Funmilola Balogun
Benedikt Löwe

Independence Results for Finite Set Theories in Well-Founded Locally Finite Graphs

Abstract. We consider all combinatorially possible systems corresponding to subsets of finite set theory (i.e., Zermelo-Fraenkel set theory without the axiom of infinity) and for each of them either provide a well-founded locally finite graph that is a model of that theory or show that this is impossible. To that end, we develop the technique of *axiom closure of graphs*.

Keywords: Finite set theory, Axiomatic set theory, Locally finite graphs, Independence results, End extensions.

1. Introduction

Relative independence results in set theory play a crucial role in the field: a large part of current set theoretic research deals with the determination of the relative consistency of set-theoretic statements, either independent of the axioms of ZFC or fragments of ZFC. The most prominent such programme is the study of relative strength of fragments of the axiom of choice, i.e., statements A such that $\mathsf{ZFC} \vdash \mathsf{A}$, but $\mathsf{ZF} \nvdash \mathsf{A}$; cf., e.g., [3,8,9]. Set theorists have also looked at systems weaker than ZF; cf., e.g., [2,5–7,12,13].

Finite set theory FST consists of all of the axioms of ZF without the axiom of infinity and has the model **HF** of hereditarily finite sets as its canonical model. In this paper, we shall consider all combinatorially possible systems corresponding to subsets of the axioms of finite set theory and develop a general technique called *axiom closure of graphs*; for each of the $2^6 = 64$ combinatorially possible systems we shall either show that it cannot hold in a transitive submodel of the hereditarily finite sets or provide a concrete model in which it holds (cf. Table 1).

Presented by **Heinrich Wansing**; *Received* May 20, 2023

Studia Logica
https://doi.org/10.1007/s11225-023-10087-w

Published online: 16 February 2024

Table 1. Summary of the results for well-founded locally finite graphs

	$\cancel{\mathtt{C}}\cancel{\mathtt{R}}$	$\cancel{\mathtt{C}}\mathtt{R}$	$\mathtt{C}\cancel{\mathtt{R}}$	$\mathtt{CR}$
$\cancel{\mathtt{S}}\cancel{\mathtt{D}}\cancel{\mathtt{U}}\cancel{\mathtt{P}}$	$\mathbf{G}^\star$	**Prop. 4**	$\mathbf{G}^\#$	**Prop. 9**
$\mathtt{S}\cancel{\mathtt{D}}\cancel{\mathtt{U}}\cancel{\mathtt{P}}$	$\mathbf{G}^\star_{\mathtt{S}}$	**Prop. 4**	$\mathbf{G}^\#_{\mathtt{S}}$	**Prop. 9**
$\cancel{\mathtt{S}}\mathtt{D}\cancel{\mathtt{U}}\cancel{\mathtt{P}}$	**Prop. 3**	**Prop. 4**	**Prop. 3**	**Prop. 3**
$\cancel{\mathtt{S}}\cancel{\mathtt{D}}\mathtt{U}\cancel{\mathtt{P}}$	$\mathbf{G}^3$	**Prop. 4**	$\mathbf{G}^2$	$\mathbf{G}^1$
$\cancel{\mathtt{S}}\cancel{\mathtt{D}}\cancel{\mathtt{U}}\mathtt{P}$	$\mathbf{G}^\star_{\mathtt{P}}$	**Prop. 4**	**Prop. 10**	**Prop. 9**
$\mathtt{SD}\cancel{\mathtt{U}}\cancel{\mathtt{P}}$	$\mathbf{G}^5_{\mathtt{D}}$	**Prop. 4**	$\mathbf{G}^4_{\mathtt{D}}$	$\mathbf{G}^1_{\mathtt{D}}$
$\mathtt{S}\cancel{\mathtt{D}}\mathtt{U}\cancel{\mathtt{P}}$	$\mathbf{G}^4_{\mathtt{S}}$	**Prop. 4**	$\mathbf{G}^3_{\mathtt{S}}$	$\mathbf{G}^1_{\mathtt{S}}$
$\mathtt{S}\cancel{\mathtt{D}}\cancel{\mathtt{U}}\mathtt{P}$	$\mathbf{G}^\star_{\mathtt{SP}}$	**Prop. 4**	*Prop. 20*	**Prop. 9**
$\cancel{\mathtt{S}}\mathtt{DU}\cancel{\mathtt{P}}$	**Prop. 3**	**Prop. 4**	**Prop. 3**	**Prop. 3**
$\cancel{\mathtt{S}}\mathtt{D}\cancel{\mathtt{U}}\mathtt{P}$	**Prop. 3**	**Prop. 4**	**Prop. 3**	**Prop. 3**
$\cancel{\mathtt{S}}\cancel{\mathtt{D}}\mathtt{UP}$	$\mathbf{G}^1_{\mathtt{P}}$	**Prop. 4**	**Prop. 10**	**Prop. 10**
$\cancel{\mathtt{S}}\mathtt{DUP}$	**Prop. 3**	**Prop. 4**	**Prop. 3**	**Prop. 3**
$\mathtt{S}\cancel{\mathtt{D}}\mathtt{UP}$	$\mathbf{G}^1_{\mathtt{SP}}$	**Prop. 4**	*Prop. 20*	**Prop. 11**
$\mathtt{SD}\cancel{\mathtt{U}}\mathtt{P}$	$\mathbf{G}^\star_{\mathtt{DP}}$	**Prop. 4**	*Prop. 20*	*Prop. 20*
$\mathtt{SDU}\cancel{\mathtt{P}}$	*Prop. 19*	**Prop. 4**	*Prop. 19*	*Prop. 19*
$\mathtt{SDUP}$	*Prop. 19*	**Prop. 4**	*Prop. 19*	**HF**

The notation for the axiom systems is defined in Section 4.1. The table either lists the graph in which the theory holds (using the notation from Sections 2.4 and 4.3) or the result that it cannot hold in a well-founded locally finite graph: inconsistencies are in **boldface**, impossibility results for transitive submodels of **HF** are in *italics*

As mentioned, in this paper, we are studying transitive submodels of the hereditarily finite sets; in particular, all of our models are well-founded; for a discussion of not necessarily wellfounded ω-models of FST, cf. [1].

Apart from independence analyses as in this paper, the technique of *axiom closure of graphs* can serve as a useful tool in undergraduate education of axiomatic set theory: concrete manipulation of graph models allows students to develop an intuition for the meaning of defined terms such as "power set" in graphs and thereby gain a more concrete understanding of independence phenomena in set theory. This educational approach has been successfully used for student research projects in the second and third year of undergraduate programmes in mathematics [10, 11].

2. Definitions

2.1. Graph Models of Set Theory

The language of set theory $\mathcal{L}_\in$ is the first-order language with a single binary relation symbol $\in$. Its structures are sets with a binary relation, i.e., *(directed) graphs* $\mathbf{G} = (V, E)$. As usual, we write $v\,E\,w$ for $(v, w) \in E$. If $\mathbf{G} = (V, E)$ is a graph and $v \in V$ is a vertex, we call $\mathrm{Ext}_{\mathbf{G}}(v) := \{w \in$

$V\,;\, w\,E\,v\}$ the *extension of* v. We call $\deg_{\mathbf{G}}(v) := |\mathrm{Ext}_{\mathbf{G}}(v)|$ the *degree* of v.[1] If $v, w \in V$, we say that v is a $\mathbf{G}$-*subset* of w if $\mathrm{Ext}_{\mathbf{G}}(v) \subseteq \mathrm{Ext}_{\mathbf{G}}(w)$ or, equivalently, if $\mathbf{G} \models \forall z(z \in v \rightarrow z \in w)$.

