} \ldots \overset{*}{\vee} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}} \neg \alpha_t)$, and hence $\vdash \overset{*}{\forall} x_a \alpha_t \simeq (\mathsf{S}^{x_a}_{(y_1)^{\mathsf{N}}} \alpha_t \overset{*}{\wedge} \ldots \overset{*}{\wedge} \mathsf{S}^{x_a}_{(y_q)^{\mathsf{N}}} \alpha_t)$ by Axiom 7e and Rule 1.

■

PROPOSITION 10. (Rule of Conjunction)
If $\vdash \alpha_t$ and $\vdash \beta_t$, then $\vdash \alpha_t \overset{*}{\wedge} \beta_t$.

PROOF. Suppose that $\vdash \alpha_t$ and $\vdash \beta_t$. Then, we obtain $\vdash \alpha_t \simeq 1^{\mathsf{N}}$ and $\vdash \beta_t \simeq 1^{\mathsf{N}}$ by Rule 6. Take $\vdash (\alpha_t \overset{*}{\wedge} \beta_t) \simeq (\alpha_t \overset{*}{\wedge} \beta_t)$, which is an instance of Axiom 2a. By Rule 1, we get $\vdash (\alpha_t \overset{*}{\wedge} \beta_t) \simeq (1^{\mathsf{N}} \overset{*}{\wedge} 1^{\mathsf{N}})$. Since $\vdash (1^{\mathsf{N}} \overset{*}{\wedge} 1^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$ (Proposition 9(4)), it follows that $\vdash (\alpha_t \overset{*}{\wedge} \beta_t) \simeq 1^{\mathsf{N}}$ again by Rule 1. Thus, $\vdash \alpha_t \overset{*}{\wedge} \beta_t$ by Rule 7. ■

PROPOSITION 11. (Rule of Universal Instantiation) *If* $\vdash \overset{*}{\forall} x_a \beta_t$*, then* $\vdash \gamma_t$*, where* γ_t *is the result of replacing all free occurrences of* x_a *in* β_t *by some formula* α_a *such that* $\vdash \alpha_a \downarrow$*, provided that the occurrence of* x_a *in* β_t *is not in a meaningful part of* β_t *beginning with the symbols* λy_c *where* $y_c \in FreeVar(\alpha_a)$*.*

PROOF. Suppose that x_a, β_t, α_a and γ_t are so related and $\vdash \overset{*}{\forall} x_a \beta_t$. Then, $\vdash \overset{*}{\forall} x_a \beta_t \simeq 1^{\mathsf{N}}$ by Rule 7 and hence $\vdash \lambda y_a 1^{\mathsf{N}} \equiv \lambda x_a \beta_t$ by applying Rule 2 to $\vdash \overset{*}{\forall} x_a \beta_t \simeq 1^{\mathsf{N}}$ and Axiom 9c. Since $\vdash \lambda y_a 1^{\mathsf{N}} \downarrow$ (Axiom 8b), we get $\vdash \lambda y_a 1^{\mathsf{N}} \simeq \lambda x_a \beta_t$ by Rules 2 and 5 and Axiom 4a. Take $\vdash (\lambda x_a \beta_t)\alpha_a \simeq (\lambda x_a \beta_t)\alpha_a$, which is an instance of Axiom 2a (where $\vdash \alpha_a \downarrow$). It follows that $\vdash (\lambda x_a \beta_t)\alpha_a \simeq (\lambda y_a 1^{\mathsf{N}})\alpha_a$ by Rule 1. Thus, $\vdash ((\lambda y_a 1^{\mathsf{N}})\alpha_a) \simeq 1^{\mathsf{N}}$ by applying Rule 2 to an instance of Axiom 1d and hence $\vdash (\lambda x_a \beta_t)\alpha_a \simeq 1^{\mathsf{N}}$ by Rule 1. Then, $\vdash (\lambda x_a \beta_t)\alpha_a$ by Rule 7. Because $\vdash (\lambda x_a \beta_t)\alpha_a \simeq \gamma_t$ (which results from applying Rule 2 to an instance of Axiom 1d), we can conclude $\vdash \gamma_t$ by Rule 5. ∎

PROPOSITION 12. (*Rule of Substitution for Free Variables*) *If* β_t*,* x_a*,* α_a *and* γ_t *are related as in the hypothesis of Rule 11, and if* $\vdash \beta_t$ *and* $\vdash \alpha_a \downarrow$*, then* $\vdash \gamma_t$*.*

PROOF. Immediate by Rules 3 and 11. ∎

PROPOSITION 13. (*Rule of Propositional Cases*) *Let* $\alpha_t \in \mathsf{ME}_t^K$ *and* $x_t \in \mathsf{VAR}_t$*. If* α'_t *and* α''_t *are obtained by replacing all free occurrences of* x_t *in* α_t *by* 1^{N} *and* 0^{N} *respectively, and if* $\vdash \alpha'_t$ *and* $\vdash \alpha''_t$*, then also* $\vdash \alpha_t$*, provided that* $\vdash \alpha_t \downarrow$*.*

