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Variations on the Kripke Trick

Abstract. In the early 1960s, to prove undecidability of monadic fragments of sublogics of the predicate modal logic **QS5** that include the classical predicate logic **QCl**, Saul Kripke showed how a classical atomic formula with a binary predicate letter can be simulated by a monadic modal formula. We consider adaptations of Kripke's simulation, which we call the Kripke trick, to various modal and superintuitionistic predicate logics not considered by Kripke. We also discuss settings where the Kripke trick does not work and where, as a result, decidability of monadic modal predicate logics can be obtained.

Keywords: Predicate modal logic, Predicate superintuitionistic logic, Undecidability, Decidability, Kripke trick.

1. Introduction

Saul Kripke [22] made an observation that allowed him to prove undecidability of monadic fragments of modal predicate logics that contain all the theorems of the classical predicate logic **QCl** and are sublogics of the predicate extension **QS5** of the modal propositional logic **S5**. Namely, Kripke observed the following: if L is a modal predicate logic such that $\mathbf{QCl} \subset L \subseteq \mathbf{QS5}$ and φ is a classical predicate formula with a single binary predicate letter Q, then the modal formula obtained from φ by substituting $\Diamond(P_1(x) \wedge P_2(y))$ for $Q(x,y)$ belongs to L if, and only if, φ belongs to **QCl**. Since the classical logic of a single binary predicate is undecidable [4], Kripke's observation implies that, if L is a modal predicate logic satisfying $\mathbf{QCl} \subset L \subseteq \mathbf{QS5}$, then the monadic fragment of L is undecidable. This construction, with an additional observation that the formula $\Diamond(P_1(x) \wedge P_2(y))$ contains the same individual variables as $Q(x,y)$, has been used to establish undecidability of monadic fragments with a restricted number of individual variables of various modal, temporal, and superintuitionistic predicate logics [18, 20, 32, 35–39]; in particular, it has been used [20] to show that two-variable monadic fragments of most common modal predicate logics are undecidable.

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In this paper, we survey the uses of the Kripke trick in the literature, discussing adaptations of Kripke's original construction [22] that have been used to prove lower algorithmic bounds (undecidability, lack of recursive enumerability, non-membership in the classes of the arithmetical hierarchy, etc.) for monadic fragments of various modal and superintuitionistic predicate logics. Our focus here is on expounding the technique and its variations rather than on any specific results. Most of the stated results are not the strongest known—to strengthen them, additional techniques are needed, most often those to do with simulating all monadic atomic formulas of the language by atomic formulas with a single fixed monadic predicate letter.[1] When expounding the variations of the Kripke trick, we focus on the essentials, peeling away unnecessary complications arising from the more complex arguments within which the uses of the trick are often embedded. We hope that this survey will be of use to researchers in, and newcomers to, the field of non-classical predicate logics, enabling them to put the Kripke trick to novel uses. We also briefly discuss how blocking the trick results in decidable fragments.

The paper is structured as follows. Section 2 contains preliminaries on modal and superintuitionistic predicate logics. In Section 3, we recall Kripke's original proof. In Section 4, we consider adaptations of Kripke's proof to common logics conservatively extending **QCl**, but not included into **QS5**. In Section 5, we do the same for logics of finite domains, which are not conservative extensions of **QCl**. In Section 6, we discuss the adaptation of the Kripke trick to modal logics of frames with only finitely many worlds. In Section 7, we discuss how the trick can be carried out with formulas of one monadic predicate letter, rather than two, as in Kripke's original proof. Section 8 briefly discusses the extension of the Kripke trick to simulation of atomic formulas with n-ary, for $n > 2$, predicate letters. In Section 9, we show how the trick can be used inside modal formulas. In Section 10, we consider the adaptation of the trick to superintuitionistic logics. We conclude, in Section 11, with a discussion of situations where the Kripke trick is blocked.

[1] For examples of such techniques we refer the reader to our earlier articles [28, 32, 35, 36, 38, 39]; these techniques generalise to predicate logics methods originally developed for simulating proposition letters in modal and superintuitionistic propositional logics [2, 5, 16, 29, 34, 40, 41].

2. Preliminaries

2.1. Syntax

We shall be considering (first-order) predicate logics in two languages: the non-modal language $\mathcal{L}$, which is the language of the classical and superintuitionistic logics, and the modal language $\mathcal{ML}$, which is the language of modal logics. The language $\mathcal{L}$ contains the following expressions[2]: a countable set *Var* of *individual variables*; for every $n \in \mathbb{N}$, a countable set of *predicate letters* of arity n (nullary predicate letters are also called *proposition letters*); the propositional constant $\bot$; binary propositional connectives $\to$, $\wedge$, and $\vee$; and quantifier symbols $\forall$ and $\exists$. The language $\mathcal{ML}$, in addition, contains a unary modal operator $\Box$. We shall not be considering languages with individual constants, function symbols, or equality; this is unnecessary since we are concerned with lower algorithmic bounds, which immediately apply to more expressive languages. Formulas of $\mathcal{L}$ (or, $\mathcal{L}$-formulas) are defined by the grammar

$$\varphi ::= P(x_1, \ldots, x_n) \mid \bot \mid (\varphi \to \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \forall x\, \varphi \mid \exists x\, \varphi,$$

and formulas of $\mathcal{ML}$ (or, $\mathcal{ML}$-formulas) by the grammar

$$\varphi ::= P(x_1, \ldots, x_n) \mid \bot \mid (\varphi \to \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \Box\varphi,$$

where P is an n-ary predicate letter and $x, x_1, \ldots, x_n \in \mathit{Var}$. From now on, we identify the languages $\mathcal{L}$ and $\mathcal{ML}$ with their sets of formulas.

In both languages, $\neg$ and $\top$ are standard abbreviations: $\neg\varphi = \varphi \to \bot$ and $\top = \neg\bot$. In $\mathcal{ML}$, $\Diamond\varphi$ is a standard abbreviation for $\neg\Box\neg\varphi$. When omitting parentheses, we assume that $\wedge$ and $\vee$ bind tighter than $\to$. Formulas of the form $P(x_1, \ldots, x_n)$ are called *atomic*. If φ is a formula, the set of subformulas of φ is denoted by $\mathit{sub}\,\varphi$, the set of the individual variables of φ is denoted by $\mathit{var}\,\varphi$, and the set of free individual variables of φ, also called the *parameters* of φ, is denoted by $\mathit{par}\,\varphi$. We write $\varphi(x_1, \ldots, x_n)$ to mean that the parameters of φ are among $x_1, \ldots, x_n$. If $\mathit{par}\,\varphi = \varnothing$, then φ is *closed*. If Γ is a set of formulas, $\mathtt{cf}\,\Gamma$ denotes the set of closed formulas from Γ. A formula containing neither quantifier symbols nor predicate letters other than nullary ones is called *propositional*. A formula containing only monadic and nullary predicate letters is called *monadic*.

[2] These expressions are often treated as symbols of an alphabet. Since we are concerned with algorithmic issues, we assume that our languages contain only finitely many symbols, which are used to encode all the expressions of the language.

2.2. Kripke Semantics for Modal Logics

Prior to defining modal predicate logics, we describe the semantic apparatus we shall be using for them. The standard Kripke semantics for modal predicate logics comes in two varieties: expanding domains semantics, discussed in Section 2.2.1, and varying domains semantics, discussed in Section 2.2.2; yet another variety, constant domain semantics, can be viewed as a special case of either (we view it as a special case of the expanding domains semantics). These two semantics generate two different kinds of logics; in particular, the same propositional modal logic extends, under the two approaches, to distinct predicate modal logics.

Both predicate Kripke semantics build upon Kripke semantics for propositional logics, whose main concept is that of a *Kripke frame*, which is a pair $\mathfrak{F} = \langle W, R\rangle$, where W is a non-empty set of *possible worlds* and R is a binary *accessibility relation* on W. Speaking of Kripke frames, we use the standard functional notation $R(w)$ for $\{w' \in W : wRw'\}$; thus, $w' \in R(w)$ means the same as wRw'. Both kinds of predicate Kripke semantics equip worlds of Kripke frames with domains of individuals, but make different assumptions about the relationships among those domains; the two semantics also differ in definitions of interpretations in models and of truth at a possible world of a model.

2.2.1. Expanding Domains Semantics

The expanding domains Kripke semantics makes the assumption that domains of possible worlds expand (more precisely, do not decrease) along the accessibility relation; moreover, it interprets formulas with respect to domains of worlds and defines validity both for closed formulas and formulas with parameters; consequently, $\mathcal{L}$-fragments of the resultant logics coincide with the classical predicate logic **QCl**.

An *augmented frame with expanding domains* (or, simply, an *augmented frame*) is a pair $\mathfrak{F}_D = \langle \mathfrak{F}, D\rangle$, where $\mathfrak{F}$ is a Kripke frame and D is a *domain function* sending each $w \in W$ to a non-empty subset of some set of individuals; the function D is required to satisfy the *expanding domains condition*: for every $w, w' \in W$,

$$wRw' \quad\Longrightarrow\quad D(w) \subseteq D(w'). \tag{2.1}$$

The set $D(w)$, also denoted by D_w, is the *domain of the world* w. Sets of the form D_w are also called *local domains*. If $\mathfrak{F} = \langle W, R\rangle$, we also write $\mathfrak{F}_D = \langle W, R, D\rangle$. We define $D^+ = \bigcup\{D_w : w \in W\}$; the set D^+ is the *domain of the augmented frame* $\mathfrak{F}_D$, or the *global domain* of $\mathfrak{F}_D$.

A *subframe* of an augmented frame $\langle W, R, D\rangle$ is an augmented frame $\langle W', R', D'\rangle$ where W' is a non-empty subset of W, and where $R' = R \restriction W'$ and $D' = D \restriction W'$.

We next define two classes of augmented frames with expanding domains that are of particular interest. We say that an augmented frame $\mathfrak{F}_D = \langle W, R, D\rangle$ is an *augmented frame with locally constant domains* if it satisfies the *local constancy condition*: for every $w, w' \in W$,

$$wRw' \quad \Longrightarrow \quad D(w) = D(w'), \tag{2.2}$$

and we say that $\mathfrak{F}_D$ is an *augmented frame with a globally constant domain* if it satisfies the *global constancy condition*: for every $w, w' \in W$,

$$D(w) = D(w'). \tag{2.3}$$

If $\mathfrak{F}_D$ is an augmented frame with a globally constant domain $\mathcal{D}$, then, following [13], we also denote $\mathfrak{F}_D$ by $\mathfrak{F} \odot \mathcal{D}$.

A *predicate Kripke model with expanding domains* (or, simply, a *Kripke model*) is a tuple $\mathfrak{M} = \langle \mathfrak{F}_D, I\rangle$, where $\mathfrak{F}_D = \langle W, R, D\rangle$ is an augmented frame with expanding domains and I is an *interpretation of predicate letters* sending a world $w \in W$ and an n-ary predicate letter P to an n-ary relation $I(w, P)$ on D_w; we also write $P^{I,w}$ for $I(w, P)$ and $\langle W, R, D, I\rangle$ for $\langle \mathfrak{F}_D, I\rangle$. We note that, if a predicate letter P is nullary (i.e., if P is a proposition letter), then $P^{I,w} \subseteq D^0_w = \{\langle\rangle\}$, i.e., either $P^{I,w} = \varnothing$ or $P^{I,w} = \{\langle\rangle\}$. The former corresponds to assigning the truth value 'false', and the latter to assigning the truth value 'true', to a proposition letter in Kripke semantics for propositional logics.

An *assignment* in an augmented frame $\langle W, R, D\rangle$ is a map $g\colon \mathit{Var} \to D^+$. Observe that, since variables never take on values from outside of D^+, individuals outside of D^+ are irrelevant to the evaluation of formulas. If g and g' are assignments such that $g'(y) = g(y)$ whenever $y \neq x$, we write $g' \stackrel{x}{=} g$.

The truth of an $\mathcal{ML}$-formula φ at a world w of a model $\mathfrak{M}$ under an assignment g is defined by recursion:

- $\mathfrak{M}, w \models^g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n)\rangle \in P^{I,w}$;
- $\mathfrak{M}, w \not\models^g \bot$;
- $\mathfrak{M}, w \models^g \varphi_1 \wedge \varphi_2$ if $\mathfrak{M}, w \models^g \varphi_1$ and $\mathfrak{M}, w \models^g \varphi_2$;
- $\mathfrak{M}, w \models^g \varphi_1 \vee \varphi_2$ if $\mathfrak{M}, w \models^g \varphi_1$ or $\mathfrak{M}, w \models^g \varphi_2$;
- $\mathfrak{M}, w \models^g \varphi_1 \to \varphi_2$ if $\mathfrak{M}, w \not\models^g \varphi_1$ or $\mathfrak{M}, w \models^g \varphi_2$;
- $\mathfrak{M}, w \models^g \Box\varphi_1$ if $\mathfrak{M}, w' \models^g \varphi_1$ whenever $w' \in R(w)$;

- $\mathfrak{M}, w \models^g \exists x\, \varphi_1$ if $\mathfrak{M}, w \models^{g'} \varphi_1$, for some g' such that $g' \stackrel{x}{=} g$ and $g'(x) \in D(w)$;
- $\mathfrak{M}, w \models^g \forall x\, \varphi_1$ if $\mathfrak{M}, w \models^{g'} \varphi_1$ whenever $g' \stackrel{x}{=} g$ and $g'(x) \in D(w)$.

If $\mathfrak{F}_D$ is an augmented frame with expanding domains and $\mathfrak{M}$ is a predicate Kripke model with expanding domains, then

- we say that a formula φ is true at a world w of $\mathfrak{M}$, and write $\mathfrak{M}, w \models \varphi$, if $\mathfrak{M}, w \models^g \varphi$, for every assignment g such that $g(x) \in D_w$ whenever $x \in par\, \varphi$;
- we say that a formula φ is true in $\mathfrak{M}$, and write $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, w \models \varphi$ whenever $w \in W$;
- we say that a formula φ is valid on $\mathfrak{F}_D$, and write $\mathfrak{F}_D \models \varphi$, if $\mathfrak{M} \models \varphi$ whenever $\mathfrak{M}$ is a model over $\mathfrak{F}_D$;
- we say that a formula φ is valid on a Kripke frame $\mathfrak{F}$, and write $\mathfrak{F} \models \varphi$, if φ is valid on every augmented frame with expanding domains over $\mathfrak{F}$;
- we say that a formula φ is valid on a class $\mathscr{C}$ of augmented frames, and write $\mathscr{C} \models \varphi$, if φ is valid on every augmented frame from $\mathscr{C}$.

A set Γ of formulas is valid on an augmented frame, a Kripke frame, or a class of frames if the corresponding relation holds for every formula from Γ.

As is well known, the global constancy condition (2.3) is not definable by modal predicate formulas: it is easy to check, using generated subframes, that the class of augmented frames satisfying (2.3) induces the same validities as the class of augmented frames satisfying (2.2). One the other hand, condition (2.2) is definable by the *Barcan formula*

$$bf \quad = \quad \forall x\, \Box P(x) \to \Box \forall x\, P(x);$$

i.e., if $\mathfrak{F}_D$ is an augmented frame with expanding domains, then $\mathfrak{F}_D \models bf$ if, and only if, $\mathfrak{F}_D$ has locally constant domains [13, Proposition 3.4.2]. Therefore, we only define logics of classes of augmented frames with expanding and with locally constant domains.

First, if $\mathscr{C}$ is a class of augmented frames with expanding domains, we define

$$\mathtt{ML}\,\mathscr{C} \quad = \quad \{\varphi \in \mathcal{ML} : \mathscr{C} \models \varphi\}.$$

If $\mathscr{C}$ is a class of Kripke frames, then $\mathtt{aug}^e\,\mathscr{C}$ and $\mathtt{aug}^c\,\mathscr{C}$ denote, respectively, the class of all augmented frames with expanding domains and the class of all augmented frames with locally constant domains over Kripke frames from $\mathscr{C}$.

Now, if $\mathscr{C}$ is a class of Kripke frames, we define

$$\begin{array}{rcl} \mathtt{ML}^e\,\mathscr{C} & = & \mathtt{ML}\,\mathtt{aug}^e\,\mathscr{C}; \\ \mathtt{ML}^c\,\mathscr{C} & = & \mathtt{ML}\,\mathtt{aug}^c\,\mathscr{C}. \end{array}$$

2.2.2. Varying Domains Semantics In contrast to the expanding domains semantics, the varying domains semantics makes no assumptions about relationships among domains of possible worlds. This requires a different definition of interpretations in models so that modal formulas with parameters can be evaluated at worlds; this leads to a different definition of truth of a formula at a world. As a result, in this semantics, some classical validities with parameters are refutable.

An *augmented frame with varying domains* is a pair $\mathfrak{F}_D = \langle \mathfrak{F}, D \rangle$, where $\mathfrak{F}$ is a Kripke frame and D is a function sending each $w \in W$ to a non-empty subset of some set of individuals. No relationship among domains of worlds is assumed a priori; in particular, (2.1) is not assumed. The notation D_w and D^+ has the same meaning as in the expanding domains semantics.