A graph is called *extensional* if for $v, w \in V$ with $\mathrm{Ext}_{\mathbf{G}}(v) = \mathrm{Ext}_{\mathbf{G}}(w)$, we have $v = w$. It is called *well-founded* if for every non-empty $A \subseteq V$ there is a $v \in A$ such that such that $\mathrm{Ext}_{\mathbf{G}}(v) \cap A = \varnothing$.

If κ is a cardinal, we say that a graph is *locally κ-small* if for all $v \in V$, $\deg_{\mathbf{G}}(v) < \kappa$. If $\kappa = \aleph_0$, we say that the graph is *locally finite*.[2]

By Mostowski's collapsing theorem, a locally κ-small, extensional, well-founded graph is isomorphic to a transitive subset of the set $\mathbf{H}_\kappa$ of sets of hereditary size less than κ. In particular, every locally finite, extensional, well-founded graph is isomorphic to a transitive subset of the set $\mathbf{HF} := \mathbf{H}_{\aleph_0} = \mathbf{V}_\omega$ of hereditarily finite sets.

A graph $\mathbf{G} = (V, E)$ is a substructure of another graph $\mathbf{G}' = (V', E')$ if $E' \cap V \times V = E$. We say that $\mathbf{G}$ is *transitive* in $\mathbf{G}'$ if for every $v \in V$ and $w\,E'\,v$, we have that $w \in V$; alternatively, we say that $\mathbf{G}'$ is an *end extension* of $\mathbf{G}$. We say that a formula φ with n free variables is *absolute for end extensions* if for any $\mathbf{G}$ transitive in $\mathbf{G}'$ and any n-tuple $(v_1, ..., v_n) \in V^n$, we have that $\mathbf{G} \models \varphi(v_1, ..., v_n)$ if and only if $\mathbf{G}' \models \varphi(v_1, ..., v_n)$. It is well-known that Δ_0 formulae are absolute for end extensions [4, Section IV.3].

If $(\mathbf{G}_n; n \in \mathbb{N})$ is a sequence of graphs $\mathbf{G}_n = (V_n, E_n)$ such that for $n < m$, $\mathbf{G}_n$ is transitive in $\mathbf{G}_m$, then $\mathbf{G}_\omega := (V_\omega, E_\omega)$ with $V_\omega := \bigcup_{n\in\mathbb{N}} V_n$ and $E_\omega := \bigcup_{n\in\mathbb{N}} E_n$ is called the *limit* of the sequence and is an end extension of all graphs in the sequence.

2.2. Axioms of Set Theory

As mentioned in Section 1, we shall be studying finite set theory FST consisting of the axioms of ZF without the axiom of infinity. We shall adopt the theory consisting of the empty set axiom, the axiom of extensionality, and the axiom of foundation as our base theory, denoted by BST, in order to exclude trivial and pathological special cases:

$$\begin{array}{rl} \mathsf{Ext} & \forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)), \\ \mathsf{Empty} & \exists x \forall z (z \notin x), \text{ and} \\ \mathsf{Found} & \forall x \exists y \forall z \neg (z \in x \wedge z \in y). \end{array}$$

[1] More precisely, it is the *in-degree* of v, but we shall not use the out-degree in this paper.

[2] To get a grasp of the notion of smallness, observe that locally 2-small, well-founded, and extensional graphs are linear chains of vertices of length $\leq \omega$.

The axiom of extensionality expresses the graph theoretic property of extensionality: a graph $\mathbf{G}$ is extensional if and only if $\mathbf{G} \models \mathsf{Ext}$. The axiom of foundation implies that there are no cycles in the graph, in particular, any finite graph satisfying the axiom of foundation is well-founded. Note that if $(\mathbf{G}_n;\, n \in \omega)$ is a sequence such that each $\mathbf{G}_n$ is a well-founded graph which is an end extension of all previous graphs, then the limit $\mathbf{G}_\omega$ is well-founded as well.

As usual, a formula Ψ in $n+1$ free variables is called *functional* in a model M if for any given tuple $(x_1, ..., x_n)$, there is at most one y such that $\Psi(y, x_1, ..., x_n)$; in other words, Ψ defines an n-ary partial class function in M.

An axiom A is called *functional* and *n-ary* if there is a formula Φ with $n+1$ free variables called its *functional description* such that

$$\mathsf{A} := \forall x_1 \cdots \forall x_n \exists y \Delta_\Phi(y, x_1, ..., x_n)$$

where $\Delta_\Phi(y, x_1, ..., x_n) := \forall z(z \in y \leftrightarrow \Phi(z, x_1, ..., x_n))$. The formula Δ_Φ is called the *Φ-comprehension*; in models of Ext, the formula Δ_Φ is functional.

All single axioms of FST except for Ext and Found are functional. The axiom schema of separation Sep consists of functional axioms Sep_φ for every formula φ. The axiom schema of replacement Repl consists of axioms Repl_φ for every formula φ with $n+2$ free variables given by

$$(\forall x_1 \ldots \forall x_n \forall u \forall z \forall z' \varphi(u, z, x_1, \ldots, x_n) \wedge \varphi(u, z', x_1, \ldots, x_n) \to z = z') \to \mathsf{R}_\varphi$$

where R_φ is functional. Note that if φ provably satisfies the antecedent of Repl_φ, then Repl_φ and R_φ are equivalent (and hence Repl_φ is functional). In the following, we shall give the functional descriptions for all these axioms.[3]

$$\begin{array}{rl}
\mathsf{Empty} & \Phi_\varnothing(z) \iff \bot, \\
\mathsf{Singleton} & \Phi_{\mathtt{S}}(z, x_1) \iff z = x_1, \\
\mathsf{Pair} & \Phi_{\mathtt{D}}(z, x_1, x_2) \iff z = x_1 \vee z = x_2, \\
\mathsf{Union} & \Phi_{\mathtt{U}}(z, x_1) \iff \exists w(w \in x_1 \wedge z \in w), \\
\mathsf{Powerset} & \Phi_{\mathtt{P}}(z, x_1) \iff \forall w(w \in z \to w \in x_1), \\
\mathsf{Sep}_\varphi & \Phi_{\mathtt{C},\varphi}(z, x_1, ..., x_n, x_{n+1}) \iff z \in x_{n+1} \wedge \varphi(z, x_1, ..., x_n), \text{ and} \\
\mathsf{R}_\varphi & \Phi_{\mathtt{R},\varphi}(z, x_1, ..., x_n, x_{n+1}) \iff \exists u(u \in x_{n+1} \wedge \varphi(u, z, x_1, ..., x_n)).
\end{array}$$

[3]Here $\mathtt{D}$ and $\mathtt{C}$ stand for “doubleton” and “comprehension” to avoid the notational clash between “pairing” & “power set” and “singleton” & “separation”, respectively.

We call the axioms Singleton, Pair, Union, and Powerset *basic axioms* to distinguish them from the axiom schemata Sep and Repl. If for any functional description Φ, the comprehension Δ_Φ is Δ_0, then the formula

$$u \in v \leftrightarrow \Delta_\Phi(u, v_1, ..., v_n)$$

is absolute for end extensions. In the cases of the axioms Empty, Singleton, and Pair, the formulae $\Phi_\varnothing$, Φ_{S}, and Φ_{D} are quantifier-free, so their comprehensions are Δ_0; in the case of Union, the comprehension $\Delta_{\Phi_{\mathsf{U}}}$ is equivalent to a Δ_0-formula (provably in predicate logic).