PROOF. Suppose that α_t, x_t, α'_t and α''_t are so related, and $\vdash \alpha'_t$ and $\vdash \alpha''_t$. By Rule 5, $\vdash \alpha'_t \downarrow$ and $\vdash \alpha''_t \downarrow$, so by applying Rule 2 to instances of Axiom 1d, we also get $\vdash (\lambda x_t \alpha_t)1^{\mathsf{N}} \simeq \alpha'_t$ and $\vdash (\lambda x_t \alpha_t)0^{\mathsf{N}} \simeq \alpha''_t$. Hence, $\vdash (\lambda x_t \alpha_t)1^{\mathsf{N}}$ and $\vdash (\lambda x_t \alpha_t)0^{\mathsf{N}}$ by Rule 5 and $\vdash (\lambda x_t \alpha_t)1^{\mathsf{N}} \overset{*}{\wedge} (\lambda x_t \alpha_t)0^{\mathsf{N}}$ by Rule 10. Take

$$\vdash ((\lambda x_a \alpha_t)1^{\mathsf{N}} \overset{*}{\wedge} (\lambda x_t \alpha_t)0^{\mathsf{N}}) \simeq \overset{*}{\forall} x_t((\lambda x_a \alpha_t)x_t),$$

which is the result of applying Rule 12 to Axiom 1b. Thus, $\vdash \overset{*}{\forall} x_t((\lambda x_t \alpha_t)x_t)$ by Rule 5. Because $\vdash x_t \downarrow$ (Axiom 8c), Rule 2 applied to $\vdash x_t \downarrow$ and to an instance of Axiom 1d gives us $\vdash (\lambda x_t \alpha_t)x_t \simeq \alpha_t$. It follows that $\vdash \forall x_t \alpha_t$ by Rule 1 and hence $\vdash \alpha_t$ by Rule 11, as $\vdash \alpha_t \downarrow$ by hypothesis. ∎

5.2. Soundness and Useful Metatheorems

THEOREM 14. (Soundness) *For every* $\alpha_t \in \mathsf{ME}_t^K$*, if* $\vdash \alpha_t$*, then* $\models \alpha_t$*.*

Proof. A straightforward verification shows that (1) every axiom and axiom schema of $\mathbf{PT}_K$ is valid and (2) the rules of inference 1, 2, 3, 4 and 5 preserve validity.

We now prove the soundness of Axiom 1b (a way of expressing that $\mathcal{D}_t^K$ contains the elements 0 and 1, and no others) in order to display how partial propositional types are interpreted semantically.

- $\vdash (g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}}) \simeq \overset{*}{\forall} x_t(g_{tt}x_t)$.

Suppose firstly that $[[(g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}})]]^{\mathcal{M},g} = 1$ for an arbitrary assignment g. It follows that $g(g_{tt})$ must be 11 and consequently $[[\lambda x_a 1^{\mathsf{N}} \equiv \lambda x_t(g_{tt}x_t)]]^{\mathcal{M},g} = 1$, so $[[\overset{*}{\forall} x_t(g_{tt}x_t)]]^{\mathcal{M},g} = 1$. Therefore, $[[(g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}}) \simeq \overset{*}{\forall} x_t(g_{tt}x_t)]]^{\mathcal{M},g} = 1$.

Suppose now that $[[(g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}})]]^{\mathcal{M},g} = 0$ for an arbitrary assignment g. The candidates for $g(g_{tt})$ are 01, 10, 00, $0*$ and $*0$. Thus, we see that there exists some $x \in \mathcal{D}_t^K$ such that $[[\lambda x_t(g_{tt}x_t)]]^{\mathcal{M},g}(x) = 0$, so $[[\overset{*}{\forall} x_t(g_{tt}x_t)]]^{\mathcal{M},g} = 0$. Therefore, $[[(g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}}) \simeq \overset{*}{\forall} x_t(g_{tt}x_t)]]^{\mathcal{M},g} = 1$.

Finally, suppose that $[[(g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}})]]^{\mathcal{M},g} = *$ for an arbitrary assignment g. The candidates for $g(g_{tt})$ are $1*$, $*1$ and $**$. Hence, $[[\lambda x_t(g_{tt}x_t)]]^{\mathcal{M},g}$ is either the empty function of type tt or a partial function f from $\mathcal{D}_t^K$ to $\mathcal{D}_t^K$ such that, for every $x \in \mathcal{D}_t^K$, if $x \in Def(f)$, then $f(x) = 1$. Therefore, $[[\overset{*}{\forall} x_t(g_{tt}x_t)]]^{\mathcal{M},g} = *$, so $[[(g_{tt}1^{\mathsf{N}} \overset{*}{\wedge} g_{tt}0^{\mathsf{N}}) \simeq \overset{*}{\forall} x_t(g_{tt}x_t)]]^{\mathcal{M},g} = 1$. ∎

We are ready to prove some results that are needed to prove completeness constructively. Firstly, we show that every tautology[4] is a formal theorem and also that every closed expression which is a name is defined:

Theorem 15. (P-Completeness) *Every tautology α_t is a formal theorem.*

Proof. The proof is by induction on the number of free variables in α_t.

1. Let α_t be a tautology containing no free variables:
   (a) Let α_t be 1^{N}. Then, $\vdash \lambda x_t x_t \simeq \lambda x_t x_t$ is an instance of Axiom 2a. Since $\vdash \lambda x_t x_t \downarrow$ (Axiom 8b), $\vdash \lambda x_t x_t \equiv \lambda x_t x_t$ by Rules 2 and 5 and Axiom 4a. By the definition of 1^{N}, we get $\vdash 1^{\mathsf{N}}$.
   (b) Let α_t be $\neg 0^{\mathsf{N}}$. Since $\vdash 0^{\mathsf{N}} \downarrow$ (Axiom 8c), we can apply Rule 2 to an instance of Axiom 1d obtaining $\vdash \lambda x_t(0^{\mathsf{N}} \equiv x_t)0^{\mathsf{N}} \simeq 1^{\mathsf{N}}$. Thus, we get $\vdash \lambda x_t(0^{\mathsf{N}} \equiv x_t)0^{\mathsf{N}}$ by Rule 7, so, by the definition of $\neg$, $\vdash \neg 0^{\mathsf{N}}$.