A *Kripke model with varying domains* is a tuple $\mathfrak{M} = \langle \mathfrak{F}_D, I \rangle$, where $\mathfrak{F}_D = \langle W, R, D \rangle$ is an augmented frame with varying domains and I is an *interpretation of predicate letters* sending a world $w \in W$ and an n-ary predicate letter P to an n-ary relation $I(w, P)$ on D^+, rather than on D_w, as in the expanding domains semantics. An assignment is defined as in the expanding domains semantics.

The truth relation $\models_{var}$ between models, worlds, assignments, and $\mathcal{ML}$-formulas is defined analogously to the definition of the relation $\models$ for the expanding domains semantics: we simply replace $\models$ with $\models_{var}$ in the truth clauses defining $\models$. Truth at a world and validity are, however, defined differently: a formula φ is considered true at a world w of a model $\mathfrak{M}$ if $\mathfrak{M}, w \models^g_{var} \varphi$ holds for every assignment g, regardless of the values of g on variables from $par\,\varphi$; in other words, these values are not required, in contrast to the expanding domains semantics, to belong to D_w. As a result, some classical validities with parameters, e.g., $\forall x\, P(x) \to P(x)$, are not valid in the varying domains semantics. Analogously to the expanding domains semantics, we say that a formula φ is *valid* on an augmented frame $\mathfrak{F}_D$ if φ is true at every world of every model over $\mathfrak{F}_D$; we say that φ is valid on a class $\mathscr{C}$ of augmented frames, and write $\mathscr{C} \models_{var} \varphi$, if it is valid on every augmented frame from $\mathscr{C}$. We say that φ is valid on a Kripke frame $\mathfrak{F}$, and write $\mathfrak{F} \models_{var} \varphi$, if φ is valid on every augmented frame over $\mathfrak{F}$.

If $\mathscr{C}$ is a class of augmented frames with varying domains, we define

$$\mathtt{vML}\,\mathscr{C} \quad = \quad \{\varphi \in \mathcal{ML} : \mathscr{C} \models_{var} \varphi\}.$$

For more background on the varying domains semantics, we refer the reader to [8].

For our purposes, it suffices to make the following observations about the varying domains semantics to be able, in the rest of the paper, not to pay special attention to the logics arising from it.

First, when it comes to closed $\mathcal{L}$-formulas, exactly those that are classically valid are valid on Kripke frames in the varying domains semantics:

Proposition 2.1. *Let $\mathfrak{F}$ be a Kripke frame and φ a closed $\mathcal{L}$-formula. Then*

$$\mathfrak{F} \models_{var} \varphi \quad \Longleftrightarrow \quad \varphi \in \mathbf{QCl}.$$

Proof. Straightforward. ■

Second, on augmented frames satisfying (2.1), the same closed $\mathcal{ML}$-formulas are valid in the varying domains semantics as in the expanding domains semantics:

Proposition 2.2. *Let $\mathfrak{F}_D$ be an augmented frame satisfying (2.1) and let φ be a closed $\mathcal{ML}$-formula. Then*

$$\mathfrak{F}_D \models_{var} \varphi \quad \Longleftrightarrow \quad \mathfrak{F}_D \models \varphi.$$

Proof. Straightforward. ■

From Proposition 2.2, we immediately obtain the following:

Corollary 2.3. *Let $\mathscr{C}$ be a class of Kripke frames. Then*

$$\texttt{cf}\,\texttt{ML}^e\,\mathscr{C} \;=\; \texttt{cf}\,\texttt{vML}\,\texttt{aug}^e\,\mathscr{C} \quad \textit{and} \quad \texttt{cf}\,\texttt{ML}^c\,\mathscr{C} \;=\; \texttt{cf}\,\texttt{vML}\,\texttt{aug}^c\,\mathscr{C}.$$

Corollary 2.4. *Let $\mathfrak{F}$ be a Kripke frame and φ a closed $\mathcal{ML}$-formula. Then*

$$\mathfrak{F} \models_{var} \varphi \quad \Longrightarrow \quad \mathfrak{F} \models \varphi.$$

The converse of Corollary 2.4 is not true: it is well known and easy to check that the converse Barcan formula $\mathit{cbf} = \Box\forall x\, P(x) \to \forall x\, \Box P(x)$ is valid on every Kripke frame in the expanding domains semantics, but not in the varying domains semantics; in fact, in the latter, *cbf* is valid precisely on those augmented frames that satisfy (2.1).

If $\mathscr{C}$ is a class of Kripke frames, we denote by $\texttt{aug}^v\,\mathscr{C}$ the class of augmented frames with varying domains over Kripke frames from $\mathscr{C}$ and define

$$\texttt{ML}^v\,\mathscr{C} \quad = \quad \texttt{vML}\,\texttt{aug}^v\,\mathscr{C}.$$

We now make an observation that will allow us not to consider explicitly the logics of varying domains in the rest of the paper:

Proposition 2.5. *Let $\mathscr{C}$ be a class of Kripke frames. Then*

$$\texttt{cf}\,\mathbf{QCl} \subseteq \texttt{cf}\,\texttt{ML}^v\,\mathscr{C} \subseteq \texttt{cf}\,\texttt{ML}^e\,\mathscr{C} \subseteq \texttt{cf}\,\texttt{ML}^c\,\mathscr{C}.$$

Proof. The first inclusion holds by Proposition 2.1. The second inclusion holds by Corollary 2.4. The third inclusion follows from the inclusion $\texttt{aug}^c\,\mathscr{C} \subseteq \texttt{aug}^e\,\mathscr{C}$. ■

Since the reductions considered throughout the paper are defined only for closed formulas, Proposition 2.5 allows us to work, most of the time, exclusively with the constant domains semantics. Since the constructions we use to obtain results for logics of constant and expanding domains automatically yield results for logics of varying domains, logics of varying domains are not considered explicitly.

2.3. Predicate Modal Logics

2.3.1. Predicate Modal Logics of Expanding Domains By a *normal modal predicate logic* we mean a set L of $\mathcal{ML}$-formulas that includes **QCl** and the minimal normal propositional modal logic **K** and is closed under Modus Ponens, Substitution, Generalisation (if $\varphi \in L$, then $\forall x\,\varphi \in L$), and Necessitation (if $\varphi \in L$, then $\Box\varphi \in L$). Closure under Substitution shall be of particular importance to us, but instead of giving the technically involved general definition of predicate Substitution, which can be found in [13, Subsection 2.5], we describe here a simple special case we will use. Suppose that φ is a formula, $Q(x_1,\ldots,x_n)$ is an atomic formula, and ψ is a formula with $par\,\psi = \{x_1,\ldots,x_n\}$. Substituting ψ for $Q(x_1,\ldots,x_n)$ in φ amounts to replacing every subformula $Q(y_1,\ldots,y_n)$ of φ with a formula obtained from ψ by replacing each x_i with y_i; for example, if

$$\varphi \quad = \quad \forall x\forall y\forall z\,(Q(x,y)\wedge Q(y,z)\to Q(x,z)),$$

then substituting $\Diamond(P_1(x_1)\wedge P_2(x_2))$ for $Q(x_1,x_2)$ gives us the formula

$$\forall x\forall y\forall z\,\big(\Diamond(P_1(x)\wedge P_2(y))\wedge\Diamond(P_1(y)\wedge P_2(z))\to\Diamond(P_1(x)\wedge P_2(z))\big).$$

Let $\mathscr{C}$ be a class of augmented frames with expanding domains, as defined in Section 2.2.1; it is well known that the set $\texttt{ML}\,\mathscr{C}$ is a normal modal predicate logic. Hence, the following holds:

Proposition 2.6. *Let $\mathscr{C}$ be a class of Kripke frames. The sets $\texttt{ML}^e\,\mathscr{C}$ and $\texttt{ML}^c\,\mathscr{C}$ are normal modal predicate logics.*

If $\mathscr{C}$ is a class of augmented frames and L is a normal modal predicate logic such that $L = \texttt{ML}\,\mathscr{C}$, then L is said to be *determined* by $\mathscr{C}$. A logic is *Kripke complete* (in the expanding domains semantics) if it is determined by some class of augmented frames with expanding domains; otherwise, the logic is said to be *Kripke incomplete* (in the expanding domains semantics). Often, it makes sense to lift the notion of determination to the level of Kripke frames: if $\mathscr{C}$ is a class of Kripke frames and $L = \mathtt{ML}^e\mathscr{C}$, then L is said to be *determined* by $\mathscr{C}$.

The smallest normal modal predicate logic is called **QK**. It is well known [13, 19] that **QK** coincides with the set of $\mathcal{ML}$-formulas valid, in the expanding domains semantics, on every Kripke frame. We next introduce some notation for logics. If L is a normal modal predicate logic and Γ a set of $\mathcal{ML}$-formulas, then $L \oplus \Gamma$ denotes the smallest normal modal predicate logic that includes $L \cup \Gamma$; we write $L \oplus \varphi$ instead of $L \oplus \{\varphi\}$. Similarly, if L is a propositional normal modal logic and Γ is a set of propositional formulas, then $L \oplus \Gamma$ denotes the smallest propositional normal modal logic that includes $L \cup \Gamma$. If L is a normal modal predicate logic, then $L.\mathbf{bf}$ denotes the logic $L \oplus bf$. If L is a normal modal propositional logic, then $\mathbf{Q}L$ denotes the minimal normal predicate extension of L, i.e., the logic $\mathbf{QK} \oplus L$. The following four useful facts about logics of the form $\mathbf{Q}L$ are easy to verify:

Proposition 2.7. *If L is a propositional modal logic and Γ a set of propositional formulas, then $\mathbf{Q}L \oplus \Gamma = \mathbf{Q}(L \oplus \Gamma)$.*

Proposition 2.8. *If L and L' are propositional modal logics such that $L \subseteq L'$, then $\mathbf{Q}L \subseteq \mathbf{Q}L'$.*

Proposition 2.9. *Let $\mathfrak{F}_D$ be an augmented frame with expanding domains and let L be a modal propositional logic. Then*

$$\mathfrak{F} \models L \quad \Longleftrightarrow \quad \mathfrak{F}_D \models \mathbf{Q}L.$$

Corollary 2.10. *Let $\mathfrak{F}$ be a Kripke frame and let L be a modal propositional logic. Then*

$$\mathfrak{F} \models L \quad \Longleftrightarrow \quad \mathfrak{F} \models \mathbf{Q}L.$$

Recall that, if L_1 and L_2 are logics in languages, respectively, $\mathcal{L}_1$ and $\mathcal{L}_2$, with $\mathcal{L}_1 \subseteq \mathcal{L}_2$, then L_2 is a *conservative extension* of L_1 if the $\mathcal{L}_1$-fragment of L_2 coincides with L_1. If $\mathfrak{M} = \langle W, R, D, I\rangle$ is a Kripke model with expanding domains, $w \in W$, and I_w is the function defined by $I_w(P) = I(w, P)$, for every predicate letter P, then the pair $\langle D_w, I_w\rangle$ is a classical model; hence, every logic determined by a class of augmented frames with expanding domains where no requirements other than (2.1) are placed on domains is a conservative extension of **QCl** (observe that this is not true for varying domains).

To make the paper self-contained, we recall a number of facts about minimal predicate extensions of common propositional modal logics:[3]

- **QT** is determined by the class of reflexive Kripke frames;
- **QK4** is determined by the class of transitive Kripke frames;
- **QS4** is determined by the class of reflexive, transitive Kripke frames;
- **QGL** and **QGrz** are Kripke incomplete [26, 32]; their Kripke frames are, respectively, strict and non-strict Noetherian partial orders, i.e., partial orders where every non-empty subset has a maximal element;
- If $L = \mathtt{ML}\,\mathscr{C}$, for some class $\mathscr{C}$ of Kripke frames, then $L.\mathbf{3} = \mathtt{ML}\,\mathscr{C}_{lin}$, where $\mathscr{C}_{lin}$ is the subclass of $\mathscr{C}$ containing all frames satisfying the condition $\forall x \forall y \forall z\,(xRy \wedge xRz \to yRz \vee zRy \vee y = z)$;
- **QK4.3.D.X** is determined by the class of serial, dense, strict partial orders; it is also determined by the Kripke frame $\langle \mathbb{Q}, \leqslant \rangle$ (the rationals with the usual non-strict order) [7];
- $\mathbf{QAlt}_n$, where $n \in \mathbb{N}$, is determined by the class of n-alternative Kripke frames, i.e., those where $|R(w)| \leqslant n$, for every world w of the frame [46];
- **QK5** is Kripke incomplete [44]; its Kripke frames are those where the accessibility relation is Euclidean, i.e., satisfies the condition $\forall x \forall y \forall z\,(xRy \wedge xRz \to yRz \wedge zRy)$;
- **QK4.5** is Kripke incomplete [44]; its Kripke frames are those where the accessibility relation is both Euclidean and transitive;
- **QKB** is determined by the class of symmetric Kripke frames;
- **QKTB** is determined by the class of reflexive, symmetric Kripke frames;
- **QS5** is determined by the class of Kripke frames where the accessibility relation is an equivalence.[4]

We note that, since $bf \in$ **QKB**, augmented frames validating logics extending **QKB** (these logics include **QKTB** and **QS5**), have locally constant domains. Since, as we have noted, conditions (2.2) and (2.3) give rise to the same validities in the expanding domains semantics, it suffices to consider, when working with extensions of **QKB**, augmented frames of the form $\mathfrak{F} \odot \mathcal{D}$.

[3] The statements about **QT**, **QK4**, and **QS4** are instances of [13, Theorem 6.1.29]:; the statements about **QKB**, **QKTB**, and **QS5** are instances of [13, Theorem 7.4.7], originally proven by Tanaka and Ono [51].

[4] The logic **QS5** is also determined by the class of Kripke frames with a universal accessibility relation.

This, in particular, is what Kripke does in his proof [22] recounted in Section 3.

2.3.2. Predicate Modal Logics of Varying Domains Logics of varying domains were introduced by Kripke [23][5] and were extensively studied in the philosophically motivated literature on modal logic [8, 14, 15]. They are usually defined as sets of formulas valid on classes of augmented frames with varying domains. Axiomatic treatments of such logics have been proposed by Kripke [23] and Hughes and Cresswell [19]. Due to Proposition 2.5, we will not explicitly consider logics of varying domains in the rest of the paper, even though the constructions and results we present apply to them, as well. When wishing to emphasize that the presented techniques apply to logics of varying domains as well as to logics of expanding domains, we use the term *modal predicate logic* to mean any set of $\mathcal{ML}$-formulas that includes $\mathtt{cf}\,\mathbf{QCl}$ and is closed under Modus Ponens, Substitution, Generalisation, and Necessitation. This general notion encompasses both logics of expanding domains and logics of varying domains.

2.4. Superintuitionistic Logics

2.4.1. Kripke Semantics for Superintuitionistic Logics An *intuitionistic Kripke frame* is a Kripke frame $\mathfrak{F} = \langle W, R \rangle$ where R is a partial order—i.e., a reflexive, transitive, and antisymmetric binary relation—on W. An *intuitionistic augmented frame* is an augmented frame $\mathfrak{F}_D = \langle \mathfrak{F}, D \rangle$ with expanding domains such that $\mathfrak{F}$ is an intuitionistic Kripke frame. An *intuitionistic Kripke model* is a Kripke model with expanding domains $\mathfrak{M} = \langle W, R, D, I \rangle$ where $\langle W, R, D \rangle$ is an intuitionistic augmented frame and the interpretation I satisfies the *heredity condition*: for every $w, w' \in W$ and every predicate letter P,

$$wRw' \quad \Longrightarrow \quad I(w, P) \subseteq I(w', P). \tag{2.4}$$

An *assignment* is a map $g\colon \mathit{Var} \to D^+$. The truth of an $\mathcal{L}$-formula φ at a world w of a model $\mathfrak{M}$ under an assignment g is defined by recursion:

- $\mathfrak{M}, w \Vdash^g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n) \rangle \in P^{I,w}$;
- $\mathfrak{M}, w \not\Vdash^g \bot$;
- $\mathfrak{M}, w \Vdash^g \varphi_1 \wedge \varphi_2$ if $\mathfrak{M}, w \Vdash^g \varphi_1$ and $\mathfrak{M}, w \Vdash^g \varphi_2$;
- $\mathfrak{M}, w \Vdash^g \varphi_1 \vee \varphi_2$ if $\mathfrak{M}, w \Vdash^g \varphi_1$ or $\mathfrak{M}, w \Vdash^g \varphi_2$;
- $\mathfrak{M}, w \Vdash^g \varphi_1 \to \varphi_2$ if $\mathfrak{M}, w' \not\Vdash^g \varphi_1$ or $\mathfrak{M}, w' \Vdash^g \varphi_2$ whenever $w' \in R(w)$;

[5] For this reason, these logics are often called *Kripkean* modal predicate logics.

- $\mathfrak{M}, w \Vdash^g \exists x \, \varphi_1$ if $\mathfrak{M}, w \Vdash^{g'} \varphi_1$, for some g' such that $g' \stackrel{x}{=} g$ and $g'(x) \in D(w)$;
- $\mathfrak{M}, w \Vdash^g \forall x \, \varphi_1$ if $\mathfrak{M}, w' \Vdash^{g'} \varphi_1$ whenever $w' \in R(w)$, $g' \stackrel{x}{=} g$, and $g'(x) \in D(w')$.