This is not necessarily true for the other axioms; e.g., it is possible to have vertices v and w in $\mathbf{G}$ such that w is the $\mathbf{G}$-power set of v, but in an end extension $\mathbf{G}'$, there are more $\mathbf{G}'$-subsets of v and w is not the $\mathbf{G}'$-power set of v anymore (cf. the proof of Proposition 15).

For arbitrary formulae φ, the functional descriptions of the axioms Sep_φ or Repl_φ are not absolute for end extensions, but the simplest instances of separation and replacement are formed with quantifier-free formulae and thus have absolute comprehensions: consider the formula (in $n+1$ free variables)

$$\varphi(z, x_1, \ldots, x_n) \iff \bigvee_{i=1}^{n} z = x_i, \tag{*}$$

and a graph $\mathbf{G}$ with distinct vertices $v_1, \ldots, v_{n+1}$ such that $\{v_1, \ldots, v_n\} \subseteq \mathrm{Ext}_{\mathbf{G}}(v_{n+1})$. Then instantiating x_i in Sep_φ with v_i states in every end extension $\mathbf{G}'$ of $\mathbf{G}$ the existence of a vertex w with $\mathrm{Ext}_{\mathbf{G}}(w) = \{v_1, \ldots, v_n\}$. Similarly, consider the formula (in $2n+2$ free variables)

$$\psi(u, z, x_1, \ldots, x_n, y_1, \ldots, y_n) \iff \bigvee_{i=1}^{n} (u = x_i \wedge z = y_i) \tag{†}$$

and observe that ψ provably satisfies the antecedent of Repl_ψ, so Repl_ψ is a functional axiom. If $\mathbf{G}$ is a graph with distinct vertices $v_1, \ldots, v_{n+1}$ and distinct vertices $w_1, \ldots, w_n$ such that $\{v_1, \ldots, v_n\} \subseteq \mathrm{Ext}_{\mathbf{G}}(v_{n+1})$, instantiating x_i and y_i in Repl_ψ with v_i and w_i, respectively, states in every end extension $\mathbf{G}'$ of $\mathbf{G}$ the existence of a vertex w with $\mathrm{Ext}_{\mathbf{G}}(w) = \{v_1, \ldots, v_n\}$. We shall call an instantiation of the axiom schemata of separation and replacement *explicit* if it is either of the form Sep_φ with φ as in (*) or of the form Repl_ψ with ψ as in (†).

Proposition 1.

(1) If $\mathbf{G} \models \mathsf{BST} + \mathsf{Sep}$*, then for all vertices* v *and all finite* $A \subseteq \mathrm{Ext}_{\mathbf{G}}(v)$*, there is a vertex* w *such that* $A = \mathrm{Ext}_{\mathbf{G}}(w)$*. If* $\mathbf{G}$ *is locally finite, the converse holds.*

(2) If $\mathbf{G} \models \mathsf{BST} + \mathsf{Repl}$*, then if there is a vertex in* $\mathbf{G}$ *of degree* k*, then for every set* $A \subseteq V$ *with* $|A| \leq k$*, there is a vertex* w *such that* $A = \mathrm{Ext}_{\mathbf{G}}(w)$*. If* $\mathbf{G}$ *is locally finite, the converse holds.*

Proof. The explicit instances of Sep and Repl yield the forward directions of both (1) and (2), respectively.

For the converse of (1), let $\mathbf{G}$ be locally finite, $v \in V$, φ an arbitrary formula with $n+1$ free variables, and

$$A := \{w;\, \mathbf{G} \models w \in v \wedge \varphi(w, \vec{v})\} \subseteq \mathrm{Ext}_{\mathbf{G}}(v).$$

Since $\mathbf{G}$ is locally finite, A is finite, by assumption, there is a vertex w such that $A = \mathrm{Ext}_{\mathbf{G}}(w)$. This witnesses the required instance of Sep_{φ}.

Similarly, for the converse of (2), let $\mathbf{G}$ be locally finite, $v \in V$, φ an arbitrary formula with $n+2$ free variables, and

$$A := \{w;\, \mathbf{G} \models \varphi(w, \vec{v}, v)\}.$$

Since $\mathbf{G}$ is locally finite, $\mathrm{Ext}_{\mathbf{G}}(v)$ is finite, say, of cardinality k and $|A| \leq k$. The assumption implies that the required instance of Repl_{φ} holds. ■

Proposition 1 and its proof express that for locally finite graphs, the axiom schemata of Separation and Replacement hold if and only if all of their explicit instances hold.

If $\mathbf{G} \models \mathsf{BST}$, A is any n-ary functional axiom with functional description Φ, and k is a natural number, we say that A is *k-violated* if there is a $\vec{v} \in V^n$ such that $\mathbf{G} \models \neg\exists y \Delta_{\Phi}(y, \vec{v})$ and if $\mathbf{G}'$ is an end extension of $\mathbf{G}$ with vertex w such that $\mathbf{G}' \models \Delta_{\Phi}(w, \vec{v})$, then $|\mathrm{Ext}_{\mathbf{G}'}(w)| \geq k$. In words: an axiom A is k-violated in $\mathbf{G}$ if A's failure in $\mathbf{G}$ is witnessed by an instance that can only be made true in an end extension of $\mathbf{G}$ by a vertex with at least k elements.

Remark 2. By the above discussion, the axioms $\mathsf{Powerset}$, Union, Sep, and Repl are k-violated in a graph $\mathbf{G}$ if the following properties hold:

$\mathsf{Powerset}$: There is a vertex v without a power set in $\mathbf{G}$ such that v has at least k $\mathbf{G}$-subsets.

Union: There is a vertex v with $|\bigcup\{\mathrm{Ext}_{\mathbf{G}}(w);\, \mathbf{G} \models w \in v\}| \geq k$ without union in $\mathbf{G}$.

Sep: There is a vertex v and $A \subseteq \mathrm{Ext}_{\mathbf{G}}(v)$ with $|A| \geq k$ that is not instantiated in $\mathbf{G}$.

Repl: There is a vertex of degree at least k and a k-element subset $A \subseteq V$ that is not instantiated in $\mathbf{G}$.

2.3. Set Theoretic Axioms in Graph Models

We collate some basic observations about the validity of set theoretic axioms in graph models. All of the results in this section are elementary; several of them are well-known arguments that frequently feature as exercises in an axiomatic set theory course. We include them only for the sake of completeness.

PROPOSITION 3. *In* BST*,* Pair *implies* Singleton*.*

PROOF. Trivial. ∎

PROPOSITION 4. *In* BST*,* Repl *implies* Sep*.*

PROOF. Let v be an arbitrary vertex and find v' such that

$$\mathrm{Ext}_{\mathbf{G}}(v') = \{z;\, \mathbf{G} \models z \in v \wedge \varphi(z, \vec{v})\} =: A$$

for some formula φ and some sequence of vertices $\vec{v}$. Without loss of generality, we can assume that A is not empty, so let $w \in A$. Use replacement to find v' with

$$\mathrm{Ext}_{\mathbf{G}}(v') = \{z;\, \mathbf{G} \models \exists z\, ((z \in v \wedge \varphi(z, \vec{v})) \vee (\neg\varphi(z, \vec{v}) \wedge z = w))\}.$$

Clearly, $\mathrm{Ext}_{\mathbf{G}}(v') = A$. ∎

PROPOSITION 5. *No finite graph* $\mathbf{G} \models$ BST *can be a model of either* Singleton*,* Pair*, or* Powerset*. Furthermore, for every natural number k, a graph model* $\mathbf{G} \models$ BST + Powerset *has a vertex with degree* $\geq k$*.*

PROOF. Let A be one of the three axioms. Starting from the vertex v_0 representing the empty set, we can recursively define v_{n+1} to be the result of applying A to v_n. It is easy to show by induction that this forms an infinite sequence of distinct vertices.