[4]See Definition 11.

(c) Let α_t be $\alpha_t \overset{*}{\vee} 1^{\mathsf{N}}$. Take $\vdash (\alpha_t \overset{*}{\vee} 1^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$ (Axiom 7a). Then, $\vdash \alpha_t \overset{*}{\vee} 1^{\mathsf{N}}$ by Rule 7.

(d) Let α_t be $1^{\mathsf{N}} \equiv 1^{\mathsf{N}}$. Immediate by applying Rule 6 to (1a).

2. Let α_t be a tautology containing some free variable, say x_t. Let α'_t be the result of replacing each occurrence of x_a in α_t by 1^{N} and α''_t by 0^{N}. Both α'_t and α''_t are tautologies, because $[[\alpha_t]]^{\mathcal{M},g} = 1$ for all assignments g, including those, g' and g'', where $g'(x_t) = 1$ and $g''(x_t) = 0$. It follows, by induction hypothesis, that $\vdash \alpha'_t$ and $\vdash \alpha''_t$. Since $\vdash \alpha_t \downarrow$ (Axiom 8c), we obtain $\vdash \alpha_t$ by Rule 13.

■

THEOREM 16. (Names are defined) *For any $a \in \mathsf{TYPES}$ and each $x \in \mathcal{D}_a^K$, $\vdash x^{\mathsf{N}} \downarrow$.*

PROOF. The proof is by induction.

1. *Base case*: $\mathcal{D}_t^K$. $\vdash 1^{\mathsf{N}} \downarrow$ and $\vdash 0^{\mathsf{N}} \downarrow$ (instances of Axiom 8c).
2. *Inductive step*: $\mathcal{D}_{ab}^K$. For any $f \in \mathcal{D}_{ab}^K$,

$$f^{\mathsf{N}} = \lambda x_a(\imath z_b((x_a \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_1))^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((x_a \equiv y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f(y_q))^{\mathsf{N}})))$$

by Theorem 2. Thus, $\vdash f^{\mathsf{N}} \downarrow$ by Axiom 8b, as desired.

■

Secondly, we prove Theorem 17, which is essential for the whole strategy of Lemma 20:

THEOREM 17. *For any $c \in \mathsf{TYPES}$ and $z_1, \ldots, z_q$ all the elements of $\mathcal{D}_c^K$, it holds that:*

1. $\vdash \neg(z_i^{\mathsf{N}} \equiv z_j^{\mathsf{N}})$ *if $i \neq j$.*
2. *If $c = ab$, then for any $y \in \mathcal{D}_a^K$, we have $\vdash (z_i^{\mathsf{N}} y^{\mathsf{N}}) \simeq (z_i y)^{\mathsf{N}}$.*

PROOF. The proof is by induction.

1. *Base case*: $\mathcal{D}_t^K$. $\neg(1^{\mathsf{N}} \equiv 0^{\mathsf{N}})$ is a tautology, so $\vdash \neg(1^{\mathsf{N}} \equiv 0^{\mathsf{N}})$ by Theorem 15.
2. *Inductive step*: $\mathcal{D}_{ab}^K$. Let $f_1, \ldots, f_p$ be a list of the distinct elements of $\mathcal{D}_{ab}^K$ and $y_1, \ldots, y_q$ a list of the distinct elements of $\mathcal{D}_a^K$.

- We start by proving (2).

Since $\vdash y_j^{\mathsf{N}}\downarrow$ (Theorem 16), from Axiom 1d, Rule 2 and Theorem 2 we have:

$$\vdash (f_i^{\mathsf{N}} y_j^{\mathsf{N}}) \simeq \imath z_b((y_j^{\mathsf{N}} \equiv y_1^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f_i y_1)^{\mathsf{N}}) \overset{*}{\vee} \ldots \overset{*}{\vee} ((y_j^{\mathsf{N}} \equiv y_q^{\mathsf{N}}) \overset{*}{\wedge} (z_b \equiv (f_i y_q)^{\mathsf{N}}))).$$

Because $\vdash \neg(y_j^{\mathsf{N}} \equiv y_k^{\mathsf{N}})$ for $j \neq k$ by induction hypothesis concerning $\mathcal{D}_a^K$, we get

$$\vdash (f_i^{\mathsf{N}} y_j^{\mathsf{N}}) \simeq \imath z_b(z_b \equiv (f_i y_j)^{\mathsf{N}}) \text{ (I.H)}$$

Now, there are two possibilities:

(a) Firstly, suppose that $[[f_i^{\mathsf{N}}]]^{\mathcal{M},g}$ is the partial function f_i in $\mathcal{D}_{ab}^K$ such that $f_i([[y_j^{\mathsf{N}}]]^{\mathcal{M},g}) = *$. In this case, $(f_i y_j)^{\mathsf{N}} = U_b$ by Definition 13. Since $\vdash U_b \uparrow$ by Proposition 9(1), $\vdash \imath z_b(z_b \equiv U_b)\uparrow$ by Rule 2 and Axiom 10c. Now, by the definition of $\uparrow$, we get $\vdash \imath z_b(z_b \equiv (f_i y_j)^{\mathsf{N}}) \simeq U_b$ and hence $\vdash (f_i^{\mathsf{N}} y_j^{\mathsf{N}}) \simeq U_b$ by applying Rule 1 to (I.H). Therefore, $\vdash (f_i^{\mathsf{N}} y_j^{\mathsf{N}}) \simeq (f_i y_j)^{\mathsf{N}}$, as $(f_i y_j)^{\mathsf{N}} = U_b$.