It follows from (2.4) and the definition of the relation $\Vdash$ that truth of $\mathcal{L}$-formulas in Kripke models is hereditary:

$$wRw' \;\&\; \mathfrak{M}, w \Vdash^g \varphi \implies \mathfrak{M}, w' \Vdash^g \varphi. \tag{2.5}$$

Definitions of truth and validity are similar to those for expanded domains semantics for $\mathcal{ML}$. We say that an $\mathcal{L}$-formula φ is *true at a world* w of a model $\mathfrak{M}$, and write $\mathfrak{M}, w \Vdash \varphi$, if $\mathfrak{M}, w \Vdash^g \varphi$, for every g assigning to the parameters of φ elements of D_w. We say that a formula φ is *true* in a model $\mathfrak{M}$, and write $\mathfrak{M} \Vdash \varphi$, if $\mathfrak{M}, w \Vdash \varphi$, for every world w of $\mathfrak{M}$. We say that a formula φ is *valid on an augmented frame* $\mathfrak{F}_D$, and write $\mathfrak{F}_D \Vdash \varphi$, if $\mathfrak{M} \Vdash \varphi$, for every model $\mathfrak{M}$ over $\mathfrak{F}_D$. We say that a formula φ is *valid on a Kripke frame* $\mathfrak{F}$, and write $\mathfrak{F} \Vdash \varphi$, if φ is valid on every augmented frame over $\mathfrak{F}$. We say that a formula φ is *valid on a class* $\mathscr{C}$ *of augmented frames*, and write $\mathscr{C} \models \varphi$, if φ is valid on every member of $\mathscr{C}$; similarly for classes of Kripke frames. A set Γ of formulas is valid on an augmented frame, a Kripke frame, or a class of frames if the corresponding relation holds for every formula from Γ.

Observe that, if $\mathfrak{F} = \langle W, R \rangle$ is an intuitionistic Kripke frame where W is a singleton and φ is an $\mathcal{L}$-formula, then $\mathfrak{F} \Vdash \varphi$ if, and only if, $\varphi \in \mathbf{QCl}$.

2.4.2. Superintuitionistic Logics The intuitionistic predicate logic **QInt** is the set of $\mathcal{L}$-formulas valid on every intuitionistic Kripke frame. The logic **QInt** can also be defined through a Hilbert-style deductive calculus with a finite set of axioms [13, 56]. A *superintuitionistic predicate logic* is a set of $\mathcal{L}$-formulas that includes **QInt** and is closed under Modus Ponens, Substitution, and Generalisation. As with modal logics, we only use a simple special case of Substitution described in Section 2.3.1. If L is a superintuitionistic predicate logic and Γ is a set of $\mathcal{L}$-formulas, then $L + \Gamma$ denotes the smallest superintuitionistic logic that includes $L \cup \Gamma$. If L is a propositional superintuitionistic logic, $\mathbf{Q}L$ denotes the minimal predicate extension of L, i.e., the logic $\mathbf{QInt} + L$.

A superintuitionistic predicate logic coinciding with the set of all formulas valid on a class $\mathscr{C}$ of augmented frames is said to be *determined by* $\mathscr{C}$. A logic determined by some class of augmented frames is said to be *Kripke complete*.

We recall that the formula $cd = \forall x\,(P(x) \vee q) \to \forall x P(x) \vee q$, where P is a unary, and q nullary, predicate letter, is valid on an augmented frame $\mathfrak{F}_D$ if, and only if, $\mathfrak{F}_D$ has locally constant domains. If L is a superintuitionistic predicate logic, then $L.\mathbf{cd}$ denotes the logic $L + cd$.

2.5. Some Facts About Classical Logic

Throughout the paper, we consider the classical predicate logic **QCl** in the language without equality. We denote by $\mathbf{QCl}_{fin}$ the classical theory, in the language without equality, of models with finite domains; we note that $\mathbf{QCl}_{fin}$ is, in fact, a superintuitionistic predicate logic; in particular, it is closed under Substitution. We now recall two well-known [1, 6, 50, 53] facts about the computational properties of **QCl** and $\mathbf{QCl}_{fin}$ (we note that Σ^0_1-completeness entails undecidability and that Π^0_1-completeness entails lack of recursive enumerability):

PROPOSITION 2.11. *The logic* **QCl** *is* Σ^0_1*-complete in languages with a single binary predicate letter and three individual variables.*

PROPOSITION 2.12. *The logic* $\mathbf{QCl}_{\mathit{fin}}$ *is* Π^0_1*-complete in languages with a single binary predicate letter and three individual variables.*

3. Kripke's Proof

We now briefly recount Kripke's proof [22]; we closely follow the original text, changing only insignificant, stylistic detail. Throughout the rest of the paper, we assume that Q is a binary predicate letter, that P_1 and P_2 are monadic predicate letters, and that, if ψ is an $\mathcal{L}$-formula, then ψ^* is an $\mathcal{ML}$-formula obtained from ψ by substituting $\Diamond(P_1(x) \wedge P_2(y))$ for $Q(x, y)$.

PROPOSITION 3.1. (Kripke [22]) *For every closed* $\mathcal{L}$*-formula* φ *containing no predicate letters other than a binary letter* Q *and every modal predicate logic* L *such that* $L \subseteq \mathbf{QS5}$*,*

$$\varphi \in \mathbf{QCl} \quad\Longleftrightarrow\quad \varphi^* \in L.$$

PROOF. ($\Rightarrow$) Assume that $\varphi \in \mathbf{QCl}$. Since φ is closed and, by definition of modal predicate logic, $\mathtt{cf}\,\mathbf{QCl} \subseteq L$, surely $\varphi \in L$. Since φ^* is a substitution instance of φ and L is closed under Substitution, $\varphi^* \in L$.

($\Leftarrow$) Assume that $\varphi \notin \mathbf{QCl}$. By the Löwenheim–Skolem theorem, there exists a classical model μ with the domain $\mathbb{N}$ such that $\mu \not\models \varphi$. Let $\mathfrak{F} = \langle \mathbb{N}, \mathbb{N} \times \mathbb{N}\rangle$. Define an interpretation I on the augmented frame

$\mathfrak{F} \odot \mathbb{N}$ so that, for every $n \in W$,

$$\begin{array}{rcl} I(n, P_1) & = & \{\langle m \rangle : \mu \models Q(m, n)\}; \\ I(n, P_2) & = & \{\langle n \rangle\}, \end{array}$$

and put $\mathfrak{M} = \langle \mathfrak{F} \odot \mathbb{N}, I \rangle$. Then $\mathfrak{M}$ is a Kripke model.

By definition of $\mathfrak{M}$, for every $n, a, b \in \mathbb{N}$,

$$\mathfrak{M}, n \models P_1(a) \wedge P_2(b) \iff b = n \text{ and } \mu \models Q(a, b).$$

Since the accessibility relation of $\mathfrak{F}$ is universal, it follows that, for every $n, a, b \in \mathbb{N}$,

$$\mathfrak{M}, n \models \Diamond(P_1(a) \wedge P_2(b)) \iff \mu \models Q(a, b). \tag{3.1}$$

Straightforward induction, using (3.1) as a basis, shows that, for every $n \in \mathbb{N}$, every assignment $g\colon \mathit{Var} \to \mathbb{N}$, and every $\psi \in \mathit{sub}\, \varphi$,

$$\mathfrak{M}, n \models^g \psi^* \iff \mu \models^g \psi.$$

Since $\mu \not\models \varphi$, it follows that $\mathfrak{M}, n \not\models \varphi^*$ whenever $n \in \mathbb{N}$. Thus, $\mathfrak{M} \not\models \varphi^*$. On the other hand, $\mathfrak{M} \models$ **QS5**.[6] Hence, $\varphi^* \notin$ **QS5**. Since, by assumption, $L \subseteq$ **QS5**, it follows that $\varphi^* \notin L$. ■

Kripke uses the fact proven in Proposition 3.1 to conclude that, if L is a modal predicate logic such that **QCl** $\subseteq L \subseteq$ **QS5**, then the monadic fragment of L is undecidable. Since $\mathit{var}\, \varphi^* = \mathit{var}\, \varphi$, Proposition 3.1, in view of Proposition 2.11, in fact, yields a stronger result:

THEOREM 3.2. (Kripke [22]) *Every sublogic of* **QS5** *that includes* `cf` **QCl** *is* Σ^0_1*-hard in languages with two monadic predicate letters and three individual variables.*

PROOF. Immediate from Propositions 2.11 and 3.1 since the function $f\colon \varphi \mapsto \varphi^*$ is recursive. ■

Remark 3.3. Kripke states the results of Proposition 3.1 and Theorem 3.2 for logics that include **QCl**, rather than `cf` **QCl**. We restated them with `cf` **QCl** to explicitly cover logics of varying domains.

The simplicity of Kripke's proof partially stems from its using only constant domains semantics and not relying on any completeness results for modal logics: the proof draws only on completeness of **QCl** and on soundness of **QS5** with respect to models with a universal accessibility relation.

[6] Kripke [22] does not explicitly justify this claim; we note that it follows from the soundness of **QS5** with respect to Kripke models with a universal accessibility relation; completeness is not required.

We note that Kripke's proof substantially relies on closure of modal predicate logics under Substitution. We also briefly comment on the use of the Löwenheim–Skolem theorem in the proof. The theorem is invoked to assume that the domain of the classical model refuting the formula φ is the set $\mathbb{N}$ of natural numbers. It is easy to see, however, that this assumption is superfluous to Kripke's argument: just go back to the proof of Proposition 3.1, omit the appeal to the Löwenheim–Skolem theorem, and replace, throughout the rest of the proof, $\mathbb{N}$ with the domain of an arbitrary model refuting φ—the proof still stands. Why then invoke the Löwenheim–Skolem theorem? Suppose we wished to prove a statement stronger than Proposition 3.1—namely, that a single augmented frame with a constant domain, and hence a single Kripke frame, suffices to refute substitution instances of all formulas not in **QCl**, so that, given a formula $\varphi \notin \mathbf{QCl}$, we only need to define a suitable Kripke model, over the said augmented frame, refuting φ^*. For that, we could have started with any infinite set X and considered the augmented frame $\langle X, X \times X\rangle \odot X$, but the Löwenheim–Skolem theorem gives us a convenient choice for X, the naturals. This line of thinking has been developed by Hughes and Cresswell, as discussed in the next section, but seems to be implicit in Kripke's paper.

4. Logics that are not Sublogics of QS5

The direction ($\Leftarrow$) of the proof of Proposition 3.1 proceeds by defining a countermodel for the formula φ^* over an augmented frame with a universal accessibility relation and a constant domain; hence, Proposition 3.1 applies only to sublogics of **QS5**. The universality of the accessibility relation and the use of the constant domain semantics are, however, not essential to the proof. As observed by Hughes and Cresswell [19, pp. 271–272], who develop Kripke's ideas, the proof can be easily adapted to apply to every logic that admits a Kripke frame with a world that sees infinitely many worlds.

We say that a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ satisfies the *Kripke–Hughes–Cresswell condition* (for short, KHC) if there exist $W_0 \subseteq W$ and $w_0 \in W$ such that

$$|W_0| = \aleph_0 \quad \text{and} \quad \{w_0\} \times W_0 \subseteq R.$$

Proposition 4.1. *Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame satisfying KHC. Then, for every closed $\mathcal{L}$-formula φ containing no predicate letters other* *than a binary letter* Q*,*

$$\varphi \in \mathbf{QCl} \quad \Longleftrightarrow \quad \varphi^* \in \mathtt{ML}^c\,\mathfrak{F}.$$

Proof. ($\Rightarrow$) As in the proof of Proposition 3.1.

($\Leftarrow$) Suppose that $\varphi \notin \mathbf{QCl}$. Then, by the Löwenheim–Skolem theorem, there exists a classical model μ with the domain $\mathbb{N}$ such that $\mu \not\models \varphi$. Since $\mathfrak{F}$ satisfies KHC, there exist $W_0 \subseteq W$ and $w_0 \in W$ such that $|W_0| = \aleph_0$ and $\{w_0\} \times W_0 \subseteq R$. We assume, without a loss of generality, that $W_0 = \mathbb{N}$. Define an interpretation I on the augmented frame $\mathfrak{F} \odot \mathbb{N}$ so that, for every $n \in W_0$,

$$\begin{array}{rcl} I(n, P_1) & = & \{\langle m\rangle : \mu \models Q(m,n)\}; \\ I(n, P_2) & = & \{\langle n\rangle\}, \end{array}$$

and $I(w, P_1) = I(w, P_2) = \varnothing$ whenever $w \notin W_0$. Set $\mathfrak{M} = \langle \mathfrak{F} \odot \mathbb{N}, I\rangle$. Then $\mathfrak{M}$ is a Kripke model and, for every $a, b \in \mathbb{N}$,

$$\mathfrak{M}, w_0 \models \Diamond(P_1(a) \wedge P_2(b)) \quad \Longleftrightarrow \quad \mu \models Q(a,b).$$

Hence, $\mathfrak{M}, w_0 \not\models \varphi^*$. Since $\mathfrak{F} \odot \mathbb{N} \in \mathtt{aug}^c\,\mathfrak{F}$, surely $\varphi^* \notin \mathtt{ML}^c\,\mathfrak{F}$. ■

Corollary 4.2. *Let* $\mathfrak{F}$ *be a Kripke frame satisfying KHC and let* φ *be a closed* $\mathcal{L}$*-formula* φ *containing no predicate letters other than a binary letter* Q*. Then the following conditions are equivalent:*

(1) $\varphi \in \mathbf{QCl}$;

(2) $\varphi^* \in \mathtt{ML}^v\,\mathfrak{F}$;

(3) $\varphi^* \in \mathtt{ML}^e\,\mathfrak{F}$;

(4) $\varphi^* \in \mathtt{ML}^c\,\mathfrak{F}$.

Proof. (1) $\Rightarrow$ (2): By Proposition 2.5, $\varphi \in \mathtt{ML}^v\,\mathfrak{F}$. Hence, by Substitution, $\varphi^* \in \mathtt{ML}^v\,\mathfrak{F}$. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4): These implications hold by Proposition 2.5. (4) $\Rightarrow$ (1): This implication is given by Proposition 4.1. ■

We say that a modal predicate logic L is *KHC-friendly* if there exists a Kripke frame $\mathfrak{F}$ such that $\mathfrak{F}$ satisfies KHC and $\mathfrak{F} \odot \mathbb{N} \models L$ (equivalently, $\mathfrak{F} \odot \mathbb{N} \models_{var} L$).

Theorem 4.3. *Every KHC-friendly modal predicate logic that includes* $\mathtt{cf}\,\mathbf{QCl}$ *is* Σ^0_1*-hard in languages with two monadic predicate letters and three individual variables.*

Proof. Immediate from Proposition 2.11 and Corollary 4.2. ■

Theorem 4.3 entails undecidability, in languages with only two monadic predicate letters and three variables, of important logics not covered by Theorem 3.2. If L is a propositional modal logic, we denote by $\mathtt{KF}\,L$ the class of Kripke frames validating L.

Corollary 4.4. *Let L_0 be one of the propositional logics* **GL.3**, **Grz.3**, *or* **K4.3.D.X**. *Then every modal predicate logic L satisfying the condition $\mathtt{cf}\,\mathbf{QCl} \subseteq L \subseteq \mathtt{ML}^c\,\mathtt{KF}\,L_0$ is Σ^0_1-hard in languages with two monadic predicate letters and three individual variables.*

Proof. For each logic L_0 from the statement of the corollary, the predicate logic $\mathtt{ML}^c\,\mathtt{KF}\,L_0$ is KHC-friendly. ■

Here are some examples of logics covered by Theorem 4.3, but not by Theorem 3.2 (the list is not meant to be exhaustive):

Corollary 4.5. *Let L be one of the propositional logics* **GL**, **GL.3**, **Grz**, **Grz.3**, **S4.3**, **K4.3**, *and* **K4.3.D.X**. *Then* $\mathbf{Q}L$ *and* $\mathbf{Q}L.\mathbf{bf}$ *are both Σ^0_1-hard in languages with two monadic predicate letters and three individual variables.*

Corollary 4.6. *Let $A \in \{\mathbb{N}, \mathbb{Q}, \mathbb{R}\}$ and let $\mathfrak{A}$ be either $\langle A, < \rangle$ or $\langle A, \leqslant \rangle$. Then $\mathtt{ML}^v\,\mathfrak{A}$, $\mathtt{ML}^e\,\mathfrak{A}$, and $\mathtt{ML}^c\,\mathfrak{A}$ are all Σ^0_1-hard in languages with two monadic predicate letters and three individual variables.*[7]

Thus, many predicate modal logics—including the sublogics of **QS5** considered in Kripke's original proof—are KHC-friendly. Many, but not all: non-examples include $\mathbf{QAlt}_n$, where $n \in \mathbb{N}$ (as we shall see in Section 11, if $L = \mathbf{QAlt}_n$, then the monadic fragment of L is decidable); they also include logics of augmented frames with finite local domains and logics of augmented frames with finitely many possible worlds, both of which we shall show, in Sects. 5 and 6, to be undecidable.[8]

5. Logics of Augmented Frames with Finite Local Domains

Logics of augmented frames with finite local domains are not KHC-friendly since they conservatively extend $\mathbf{QCl}_{\mathit{fin}}$, which is not valid on augmented

[7] We note that $\mathtt{ML}^e\langle \mathbb{Q}, \leqslant \rangle = \mathbf{QS4.3}$ and $\mathtt{ML}^e\langle \mathbb{Q}, < \rangle = \mathbf{KS4.3.D.X}$ [7].