In the case of A = Powerset, a standard induction shows that for all natural numbers i, if $x \mathrel{E} v_i$, then x is a $\mathbf{G}$-subset of v_i.[4] Therefore, for all i, we have that v_i is a $\mathbf{G}$-subset of v_{i+1}. We use this to prove by induction that for all k, $\{v_0, ..., v_{k-1}\} \subseteq \mathrm{Ext}_{\mathbf{G}}(v_k)$, so $\deg_{\mathbf{G}}(v_k) \geq k$. ∎

[4] The induction step of this argument is usually phrased as "the power set of a transitive set is transitive".

In the proofs of Propositions 6 & 7, we shall be utilising the characterisation of Sep and Repl in locally finite graphs given in Proposition 1.

Proposition 6. *If* $\mathbf{G} \models \mathsf{BST}$ *is locally* 2*-small, then*

(i) Union *and* Sep *hold and*

(ii) if Singleton *holds, then* Repl *holds.*

Proof. Being locally 2-small means that all vertices have degree at most one, i.e., are empty or singletons. The only possible subsets of a singleton are the empty set and itself; thus (assuming Empty), no singleton can ever witness $\neg$Sep; hence, Sep holds. The union of the empty set is the empty set and the union of a singleton is its unique element, so Union holds. Claim *(ii)* follows directly from Proposition 1. ■

Proposition 7. *If* $\mathbf{G} \models \mathsf{BST}$ *is locally* 3*-small, then*

(i) if Singleton *holds, then* Sep *holds and*

(ii) if Pair *holds, then* Repl *holds.*

Proof. Being locally 3-small means that all vertices have degree at most two, so each set can have at most four $\mathbf{G}$-subsets: the empty set, at most two singletons, and itself. The assumption of Singleton implies that they all exist in $\mathbf{G}$. Claim *(ii)* follows directly from Proposition 1. ■

A similar argument shows that if $\mathbf{G} \models \mathsf{BST}$ is locally 4-small and Pair holds, then Sep holds; furthermore, if $\mathbf{G} \models \mathsf{BST}$ is locally finite and Pair and Union hold, then Repl holds (cf. the proof of Proposition 19).

Proposition 8. *If* $\mathbf{G} \models \mathsf{BST} + \mathsf{Repl}$ *is not* 2*-small, then* $\mathbf{G} \models \mathsf{Pair}$.

Proof. Not being 2-small means that there is a set of size at least two; the claim follows directly from Proposition 1 *(2)*. ■

Proposition 9. *In* BST*,* Repl *implies* Pair $\vee$ Union*.*

Proof. Let $\mathbf{G} \models \mathsf{BST}$. If $\mathbf{G}$ is locally 2-small, then Union holds by Proposition 6; if $\mathbf{G}$ is not locally 2-small, then Pair holds by Proposition 8. ■

Proposition 10. *In* BST*,* Sep + Powerset *implies* Singleton*.*

Proof. Let $\mathbf{G} \models \mathsf{BST}$. If v is a vertex and w is the power set of v in $\mathbf{G}$, then $\mathbf{G} \models v \in w$. Thus, we can now separate the singleton of v from w by an explicit instance of separation.

Proposition 11. *In* BST*,* Repl + Powerset *implies* Pair*.*

Proof. By Proposition 8, we only need to find a set of size at least two. This can be obtained by applying the power set axiom twice to the empty set.

2.4. Some Concrete Finite Graphs

Finite extensional graphs are isomorphic to a unique transitive element of **HF**, so it is possible to specify a graph by its Mostowski image.

The simplest possible graph is $\mathbf{G}^1$, the unique graph whose Mostowski image is $1 = \{0\}$. This is the graph with a single vertex and no edges. It is easy to check that

$$\mathbf{G}^1 \models \mathsf{BST} + \neg\mathsf{Singleton} + \neg\mathsf{Pair} + \mathsf{Union} + \neg\mathsf{Powerset} + \mathsf{Sep} + \mathsf{Repl}.$$

The next graph is $\mathbf{G}^2$, the unique graph whose Mostowski image is $2 = \{0,1\}$. This is a finite 2-small graph where we cannot replace 0 by 1 in the set $\{0\} = 1$, thus by Proposition 6,

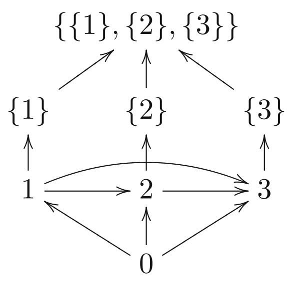
$$\mathbf{G}^2 \models \mathsf{BST} + \neg\mathsf{Singleton} + \neg\mathsf{Pair} + \mathsf{Union} + \neg\mathsf{Powerset} + \mathsf{Sep} + \neg\mathsf{Repl}.$$

Continuing along the natural numbers, the next graph is $\mathbf{G}^3$, the unique graph whose Mostowski image is $3 = \{0,1,2\}$. In this graph, we cannot separate $\{1\}$ from $2 = \{0,1\}$, so Sep fails. Hence,

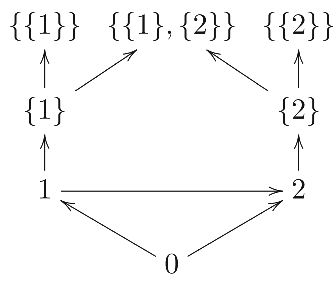
$$\mathbf{G}^3 \models \mathsf{BST} + \neg\mathsf{Singleton} + \neg\mathsf{Pair} + \mathsf{Union} + \neg\mathsf{Powerset} + \neg\mathsf{Sep} + \neg\mathsf{Repl}.$$

The same is true for $\mathbf{G}^4$ and $\mathbf{G}^5$, the unique graphs whose Mostowski images are $4 = \{0,1,2,3\}$ or $5 = \{0,1,2,3,4\}$, respectively. All examples of this type, i.e., graphs isomorphic to an ordinal, will satisfy Union, since $\bigcup \alpha+1 = \alpha$ and $\bigcup \lambda = \lambda$ for limit ordinals λ.

Moving to examples that violate Union, we define the graphs $\mathbf{G}^\star$ and $\mathbf{G}^\#$ as follows (as usual, $0 = \varnothing$, $1 = \{0\}$, $2 = \{0,1\}$, and $3 = \{0,1,2\}$).