(b) Suppose that $[[f_i^{\mathsf{N}}]]^{\mathcal{M},g}$ is the function f_i in $\mathcal{D}_{ab}^K$ such that $f_i([[y_j^{\mathsf{N}}]]^{\mathcal{M},g}) = \theta$, where $\theta \in \mathcal{D}_b^K$. In this case, $(f_i y_j)^{\mathsf{N}}$ is the name of θ, so $\vdash (f_i y_j)^{\mathsf{N}}\downarrow$ by Theorem 16. Then, $\vdash \imath z_b(z_b \equiv (f_i y_j)^{\mathsf{N}})\downarrow$ by Rule 2 and Axiom 10a, so $\vdash \lambda z_b(z_b \equiv (f_i y_j)^{\mathsf{N}}))(\imath z_b(z_b \equiv (f_i y_j)^{\mathsf{N}})) \equiv 1^{\mathsf{N}}$ by Rule 2 again and Axiom 10b. We obtain $\vdash \lambda z_b(z_b \equiv (f_i y_j)^{\mathsf{N}}))(\imath z_b(z_b \equiv (f_i y_j)^{\mathsf{N}}))$ by Rule 6. Since $\vdash \imath z_b(z_b \equiv (f_i y_j)^{\mathsf{N}})\downarrow$, from this it follows that $\vdash \imath z_b(z_b \equiv (f_i y_j)^{\mathsf{N}}) \simeq (f_i y_j)^{\mathsf{N}}$ by Axiom 1d and Rule 2. Therefore, $\vdash (f_i^{\mathsf{N}} y_j^{\mathsf{N}}) \simeq (f_i y_j)^{\mathsf{N}}$ by applying Rule 1 to (I.H), as desired.

In consequence, $\vdash (f_i^{\mathsf{N}} y_j^{\mathsf{N}}) \simeq (f_i y_j)^{\mathsf{N}}$.

- We turn next to the proof of (1).

Notice that, if $i \neq j$, then for some $k \in \{1, ..., p\}$ it holds that $(f_i y_k) \neq (f_j y_k)$. Then, by induction hypothesis concerning $\mathcal{D}_b^K$, we know that $\vdash \neg((f_i y_k)^{\mathsf{N}} \equiv (f_j y_k)^{\mathsf{N}})$. Since $\vdash (f_i y_k)^{\mathsf{N}}\downarrow$ and $\vdash (f_j y_k)^{\mathsf{N}}\downarrow$ by Theorem 16, $\vdash ((f_i y_k)^{\mathsf{N}} \simeq (f_j y_k)^{\mathsf{N}}) \simeq ((f_i y_k)^{\mathsf{N}} \equiv (f_j y_k)^{\mathsf{N}})$ by Rule 2 and Axiom 4a. In consequence, $\vdash \neg((f_i y_k)^{\mathsf{N}} \simeq (f_j y_k)^{\mathsf{N}})$ by Rule 8. Because we proved already that $\vdash (f_i^{\mathsf{N}} y_k^{\mathsf{N}}) \simeq (f_i y_k)^{\mathsf{N}}$ and $\vdash (f_j^{\mathsf{N}} y_k^{\mathsf{N}}) \simeq (f_j y_k)^{\mathsf{N}}$, we get $\vdash \neg((f_i^{\mathsf{N}} y_k^{\mathsf{N}}) \simeq (f_j^{\mathsf{N}} y_k^{\mathsf{N}}))$ by Rule 1. Therefore, $\vdash \overset{*}{\exists} y_a(\neg((f_i^{\mathsf{N}} y_a) \simeq (f_j^{\mathsf{N}} y_a)))$ by Rule 4, as $\vdash y_k^{\mathsf{N}}\downarrow$ (by Theorem 16) and $\vdash \neg\overset{*}{\forall} y_a((f_i^{\mathsf{N}} y_a) \simeq (f_j^{\mathsf{N}} y_a))$ by Rules 7 and 1 and Axioms 5a and 9b. Thus, we obtain the desired $\vdash \neg(f_i \equiv f_j)$ by applying Rule 8 to Axiom 1c.

■

Finally, we also prove Propositions 18 and 19, which are needed to establish cases (6) and (7) of Lemma 20, respectively.

PROPOSITION 18. $\vdash (([[\alpha_a]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_a]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq ([[\alpha_a \simeq \beta_a]]^{\mathcal{M},g})^{\mathsf{N}}$.

PROOF. The proof is by induction.

1. *Base case*: $\mathcal{D}_t^K$.

(a) If $([[\alpha_t \simeq \beta_t]]^{\mathcal{M},g})^{\mathsf{N}} = 1^{\mathsf{N}}$, then $[[\alpha_t]]^{\mathcal{M},g} = [[\beta_t]]^{\mathcal{M},g}$ and therefore $\vdash ([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$ is an instance of Axiom 2a. Consequently, we get $\vdash (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq \mathsf{T}^{\mathsf{N}}$ by Rule 7.

(b) If $([[\alpha_t \simeq \beta_t]]^{\mathcal{M},g})^{\mathsf{N}} = 0^{\mathsf{N}}$, then $[[\alpha_t]]^{\mathcal{M},g} \neq [[\beta_t]]^{\mathcal{M},g}$. Hence, it follows that $\vdash (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$ is either an instance of one of the axioms of group 3 or follows from such an instance and Axiom 2b by applying Rule 1.