[8] We briefly note that, since the proof of Proposition 4.1 involves evaluating formulas only at a single world, it can be adapted, in a straightforward manner, to quasi-normal and non-normal modal predicate logics; such logics are not considered in this paper, and we leave details to the interested reader.

frames with infinite domains. In this section, we consider how Kripke's proof can be adapted to such logics and state a property guaranteeing that a logic is amenable to the adaptation.

First, we define the logics considered in this section. We say that an augmented frame (with varying, expanding, or constant domains) $\mathfrak{F}_D = \langle W, R, D\rangle$ is *locally finite* if $|D_w| \in \mathbb{N}^+$ whenever $w \in W$. If $\mathscr{C}$ is a class of Kripke frames, then we define $\mathtt{aug}^v_{fin}\,\mathscr{C}$, $\mathtt{aug}^e_{fin}\,\mathscr{C}$, and $\mathtt{aug}^c_{fin}\,\mathscr{C}$ to be the subclasses of, respectively, $\mathtt{aug}^v\,\mathscr{C}$, $\mathtt{aug}^e\,\mathscr{C}$, and $\mathtt{aug}^c\,\mathscr{C}$ containing all locally finite augmented frames from the respective class. We, lastly, define the following sets of $\mathcal{ML}$-formulas:

$$\begin{array}{rcl} \mathtt{ML}^c_{fin}\,\mathscr{C} & = & \mathtt{ML}\,\mathtt{aug}^c_{fin}\,\mathscr{C}; \\ \mathtt{ML}^e_{fin}\,\mathscr{C} & = & \mathtt{ML}\,\mathtt{aug}^e_{fin}\,\mathscr{C}; \\ \mathtt{ML}^v_{fin}\,\mathscr{C} & = & \mathtt{vML}\,\mathtt{aug}^v_{fin}\,\mathscr{C}. \end{array}$$

These sets of formulas are predicate modal logics; in particular, they are closed under Substitution; furthermore, $\mathtt{ML}^c_{fin}\,\mathscr{C}$ and $\mathtt{ML}^e_{fin}\,\mathscr{C}$ are normal.

Proposition 5.1. *Let $\mathscr{C}$ be a class of Kripke frames. Then $\mathtt{ML}^c_{fin}\,\mathscr{C}$ and $\mathtt{ML}^e_{fin}\,\mathscr{C}$ are conservative extensions of $\mathbf{QCl}_{fin}$ and, moreover,*

$$\mathtt{cf}\,\mathbf{QCl}_{fin} \subseteq \mathtt{cf}\,\mathtt{ML}^v_{fin}\,\mathscr{C} \subseteq \mathtt{cf}\,\mathtt{ML}^e_{fin}\,\mathscr{C} \subseteq \mathtt{cf}\,\mathtt{ML}^c_{fin}\,\mathscr{C}.$$

Proof. Similar to the proof of Proposition 2.5. ■

We seek a weakening of KHC that would play the same role for locally finite augmented frames. It turns out that the condition we seek is very similar to the condition formulated by Skvortsov in the context of superintuitionistic predicate logics of frames with finitely many possible worlds[9] [48, Corollary 3], cf. [33, Lemma 3.3]: We say that a class $\mathscr{C}$ of Kripke frames satisfies the *weak Kripke–Hughes–Cresswell condition* (for short, wKHC) if, for every $n \in \mathbb{N}^+$, there exists a Kripke frame $\langle W, R\rangle \in \mathscr{C}$ with $W_0 \subseteq W$ and $w_0 \in W$ such that

$$|W_0| = n \quad \text{and} \quad \{w_0\} \times W_0 \subseteq R.$$

Observe that, if a class of Kripke frames contains a frame satisfying KHC, then the class satisfies wKHC, but the converse is not always true.

Proposition 5.2. *Let $\mathscr{C}$ be a class of Kripke frames satisfying wKHC. Then, for every closed $\mathcal{L}$-formula φ containing no predicate letters other than a binary letter Q,*

$$\varphi \in \mathbf{QCl}_{fin} \iff \varphi^* \in \mathtt{ML}^c_{fin}\,\mathscr{C}.$$

[9] Note, however, that Skvortsov's set-up differs from the one studied in this section.

PROOF. ($\Rightarrow$) Assume that $\varphi \in \mathbf{QCl}_{\mathit{fin}}$. By Proposition 5.1, $\varphi \in \mathtt{cf}\,\mathtt{ML}^c_{\mathit{fin}}\,\mathscr{C}$. Since $\mathtt{ML}^c_{\mathit{fin}}\,\mathscr{C}$ is closed under Substitution, surely $\varphi^* \in \mathtt{cf}\,\mathtt{ML}^c_{\mathit{fin}}\,\mathscr{C}$.

($\Leftarrow$) Assume that $\varphi \notin \mathbf{QCl}_{\mathit{fin}}$. Then there exists $n \in \mathbb{N}$ and a classical model μ with the domain $\{1, \ldots, n\}$ such that $\mu \not\models \varphi$. Since $\mathscr{C}$ satisfies wKHC, it contains a Kripke frame $\mathfrak{F} = \langle W, R\rangle$ with $W_0 \subseteq W$ and $w_0 \in W$ such that $|W_0| = n$ and $\{w_0\} \times W_0 \subseteq R$. We assume, without a loss of generality, that $W_0 = \{1, \ldots, n\}$. Define an interpretation I on $\mathfrak{F}\odot\{1, \ldots, n\}$ so that, for every $k \in \{1, \ldots, n\}$,

$$\begin{array}{lcl} I(k, P_1) & = & \{\langle m\rangle : \mu \models Q(m, k)\}; \\ I(k, P_2) & = & \{\langle k\rangle\}, \end{array}$$

and $I(w, P_1) = I(w, P_2) = \varnothing$ whenever $w \notin W_0$. Set $\mathfrak{M} = \langle \mathfrak{F}\odot\{1, \ldots, n\}, I\rangle$. Then $\mathfrak{M}$ is a Kripke model and, for every $m, k \in \{1, \ldots, n\}$,

$$\mathfrak{M}, w_0 \models \Diamond(P_1(m) \wedge P_2(k)) \iff \mu \models Q(m, k).$$

Hence, $\mathfrak{M}, w_0 \not\models \varphi^*$, and so $\varphi^* \notin \mathtt{ML}^c_{\mathit{fin}}\,\mathscr{C}$. ■

COROLLARY 5.3. *Let $\mathscr{C}$ be a class of Kripke frames satisfying wKHC and let φ be a closed $\mathcal{L}$-formula φ containing no predicate letters other than a binary letter Q. Then the following conditions are equivalent:*

(1) $\varphi \in \mathbf{QCl}_{\mathit{fin}}$;

(2) $\varphi^* \in \mathtt{ML}^v_{\mathit{fin}}\,\mathscr{C}$;

(3) $\varphi^* \in \mathtt{ML}^e_{\mathit{fin}}\,\mathscr{C}$;

(4) $\varphi^* \in \mathtt{ML}^c_{\mathit{fin}}\,\mathscr{C}$.

PROOF. Analogous to the proof of Corollary 4.2. ■

THEOREM 5.4. *Let $\mathscr{C}$ be a class of Kripke frames satisfying wKHC. Then the logics $\mathtt{ML}^c_{\mathit{fin}}\,\mathscr{C}$, $\mathtt{ML}^e_{\mathit{fin}}\,\mathscr{C}$, and $\mathtt{ML}^v_{\mathit{fin}}\,\mathscr{C}$ are all Π^0_1-hard in languages with two monadic predicate letters and three individual variables.*

PROOF. Immediate from Proposition 2.12 and Corollary 5.3. ■

Since Π^0_1-hardness means lack of recursive enumerability, logics from Theorem 5.4 are not recursively enumerable.

COROLLARY 5.5. *Let L be a propositional modal logic such that $\mathtt{KF}\,L$ satisfies wKHC. Then the logics $\mathtt{ML}^c_{\mathit{fin}}\,\mathtt{KF}\,L$, $\mathtt{ML}^e_{\mathit{fin}}\,\mathtt{KF}\,L$, and $\mathtt{ML}^v_{\mathit{fin}}\,\mathtt{KF}\,L$ are all Π^0_1-hard in languages with two monadic predicate letters and three individual variables.*

PROOF. Immediate from Theorem 5.4. ■

COROLLARY 5.6. *Let L_0 be one of the propositional logics* **S5**, **GL.3**, **Grz.3**, *or* **K4.3.D.X**. *Then every predicate modal logic L satisfying the condition $\mathtt{cf}\,\mathbf{QCl}_{\mathit{fin}} \subseteq L \subseteq \mathtt{ML}^c_{\mathit{fin}}\,\mathtt{KF}\,L_0$ is Π^0_1-hard in languages with two monadic predicate letters and three individual variables.*

PROOF. Observe that, for each propositional logic L_0 from the statement of the corollary, the class $\mathtt{KF}\,L_0$ contains a frame satisfying KHC; hence, $\mathtt{KF}\,L_0$ satisfies wKHC. ∎

6. Logics of Augmented Frames with Finitely Many Worlds

Neither Kripke [22] nor Hughes and Cresswell [19] consider logics determined by frames with only finitely many possible worlds. In propositional modal logic, the finite frame property (ffp) typically leads to decidability (recall that a logic has the ffp if it is determined by a class of Kripke frames each with finitely many possible worlds). More precisely, every recursively enumerable propositional logic with the ffp is decidable [17].[10] In predicate modal logic, as we shall see, the situation is very different: applying the Kripke trick to logics determined by augmented frames with finitely many worlds, we can show that such logics are not recursively enumerable.

First, we define the logics considered in this section. If $\mathscr{C}$ is a class of Kripke frames, we define $\mathtt{wfin}\,\mathscr{C} = \{\langle W, R\rangle \in \mathscr{C} : |W| \in \mathbb{N}^+\}$ and then define

$$\begin{array}{rcl} \mathtt{ML}^c_{\mathit{wfin}}\,\mathscr{C} & = & \mathtt{ML}\,\mathtt{aug}^c\,\mathtt{wfin}\,\mathscr{C}; \\ \mathtt{ML}^e_{\mathit{wfin}}\,\mathscr{C} & = & \mathtt{ML}\,\mathtt{aug}^e\,\mathtt{wfin}\,\mathscr{C}; \\ \mathtt{ML}^v_{\mathit{wfin}}\,\mathscr{C} & = & \mathtt{vML}\,\mathtt{aug}^v\,\mathtt{wfin}\,\mathscr{C}. \end{array}$$

We now introduce the condition playing the role of KHC for these logics. This is exactly the condition set out by Skvortsov [48, Corollary 3] for superintuitionistic logics. We, therefore, say that a class $\mathscr{C}$ of Kripke frames is a *Skvortsov class* if its subclass $\mathtt{wfin}\,\mathscr{C}$ satisfies wKHC.

PROPOSITION 6.1. *Let $\mathscr{C}$ be a Skvorstov class of Kripke frames. Then the logics $\mathtt{ML}^c_{\mathit{wfin}}\,\mathscr{C}$ and $\mathtt{ML}^e_{\mathit{wfin}}\,\mathscr{C}$ are conservative extensions of* **QCl** *and, moreover,*

$$\mathtt{cf}\,\mathbf{QCl} \subseteq \mathtt{cf}\,\mathtt{ML}^v_{\mathit{wfin}}\,\mathscr{C} \subseteq \mathtt{cf}\,\mathtt{ML}^e_{\mathit{wfin}}\,\mathscr{C} \subseteq \mathtt{cf}\,\mathtt{ML}^c_{\mathit{wfin}}\,\mathscr{C}.$$

[10] In general neither the ffp [55], nor even the linear size frame property [42,49], guarantees decidability of propositional modal or superintuitionistic logics; examples of undecidable propositional logics with the ffp are, however, rather contrived.

Proof. Similar to the proof of Proposition 2.5. ∎

The key observation in this section is that, if $\mathscr{C}$ is a Skvortsov class of Kripke frames, then $\mathbf{QCl}_{\mathit{fin}}$ can be recursively embedded into $\mathtt{ML}^c_{\mathit{wfin}}\mathscr{C}$ by simulating finite local domains on the possibly infinite local domains of augmented frames over Kripke frames from $\mathscr{C}$.

To minimize the number of monadic letters used in the simulation presented in this section, we will be using the monadic predicate letter P_2 for two purposes: to simulate, along with P_1, the binary letter Q, and to induce equivalence relations on local domains. First, we define

$$\begin{array}{rcl} C & = & \forall x \forall y \left(\Box(P_2(x) \leftrightarrow P_2(y)) \to \forall z\,(Q(x,z) \to Q(y,z))\right. \\ & & \left.\wedge\ \forall z\,(Q(z,x) \to Q(z,y))\right). \end{array}$$

Recall that the translation $\cdot^*$ was defined in Section 3.

Proposition 6.2. *Let $\mathscr{C}$ be a Skvortsov class of Kripke frames. Then, for every closed $\mathcal{L}$-formula φ containing no predicate letters other than a binary letter Q,*

$$\varphi \in \mathbf{QCl}_{\mathit{fin}} \iff C^* \to \varphi^* \in \mathtt{ML}^c_{\mathit{wfin}}\mathscr{C}.$$

Proof. ($\Rightarrow$) Assume that $C^* \to \varphi^* \notin \mathtt{ML}^c_{\mathit{wfin}}\mathscr{C}$. Then there exist a Kripke frame $\mathfrak{F} = \langle W, R\rangle \in \mathscr{C}$ with a finite W, a model $\mathfrak{M} = \langle \mathfrak{F} \odot \mathcal{D}, I\rangle$, and $w_0 \in W$ such that

$$\mathfrak{M}, w_0 \models C^* \quad \text{and} \quad \mathfrak{M}, w_0 \not\models \varphi^*.$$

We obtain a classical model with a finite domain refuting φ. We define a binary relation $\approx$ on the set $\mathcal{D}$ by

$$a \approx b \iff \mathfrak{M}, w_0 \models \Box(P_2(a) \leftrightarrow P_2(b)).$$

It should be clear that $\approx$ is an equivalence on $\mathcal{D}$. Set $[a] = \{b \in \mathcal{D} : a \approx b\}$ and $\bar{\mathcal{D}} = \{[a] : a \in \mathcal{D}\} = \mathcal{D}/\!\approx$. We show that the set $\bar{\mathcal{D}}$ is finite. Put, for every $a \in \mathcal{D}$,

$$\mathcal{V}(a) = \{w \in R(w_0) : \mathfrak{M}, w \models P_2(a)\},$$

and set

$$\bar{\mathcal{V}} = \{\mathcal{V}(a) : a \in \mathcal{D}\}.$$

The definition of $\approx$ implies that $\mathcal{V}(a) = \mathcal{V}(b)$ if, and only if, $a \approx b$, i.e., there exists a bijection between $\bar{\mathcal{V}}$ and $\bar{\mathcal{D}}$, and so $|\bar{\mathcal{V}}| = |\bar{\mathcal{D}}|$. Surely, $|\bar{\mathcal{V}}| \leqslant 2^{|R(w_0)|} \leqslant 2^{|W|}$. Since, by assumption, W is finite, it follows that $\bar{\mathcal{D}}$ is finite.

Since $\mathfrak{M}, w_0 \models C^*$, the relation $\approx$ is a congruence with respect to the binary relation $\{\langle a, c\rangle : \mathfrak{M}, w_0 \models \Diamond(P_1(a) \wedge P_2(c))\}$ on $\mathcal{D}$. Hence, the following definition of a classical model $\mu = \langle \bar{\mathcal{D}}, \mathcal{I}\rangle$ is sound: for every $a, c \in D(w_0)$,

$$\langle [a], [c]\rangle \in \mathcal{I}(Q) \quad \leftrightharpoons \quad \text{there exists}\, w \in R(w_0) \text{ with } \mathfrak{M}, w \models P_1(a) \wedge P_2(c).$$

We next show that $\mu \not\models \varphi$. We say that assignments $\bar{g}$ in μ and g in $\mathfrak{M}$ *agree* if $\bar{g}(x) = [g(x)]$, for every x. Straightforward induction, with the definition of $\mathcal{I}(Q)$ as a basis, shows that $\mu \models^{\bar{g}} \psi$ if, and only if, $\mathfrak{M}, w_0 \models^g \psi$, for every $\psi \in sub\,\varphi$ and every pair of agreeing assignments $\bar{g}$ and g. Hence, $\mu \not\models \varphi$, and so $\varphi \notin \mathbf{QCl}_{\mathit{fin}}$.