$\mathbf{G}^\star$



$\mathbf{G}^\#$

Note that

$$\mathbf{G}^\star \models \mathsf{BST} + \neg\mathsf{Singleton} + \neg\mathsf{Pair} + \neg\mathsf{Union} + \neg\mathsf{Powerset} + \neg\mathsf{Sep} + \neg\mathsf{Repl} \text{ and}$$
$$\mathbf{G}^\# \models \mathsf{BST} + \neg\mathsf{Singleton} + \neg\mathsf{Pair} + \neg\mathsf{Union} + \neg\mathsf{Powerset} + \mathsf{Sep} + \neg\mathsf{Repl}.$$

Table 2. Level of violation of axioms in the finite graphs discussed in Section 2.4

	Union	Powerset	Sep	Repl
$\mathbf{G}^1$	–	1-violated	–	–
$\mathbf{G}^2$	–	2-violated	–	1-violated
$\mathbf{G}^3$	–	3-violated	1-violated	2-violated
$\mathbf{G}^4$	–	4-violated	2-violated	3-violated
$\mathbf{G}^5$	–	5-violated	3-violated	4-violated
$\mathbf{G}^\star$	3-violated	5-violated	2-violated	3-violated
$\mathbf{G}^\#$	2-violated	3-violated	–	2-violated

Axioms that hold in the graph are marked by "–"

For each of our finite graphs, using Remark 2, we identify the level of violation of those axioms not true in it in Table 2. It is easy to check that the obvious choice of vertex in these graphs will provide the level of violation given.

3. Closure Graphs

3.1. Graph Operations and Closure Graphs

A *graph operation* is an operation $\mathbf{C}$ that assigns to each graph $\mathbf{G}$ an end extension $\mathbf{C}(\mathbf{G})$. We say that it is *extensional* if it preserves extensionality, i.e., whenever $\mathbf{G}$ is extensional, so is $\mathbf{C}(\mathbf{G})$; we say that it is *finite* if it preserves finiteness, i.e., whenever $\mathbf{G}$ is finite, so is $\mathbf{C}(\mathbf{G})$. If $\Gamma := (\mathbf{C}_n\,;\, n \in \omega)$ is a sequence of graph operations, we call Γ extensional or finite if every graph operation occurring in it is extensional or finite, respectively. We define the *closure graph of* $\mathbf{G}$ *with respect to* Γ by recursion:

$$\mathbf{G}_0 := \mathbf{G},$$
$$\mathbf{G}_{n+1} := \mathbf{C}_n(\mathbf{G}_n),$$

and $\mathbf{G}_\Gamma$ is the limit of the sequence $(\mathbf{G}_n;\, n \in \omega)$.

Proposition 12. *If $\mathbf{G}$ and Γ are extensional and finite, then $\mathbf{G}_\Gamma$ is extensional, well-founded, and locally finite, hence isomorphic to a transitive subset of* $\mathbf{HF}$.

Proof. The claim follows directly from the definitions by induction and the fact that limits preserve well-foundedness.

3.2. Axiom Closures

For functional axioms, the comprehension Δ_Φ defines a total (class) function. An *axiom closure* of a graph $\mathbf{G}$ is a graph obtained from $\mathbf{G}$ as a closure graph $\mathbf{G}_\Gamma$ of a graph operation that corresponds to adding the witnesses needed to make all $\mathbf{G}$-instances of the axiom A true. The intuitive idea of an axiom closure is the following:

> Given a graph $\mathbf{G} = (V, E)$, check whether Δ_Φ already defines a function; if not, then there are tuples $\vec{v}$ where there is no vertex w such that $\mathbf{G} \models \Delta_\Phi(w, \vec{v})$. Now add a new vertex to become the witness for that formula in a way that preserves extensionality.

In the following, we shall make this precise. If $\mathbf{G} = (V, E)$ is a graph, Φ is a formula with $n+1$ free variables, and $\vec{v} := (v_1, ..., v_n) \in V^n$ is an n-tuple, we say that $\vec{v}$ is *Φ-realised in* $\mathbf{G}$ if

$$\mathbf{G} \models \exists y \Delta_\Phi(y, v_1, ..., v_n);$$

otherwise, we say that $\vec{v}$ is *Φ-omitted in* $\mathbf{G}$.

As mentioned, we should like to add new vertices to $\mathbf{G}$ that make sure that every tuple that is Φ-omitted in $\mathbf{G}$ becomes Φ-realised in $\mathbf{C}(\mathbf{G})$. But we need to make sure that we do not break extensionality: it might be that two Φ-omitted tuples $\vec{v} \neq \vec{w}$ both require the same Φ-comprehension to exist and we need to make sure that we do not add two distinct new vertices with the same extensions.[5] Thus, we define an equivalence relation $\sim_\Phi$ on V^n by

$$\vec{v} \sim_\Phi \vec{w} :\Longleftrightarrow \mathbf{G} \models \forall z(\Phi(z, \vec{v}) \leftrightarrow \Phi(z, \vec{w})).$$

Clearly, if a tuple is Φ-omitted, then so is every $\sim_\Phi$-equivalent tuple. Thus, let $O_{\mathbf{G},\Phi}$ be a set of n-tuples that has exactly one representative from each $\sim_\Phi$-equivalence class of Φ-omitted tuples. Let $V' := \{w_{\vec{v}}\,;\, \vec{v} \in O_{\mathbf{G},\Phi}\}$ be a set of pairwise distinct new vertices, i.e., $V \cap V' = \varnothing$. Then define $\mathbf{C}_\Phi(\mathbf{G}) := (V^*, E^*)$ where $V^* := V \cup V'$ and

$$E^* := E \cup \{(z, w_{\vec{v}}); \vec{v} \in O_{\mathbf{G},\Phi} \text{ and } \mathbf{G} \models \Phi(z, \vec{v})\}.$$

By construction, $\mathbf{C}_\Phi$ is an extensional graph operation. If V is finite, then so is V^n, and so $\mathbf{C}_\Phi$ is a finite graph operation. If Φ is one of our formulae defining the basic axioms of set theory, we also write $\mathbf{C}_\mathtt{S}$, $\mathbf{C}_\mathtt{D}$, $\mathbf{C}_\mathtt{U}$, or $\mathbf{C}_\mathtt{P}$ for $\mathbf{C}_{\Phi_\mathtt{S}}$, $\mathbf{C}_{\Phi_\mathtt{D}}$, $\mathbf{C}_{\Phi_\mathtt{U}}$, or $\mathbf{C}_{\Phi_\mathtt{P}}$, respectively.

---

[5] A concrete example is the formula $\Phi_\mathtt{D}$: if $v \neq w$, then both $\Delta_{\Phi_\mathtt{D}}(\cdot, v, w)$ and $\Delta_{\Phi_\mathtt{D}}(\cdot, w, v)$ define the unordered pair $\{v, w\}$.

To illustrate this construction, consider the unary functional axiom $\mathsf{Singleton}$ with functional description $\Phi_{\mathsf{S}}(z, x_1) \iff z = x_1$. Let $\mathbf{G} = (V, E)$ be any graph. We consider every single vertex $v \in V$ and check whether the singleton of v exists in $\mathbf{G}$, i.e., whether there is a vertex w such that $\mathrm{Ext}_{\mathbf{G}}(w) = \{v\}$. A vertex whose singleton exists is *realised*; otherwise it is *omitted*. Note that no two vertices are $\sim_{\Phi_{\mathsf{S}}}$-equivalent, so $O_{\mathbf{G},\Phi_{\mathsf{S}}}$ is just the set of omitted vertices. For each omitted vertex v, we now add a new vertex w_v and an edge (v, w_v) and obtain $\mathbf{C}_{\mathsf{S}}(\mathbf{G})$ such that every vertex in $\mathbf{G}$ has a singleton in $\mathbf{C}_{\mathsf{S}}(\mathbf{G})$. However, $\mathbf{C}_{\mathsf{S}}(\mathbf{G})$ will contain new vertices that do not have singletons yet (e.g., any newly added w_v).