2. *Inductive step*: $\mathcal{D}_{ab}^K$.

(a) If $([[\alpha_{ab} \simeq \beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} = 1^{\mathsf{N}}$, then $[[\alpha_{ab}]]^{\mathcal{M},g} = [[\beta_{ab}]]^{\mathcal{M},g}$, so the argument for (1a) works here.

(b) If $([[\alpha_{ab} \simeq \beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} = 0^{\mathsf{N}}$, then $[[\alpha_{ab}]]^{\mathcal{M},g} \neq [[\beta_{ab}]]^{\mathcal{M},g}$. This means that there exists some $x \in \mathcal{D}_a^K$ such that $[[\alpha_{ab}]]^{\mathcal{M},g}(x) = y$ and $[[\beta_{ab}]]^{\mathcal{M},g}(x) = z$, with $y, z \in \mathcal{D}_b^K$ and $y \neq z$. Now, by induction hypothesis, we have

$$\vdash (([[\alpha_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}}) \simeq ([[\alpha_{ab}(x)^{\mathsf{N}} \simeq \beta_{ab}(x)^{\mathsf{N}}]]^{\mathcal{M},g})^{\mathsf{N}}$$

and hence $\vdash (([[\alpha_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$. Therefore, $\vdash \neg(([[\alpha_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$ by Rule 2 and Axiom 5b, so $\vdash \neg(([[\alpha_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g}(x))^{\mathsf{N}})$ by Rule 7. By Theorem 17(2), $\vdash \neg(([[\alpha_{ab}]]^{\mathcal{M},g})^{\mathsf{N}}(x)^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}}(x)^{\mathsf{N}})$ and consequently

$$\vdash \neg\forall x_a(([[\alpha_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} x_a \simeq ([[\beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} x_a)$$

by Rules 4 (for $\vdash (x)^{\mathsf{N}} \downarrow$ by Theorem 16), 7 and 1 and Axioms 5a and 9b. Therefore, $\vdash \neg(([[\alpha_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} \equiv ([[\beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}})$ by applying Rule 8 to Axiom 1c, so we obtain $\vdash \neg(([[\alpha_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}})$ by Axiom 4a, Theorem 16 again and Rules 2 and 5. Because $\vdash \neg(([[\alpha_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq \mathsf{T}^{\mathsf{N}}$ by Rule 7, it follows by Axioms 5a and 5d and Rules 5 and 1 that

$$\vdash (([[\alpha_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_{ab}]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq \mathsf{F}^{\mathsf{N}},$$

as desired.

■

PROPOSITION 19. $\vdash (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \overset{*}{\vee} ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq ([[\alpha_t \overset{*}{\vee} \beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$.

PROOF.

1. If $([[\alpha_t \overset{*}{\vee} \beta_t]]^{\mathcal{M},g})^{\mathsf{N}} = 1^{\mathsf{N}}$, then either $[[\alpha_t]]^{\mathcal{M},g}$ or $[[\beta_t]]^{\mathcal{M},g}$ is 1. In consequence, $\vdash (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \overset{*}{\vee} ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$ is an instance of Axiom 7a or follows from such an instance and Axiom 2b by applying Rule 1.
2. If $([[\alpha_t \overset{*}{\vee} \beta_t]]^{\mathcal{M},g})^{\mathsf{N}} = 0^{\mathsf{N}}$, then $[[\alpha_t]]^{\mathcal{M},g} = [[\beta_t]]^{\mathcal{M},g} = 0$. Clearly, we see that $\vdash (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \overset{*}{\vee} ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$ is Axiom 7b.
3. If $([[\alpha_t \overset{*}{\vee} \beta_t]]^{\mathcal{M},g})^{\mathsf{N}} = U_t$ (see Definition 14), then $[[\alpha_t]]^{\mathcal{M},g} = [[\beta_t]]^{\mathcal{M},g} = *$ or one is $*$ and the other 0. In the first case, $\vdash (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \overset{*}{\vee} ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq U_t$ is Axiom 7d. In the second case, $\vdash (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \overset{*}{\vee} ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}) \simeq U_t$ is Axiom 7c or follows from it, Axiom 2b and Rule 1.

■

6. Completeness

The method of proof for the completeness of **PT** is rather different from the one Henkin used to prove it for first-order logic and Church's Type Theory. In this case, the proof is constructive, as it is based on the Nameability theorem [12, 341-43]. To prove completeness for $\mathbf{PT}_K$, we will follow Henkin's strategy in [12], so first we have to give a proof of the following Lemma (and completeness easily follows):

LEMMA 20. *Let* $\alpha_c \in \mathsf{ME}_c^K$. *Then,* $\vdash \alpha_c^{(g)} \simeq ([[\alpha_c]]^{\mathcal{M},g})^{\mathsf{N}}$.

PROOF. The proof is by induction on the length of α_c.