($\Leftarrow$) Assume that $\varphi \notin \mathbf{QCl}_{\mathit{fin}}$. Then, there exist $n \in \mathbb{N}$ and a classical model μ with the domain $\{1, \dots, n\}$ such that $\mu \not\models \varphi$. Since $\mathscr{C}$ is a Skvortsov class, there exists a (finite) Kripke frame $\mathfrak{F} = \langle W, R\rangle \in \mathscr{C}$ with $W_0 \subseteq W$ and $w_0 \in W$ such that $|W_0| = n$ and $\{w_0\} \times W_0 \subseteq R$. We assume, without a loss of generality, that $W_0 = \{1, \dots, n\}$. Define an interpretation I on the augmented frame $\mathfrak{F} \odot \{1, \dots, n\}$ so that, for every $k \in \{1, \dots, n\}$,

$$\begin{array}{rcl} I(k, P_1) & = & \{\langle m\rangle : \mu \models Q(m, k)\}; \\ I(k, P_2) & = & \{\langle k\rangle\}, \end{array}$$

and $I(w, P_1) = I(w, P_2) = \varnothing$ whenever $w \notin W_0$. Set $\mathfrak{M} = \langle \mathfrak{F} \odot \{1, \dots, n\}, I\rangle$. Then, for every $a, b \in D(w_0)$,

$$\mathfrak{M}, w_0 \models \Box(P_2(a) \leftrightarrow P_2(b)) \quad \Longleftrightarrow \quad a = b.$$

Hence, $\mathfrak{M}, w_0 \models C^*$. On the other hand, for every $m, k \in \{1, \dots, n\}$,

$$\mathfrak{M}, w_0 \models \Diamond(P_1(m) \wedge P_2(k)) \quad \Longleftrightarrow \quad \mu \models Q(m, k),$$

and so, by straightforward induction, for every subformula $\psi \in sub\,\varphi$ and every assignment g,

$$\mathfrak{M}, w_0 \models^g \psi^* \quad \Longleftrightarrow \quad \mu \models^g \psi.$$

Hence, $\mathfrak{M}, w_0 \not\models \varphi^*$, and so $C^* \to \varphi^* \notin \mathtt{ML}^c_{\mathit{wfin}}\mathscr{C}$. ■

COROLLARY 6.3. *Let $\mathscr{C}$ be a Skvortsov class of Kripke frames and let φ be a closed $\mathcal{L}$-formula containing no predicate letters other than a binary letter Q. Then the following conditions are equivalent:*

(1) $\varphi \in \mathbf{QCl}_{\mathit{fin}}$;

(2) $C^* \to \varphi^* \in \mathtt{ML}^v_{\mathit{wfin}}\mathscr{C}$;

(3) $C^* \to \varphi^* \in \mathtt{ML}^e_{\mathit{wfin}}\mathscr{C}$;

(4) $C^* \to \varphi^* \in \mathtt{ML}^c_{\mathit{wfin}}\mathscr{C}$.

Proof. (1) ⇒ (2): Observe that in the proof of the implication (⇒) from Proposition 6.2 we did not rely on the constancy of domains. Hence, the proof can be repeated for $\mathtt{ML}^{v}_{\mathit{wfin}}\mathscr{C}$.

(2) ⇒ (3) and (3) ⇒ (4): These implications hold by Proposition 6.1.

(4) ⇒ (1): This implication is given by Proposition 6.2. ■

Theorem 6.4. *Let $\mathscr{C}$ be a Skvortsov class of Kripke frames. Then, the logics $\mathtt{ML}^{e}_{\mathit{wfin}}\mathscr{C}$, $\mathtt{ML}^{c}_{\mathit{wfin}}\mathscr{C}$, and $\mathtt{ML}^{v}_{\mathit{wfin}}\mathscr{C}$ are Π^0_1-hard in languages with two monadic predicate letters and three individual variables.*

Proof. Immediate from Proposition 2.12 and Corollary 6.3. ■

Corollary 6.5. *Let L be a propositional modal logic such that $\mathtt{KF}\,L$ is a Skvortsov class of Kripke frames. Then the logics $\mathtt{ML}^{c}_{\mathit{wfin}}\mathtt{KF}\,L$, $\mathtt{ML}^{e}_{\mathit{wfin}}\mathtt{KF}\,L$, and $\mathtt{ML}^{v}_{\mathit{wfin}}\mathtt{KF}\,L$ are all Π^0_1-hard in languages with two monadic predicate letters and three individual variables.*

Proof. Immediate from Theorem 6.4. ■

Corollary 6.6. *Let $L_0 \in \{\mathbf{S5}, \mathbf{GL.3}, \mathbf{Grz.3}, \mathbf{K4.3.D.X}\}$. Then every sublogic of $\mathtt{ML}^{c}_{\mathit{wfin}}\mathtt{KF}\,L_0$ that includes $\mathbf{QCl}_{\mathit{fin}}$ is Π^0_1-hard in languages with two monadic predicate letters and three individual variables.*

Proof. Observe that, for each propositional logic L_0 from the statement of the corollary, the class $\mathtt{KF}\,L_0$ is Skvortsov. ■

7. Kripke Trick with a Single Predicate Letter

The Kripke trick itself, as well as all the modifications discussed thus far, use two monadic predicate letters. Kripke's primary concern was proving undecidability of monadic fragments of modal predicate logics, regardless of the number of predicate letters involved. More recently, finer analysis of computational properties of modal predicate logics has become of interest, partly as a result of the realisation of the immense expressive power of modal predicate languages and partly due to the analysis of the expressive power of propositional logics through embeddings into very restricted fragments of modal predicate logics [10,58]. While techniques for simulating atomic formulas involving an arbitrary number of monadic letters by atomic formulas with a single fixed monadic letter are outside of the scope of this paper, we show, in this section, how the Kripke trick itself can be implemented with a single monadic predicate letter, rather than two.

It should be clear that, if a modal predicate logic L admits an augmented frame with infinite domains over an irreflexive tree of height 3 that has an infinite branching factor, then, in L, the formula $Q(x,y)$ can be simulated with $\Diamond(P(x) \wedge \Diamond P(y))$—the proof is similar to the proof of Proposition 4.1. This does not, however, work for logics proving $\Box p \to p$ and, hence, requiring reflexive frames: if a model $\mathfrak{M}$ is based on a reflexive frame, then $\mathfrak{M}, w \models \Diamond(P(a) \wedge \Diamond P(b))$ implies $\mathfrak{M}, w \models \Diamond(P(a) \wedge \Diamond P(a))$, for every world w of $\mathfrak{M}$; hence, the classical model of the $\mathcal{L}$-formula into which the substitution of $\Diamond(P(x) \wedge \Diamond P(y))$ for $Q(x,y)$ is made has to validate the formula $\forall x \forall y\, (Q(x,y) \to Q(x,x))$. Thus, over reflexive frames, $\Diamond(P(x) \wedge \Diamond P(y))$ does not have the same meaning as $Q(x,y)$ if Q is allowed to stand for an arbitrary binary predicate.

We can, nevertheless, recursively reduce to modal logics with a single monadic predicate letter a classical first-order theory of a binary predicate with special properties. A suitable candidate for such a theory is the first-order theory **SIB** of relational structures with a symmetric and irreflexive binary relation, which is known to be undecidable with only three individual variables [21,27,31]:

Proposition 7.1. *The theory* **SIB** *is undecidable with three individual variables.*

We define

$$sib \quad = \quad \forall x \forall y\, (Q(x,y) \to Q(y,x)) \wedge \forall x\, \neg Q(x,x).$$

Let ψ be an $\mathcal{L}$-formula, let P be a monadic predicate letter, and let ψ° be an $\mathcal{ML}$-formula obtained from ψ by substituting $\neg\Diamond(P(x) \wedge P(y))$ for $Q(x,y)$. Then the following holds [30] (cf. [9], Chapter 11, Theorem III):

Proposition 7.2. *Let $\mathfrak{F}$ be a Kripke frame satisfying KHC. Then, for every closed $\mathcal{L}$-formula φ containing no predicate letters other than a binary letter Q,*

$$\varphi \in \mathbf{SIB} \quad \Longleftrightarrow \quad sib^\circ \to \varphi^\circ \in \mathtt{ML}^c\,\mathfrak{F}.$$

Proof. ($\Rightarrow$) Assume that $\varphi \in \mathbf{SIB}$. Then $sib \to \varphi \in \mathbf{QCl}$, and so $sib \to \varphi \in \mathtt{ML}^c\,\mathfrak{F}$. Since $\mathtt{ML}^c\,\mathfrak{F}$ is closed under Substitution, surely it follows that $sib^\circ \to \varphi^\circ \in \mathtt{ML}^c\,\mathfrak{F}$.

($\Leftarrow$) Assume that $\varphi \notin \mathbf{SIB}$. Then $sib \to \varphi \notin \mathbf{QCl}$, and so, by the Löwenheim–Skolem theorem, there exists a classical model $\mu = \langle \mathbb{N}, \mathcal{I} \rangle$ such that $\mu \not\models sib \to \varphi$. Since $\mu \models sib$, the relation $\mathcal{I}(Q)$ is symmetric and irreflexive. Since $\mathfrak{F}$ satisfies KHC, there exist $W_0 \subseteq W$ and $w_0 \in W$ such that $|W_0| = \aleph_0$ and $\{w_0\} \times W_0 \subseteq R$. We assume, without a loss of generality, that $W_0 = \mathbb{N} \times \mathbb{N}$. Define an interpretation I on the augmented frame $\mathfrak{F} \odot \mathbb{N}$

so that, for every $n, m \in \mathbb{N}$,

$$I(\langle n, m\rangle, P) \quad = \quad \begin{cases} \{\langle n\rangle, \langle m\rangle\} & \text{if} \quad \mu \not\models Q(n,m); \\ \varnothing & \text{otherwise,} \end{cases}$$

and $I(w, P) = \varnothing$ whenever $w \in W \backslash W_0$. Set $\mathfrak{M} = \langle \mathfrak{F} \odot \mathbb{N}, I\rangle$. Since $\mathcal{I}(Q)$ is symmetric and irreflexive, for every $n, m \in \mathbb{N}$,

$$\mathfrak{M}, w_0 \models \Diamond(P(n) \wedge P(m)) \quad \iff \quad \mu \not\models Q(n,m). \tag{7.1}$$

Straightforward induction, using (7.1) as a basis, shows that, for every $\psi \in \mathit{sub}\,\varphi$ and every assignment g,

$$\mathfrak{M}, w_0 \models^g \psi^\circ \quad \iff \quad \mu \models^g \psi.$$

Since $\mu \not\models \varphi$, it follows that $\mathfrak{M}, w_0 \not\models \varphi^\circ$. Since $\mathcal{I}(Q)$ is symmetric and irreflexive, the binary relation

$$\{\langle a, b\rangle \in \mathbb{N} \times \mathbb{N} : \mathfrak{M}, w_0 \models \neg\Diamond(P(a) \wedge P(b))\}$$

is also symmetric and irreflexive; hence, $\mathfrak{M}, w_0 \models \mathit{sib}^\circ$. Therefore, $\mathfrak{M}, w_0 \not\models \mathit{sib}^\circ \to \varphi^\circ$, and so $\mathit{sib}^\circ \to \varphi^\circ \notin \mathtt{ML}^c\,\mathfrak{F}$. ■

Corollary 7.3. *Let $\mathfrak{F}$ be a Kripke frame satisfying KHC and let φ be a closed $\mathcal{L}$-formula φ containing no predicate letters other than a binary letter Q. Then the following conditions are equivalent:*

(1) $\varphi \in \mathbf{QCl}$;

(2) $\mathit{sib}^\circ \to \varphi^\circ \in \mathtt{ML}^v\,\mathfrak{F}$;

(3) $\mathit{sib}^\circ \to \varphi^\circ \in \mathtt{ML}^e\,\mathfrak{F}$;

(4) $\mathit{sib}^\circ \to \varphi^\circ \in \mathtt{ML}^c\,\mathfrak{F}$.

Proof. (1) ⇒ (2): Observe that in the proof of the implication (⇒) from Proposition 7.2 we did not rely on the constancy of domains. Hence, the proof can be repeated for $\mathtt{ML}^v\,\mathscr{C}$.

(2) ⇒ (3) and (3) ⇒ (4): These implications hold by Proposition 2.5.

(4) ⇒ (1): This implication holds by Proposition 7.2. ■

Theorem 7.4. *Every KHC-friendly modal predicate logic is Σ^0_1-hard in languages with one monadic predicate letter and three individual variables.*

Proof. Immediate from Proposition 7.1 and Corollary 7.3. ■

Corollary 7.5. *Let $L_0 \in \{\mathbf{QS5}, \mathbf{QGL.3.bf}, \mathbf{QGrz.3.bf}, \mathbf{QK4.3.D.X.bf}\}$ and L be a modal predicate logic with $\mathtt{cf}\,\mathbf{QCl} \subseteq L \subseteq L_0$. Then, L is Σ^0_1-hard in languages with only one monadic predicate letter and three individual variables.*

8. Simulation of Predicate Letters of Higher Arity

We briefly note that Kripke's idea of simulating classical atomic formulas with a binary predicate letter by modal formulas with two monadic letters is easily generalisable: as observed by Alexander Chagrov, we can simulate $S(x_1, \dots, x_n)$, where S is an n-ary predicate letter, with the formula $\Diamond(P_1(x_1) \wedge \dots \wedge P_n(x_n))$. This enables us to easily and effortlessly simulate all classical atomic formulas with monadic modal formulas. Then, applying techniques for simulating an arbitrary number of monadic atomic formulas by formulas with a single fixed monadic letter [28,35,36,39], we can simulate every classical atomic formula by modal formulas with a single monadic predicate letter; note that simulating atomic formulas with n-ary predicate letters, for $n \geqslant 3$, within **QCl** itself is rather cumbersome [3, Chapter 21]. We come back to this theme in Section 9 and, in the context of superintuitionistic logics, in Section 10.

9. Kripke Trick for Modal Formulas

So far, we have seen how variations of the Kripke trick are applied to classical formulas; this is useful for the transfer of algorithmic lower bounds from non-modal to modal logics. We might, however, want to reduce sets of modal formulas containing binary predicate letters to sets of monadic modal formulas; for that, it is useful to be able to apply the Kripke trick to formulas with modalities.

The substitution of $\Diamond(P_1(x) \wedge P_2(y))$ for $Q(x, y)$ used so far, if applied to a modal formula, is not guaranteed to produce a **QK**-equivalent formula: since **QK** admits frames containing worlds that do not see any worlds, it follows that

$$\forall x \forall y\, Q(x, y) \to \Diamond\top \;\notin\; \mathbf{QK}.$$

On the other hand, surely

$$\forall x \forall y\, \Diamond(P_1(x) \wedge P_2(y)) \to \Diamond\top \;\in\; \mathbf{QK}.$$

This problem does not arise when applying the Kripke trick to a classical formula φ: then the resultant $\mathcal{ML}$-formula φ^* contains modalities only in formulas of the form $\Diamond(P_1(x) \wedge P_2(y))$ substituted for atomic formulas of the form $Q(x, y)$; hence, in the proof of the correctness of the substitution, the truth status of subformulas of φ^* other than $\Diamond(P_1(x) \wedge P_2(y))$ at the world refuting φ^* is independent of the truth status of any formulas at any other worlds. To avoid undesirable effects in the modal setting, we can use *relativization.*

We first explain how relativization works when applied to arbitrary modal formulas. Along the way, we show that, for some subframe logics, combining relativization with the Kripke trick suffices to prove their embeddability into their own monadic fragments. We recall that a modal predicate logic L is *subframe* if, for every augmented frame with expanding domains $\mathfrak{F}_D$, the following holds: if $\mathfrak{F}_D \models L$ and $\mathfrak{F}'_D$ is a subframe of $\mathfrak{F}_D$, then $\mathfrak{F}'_D \models L$.

Let φ be an $\mathcal{ML}$-formula and p a proposition letter not in φ. We recursively define the p-relativization φ^p of φ:

$$
\begin{array}{rcll}
\varphi^p & = & \varphi & \text{if}\, \varphi \text{ is atomic;} \\
(\varphi_1 \circ \varphi_2)^p & = & \varphi_1^p \circ \varphi_2^p & \text{if}\, \circ \in \{\wedge, \vee, \rightarrow\}; \\
(\Box \varphi_1)^p & = & \Box (p \rightarrow \varphi_1^p); & \\
(\mathcal{Q}\, x\, \varphi_1)^p & = & \mathcal{Q}\, x\, \varphi_1^p & \text{if}\, \mathcal{Q} \in \{\forall, \exists\}.
\end{array}
$$

LEMMA 9.1. *For every $\mathcal{ML}$-formula φ and every Kripke complete subframe normal modal predicate logic L,*

$$\varphi \in L \quad \Longleftrightarrow \quad p \rightarrow \varphi^p \in L.$$

PROOF. Suppose that $p \rightarrow \varphi^p \notin L$. Since L is Kripke complete, there exists a model $\mathfrak{M} = \langle W, R, D, I \rangle$ and $w_0 \in W$ such that $\langle W, R, D \rangle \models L$ and $\mathfrak{M}, w_0 \not\models p \rightarrow \varphi^p$. Set

$$W' \quad = \quad \{w \in W : \mathfrak{M}, w \models p\}.$$

Since $\mathfrak{M}, w_0 \models p$, surely $W' \neq \varnothing$. We define

$$R' = R \restriction W', \quad D' = D \restriction W', \quad I' = I \restriction W', \quad \text{and} \quad \mathfrak{M}' = \langle W', R', D', I' \rangle.$$

Then $\mathfrak{M}'$ is a Kripke model such that $\mathfrak{M}', w_0 \not\models \varphi$; hence, $\langle W', R', D' \rangle \not\models \varphi$. On the other hand, since L is subframe, $\langle W', R', D' \rangle \models L$. Hence, $\varphi \notin L$.