If $\mathbf{G} = \mathbf{G}^1$ and $\Gamma = (\mathbf{C}_n\,; n \in \omega)$ is the sequence such that $\mathbf{C}_n = \mathbf{C}_{\mathsf{S}}$ for all $n \in \omega$, then the closure graph $\mathbf{G}_\Gamma$ is isomorphic to the Zermelo natural numbers $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, ...\}$. Every vertex is either empty or a singleton.[6]

If $\mathbf{C}$ is a graph operation and $\Gamma = (\mathbf{C}_n; n \in \omega)$ is a sequence of graph operations, we say that $\mathbf{C}$ *occurs unboundedly in* Γ if the set $\{n \in \omega; \mathbf{C}_n = \mathbf{C}\}$ is unbounded.

PROPOSITION 13. *Let* $\mathbf{G} \models \mathsf{BST}$*,* A *an n-ary functional axiom with functional description Φ, and Γ a sequence of graph operations such that $\mathbf{C}_\Phi$ occurs unboundedly in Γ. Suppose that both Φ and Δ_Φ are absolute for end extensions. Then* $\mathbf{G}_\Gamma \models \mathsf{A}$.

PROOF. We use $\mathbf{G}_k = (V_k, E_k)$ to denote the stages of the closure construction and $\mathbf{G}_\Gamma = (V_\Gamma, E_\Gamma)$ to denote the closure graph. Let $\vec{v}$ be an n-tuple in V_Γ. We have to show that it is Φ-realised in $\mathbf{G}_\Gamma$. Find N large enough such that all elements of $\vec{v}$ occur in V_N. By the assumption, there is some stage $K > N$ where $\Gamma(K) = \mathbf{C}_\Phi$, but this means that we find $y \in V_{K+1}$ such that

$$\mathbf{G}_{K+1} \models z \in y \iff \mathbf{G}_K \models \Phi(z, \vec{v}).$$

Absoluteness of Φ lifts this to

$$\begin{aligned} &\mathbf{G}_{K+1} \models \forall z(z \in y \leftrightarrow \Phi(z, \vec{v})), \text{ i.e.,} \\ &\mathbf{G}_{K+1} \models \Delta_\Phi(y, \vec{v}). \end{aligned}$$

Now absoluteness of Δ_Φ gives $\mathbf{G}_\Gamma \models \Delta_\Phi(y, \vec{v})$, so y still Φ-realises $\vec{v}$ in $\mathbf{G}_\Gamma$. ∎

[6] Similarly, in order to get familiar with the definition of closure graphs, the reader might find it illuminating to check that if $\mathbf{G} = \mathbf{G}^1$ and $\Gamma = (\mathbf{C}_n\,; n \in \omega)$ with $\mathbf{C}_n = \mathbf{C}_{\mathsf{P}}$ for all $n \in \omega$, then the closure graph $\mathbf{G}_\Gamma$ is isomorphic to the von Neumann natural numbers.

Corollary 14. *Let* $\mathbf{G} \models \mathsf{BST}$ *and* Γ *a sequence of graph operations such that* $\mathbf{C}_{\mathtt{S}}$, $\mathbf{C}_{\mathtt{D}}$, *or* $\mathbf{C}_{\mathtt{U}}$ *occurs cofinally in* Γ. *Then* $\mathbf{G}_\Gamma$ *satisfies* Singleton, Pair, *or* Union, *respectively.*

Proof. Follows directly from Proposition 13 and the fact that all corresponding formulae are absolute for end extensions. ■

Note that Proposition 13 does not apply to the axiom Powerset, since the comprehension of its functional description $\Delta_{\Phi_{\mathtt{P}}}$ is not absolute: as mentioned, an end extension $\mathbf{G}'$ of a graph $\mathbf{G}$ can contain more $\mathbf{G}'$-subsets of a given vertex $v \in V$.

Proposition 15. *Let* $\mathbf{G} \models \mathsf{BST}$ *be finite and* Γ *an extensional and finite sequence of graph operations such that* $\mathbf{C}_{\mathtt{P}}$ *occurs cofinally in* Γ. *Then* $\mathbf{G}_\Gamma$ *satisfies* Powerset.

Proof. We use the notation from the proof of Proposition 13. By Proposition 12, we know that $\mathbf{G}_\Gamma$ is locally finite; thus for any $v \in V_\Gamma$, we have that $\deg_{\mathbf{G}_\Gamma}(v) = n$ for some natural number n. This means that there are at most 2^n many $\mathbf{G}_\Gamma$-subsets of v. But this means that there is some N such that all $\mathbf{G}_\Gamma$-subsets of v have appeared before stage N of the construction. By our assumption, there is some $K > N$ such that $\Gamma(K) = \mathbf{C}_{\mathtt{P}}$, but then the $\mathbf{G}_{K+1}$-power set of v is the $\mathbf{G}_\Gamma$-power set of v. ■

4. Consistency and Inconsistency Results

4.1. Notational Set-up

In this section, we consider all $2^6 = 64$ possible combinations of Singleton, Pair, Union, Powerset, Sep, and Repl and their negations and check whether they can hold in a transitive submodel of **HF**, or equivalently, in an extensional, well-founded, locally finite graph model. In order to keep the notation brief, we use the labels $\mathtt{S}$, $\mathtt{D}$, $\mathtt{U}$, $\mathtt{P}$, $\mathtt{C}$, and $\mathtt{R}$ for the six axioms and $\not\mathtt{S}$, $\not\mathtt{D}$, $\not\mathtt{U}$, $\not\mathtt{P}$, $\not\mathtt{C}$, and $\not\mathtt{R}$ for their negations.[7]

By Proposition 4, Repl implies Sep, excluding all $2^4 = 16$ combinations with $\not\mathtt{C}\mathtt{R}$. Proposition 3 makes any theory containing $\not\mathtt{S}\mathtt{D}$ trivially inconsistent, immediately removing $12 = 2^2 \times 3$ additional theories from the list. Furthermore, Proposition 9 means that any theory with $\not\mathtt{D}\not\mathtt{U}\mathtt{R}$ cannot hold, removing four more theories; Proposition 10 means that any theory with $\not\mathtt{S}\mathtt{P}\mathtt{C}$ cannot hold, removing three more theories; finally, Proposition 11 means that

[7] Cf. Footnote 3 for the notations $\mathtt{D}$ and $\mathtt{C}$.

any theory with $\mathtt{\xcancel{D}PR}$ cannot hold, removing another theory. These basic axiomatic connections from Section 2.3 remove 36 of our 64 combinatorially possible theories; consequently, 28 remain.