1. Let α_c be a variable x_a. In this case, $[[x_a]]^{\mathcal{M},g} \in \mathcal{D}_a^K$ for every assignment g and $x_a^{(g)} = ([[x_a]]^{\mathcal{M},g})^{\mathsf{N}}$ by Definition 15. Therefore, $\vdash x_a^{(g)} \simeq ([[x_a]]^{\mathcal{M},g})^{\mathsf{N}}$ is an instance of Axiom 2a.
2. Let α_c be a primitive constant $\mathsf{Q}_{a\langle at\rangle}$. We have to show that $\vdash \mathsf{Q}_{a\langle at\rangle} \simeq q^{\mathsf{N}}$, where $q \in \mathcal{D}_{a\langle at\rangle}$, because $[[\mathsf{Q}_{a\langle at\rangle}]]^{\mathcal{M},g} = q$ and $\mathsf{Q}_{a\langle at\rangle}^{(g)} = \mathsf{Q}_{a\langle at\rangle}$. Suppose that $y_1, \ldots, y_m$ are distinct and are all the elements of $\mathcal{D}_a^K$. By Axiom 2a, $\vdash x_a \simeq x_a$. Since $\vdash x_a \downarrow$ (Axiom 8c), it holds that $\vdash x_a \equiv x_a$ by Rules 2, 5 and Axiom 4a. By induction hypothesis, $\vdash x_a^{(g)} \simeq ([[x_a]]^{\mathcal{M},g})^{\mathsf{N}}$, so, assuming that $([[x_a]]^{\mathcal{M},g})^{\mathsf{N}} = y_i^{\mathsf{N}}$, we get $\vdash y_i^{\mathsf{N}} \equiv y_i^{\mathsf{N}}$ by Rule 1. Therefore, $\vdash (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}}) y_i^{\mathsf{N}}$ by the definition of

$\equiv$, so $\vdash (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}}) y_i^{\mathsf{N}} \simeq 1^{\mathsf{N}}$ by Rule 7. We also know that $\vdash \neg(y_i^{\mathsf{N}} \equiv y_j^{\mathsf{N}})$ by Theorem 17(1) and hence $\vdash \neg((\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}}) y_j^{\mathsf{N}})$ by the definition of $\equiv$, so $\vdash (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}}) y_j^{\mathsf{N}} \simeq 0^{\mathsf{N}}$ by Rules 7, 5 and 1 and Axioms 5a and 5d. In other words, $\vdash (q^{\mathsf{N}} y_i^{\mathsf{N}}) y_i^{\mathsf{N}} \simeq 1^{\mathsf{N}}$ and $\vdash (q^{\mathsf{N}} y_i^{\mathsf{N}}) y_j^{\mathsf{N}} \simeq 0^{\mathsf{N}}$, so $\vdash (q^{\mathsf{N}} y_i^{\mathsf{N}} y_i^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$ and $\vdash (q^{\mathsf{N}} y_i^{\mathsf{N}} y_j^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$ by Theorem 17(2).

In particular, $\vdash (\mathsf{Q}_{a\langle at\rangle} y_1^{\mathsf{N}}) y_1^{\mathsf{N}} \simeq 1^{\mathsf{N}}$ and $\vdash (q^{\mathsf{N}} y_1^{\mathsf{N}} y_1^{\mathsf{N}}) \simeq 1^{\mathsf{N}}$. By Axiom 2b and Rules 7 and 5, we get $\vdash (\mathsf{Q}_{a\langle at\rangle} y_1^{\mathsf{N}} y_1^{\mathsf{N}}) \simeq (q^{\mathsf{N}} y_1^{\mathsf{N}} y_1^{\mathsf{N}})$ by Rule 1. Analogously, from $\vdash (\mathsf{Q}_{a\langle at\rangle} y_2^{\mathsf{N}} y_1^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$ and $\vdash (q^{\mathsf{N}} y_2^{\mathsf{N}} y_1^{\mathsf{N}}) \simeq 0^{\mathsf{N}}$, it follows that $\vdash (\mathsf{Q}_{a\langle at\rangle} y_2^{\mathsf{N}} y_1^{\mathsf{N}}) \simeq (q^{\mathsf{N}} y_2^{\mathsf{N}} y_1^{\mathsf{N}})$ again by Rule 1. In consequence we get, for each $i \in \{1, ..., m\}$,

$$\vdash (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}} y_1^{\mathsf{N}}) \simeq (q^{\mathsf{N}} y_i^{\mathsf{N}} y_1^{\mathsf{N}}) \overset{*}{\wedge} \ldots \overset{*}{\wedge} (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}} y_m^{\mathsf{N}}) \simeq (q^{\mathsf{N}} y_i^{\mathsf{N}} y_m^{\mathsf{N}})$$

by Rule 10. Thus, $\vdash \overset{*}{\forall} x_a((\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}} x_a) \simeq (q^{\mathsf{N}} y_i^{\mathsf{N}} x_a))$ by Proposition 9(7) and Rule 5. Then, we get $\vdash (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}}) \equiv (q^{\mathsf{N}} y_i^{\mathsf{N}})$ by applying Rule 5 to an instance of Axiom 1c. By Theorem 17(2) and Rule 1, $\vdash (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}}) \equiv (q y_i)^{\mathsf{N}}$ and it holds that $\vdash (q y_i)^{\mathsf{N}} \downarrow$ (Theorem 16). In consequence, we can apply Rules 2 and 5 to an instance of Axiom 4a obtaining $\vdash (\mathsf{Q}_{a\langle at\rangle} y_i^{\mathsf{N}}) \simeq (q^{\mathsf{N}} y_i^{\mathsf{N}})$. Because this holds for each $i \in \{1, ..., m\}$, we can conclude the desired $\vdash \mathsf{Q}_{a\langle at\rangle} \simeq q^{\mathsf{N}}$.

3. Let α_c be a primitive constant U_a. In this case, $U_a^{(g)} = U_a$ by Definition 15 and $([[U_a]]^{\mathcal{M},g})^{\mathsf{N}} = U_a$ by Definition 14. Thus, $\vdash U_a^{(g)} \simeq ([[U_a]]^{\mathcal{M},g})^{\mathsf{N}}$ is an instance of Axiom 2a.