Conversely, suppose that $p \rightarrow \varphi^p \in L$. Let φ' be obtained from $p \rightarrow \varphi^p$ by substituting $\top$ for p. By Substitution, $\varphi' \in L$. Since $(\varphi \leftrightarrow \varphi') \in \mathbf{QK}$, it follows that $\varphi \in L$. ∎

For some subframe logics, namely for **QK**, **QKT**, **QKB**, and **QKBT**, Lemma 9.1 enables us to reproduce the Kripke trick for an arbitrary $\mathcal{ML}$-formula:

PROPOSITION 9.2. *Let $L \in \{\mathbf{QK}, \mathbf{QKT}, \mathbf{QKB}, \mathbf{QKBT}\}$. Then, for every $\mathcal{ML}$-formula φ,*

$$\varphi \in L \quad \Longleftrightarrow \quad p \rightarrow (\varphi^p)^* \in L.$$

PROOF. ($\Rightarrow$) Suppose that $\varphi \in L$. Then, by Lemma 9.1 and Substitution, $(p \to \varphi^p)^* \in L$, i.e., $p \to (\varphi^p)^* \in L$.

($\Leftarrow$) We first give a proof for **QK** and then point out the adjustments necessary for the other logics. Suppose that $\varphi \notin$ **QK**. Then there exists a Kripke model $\mathfrak{M} = \langle W, R, D, I \rangle$ and $w_0 \in W$ such that $\mathfrak{M}, w_0 \not\models \varphi$. Let $\mathfrak{M}^p = \langle W, R, D, I^p \rangle$ be an expansion of $\mathfrak{M}$ such that $\mathfrak{M}^p \models p$ (recall that p does not occur in φ). Then $\mathfrak{M}^p, w_0 \models p$ and, as can be easily checked, $\mathfrak{M}^p, w_0 \not\models \varphi^p$. Extend the model $\mathfrak{M}^p$ as follows: let

- $W' = W \cup \{\langle w, a \rangle : w \in W, a \in D_w\}$;
- $R' = R \cup \{\langle w, \langle w, a \rangle \rangle : w \in W, a \in D_w\}$;
- $D' \colon W' \to \mathcal{P}(D^+)$ be the map defined by

$$D'(u) \quad = \quad \begin{cases} D(u) & \text{if } u \in W; \\ D(w) & \text{if } u = \langle w, a \rangle, \text{ for some } w \in W \text{ and } a \in D_w; \end{cases}$$

- I' be an interpretation on the augmented frame $\langle W', R', D' \rangle$ such that
  - $I'(w, S) = I(w, S)$ whenever $w \in W$ and S is a letter from $p \to \varphi^p$;
  - $I'(w, P_1) = I'(w, P_2) = \varnothing$ whenever $w \in W$;
  - for every $w \in W$ and every $b \in D_w$,

$$\begin{array}{lcl} I'(\langle w, b \rangle, P_1) & = & \{\langle a \rangle : \mathfrak{M}^p, w \models Q(a, b)\}; \\ I'(\langle w, b \rangle, P_2) & = & \{\langle b \rangle\}; \\ I'(\langle w, b \rangle, S) & = & \varnothing \quad \text{if } S \notin \{P_1, P_2\}; \end{array}$$

- $\mathfrak{M}' = \langle W', R', D', I' \rangle$.

Thus, in particular, $\mathfrak{M}', \langle w, b \rangle \not\models p$ whenever $w \in W$ and $b \in D_w$.

Then, for every $w \in W$ and every $a, b \in D_w$,

$$\mathfrak{M}', w \models \Diamond(P_1(a) \wedge P_2(b)) \iff \mathfrak{M}^p, w \models Q(a, b), \tag{9.1}$$

and, for every $w \in W$, every atomic subformula θ of φ^p that does not have the form $Q(x, y)$, and every assignment g,

$$\mathfrak{M}', w \models^g \theta^* \iff \mathfrak{M}', w \models^g \theta \iff \mathfrak{M}^p, w \models^g \theta. \tag{9.2}$$

Then straightforward induction on ψ shows that, for every $w \in W$, every $\psi \in sub\,\varphi$, and every assignment g,

$$\mathfrak{M}', w \models^g (\psi^p)^* \iff \mathfrak{M}^p, w \models^g \psi^p;$$

the basis of the induction follows from (9.1) and (9.2), while the inductive step for $\Box$ goes through since p is false in $\mathfrak{M}'$ outside of W and every subformula of ψ beginning with $\Box$ has the form $\Box(p \to \psi')$.

It follows that $\mathfrak{M}', w_0 \not\models (\varphi^p)^*$. Hence, $\mathfrak{M}', w_0 \not\models p \to (\varphi^p)^*$, and so $p \to (\varphi^p)^* \notin \mathbf{QK}$.

To obtain proofs for **QKT**, **QKB**, **QKTB**, amend the definition of the model $\mathfrak{M}'$ by taking, respectively, reflexive, symmetric, and symmetric reflexive closure of the relation R' as the accessibility relation of the model $\mathfrak{M}'$. ■

Using the observation of Section 8, we can recursively embed logics mentioned in Proposition 9.2 into their own monadic fragments. Suppose that $\varphi \in \mathcal{ML}$, that S is an n-ary predicate letter, and that the formula $\varphi^{\#}$ is obtained from φ by substituting $\Diamond(P_1(x_1) \wedge \cdots \wedge P_n(x_n))$ for $S(x_1, \ldots, x_n)$.

PROPOSITION 9.3. *Let* $L \in \{\mathbf{QK}, \mathbf{QKT}, \mathbf{QKB}, \mathbf{QKBT}\}$. *Then, for every* $\varphi \in \mathcal{ML}$,

$$\varphi \in L \quad \iff \quad p \to (\varphi^p)^{\#} \in L.$$

PROOF. Similar to the proof of Proposition 9.2. ■

COROLLARY 9.4. *Let* $L \in \{\mathbf{QK}, \mathbf{QKT}, \mathbf{QKB}, \mathbf{QKBT}\}$. *Then* L *is recursively embeddable into its own monadic fragment.*

Relativization by itself is not, however, sufficient for logics of transitive Kripke frames, such as **QK4** and **QS4**. Here is a simple counter-example: consider the formula

$$\varphi \quad = \quad \forall x \forall y\, (\neg Q(x, y) \to \Box \neg Q(x, y)).$$

Relativizing φ and applying the substitution of $\Diamond(P_1(x) \wedge P_2(y))$ for $Q(x, y)$ to the resultant formula gives us

$$(\varphi^p)^* \quad = \quad \forall x \forall y\, (\neg \Diamond(P_1(x) \wedge P_2(y)) \to \Box(p \to \neg \Diamond(P_1(x) \wedge P_2(y)))).$$

It should be clear that φ is refuted on the trivially transitive Kripke frame $\mathfrak{F} = \langle \{w, v\}, \{\langle w, v\rangle\}\rangle$; hence, $\varphi \notin \mathbf{QK4}$. On the other hand, as can be easily checked, $p \to (\varphi^p)^* \in \mathbf{QK4}$.

This is the effect of transitivity. In the model $\mathfrak{M}'$ from the proof of Proposition 9.2, for every $w \in W$, fresh worlds $\langle w, a\rangle$ are R'-accessible from w, but not from any $w' \in W \setminus \{w\}$. This could not be ensured were $\mathfrak{M}$ based on a transitive Kripke frame: if R were transitive, then, by transitivity, the world $\langle w, a\rangle$ could become accessible from some world u that sees w, but is distinct from w. This would imply that $\mathfrak{M}', u \models \Diamond(P_1(a) \wedge P_2(b))$ even if $\mathfrak{M}, u \not\models Q(a, b)$, thus violating (9.1) and, hence, invalidating the proof of Proposition 9.2.

If L is a Kripke complete subframe logic whose frames are transitive, the analogue of Proposition 9.2 holds provided the formula φ has the following special property: if φ is satisfiable in a model validating L, then it is satisfiable in a model $\mathfrak{M} = \langle W, R, D, I\rangle$ that validates L and, additionally, is downward hereditary with respect to the interpretation of the letter Q, i.e., for all $w, w' \in W$:

$$\mathfrak{M}, w \models Q(a,b) \ \&\ w \in R(w') \,\&\, a,b \in D(w') \implies \mathfrak{M}, w' \models Q(a,b). \quad (9.3)$$

We note that, instead of models satisfying (9.3), we could use models satisfying the upward heredity property with respect to Q: for all $w, w' \in W$ and $a, b \in D_w$,

$$\mathfrak{M}, w \models Q(a,b) \ \&\ w' \in R(w) \quad \implies \quad \mathfrak{M}, w' \models Q(a,b), \quad (9.4)$$

but then $Q(x,y)$ should be simulated by $\neg\Diamond(P_1(x) \wedge P_2(y))$ rather than by $\Diamond(P_1(x) \wedge P_2(y))$.

When working with logics of transitive Kripke frames, it is often possible to encode an undecidable problem with formulas satisfying either (9.3) or (9.4), but relativization might need extra work when dealing with logics of particular Kripke frames, such as $\langle \mathbb{N}, \leqslant\rangle$. Then the Kripke trick can be carried out by simulating formulas of the form $Q(x,y)$ using the worlds of the given frame, rather than fresh worlds appended to the frame. We briefly sketch how this can be done in the case of logics $\mathtt{ML}^e\langle \mathbb{N}, \leqslant\rangle$ and $\mathtt{ML}^c\langle \mathbb{N}, \leqslant\rangle$.[11] Here, we need to encode a suitable undecidable problem with monadic modal formulas satisfiable in a model $\mathfrak{M} = \langle \mathbb{N}, \leqslant, D, I\rangle$ satisfying both (9.3) and (9.4) and, additionally, such that with $|D^+| = \aleph_0$. Then we simulate each subformula $Q(x,y)$ of φ with monadic formulas, as follows. Since $|D^+| = \aleph_0$, we can allocate to each $w \in \mathbb{N}$ a pair from $D_w \times D_w$ so that each pair from the set $D^+ \times D^+$ is allocated to infinitely many worlds from $\mathbb{N}$. We then define a model $\mathfrak{M}'$ over $\langle \mathbb{N}, \leqslant\rangle$ so that, for every $w \in \mathbb{N}$,

$$\mathfrak{M}', w \models P_1(a) \wedge P_2(b) \quad \leftrightharpoons \quad \langle a, b\rangle \text{ is allocated to } w \text{ and } \mathfrak{M}, w \models Q(a,b).$$

Then, by (9.3) and (9.4), for every $w \in W$ and every $a, b \in D_w$,

$$\mathfrak{M}', w \models \Diamond(P_1(a) \wedge P_2(b)) \quad \iff \quad \mathfrak{M}, w \models Q(a,b).$$

Hence, $\mathfrak{M}', w \models \varphi^*$ if, and only if, $\mathfrak{M}, w \models \varphi$ whenever $w \in \mathbb{N}$.

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[11] The construction presented below is a slight generalization of that used in our proof of Π^1_1-hardness of two-variable monadic fragments of $\mathtt{ML}^e\langle \mathbb{N}, \leqslant\rangle$ and $\mathtt{ML}^c\langle \mathbb{N}, \leqslant\rangle$; details can be found in [37,39].

10. Kripke Trick for Superintuitionistic Logics

10.1. Difficulties of the Transfer from Modal Languages

The transfer of the Kripke trick from modal to superintuitionistic logics has caused some difficulties. Kripke, in his paper [22, p. 115], announced a sequel on the intuitionistic predicate logic, but it seems to have never appeared in print.[12] We briefly point out how, and why, the straightforward attempts to apply the trick to superintuitionistic logics fail.

Kontchakov et al. [20, § 4] suggest that the most obvious way to simulate $Q(x, y)$ in **QInt** would be to use $\neg\neg(P_1(x) \wedge P_2(y))$. To see why, observe that, in intuitionistic Kripke models,

$$\mathfrak{M}, w \models \neg\neg\varphi \quad \iff \quad \mathfrak{M}, w' \models \varphi, \text{for some } w' \in R(w);$$

thus, in the Kripke semantics for $\mathcal{L}$ the formula $\neg\neg\varphi$ has meaning similar to the meaning of the formula $\Diamond\varphi$ in Kripke semantics for $\mathcal{ML}$ (more precisely, intuitionistic formula $\neg\neg\varphi$ behaves like the modal formula $\Box\Diamond\varphi$, but all intuitionistic formulas, due to (2.5), can be viewed as statements about necessity). Therefore, the intuitionistic formula $\neg\neg(P_1(x)\wedge P_2(y))$ has meaning similar to that of the modal formula $\Diamond(P_1(x) \wedge P_2(y))$. The substitution of $\neg\neg(P_1(x) \wedge P_2(y))$ for $Q(x, y)$ does not, however, work in **QInt**. Indeed, consider the formula

$$A \quad = \quad \forall x\forall y\, (\neg\neg Q(x, y) \to Q(x, y)).$$

Since A is an instance of the law of double negation elimination, which is not valid intuitionistically, surely $A \notin$ **QInt**. On the other hand, substituting $\neg\neg(P_1(x) \wedge P_2(y))$ for $Q(x, y)$ into A results in the formula

$$\forall x\forall y\, (\neg\neg\neg\neg(P_1(x) \wedge P_2(y)) \to \neg\neg(P_1(x) \wedge P_2(y))),$$

which belongs to **QInt**, since $\neg\neg\neg\psi \to \neg\psi$ is intuitionistically valid.

Another idea would be to negate Kripke's formula: $\neg\Diamond(P_1(x) \wedge P_2(y))$ is equivalent to $\Box(\neg P_1(x) \vee \neg P_2(y))$, which, due to (2.5), has meaning similar to the intuitionistic formula $\neg P_1(x) \vee \neg P_2(y)$. This substitution works, at least, for the formula A. It does not, however, work in general: consider the formula

$$B \quad = \quad \forall x\forall y\forall z\, (Q(x, y) \to Q(x, z) \vee Q(z, y)),$$

[12] The undecidability of the monadic fragment of **QInt** had been established by Maslov et al. [25] using unrelated methods; a different proof, also circumventing the Kripke trick, was discovered by Gabbay [9, Chapter 14, Theorem 1].

which is refuted in a classical model with a two-element domain, and hence does not belong to **QInt**. On the other hand, as can be easily checked, its substitution instance

$$\forall x \forall y \forall z\, ((\neg P_1(x) \vee \neg P_2(y)) \rightarrow (\neg P_1(x) \vee \neg P_2(z)) \vee (\neg P_1(y) \vee \neg P_2(y)))$$

is in **QInt**. The formula B would also furnish a counterexample if we tried substituting $P_1(x) \vee P_2(y)$ for $Q(x, y)$. Another reformulation, $P_1(x) \rightarrow P_2(y)$ does not work, either: the formula

$$\forall x \forall y \forall z \forall u\, \neg(\neg Q(x, y) \wedge \neg Q(z, u) \wedge Q(x, u))$$

is refuted in the classical model $\langle D, I \rangle$ with $D = \{a, b\}$ and $I(Q) = \{\langle a, b \rangle\}$, and hence is not in **QInt**, but its substitution instance

$$\forall x \forall y \forall z \forall u\, \neg(\neg(P_1(x) \rightarrow P_2(y)) \wedge \neg(P_1(z) \rightarrow P_2(u)) \wedge (P_1(x) \rightarrow P_2(u)))$$

is in **QInt**. Thus, head-on attempts fail.

10.2. The Kripke Trick for Superintuitionistic Logics

A successful attempt to transfer the Kripke trick to superintuitionistic logics involves two ideas: relativization and avoidance of non-positive formulas.

An $\mathcal{L}$-formula φ is *positive* if it does not contain $\bot$. Observe that positive formulas do not contain $\neg$ either. We denote by $\mathcal{L}^+$ the set of positive $\mathcal{L}$-formulas and, if L is a superintuitionistic predicate logic, we define $L^+ = L \cap \mathcal{L}^+$. We say that a superintuitionistic predicate logic L is *subframe* in case, for every intuitionistic augmented frame $\mathfrak{F}_D$, if $\mathfrak{F}_D \models L$ then $\mathfrak{F}'_D \models L$ whenever $\mathfrak{F}'_D$ is a subframe of $\mathfrak{F}_D$.

We first show that, if L is a subframe Kripke complete superintuitionistic predicate logic, then there exists a recursive embedding of L into L^+. We denote by $\bar{\forall}\chi$ the universal closure of a formula χ (we assume, without a loss of generality, that each formula has a unique universal closure). For an $\mathcal{L}$-formula φ and a proposition letter f not in φ, we denote by φ^f the result of substituting f for $\bot$ in φ and define

$$F \quad = \quad \bar{\forall} \bigwedge_{\psi \in sub\, \varphi} (f \rightarrow \psi^f).$$

Proposition 10.1. *Let L be a subframe Kripke complete superintuitionistic predicate logic and let φ be an $\mathcal{L}$-formula. Then*

$$\varphi \in L \quad \Longleftrightarrow \quad F \rightarrow \varphi^f \in L.$$

Proof. ($\Leftarrow$) Assume that $F \rightarrow \varphi^f \in L$. Let $\psi^\bot$ be the result of substituting $\bot$ for f in an $\mathcal{L}$-formula ψ. By Substitution, $(F \rightarrow \varphi^f)^\bot \in L$, i.e.,

$F^{\perp} \to \varphi \in L$. Since $\bot \to \psi \in \mathbf{QInt}$, for every formula ψ, surely $F^{\perp} \in L$. Hence, by Modus Ponens, $\varphi \in L$.