4.2. Finite Graphs

By Proposition 5, only theories containing $\mathtt{\xcancel{S}\xcancel{D}\xcancel{P}}$ can hold in finite models. Using our examples from Section 2.4, we see that

$$\mathbf{G}^1 \models \mathtt{\xcancel{S}\xcancel{D}U\xcancel{P}CR}$$

$$\mathbf{G}^2 \models \mathtt{\xcancel{S}\xcancel{D}U\xcancel{P}C\xcancel{R}},$$

$$\mathbf{G}^3 \models \mathtt{\xcancel{S}\xcancel{D}U\xcancel{P}\xcancel{C}\xcancel{R}},$$

$$\mathbf{G}^\star \models \mathtt{\xcancel{S}\xcancel{D}\xcancel{U}\xcancel{P}C\xcancel{R}}, \text{ and}$$

$$\mathbf{G}^\# \models \mathtt{\xcancel{S}\xcancel{D}\xcancel{U}\xcancel{P}\xcancel{C}\xcancel{R}}.$$

4.3. Graphs Obtained by Axiom Closures

We shall consider a number of different sequences of graph operations: $\mathit{\Gamma}_{\mathtt{S}}$, $\mathit{\Gamma}_{\mathtt{D}}$, and $\mathit{\Gamma}_{\mathtt{P}}$ are the sequences where each graph operation is $\mathbf{C}_{\mathtt{S}}$, $\mathbf{C}_{\mathtt{D}}$, or $\mathbf{C}_{\mathtt{P}}$, respectively. For $\mathtt{X}, \mathtt{Y} \in \{\mathtt{S}, \mathtt{D}, \mathtt{P}\}$, we write $\mathit{\Gamma}_{\mathtt{XY}}$ for the sequence where all even-numbered graph operations are $\mathbf{C}_{\mathtt{X}}$ and all odd-numbered graph operations are $\mathbf{C}_{\mathtt{Y}}$. If $\mathbf{G}$ is any graph, we write $\mathbf{G}_{\mathtt{X}}$ or $\mathbf{G}_{\mathtt{XY}}$ for the closure graph by $\mathit{\Gamma}_{\mathtt{X}}$ or $\mathit{\Gamma}_{\mathtt{XY}}$, respectively.

Lemma 16. *If* A *is* 2*-violated in* $\mathbf{G}$*, then* $\mathbf{G}_{\mathtt{S}} \models \neg\mathsf{A}$*; If* A *is* 3*-violated in* $\mathbf{G}$*, then* $\mathbf{G}_{\mathtt{D}} \models \neg\mathsf{A}$*.*

Proof. Note that the graph operation $\mathbf{C}_{\mathtt{S}}$ only adds new vertices of degree one and the graph operation $\mathbf{C}_{\mathtt{D}}$ only adds new vertices of degree at most two, so no newly added vertex can ever be the witness to the instance of A that is violated in $\mathbf{G}$. ■

Proposition 17. *If* $\mathbf{G}$ *is any finite graph, then*

$$\mathbf{G}_{\mathtt{P}} \models \mathtt{\xcancel{S}\xcancel{D}P\xcancel{C}\xcancel{R}},$$

$$\mathbf{G}_{\mathtt{SP}} \models \mathtt{S\xcancel{D}P\xcancel{C}\xcancel{R}}, \textit{ and}$$

$$\mathbf{G}_{\mathtt{DP}} \models \mathtt{SDP\xcancel{C}\xcancel{R}}.$$

Proof. The positive claims follow from Corollary 14 & Proposition 15. Proposition 5 implies that the graphs $\mathbf{G}_{\mathsf{P}}$, $\mathbf{G}_{\mathsf{SP}}$, and $\mathbf{G}_{\mathsf{DP}}$ are infinite and have vertices with arbitrarily large finite extensions. Because $\mathbf{G}$ is finite, there is an n such that the degrees of vertices in $\mathbf{G}$ are bounded by n and therefore any vertex in the axiom closure graph with more than n vertices must have been added by the closure operations.

All vertices added by the closure operations are of a particular form: in $\mathbf{G}_{\mathsf{P}}$, they all contain the empty set; in $\mathbf{G}_{\mathsf{SP}}$, they all contain the empty set or are singletons; and in $\mathbf{G}_{\mathsf{DP}}$, they all contain the empty set or are singletons or pairs. This means that any vertex with degree at least $n + 3$ was added by the closure operations and since it is neither a pair nor a singleton, it must contain the empty set.

These observations imply $\xcancel{\mathrm{S}}\xcancel{\mathrm{D}}$ in $\mathbf{G}_{\mathsf{P}}$: if v is any vertex added by the closure operation and w is a vertex containing v, then w was added by the closure operation and thus, it must contain the empty set. But then it cannot be the singleton of v.

Similarly, they imply $\xcancel{\mathrm{D}}$ in $\mathbf{G}_{\mathsf{SP}}$: if v_0 and v_1 are any two distinct vertices added by the closure operations and w is a set containing v_0 and v_1, then w was added by the closure operations and thus either w is a singleton or contains the empty set; in neither case can w be the pair of v_0 and v_1.

By Proposition 4, it is sufficient to show that Sep cannot hold in any of the three axiom closures. Pick any vertex that has degree $k \geq n + 4$; by the above argument, it must have the empty set as predecessor, say, its extension is $\{v_0, ..., v_{k-1}\}$ with distinct vertices where $\mathrm{Ext}_{\mathbf{G}}(v_0) = \varnothing$. If Sep holds, we can find a vertex with extension $\{v_1, ..., v_{k-1}\}$. But that has $k - 1 \geq (n + 4) - 1 = n + 3$ elements, and thus would have to contain the empty set. Contradiction! ■

Lemma 18. *(Preservation Lemma) Let* $\mathbf{G}$ *be a graph.*

(1) If $\mathbf{G} \models \mathsf{Union}$*, then so do* $\mathbf{C}_{\mathsf{S}}(\mathbf{G})$*,* $\mathbf{C}_{\mathsf{P}}(\mathbf{G})$*,* $\mathbf{G}_{\mathsf{S}}$*, and* $\mathbf{G}_{\mathsf{P}}$*.*

(2) If $\mathbf{G}$ *is finite and* $\mathbf{G} \models \mathsf{Sep}$*, then so do* $\mathbf{C}_{\mathsf{S}}(\mathbf{G})$*,* $\mathbf{C}_{\mathsf{D}}(\mathbf{G})$*,* $\mathbf{G}_{\mathsf{S}}$*, and* $\mathbf{G}_{\mathsf{D}}$*.*

Proof. To prove *(1)*, observe that the new elements added by $\mathbf{C}_{\mathsf{S}}$ (singletons of previous vertices) and $\mathbf{C}_{\mathsf{P}}$ (power sets of previous vertices) both have unions in the original graph: the union of a singleton is its unique element, the union of a power set is the set it is a power set of. For *(2)*, use Proposition 1 and the proofs of Propositions 6 & 7. ■

Singleton closure. We consider finite graphs $\mathbf{G}$ and their locally finite singleton closure $\mathbf{G}_{\mathsf{S}}$. By Corollary 14, $\mathbf{G}_{\mathsf{S}} \models \mathsf{Singleton}$; thus, by Proposition 5, $\mathbf{G}_{\mathsf{S}}$ is infinite, but it has only finitely many elements of degree more than one. As a consequence, it satisfies $\mathtt{S}\xcancel{\mathtt{D}}\xcancel{\mathtt{P}}$. Therefore, using the Preservation Lemma 18,

$\mathbf{G}^1_{\mathtt{S}} \models \mathtt{S}\xcancel{\mathtt{D}}\mathtt{U}\xcancel{\mathtt{P}}\mathtt{CR}$: use that $\mathbf{G}^1_{\mathtt{S}}$ is locally 2-small and Proposition 6 *(ii)*[8];

$\mathbf{G}^3_{\mathtt{S}} \models \mathtt{S}\xcancel{\mathtt{D}}\mathtt{U}\xcancel{\mathtt{P}}\mathtt{C}\xcancel{\mathtt{R}}$: all new vertices have degree one, thus no new vertices can be a counterexample to Sep (cf. the proof of Proposition 6); the singleton closure provides all missing separation instances in $\mathbf{G}^3$; note that Repl was 2-violated in $\mathbf{G}^3$ and use Lemma 16;