4. Let α_c be of the form $\gamma_{ab}\beta_a$. Firstly, we make the induction hypothesis that $\vdash \gamma_{ab}^{(g)} \simeq ([[\gamma_{ab}]]^{\mathcal{M},g})^{\mathsf{N}}$ and $\vdash \beta_a^{(g)} \simeq ([[\beta_a]]^{\mathcal{M},g})^{\mathsf{N}}$. Therefore, $\vdash \gamma_{ab}^{(g)} \beta_a^{(g)} \simeq ([[\gamma_{ab}]]^{\mathcal{M},g})^{\mathsf{N}} ([[\beta_a]]^{\mathcal{M},g})^{\mathsf{N}}$ and consequently

$$\vdash \gamma_{ab}^{(g)} \beta_a^{(g)} \simeq ([[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}))^{\mathsf{N}}$$

by Theorem 17(2). Now, we must distinguish four possibilities:

(a) If $[[\gamma_{ab}^{(g)}]]^{\mathcal{M},g} = *$, then $([[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}))^{\mathsf{N}} = U_b$ by Definition 13, so $\vdash \gamma_{ab}^{(g)} \beta_a^{(g)} \simeq U_b$. Because $\gamma_{ab}^{(g)} \beta_a^{(g)} = (\gamma_{ab}\beta_a)^{(g)}$ by Definition 15, we obtain $\vdash (\gamma_{ab}\beta_a)^{(g)} \simeq U_b$ and hence $\vdash (\gamma_{ab}\beta_a)^{(g)} \simeq ([[\gamma_{ab}\beta_a]]^{\mathcal{M},g})^{\mathsf{N}}$.

(b) If $[[\beta_a^{(g)}]]^{\mathcal{M},g} = *$, then $([[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}))^{\mathsf{N}} = U_b$ by Definition 13, so the argument in (4a) works here.

(c) If $[[\gamma_{ab}^{(g)}]]^{\mathcal{M},g} \in \mathcal{D}_{ab}^K$, $[[\beta_a^{(g)}]]^{\mathcal{M},g} \in \mathcal{D}_a^K$, but $[[\gamma_{ab}]]^{\mathcal{M},g}$ is not defined at $[[\beta_a]]^{\mathcal{M},g}$, then $[[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}) = *$, so $([[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}))^{\mathsf{N}} = U_b$ by Definition 13. Thus, the argument in (4a) also works here.

(d) If $[[\gamma^{(g)}_{ab}]]^{\mathcal{M},g} \in \mathcal{D}^K_{ab}$, $[[\beta^{(g)}_a]]^{\mathcal{M},g} \in \mathcal{D}^K_a$ and $[[\beta_a]]^{\mathcal{M},g} \in Def([[\gamma_{ab}]]^{\mathcal{M},g})$, then $[[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}) \in \mathcal{D}^K_b$. Clearly,

$$[[\gamma_{ab}]]^{\mathcal{M},g}([[\beta_a]]^{\mathcal{M},g}) = [[\gamma_{ab}\beta_a]]^{\mathcal{M},g}.$$

Since $\gamma^{(g)}_{ab}\beta^{(g)}_a = (\gamma_{ab}\beta_a)^{(g)}$ by Definition 15, we conclude $\vdash (\gamma_{ab}\beta_a)^{(g)} \simeq ([[\gamma_{ab}\beta_a]]^{\mathcal{M},g})^{\mathsf{N}}$, as desired.

5. Let α_c be $\lambda x_a\alpha_b$. We have to show that $\vdash (\lambda x_a\alpha_b)^{(g)} \simeq ([[\lambda x_a\alpha_b]]^{\mathcal{M},g})^{\mathsf{N}}$, because $[[\lambda x_a\alpha_b]]^{\mathcal{M},g} \in \mathcal{D}^K_{ab}$ for every assignment g.
Suppose that $y_1, \ldots, y_q$ are all the distinct elements of $\mathcal{D}^K_a$. By induction hypothesis, we have $\vdash \alpha^{(g)}_b \simeq ([[\alpha_b]]^{\mathcal{M},g})^{\mathsf{N}}$ for every assignment g.
Now, since $\vdash (y_i)^{\mathsf{N}}\downarrow$ (by Theorem 16), it follows that $\vdash (\lambda x_a(\alpha_b)^{(g_{\{x_a\}})})$ $y^{\mathsf{N}}_i \simeq (\mathsf{S}^{x_a}_{(y_i)^{\mathsf{N}}}\alpha^{(g_{\{x_a\}})}_b)$ by applying Rule 2 to Axiom 1d. Because $[[\mathsf{S}^{x_a}_{(y_i)^{\mathsf{N}}}\alpha_b]]^{\mathcal{M},g} = [[(\lambda x_a(\alpha_b)]]^{\mathcal{M},g}(y_i)$ (Definition 10) and $\lambda x_a(\alpha_b)^{(g_{\{x_a\}})}$ $= (\lambda x_a\alpha_b)^{(g)}$ (Definition 15), our induction hypothesis yields:

$$\vdash (\lambda x_a\alpha_b)^{(g)}y^{\mathsf{N}}_i \simeq ([[\lambda x_a\alpha_b]]^{\mathcal{M},g}(y_i))^{\mathsf{N}}$$

and hence

$$\vdash (\lambda x_a\alpha_b)^{(g)}y^{\mathsf{N}}_i \simeq ([[\lambda x_a\alpha_b]]^{\mathcal{M},g})^{\mathsf{N}}(y_i)^{\mathsf{N}}$$

by Theorem 17(2). Thus, $\vdash \overset{*}{\forall} y_a((\lambda x_a\alpha_b)^{(g)}y_a \simeq ([[\lambda x_a\alpha_b]]^{\mathcal{M},g})^{\mathsf{N}}y_a)$ by Rule 3 and hence $\vdash (\lambda x_a\alpha_b)^{(g)} \equiv ([[\lambda x_a\alpha_b]]^{\mathcal{M},g})^{\mathsf{N}}$ by Rule 5 and Axiom 1c. Since $\vdash (\lambda x_a\alpha_b)^{(g)}\downarrow$ (Axiom 8a) and $\vdash ([[\lambda x_a\alpha_b]]^{\mathcal{M},g})^{\mathsf{N}}\downarrow$ (Theorem 16), by Rules 2 and 5 and Axiom 4a, as desired.