($\Rightarrow$) Assume that $F \to \varphi^f \notin L$. Since L is Kripke complete, there exist an intuitionistic Kripke model $\mathfrak{M} = \langle W, R, D, I\rangle$, a world $w_0 \in W$, and an assignment g_0, with $g_0(x) \in D(w_0)$ for every $x \in par\,\varphi^f$, such that

$$\langle W, R, D\rangle \models L, \quad \mathfrak{M}, w_0 \Vdash^{g_0} F, \quad \text{and} \quad \mathfrak{M}, w_0 \not\Vdash^{g_0} \varphi^f.$$

Set $W' = \{w \in W : \mathfrak{M}, w \not\Vdash f\}$. Since $\mathfrak{M}, w_0 \Vdash F$, it follows that, for every $w \in W$, every $\psi \in sub\,\varphi$, and every assignment g such that $g(x) \in D_w$ whenever $x \in par\,\psi^f$,

$$w \in R(w_0) \;\&\; \mathfrak{M}, w \Vdash^g f \quad \Longrightarrow \quad \mathfrak{M}, w \Vdash^g \psi^f. \tag{10.1}$$

Since $w_0 \in R(w_0)$ and $\mathfrak{M}, w_0 \not\Vdash \varphi^f$, it follows, by (10.1), that $\mathfrak{M}, w_0 \not\Vdash f$. Hence $w_0 \in W'$, and so $W' \neq \varnothing$. Set

$$R' = R \restriction W', \quad D' = R \restriction D', \quad I' = I \restriction W', \quad \text{and} \quad \mathfrak{M}' = \langle W', R', D', I'\rangle.$$

Then $\mathfrak{M}'$ is an intuitionistic Kripke model. Since L is subframe, $\langle W', R', D'\rangle \models L$. We show, by induction on ψ, that, for every $w \in R'(w_0)$, every $\psi \in sub\,\varphi$, and every assignment g such that $g(x) \in D(w)$ whenever $x \in par\,\psi^f$,

$$\mathfrak{M}', w \Vdash^g \psi^f \iff \mathfrak{M}, w \Vdash^g \psi^f. \tag{10.2}$$

For atomic formulas, (10.2) holds since $\mathfrak{M}'$ is a submodel of $\mathfrak{M}$. The cases $\psi = \psi_1 \wedge \psi_2$, $\psi = \psi_1 \vee \psi_2$, and $\psi = \exists x\,\psi_1$ are straightforward.

Let $\psi = \psi_1 \to \psi_2$ and $w \in R'(w_0)$. If $\mathfrak{M}, w \Vdash^g \psi_1^f \to \psi_2^f$, then surely $\mathfrak{M}', w \Vdash^g \psi_1^f \to \psi_2^f$. Conversely, suppose that $\mathfrak{M}, w \not\Vdash^g \psi_1^f \to \psi_2^f$. Then there exists $w' \in R(w)$ such that $\mathfrak{M}, w' \Vdash^g \psi_1^f$ and $\mathfrak{M}, w' \not\Vdash^g \psi_2^f$. By transitivity, $w' \in R(w_0)$. Since $\mathfrak{M}, w' \not\Vdash^g \psi_2^f$, it follows, by (10.1), that $\mathfrak{M}, w' \not\Vdash f$, and so $w' \in W'$, by definition of W'. Hence, by inductive hypothesis, $\mathfrak{M}', w' \Vdash^g \psi_1^f$ and $\mathfrak{M}', w' \not\Vdash^g \psi_2^f$. Since $w' \in R'(w)$, it follows that $\mathfrak{M}', w \not\Vdash^g \psi_1^f \to \psi_2^f$.

Let $\psi = \forall x\,\psi_1$ and $w \in R'(w_0)$. If $\mathfrak{M}, w \Vdash^g \forall x\,\psi_1^f$, then surely $\mathfrak{M}', w \Vdash^g \forall x\,\psi_1^f$. Conversely, suppose that $\mathfrak{M}, w \not\Vdash^g \forall x\,\psi_1^f$. Then there exist $w' \in R(w)$ and $g' \stackrel{x}{=} g$ such that $g'(x) \in D(w')$ and $\mathfrak{M}, w' \not\Vdash^{g'} \psi_1^f$. By transitivity, $w' \in R(w_0)$. Thus, by (10.1), $\mathfrak{M}, w' \not\Vdash f$, and so $w' \in W'$. Hence, by inductive hypothesis, $\mathfrak{M}', w' \not\Vdash^{g'} \psi_1^f$. Since $w' \in R'(w)$, it follows that $\mathfrak{M}', w \not\Vdash^g \forall x\,\psi_1^f$.

Since $\mathfrak{M}, w_0 \not\Vdash^{g_0} \varphi^f$, by (10.2), $\mathfrak{M}', w_0 \not\Vdash^{g_0} \varphi^f$. Since $\mathfrak{M}' \Vdash f \leftrightarrow \bot$, it follows, by straightforward induction, that $\mathfrak{M}', w_0 \not\Vdash^{g_0} (\varphi^f)^{\perp}$, i.e., $\mathfrak{M}', w_0 \not\Vdash^{g_0} \varphi$. Hence, $\langle W', R', D'\rangle \not\Vdash \varphi$. On the other hand, as we have seen, $\langle W', R', D'\rangle \Vdash L$. Hence, $\varphi \notin L$. ∎

To make the paper self-contained, we recall that the superintuitionistic logic **QKC** is determined by the class of convergent intuitionistic Kripke frames [13, Theorem 6.6.20], i.e., those satisfying the classical first-order condition

$$\forall x \forall y \forall z \, (xRy \wedge xRz \rightarrow \exists u \, (yRu \wedge zRu)).$$

We next notice that all the logics between **QInt** and **QKC** have the same positive fragment, and that the same holds true for all the logics between **QInt.cd** and **QKC.cd**:

Proposition 10.2. $\mathbf{QInt}^+ = \mathbf{QKC}^+$ *and* $\mathbf{QInt.cd}^+ = \mathbf{QKC.cd}^+$.

Proof. Since $\mathbf{QInt} \subseteq \mathbf{QKC}$, surely $\mathbf{QInt}^+ \subseteq \mathbf{QKC}^+$. For the converse, suppose that $\varphi \in \mathcal{L}^+ \setminus \mathbf{QInt}^+$. Then there exists an intuitionistic Kripke model $\mathfrak{M} = \langle W, R, D, I \rangle$, a world $w_0 \in W$, and an assignment g_0, with $g_0(x) \in D(w_0)$ whenever $x \in par\, \varphi$, such that $\mathfrak{M}, w_0 \not\Vdash^{g_0} \varphi$. Assuming that $u_0 \notin W$, we define

- $W' = W \cup \{u_0\}$;
- $R' = R \cup \big((W \cup \{u_0\}) \times \{u_0\}\big)$.

Then $\langle W', R' \rangle$ is a convergent intuitionistic Kripke frame; hence, $\langle W', R' \rangle \Vdash$ **QKC**, and so $\langle W', R' \rangle \Vdash \mathbf{QKC}^+$. Define the domain function D' on W' by

$$D'(w) \quad = \quad \begin{cases} D(w) & \text{if}\, w \in W; \\ D^+ & \text{if}\, w = u_0, \end{cases}$$

and the interpretation I' on the augmented frame $\langle W', R', D' \rangle$ so that, for every n-ary predicate letter P,

$$I'(P, w) \quad = \quad \begin{cases} I(P, w) & \text{if}\, w \in W; \\ (D^+)^n & \text{if}\, w = u_0. \end{cases}$$

Then $\mathfrak{M}' = \langle W', R', D', I' \rangle$ is an intuitionistic Kripke model. Observe that, by definition of $\mathfrak{M}'$, for every $\psi \in \mathcal{L}^+$ and for every assignment g such that $g(x) \in D(u_0)$ whenever $x \in par\, \psi$,

$$\mathfrak{M}', u_0 \Vdash^g \psi. \tag{10.3}$$

We show, by induction on $\psi \in sub\, \varphi$, that, for every $w \in W$ and every assignment g such that $g(x) \in D(w)$ whenever $x \in par\, \psi$,

$$\mathfrak{M}, w \Vdash^g \psi \quad \iff \quad \mathfrak{M}', w \Vdash^g \psi. \tag{10.4}$$

The cases when ψ is an atomic formula, $\psi = \psi_1 \wedge \psi_2$, $\psi = \psi_1 \vee \psi_2$, and $\psi = \exists x \, \psi_1$ are straightforward.

Let $\psi = \psi_1 \to \psi_2$ and $w \in W$. If $\mathfrak{M}, w \not\Vdash^g \psi_1 \to \psi_2$, then surely $\mathfrak{M}', w \not\Vdash^g \psi_1 \to \psi_2$. Conversely, assume that $\mathfrak{M}, w \Vdash^g \psi_1 \to \psi_2$. Suppose that $w' \in R(w)$. If $w' \in W$, then, by inductive hypothesis, either $\mathfrak{M}', w' \not\Vdash^g \psi_1$ or $\mathfrak{M}', w' \Vdash^g \psi_2$. If $w' = u_0$, then, by (10.3), $\mathfrak{M}', w' \Vdash^g \psi_2$. Hence, in either case, $\mathfrak{M}', w \Vdash^g \psi_1 \to \psi_2$.

Let $\psi = \forall x\, \psi_1$ and $w \in W$. If $\mathfrak{M}, w \not\Vdash^g \forall x\, \psi_1$, then surely $\mathfrak{M}', w \not\Vdash^g \forall x\, \psi_1$. Conversely, assume that $\mathfrak{M}, w \Vdash^g \forall x\, \psi_1$. Suppose that $w' \in R(w)$. If $w' \in W$, then, by inductive hypothesis, $\mathfrak{M}', w' \Vdash^{g'} \psi_1$ whenever $g' \stackrel{x}{=} g$ and $g'(x) \in D'(w')$. If $w' = u_0$, then, by (10.3), $\mathfrak{M}', w' \Vdash^{g'} \psi_1$, for every assignment g'. Hence, in either case, $\mathfrak{M}', w \Vdash^g \forall x\, \psi_1$.

Since $\mathfrak{M}, w_0 \not\Vdash^{g_0} \varphi$, by (10.4), $\mathfrak{M}', w_0 \not\Vdash^{g_0} \varphi$. Hence, $\varphi \notin \mathbf{QKC}^+$. Thus, $\mathbf{QInt}^+ = \mathbf{QKC}^+$.

To see that $\mathbf{QInt.cd}^+ = \mathbf{QKC.cd}^+$, observe that, if the model $\mathfrak{M}$ is based on a locally constant augmented frame, then so is $\mathfrak{M}'$. ■

We now do the Kripke trick for superintuitionistic predicate logics. Let q and p be nullary predicate letters and let, for an $\mathcal{L}$-formula ψ, the formula $\psi^\#$ be obtained from ψ by substituting $(P_1(x) \wedge P_2(y) \to q) \vee p$ for $Q(x, y)$.

PROPOSITION 10.3. *For every formula $\varphi \in \mathcal{L}^+$ containing only a binary predicate letter Q,*

$$\varphi \in \mathbf{QInt} \iff \varphi^\# \in \mathbf{QInt}.$$

PROOF. (⇒) This implication follows by Substitution.

(⇐) Assume that $\varphi \notin \mathbf{QInt}$. Then there exists an intuitionistic Kripke model $\mathfrak{M} = \langle W, R, D, I\rangle$, a world $w_0 \in W$, and an assignment g_0, with $g_0(x) \in D(w_0)$ whenever $x \in \mathit{par}\,\varphi$, such that $\mathfrak{M}, w_0 \not\Vdash^{g_0} \varphi$. Define

$$W^* = \bigcup_{w \in W} (\{w\} \times D(w)) \quad \text{and} \quad W' = W \cup W^*.$$

We also denote the world $\langle w, a\rangle \in W^*$ by w_a. We assume, without a loss of generality, that $W \cap W^* = \varnothing$.

Define R' to be the reflexive transitive closure of the relation

$$R \cup \{\langle w, w_a\rangle : w \in W \text{ and } a \in D(w)\}.$$

Define the domain function D' on the Kripke frame $\langle W', R'\rangle$ by

$$D'(u) = \begin{cases} D(u) & \text{if}\, u \in W; \\ D(w) & \text{if}\, u = w_a, \text{ for some}\, w \in W \text{ and}\, a \in D(w). \end{cases}$$

Define the interpretation I' on the augmented frame $\langle W', R', D'\rangle$ so that

- for every $w \in W$ and every $a, b \in D(w)$,

$$\begin{array}{rcl} I'(P_1, w_a) & = & \{\langle a\rangle\}; \\ I'(P_2, w_a) & = & \{\langle b\rangle : \mathfrak{M}, w \not\Vdash Q(a,b)\}; \\ I'(p, w_a) & = & \{\langle\rangle\}; \\ I'(q, w_a) & = & \varnothing; \end{array}$$

- for every $w \in W$,

$$I'(P_1, w) \;=\; I'(P_2, w) \;=\; I'(p, w) \;=\; I'(q, w) \;=\; \varnothing.$$

Lastly, define $\mathfrak{M}' = \langle W', R', D', I'\rangle$. Clearly, the interpretation I' satisfies (2.4); hence, $\mathfrak{M}'$ is an intuitionistic Kripke model. We shall prove that $\mathfrak{M}', w_0 \not\Vdash \varphi^{\#}$. We first prove the following:

SUBLEMMA 10.4. *Let $\psi \in sub\,\varphi$, $w \in W^*$, and g be an assignment in $\mathfrak{M}'$. Then, $\mathfrak{M}', w \Vdash^g \psi^{\#}$.*

PROOF. Since φ is positive and contains no predicate letters other than Q, the formula $\psi^{\#}$ is built from formulas of the form $(P_1(x) \wedge P_2(y) \to q) \vee p$ using the logical symbols $\wedge$, $\vee$, $\to$, $\exists$, and $\forall$. Suppose that $w \in W^*$. Then $\mathfrak{M}', w \Vdash p$, by definition of I'. Therefore, each formula of the form $(P_1(x) \wedge P_2(y) \to q) \vee p$, and hence $\psi^{\#}$, is true at w under every assignment. ■

We next show that, for every $\psi \in sub\,\varphi$, every $w \in W$, and every assignment g such that $g(x) \in D(w)$ whenever $x \in par\,\psi$,

$$\mathfrak{M}, w \Vdash^g \psi \quad \iff \quad \mathfrak{M}', w \Vdash^g \psi^{\#}. \tag{10.5}$$

Assuming that $w \in W$, we proceed by induction on ψ.

Suppose that $\psi = Q(x, y)$, and so $\psi^{\#} = (P_1(x) \wedge P_2(y) \to q) \vee p$.

Assume that $\mathfrak{M}, w \not\Vdash Q(a, b)$. Then, by definition of $\mathfrak{M}'$,

$$w_a \in R'(w), \quad \mathfrak{M}', w_a \Vdash P_1(a) \wedge P_2(b), \quad \text{and} \quad \mathfrak{M}', w_a \not\Vdash q,$$

as well as $\mathfrak{M}', w \not\Vdash p$. Hence, $\mathfrak{M}', w \not\Vdash (P_1(a) \wedge P_2(b) \to q) \vee p$.

Conversely, assume that $\mathfrak{M}, w \Vdash Q(a, b)$. We show that, then, for every $u \in R'(w)$,

$$\mathfrak{M}', u \not\Vdash P_1(a) \wedge P_2(b),$$

which implies that $\mathfrak{M}', w \Vdash (P_1(a) \wedge P_2(b) \to q) \vee p$. Suppose, for the sake of contradiction, that $u \in R'(w)$ and $\mathfrak{M}', u \Vdash P_1(a) \wedge P_2(b)$. The definition of I' implies that $u = v_a$, for some $v \in R(w)$. Then the definition of I'

for P_2 implies that $\mathfrak{M}, v \not\Vdash Q(a,b)$. Hence, by (2.4), $\mathfrak{M}, w \not\Vdash Q(a,b)$, in contradiction with the assumption.

The cases $\psi = \psi_1 \vee \psi_2$, $\psi = \psi_1 \wedge \psi_1$, and $\psi = \exists x\, \psi_1$ are straightforward.

Suppose that $\psi = \psi_1 \to \psi_2$, and so $\psi^\# = \psi_1^\# \to \psi_2^\#$.

Assume that $\mathfrak{M}, w \not\Vdash^g \psi_1 \to \psi_2$. Then there exists $v \in R(w)$ such that $\mathfrak{M}, v \Vdash^g \psi_1$ and $\mathfrak{M}, v \not\Vdash^g \psi_2$. By inductive hypothesis, $\mathfrak{M}', v \Vdash^g \psi_1^\#$ and $\mathfrak{M}', v \not\Vdash^g \psi_2^\#$. Since $v \in R(w)$ and $R \subseteq R'$, it follows that $v \in R'(w)$. Hence, $\mathfrak{M}', w \not\Vdash^g \psi_1^\# \to \psi_2^\#$.

Conversely, assume that $\mathfrak{M}', w \not\Vdash^g \psi_1^\# \to \psi_2^\#$. Then there exists $v \in R'(w)$ such that $\mathfrak{M}', v \Vdash^g \psi_1^\#$ and $\mathfrak{M}', v \not\Vdash^g \psi_2^\#$. Since $\mathfrak{M}', v \not\Vdash^g \psi_2^\#$, by Sublemma 10.4, $v \notin W^*$, and so $v \in W$. Hence, by inductive hypothesis, $\mathfrak{M}, v \Vdash^g \psi_1$ and $\mathfrak{M}, v \not\Vdash^g \psi_2$. Since $w, v \in W$, $v \in R'(w)$, and $R = R' \restriction W$, it follows that $v \in R(w)$. Hence, $\mathfrak{M}, w \not\Vdash^g \psi_1 \to \psi_2$.

Suppose that $\psi = \forall x\, \psi_1$, and so $\psi^\# = \forall x\, \psi_1^\#$.

Assume that $\mathfrak{M}, w \not\Vdash^g \forall x\, \psi_1$. Then there exist $v \in R(w)$ and g' with $g' \stackrel{x}{=} g$ and $g'(x) \in D(v)$ such that $\mathfrak{M}, v \not\Vdash^{g'} \psi_1$. By inductive hypothesis, $\mathfrak{M}', v \not\Vdash^{g'} \psi_1^\#$. Since $v \in R(w)$ and $R \subseteq R'$, it follows that $v \in R'(w)$. Hence, $\mathfrak{M}', w \not\Vdash^g \forall x\, \psi_1^\#$.

Conversely, assume that $\mathfrak{M}', w \not\Vdash^g \forall x\, \psi_1^\#$. Then there exist $v \in R'(w)$ and some g' with $g' \stackrel{x}{=} g$ and $g'(x) \in D'(v)$ such that $\mathfrak{M}', v \not\Vdash^{g'} \psi_1^\#$. By Sublemma 10.4, $v \notin W^*$, and so $v \in W$. Hence, by inductive hypothesis, $\mathfrak{M}, v \not\Vdash^{g'} \psi_1$. Since $w, v \in W$, $v \in R'(w)$, and $R = R' \restriction W$, it follows that $v \in R(w)$. Hence, $\mathfrak{M}, w \not\Vdash^g \forall x\, \psi_1$.

This completes the induction, thus proving (10.5).

Since $\mathfrak{M}, w_0 \not\Vdash^{g_0} \varphi$, it follows, by (10.5), that $\mathfrak{M}', w_0 \not\Vdash^{g_0} \varphi^\#$. Hence, $\varphi^\# \notin \mathbf{QInt}$. ■

Proposition 10.5. *For every formula $\varphi \in \mathcal{L}^+$ containing only a binary predicate letter Q,*

$$\varphi \in \mathbf{QInt.cd} \quad \Longleftrightarrow \quad \varphi^\# \in \mathbf{QInt.cd}.$$

Proof. Similar to the proof of Proposition 10.3. For ($\Leftarrow$), it suffices to notice that the model $\mathfrak{M}'$ has locally constant domains whenever $\mathfrak{M}$ has locally constant domains. ■

Similarly to what happens in modal logics (see Section 8), Propositions 10.3 and 10.5 can be generalised to predicate letters of any arity greater than or equal to two. Suppose that $\varphi \in \mathcal{L}^+$ and that φ contains only an n-ary predicate letter S, and let $\varphi^\star$ be obtained from φ by substituting the formula $(P_1(x_1) \wedge \ldots \wedge P_n(x_n) \to q) \vee p$ for $S(x_1, \ldots, x_n)$.

PROPOSITION 10.6. *For every formula $\varphi \in \mathcal{L}^+$ containing only an n-ary predicate letter S,*

$$(1) \quad \varphi \in \mathbf{QInt} \iff \varphi^\star \in \mathbf{QInt};$$
$$(2) \quad \varphi \in \mathbf{QInt.cd} \iff \varphi^\star \in \mathbf{QInt.cd}.$$

PROOF. Similar to the proofs of Propositions 10.3 and 10.5. ■

It should be clear that Proposition 10.6 can be easily generalised to formulas with any number of predicate letters of any arity greater than one—all we need to do is to repeatedly apply the construction of Proposition 10.6, dealing with one predicate letter at a time. Since this construction preserves the number of individual variables in the original formula, using Proposition 10.2, we obtain the following:

PROPOSITION 10.7. *Let L be a subframe Kripke complete superintuitionistic predicate logic such that $L \subseteq \mathbf{QKC}$ and let $n \in \mathbb{N}^+$. Then*

(1) *the n-variable fragment of L is recursively embeddable into the positive monadic n-variable fragment of L;*
(2) *the n-variable fragment of $L.\mathbf{cd}$ is recursively embeddable into the positive monadic n-variable fragment of $L.\mathbf{cd}$.*

PROOF. (1): Let φ be an $\mathcal{L}$-formula with $var\,\varphi = \{x_1, \dots, x_n\}$. We assume, without a loss of generality, that φ contains a single predicate letter S of arity greater than one (otherwise, we apply the translation defined below repeatedly). Since L is subframe and Kripke complete, by Proposition 10.1,

$$\varphi \in L \iff F \to \varphi^f \in L.$$

Since $F \to \varphi^f$ and $(F \to \varphi^f)^\star$ are both positive and $L \subseteq \mathbf{QKC}$, by Propositions 10.2 and 10.6,

$$\begin{aligned} F \to \varphi^f \in L &\iff F \to \varphi^f \in \mathbf{QInt} \\ &\iff (F \to \varphi^f)^\star \in \mathbf{QInt} \\ &\iff (F \to \varphi^f)^\star \in L. \end{aligned}$$

Hence,

$$\varphi \in L \iff (F \to \varphi^f)^\star \in L.$$

Since $var(F \to \varphi^f)^\star = \{x_1, \dots, x_n\}$, the statement follows.

(2): Analogous to the proof of (1). ■

Kontchakov et al. [20, Theorem 1] have shown that **QInt** and **QInt.cd** are undecidable in languages with two variables.[13] Hence, by Proposition 10.7, we obtain the following:

THEOREM 10.8. *Positive monadic two-variable fragments of* $\mathbf{QInt}^+$ *and* $\mathbf{QInt.cd}^+$ *are both undecidable.*

We note that, to obtain an embedding of logics between **QInt** and **QKC**, and those between **QInt.cd** and **QKC.cd**, into their monadic fragments, we could have used, in the proof of Proposition 10.3, $\bot$ instead of q. The use of q allowed us to obtain an embedding into *positivite* monadic fragments. The positivity of the target fragments can be used [35] to simulate all their atomic formulas by formulas with a single fixed monadic predicate letter, which proves undecidability of all superintuitionistic predicate logics between **QInt** and **QKC.cd** in languages with two variables and a single monadic predicate letter.

The techniques described in this section can also be used to prove Π^0_1-hardness of the monadic positive fragments of all logics between $\mathbf{QInt}_{wfin}$ and $\mathbf{QKC.cd}_{wfin}$ [38]. Constructions similar to those described here can also be obtained for the predicate counterparts of Visser's propositional logics **BPL** and **FPL** [57].

11. When the Kripke Trick is Blocked

The Kripke trick lies close to the heart of undecidability of rather poor fragments of modal and superintuitionistic predicate logics: combined with techniques for simulating all monadic atomic formulas by formulas with a single fixed monadic letter, it leads to undecidability of modal and superintuitionistic predicate logics with only a single monadic predicate letter and two individual variables. Hence, identification of decidable fragments of predicate modal logics is closely related to identifying settings where the Kripke trick does not work. The trick can be blocked due to either syntactic or semantic considerations.

11.1. Syntactic Restrictions

The Kripke trick uses formulas of the form $\Diamond(P_1(x)\wedge P_2(y))$; as we have seen, in some settings, more economical formulas of the form $\neg\Diamond(P(x)\wedge P(y))$ can

[13] Undecidability of **QInt.cd** with two variables and predicate letters of arity at most one had earlier been established by Gabbay and Shehtman [11].

be used instead. Disallowing such formulas is one way to obtain decidable fragments of modal predicate logics.

11.1.1. Monodic Fragments Wolter and Zakharyaschev [58] define *monodic* fragments as those where modal operators are applied to subformulas with at most one parameter. Thus, formulas of the form $\Diamond(P_1(x) \wedge P_2(y))$ and $\Diamond(P(x) \wedge P(y))$ are disallowed since they contain two parameters in the scope of $\Diamond$. Wolter and Zakharyaschev prove that the two-variable monodic fragments [58, Theorem 5.1] and the monadic monodic fragments [58, Theorem 5.8] of logics **QK**, **QT**, **QK4**, and **QS4** are decidable; they mention [58, Remark 4.8] that their techniques can be extended to other standard modal logics.

11.1.2. One-Variable Fragments An apparently more severe restriction than monodicity is the requirement that formulas contain at most one individual variable, free or bound (one-variable formulas are, clearly, monodic). Wolter and Zakharyaschev prove [58, Theorem 5.2] that one-variable fragments of **QK**, **QT**, **QK4**, and **QS4** are decidable.

In fact, in **QK** and hence in every its extension, every monadic monodic formula is equivalent to a Boolean combination of one-variable formulas [47, Lemma 3.1]; thus, one-variable fragments are no less expressive than monadic monodic ones (observe that in one-variable formulas it suffices to use only monadic predicate letters). Decidability of a wide range of one-variable fragments follows from their close link with two-modal propositional logics known as *semiproducts* and *products*. We omit here discussion of the connection between semiproducts, products, and one-variable fragments, referring the reader to [10,12,45], and only note the following:

PROPOSITION 11.1. *Let L be a Kripke complete modal propositional logic. If the logic* $\mathbf{Q}L$ *is Kripke complete, then there exists a bijection between the one-variable fragment of* $\mathbf{Q}L$ *and the semiproduct of L with* **S5**. *Also, if the logic* $\mathbf{Q}L.\mathbf{bf}$ *is Kripke complete, then there exists a bijection between the one-variable fragment of* $\mathbf{Q}L$ *and the product of L with* **S5**.

This opens the door to the transfer of decidability results from semiproducts and products to one-variable fragments of modal predicate logics.

11.1.3. Bundled Fragments Another restriction on the use of modal operators that rules out formulas of the form $\Diamond(P_1(x) \wedge P_2(y))$ is adopted in *bundled* fragments, where a modality can only be used in combination with a quantifier; such a combination, e.g., $\forall x \Box$ or $\Box \forall x$, is called a bundle. Allowing only certain bundles, none of which permits to use $\Diamond$ without a pairing quantifier, leads to decidable fragments of **QK** [24].

11.2. Semantic Restrictions

If we are not placing any syntactic restrictions on formulas, blocking the Kripke trick should involve violation of the frame properties considered in Sections 4 and 5: Kripke frames where a world can see any number of possible worlds should be disallowed. The most obvious example of standard logics not admitting such Kripke frames are logics $\mathbf{QAlt}_n$, with $n \in \mathbb{N}$, which, recall, are logics of n-alternative Kripke frames [46, Theorem 5], i.e. Kripke frames where, for every $w \in W$:

$$|R(w)| \leqslant n. \tag{11.1}$$

It turns out that monadic fragments of logics $\mathbf{QAlt}_n$ are decidable, due to the following observation [43, Lemma 3.1]:

PROPOSITION 11.2. *If a monadic modal formula with m predicate letters is refuted on an augmented frame $\mathfrak{F}_D = \langle W, R, D\rangle$ with finite W, then it is refuted on an augmented frame $\mathfrak{F}_{\bar{D}} = \langle W, R, \bar{D}\rangle$ with $|\bar{D}^+| \leqslant 2^{|W|(m+1)}$; moreover, if $\mathfrak{F}_D$ has locally constant domains, then so does $\mathfrak{F}_{\bar{D}}$.*

PROOF. Let φ be a modal formula with monadic predicate letters $P_1, \ldots, P_m$. Let $\mathfrak{M} = \langle W, R, D, I\rangle$ be a model, with finite W, such that $\mathfrak{M}, w_0 \not\models \varphi$, for some $w_0 \in W$.

Define the binary relation $\sim$ on D^+ so that $a \sim b$ if, and only if, for every $w \in W$ and every $i \in \{1, \ldots, m\}$,

$$a \in D_w \iff b \in D_w; \tag{11.2}$$
$$\mathfrak{M}, w \models P_i(a) \iff \mathfrak{M}, w \models P_i(b). \tag{11.3}$$

It should be clear that $\sim$ is an equivalence on D^+. For every $a \in D^+$, let $\bar{a} = \{b \in D^+ : b \sim a\}$; let also $\mathcal{D} = D^+/{\sim} = \{\bar{a} : a \in D^+\}$.

To define the sought augmented frame, let, for every $w \in W$,

$$\bar{D}(w) = \{\bar{a} \in \mathcal{D} : a \in D_w\}.$$

Observe that $\bar{D}^+ = \mathcal{D}$. Then $\mathfrak{F}_{\bar{D}} = \langle W, R, \bar{D}\rangle$ is an augmented frame with expanding domains: if wRv, then, by (2.1), $D_w \subseteq D_v$, and hence, by definition of $\bar{D}$, also $\bar{D}_w \subseteq \bar{D}_v$. Moreover, by definition of $\bar{D}$, if $D_w = D_v$, then $\bar{D}_w = \bar{D}_v$. Thus, if $\mathfrak{F}_D$ has locally constant domains, then so does $\mathfrak{F}_{\bar{D}}$; this shall give us the additional claim of the proposition. We now show that $\mathfrak{F}_{\bar{D}}$ is the required augmented frame.

Let $\bar{I}$ be an interpretation on $\mathfrak{F}_{\bar{D}}$ such that, for every $w \in W$ and every $i \in \{1, \ldots, m\}$,

$$\bar{I}(w, P_i) = \{\langle \bar{a}\rangle : \langle a\rangle \in I(w, P_i)\},$$

and let $\bar{\mathfrak{M}} = \langle \mathfrak{F}_{\bar{D}}, \bar{I} \rangle$.

Straightforward induction on φ shows that, for every subformula $\psi(x_1, \ldots, x_k)$ of φ, every $w \in W$, and every $a_1, \ldots, a_k \in D_w$,

$$\mathfrak{M}, w \models \psi(a_1, \ldots, a_k) \quad \Longleftrightarrow \quad \bar{\mathfrak{M}}, w \models \psi(\bar{a}_1, \ldots, \bar{a}_k).$$

Thus, in particular, $\bar{\mathfrak{M}}, w_0 \not\models \varphi$, and so $\mathfrak{F}_{\bar{D}} \not\models \varphi$.

To prove the lemma, it remains to show that $|\bar{D}^+| \leqslant 2^{|W|(m+1)}$. Condition (11.2) defines an equivalence on D^+ partitioning it into at most $2^{|W|}$ equivalence classes. Each of those classes is further partitioned by the equivalence defined by (11.3) into at most $2^{|W|m}$ classes. The resultant partitions are exactly those induced on D^+ by $\sim$. Hence, $|\bar{D}^+| \leqslant 2^{|W|(m+1)}$. ■

Proposition 11.2 readily implies the following:

THEOREM 11.3. *For every $n \in \mathbb{N}$, the monadic fragment of the logic* $\mathbf{QAlt}_n$ *is decidable.*

PROOF. Suppose that $\varphi \notin \mathbf{QAlt}_n$. Then, $\mathfrak{M}, w_0 \not\models \varphi$, for some Kripke model $\mathfrak{M} = \langle W, R, D, I \rangle$, with $\langle W, R \rangle$ satisfying (11.1), and some $w_0 \in W$. Denote by $\mathrm{md}\,\varphi$ the modal depth of a formula φ, defined as the maximal number of nesting modalities in φ. Define the Kripke frame $\mathfrak{F}^\varphi$ by

$$\begin{array}{rcl} W^\varphi & = & R^0(w_0) \cup \ldots \cup R^{\mathrm{md}\varphi}(w_0); \\ R^\varphi & = & R \restriction W^\varphi; \\ \mathfrak{F}^\varphi & = & \langle W^\varphi, R^\varphi \rangle. \end{array}$$

Since only the worlds accessible in at most $\mathrm{md}\,\varphi$ steps from w_0 affect the truth of φ at w_0, surely $\mathfrak{F}^\varphi, w_0 \not\models \varphi$. Define the number n_φ by

$$n_\varphi \quad = \quad 1 + n + n^2 + \ldots + n^{\mathrm{md}\,\varphi}.$$

Since $\mathfrak{F}$, and hence $\mathfrak{F}^\varphi$, satisfies (11.1), W^φ contains at most n_φ worlds. Let the number of monadic predicate letters in φ be m. By Proposition 11.2, φ is refuted on an augmented frame over $\mathfrak{F}^\varphi$ with at most $2^{n_\varphi(m+1)}$ individuals.

Thus, to check if $\varphi \in \mathbf{QAlt}_n$, it is enough to check if φ is valid on every augmented frame satisfying (11.1) and containing at most n_φ worlds and $2^{n_\varphi(m+1)}$ individuals. ■

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