6. Let α_c be $\alpha_t \simeq \beta_t$. By induction hypothesis, we have $\vdash \alpha^{(g)}_t \simeq ([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}}$ and $\vdash \beta^{(g)}_t \simeq ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$. Take $\vdash (\alpha^{(g)}_t \simeq \beta^{(g)}_t) \simeq (\alpha^{(g)}_t \simeq \beta^{(g)}_t)$, which is an instance of Axiom 2a. Then, by applying Rule 1 twice, we get $\vdash (\alpha^{(g)}_t \simeq \beta^{(g)}_t) \simeq (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \simeq ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}})$ and consequently

$$\vdash (\alpha^{(g)}_t \simeq \beta^{(g)}_t) \simeq ([[\alpha_t \simeq \beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$$

by Proposition 18 and Rule 1 again. Because $(\alpha^{(g)}_t \simeq \beta^{(g)}_t) = (\alpha_t \simeq \beta_t)^{(g)}$ by Definition 15, we get $\vdash (\alpha_t \simeq \beta_t)^{(g)} \simeq ([[\alpha_t \simeq \beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$, as desired.

7. Let α_c be $\alpha_t \overset{*}{\vee} \beta_t$. By induction hypothesis, we have $\vdash \alpha^{(g)}_t \simeq ([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}}$ and $\vdash \beta^{(g)}_t \simeq ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$. Take $\vdash (\alpha^{(g)}_t \overset{*}{\vee} \beta^{(g)}_t) \simeq (\alpha^{(g)}_t \overset{*}{\vee} \beta^{(g)}_t)$, which is an instance of Axiom 2a. Then, by applying Rule 1 twice, we get $\vdash (\alpha^{(g)}_t \overset{*}{\vee} \beta^{(g)}_t) \simeq (([[\alpha_t]]^{\mathcal{M},g})^{\mathsf{N}} \overset{*}{\vee} ([[\beta_t]]^{\mathcal{M},g})^{\mathsf{N}})$ and consequently

$$\vdash (\alpha^{(g)}_t \overset{*}{\vee} \beta^{(g)}_t) \simeq ([[\alpha_t \overset{*}{\vee} \beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$$

by Proposition 19 and Rule 1 again. Because $(\alpha_t^{(g)} \overset{*}{\vee} \beta_t^{(g)}) = (\alpha_t \overset{*}{\vee} \beta_t)^{(g)}$ by Definition 15, we get $\vdash (\alpha_t \overset{*}{\vee} \beta_t)^{(g)} \simeq ([[\alpha_t \overset{*}{\vee} \beta_t]]^{\mathcal{M},g})^{\mathsf{N}}$, as desired.

■

THEOREM 21. (Completeness) *For every $\alpha_t \in \mathsf{ME}_t^K$, if $\models \alpha_t$, then $\vdash \alpha_t$.*

PROOF. If α_t is closed, $\alpha_t^{(g)} = \alpha_t$ by Definition 15. Since α_t is valid, $[[\alpha_t]]^{\mathcal{M},g}$ $= 1$ for every assignment g, so Lemma 20 gives us $\vdash \alpha_t \simeq 1^{\mathsf{N}}$. Therefore, $\vdash \alpha_t$ by Rule 7.

If α_t is not closed, the closure of α_t is a theorem of $\mathbf{PT}_K$. Let $x_{a_1}, \ldots, x_{a_n}$ be all the variables occurring free in α_t. Then, $(\overset{*}{\forall} x_{a_1}, \ldots, x_{a_n} \alpha_t)^{(g)} = \overset{*}{\forall} x_{a_1}, \ldots, x_{a_n} \alpha_t$, so by the previous argument $\vdash (\overset{*}{\forall} x_{a_1}, \ldots, x_{a_n} \alpha_t) \simeq 1^{\mathsf{N}}$. Hence, $\vdash \overset{*}{\forall} x_{a_1}, \ldots, x_{a_n} \alpha_t$ by Rule 7. ■

7. Conclusion

In this paper, we have defined a version of Henkin's Propositional Type Theory which is partial in a double sense. The hierarchy of propositional types contains partial functions and some meaningful expressions of the language, including formulas, may be undefined. This is a novelty with respect to Farmer's system (Andrew's Type Theory with undefinedess), because for him formulas must be always defined. Although Lepage's Partial Propositional Logic allows formulas to be undefined, his Type Theory lacks the Nameability theorem characterizing Henkin's original system and a construtive proof of completeness was not given.

For future work, we intend first to extend this framework to a Type Theory having a basic type for individuals. Secondly, we are interested in implementing our proof system in an automated theorem prover for higher-order logic, like Isabelle/HOL. Finally, a translation of this logic into many-sorted logic could be explored. In the meantime, we hope to have shown that, as we said above, having partial functions and undefinedness at our disposal in a Type Theory provides high benefits at low cost.

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