$\mathbf{G}^4_{\mathtt{S}} \models \mathtt{S}\xcancel{\mathtt{D}}\mathtt{U}\xcancel{\mathtt{P}}\xcancel{\mathtt{C}}\xcancel{\mathtt{R}}$: note that Sep and Repl were both 2-violated in $\mathbf{G}^4$ and use Lemma 16;

$\mathbf{G}^\star_{\mathtt{S}} \models \mathtt{S}\xcancel{\mathtt{D}}\xcancel{\mathtt{U}}\xcancel{\mathtt{P}}\xcancel{\mathtt{C}}\xcancel{\mathtt{R}}$: note that Union, Sep, and Repl are all 2-violated in $\mathbf{G}^\star$ and use Lemma 16;

$\mathbf{G}^\#_{\mathtt{S}} \models \mathtt{S}\xcancel{\mathtt{D}}\xcancel{\mathtt{U}}\xcancel{\mathtt{P}}\mathtt{C}\xcancel{\mathtt{R}}$: note that Union and Repl are both 2-violated in $\mathbf{G}^\#$ and use Lemma 16.

Pair closure. We consider finite graphs $\mathbf{G}$ and their locally finite pair closure $\mathbf{G}_{\mathtt{D}}$. By Corollary 14, $\mathbf{G}_{\mathtt{D}} \models$ Pair; thus, by Proposition 5, $\mathbf{G}_{\mathtt{D}}$ is infinite, but it has only finitely many elements of degree more than two. But for any pair, all four subsets exist, so power sets of pairs will have size four, whence Powerset cannot hold. Also, for any four distinct vertices, we can apply the pairing axiom twice to obtain a set whose union has these four vertices as extension, so Union must fail. As a consequence, the pair closures all satisfy $\mathtt{SD}\xcancel{\mathtt{U}}\xcancel{\mathtt{P}}$. We observe that

$\mathbf{G}^1_{\mathtt{D}} \models \mathtt{SD}\xcancel{\mathtt{U}}\xcancel{\mathtt{P}}\mathtt{CR}$: use that $\mathbf{G}^1_{\mathtt{D}}$ is locally 3-small and Proposition 7 *(ii)*[9];

$\mathbf{G}^4_{\mathtt{D}} \models \mathtt{SD}\xcancel{\mathtt{U}}\xcancel{\mathtt{P}}\mathtt{C}\xcancel{\mathtt{R}}$: all new vertices have degree at most two, thus no new vertices can be a counterexample to Sep (cf. the proof of Proposition 7); the pair closure provides all missing separation instances in $\mathbf{G}^4$; note that Repl was 3-violated in $\mathbf{G}^4$ and use Lemma 16;

$\mathbf{G}^5_{\mathtt{D}} \models \mathtt{SD}\xcancel{\mathtt{U}}\xcancel{\mathtt{P}}\xcancel{\mathtt{C}}\xcancel{\mathtt{R}}$: note that Sep and Repl were both 3-violated in $\mathbf{G}^5$ and use Lemma 16.

Power set closure. We consider finite graphs $\mathbf{G}$ and their locally finite pair closure $\mathbf{G}_{\mathtt{P}}$. By Proposition 17, $\mathbf{G}_{\mathtt{P}} \models \xcancel{\mathtt{S}}\xcancel{\mathtt{D}}\mathtt{P}\xcancel{\mathtt{C}}\xcancel{\mathtt{R}}$ We obtain

[8] Note that $\mathbf{G}^1_{\mathtt{S}} = \mathbf{G}^2_{\mathtt{S}}$.

[9] Note that $\mathbf{G}^1_{\mathtt{D}} = \mathbf{G}^2_{\mathtt{D}} = \mathbf{G}^3_{\mathtt{D}}$.

- $\mathbf{G}^1_{\mathtt{P}} \models \mathtt{\cancel{S}\cancel{D}UP\cancel{C}\cancel{R}}$: follows from the Preservation Lemma 18;[10]
- $\mathbf{G}^\star_{\mathtt{P}} \models \mathtt{\cancel{S}\cancel{D}\cancel{U}P\cancel{C}\cancel{R}}$: the set witnessing the 3-violation of Union in $\mathbf{G}^\star$ is $\{1,2,3\} = \bigcup\{\{1\},\{2\},\{3\}\}$; this set does not contain the empty set, so it is not added by the axiom closure, whence Union still fails.

Singleton & Power set closure. We consider finite graphs $\mathbf{G}$ and their singleton & power set closure $\mathbf{G}_{\mathtt{SP}}$. By Proposition 17, $\mathbf{G}_{\mathtt{SP}} \models \mathtt{S\cancel{D}P\cancel{C}\cancel{R}}$ We obtain

- $\mathbf{G}^1_{\mathtt{SP}} \models \mathtt{S\cancel{D}UP\cancel{C}\cancel{R}}$: follows from the Preservation Lemma 18;
- $\mathbf{G}^\star_{\mathtt{SP}} \models \mathtt{S\cancel{D}\cancel{U}P\cancel{C}\cancel{R}}$: the set witnessing the 3-violation of Union in $\mathbf{G}^\star$ is $\{1,2,3\} = \bigcup\{\{1\},\{2\},\{3\}\}$; this set is neither a singleton nor does it contain the empty set, so it is not added by the axiom closure, whence Union still fails.

Pair & Power set closure. For our analysis, we only need one pair & power set closure, the graph $\mathbf{G}^\star_{\mathtt{DP}}$. By Proposition 17, $\mathbf{G}^\star_{\mathtt{DP}} \models \mathtt{SDP\cancel{C}\cancel{R}}$. Once more, the set witnessing the 3-violation of Union in $\mathbf{G}^\star$ is $\{1,2,3\} = \bigcup\{\{1\},\{2\},\{3\}\}$; this set has degree three and does not contain the empty set, so it is not added by the axiom closure, whence Union still fails. We obtain $\mathbf{G}^\star_{\mathtt{DP}} \models \mathtt{SD\cancel{U}P\cancel{C}\cancel{R}}$.

4.4. Obtaining All Hereditarily Finite Sets

In Section 4.1, we excluded 36 of the 64 combinatorially possible theories as inconsistent; in Section 4.2, we gave 5 finite graphs for different possible theories; and in 4.3, we provided 13 closure graphs. As a consequence $10 = 64 - (36 + 5 + 13)$ theories remain. One of these is $\mathsf{FST} = \mathtt{SDUPCR}$ which holds in $\mathbf{HF}$; the other nine cannot be valid in a transitive substructure of $\mathbf{HF}$, as we shall show in this section.

PROPOSITION 19. *Any subset of* $\mathbf{HF}$ *that contains* $\varnothing$ *and is closed under pairing and union is equal to* $\mathbf{HF}$*.*

PROOF. We prove this by $\in$-induction on $\mathbf{HF}$. Let $\varnothing \in M \subseteq \mathbf{HF}$ be closed under pairing and union. This implies that it is closed under binary union, i.e., if $x, y \in M$, then $x \cup y = \bigcup\{x, y\} \in M$. Assume towards a contradiction that $x \in \mathbf{HF}$ is minimal such that $x \notin M$. Since x is finite, let $x = \{x_0, ..., x_n\} \subseteq M$. By closure under pairing, for each $i \leq n$, we have that $\{x_i\} \in M$. Recursively define $x^*_0 := \{x_0\}$ and $x^*_{i+1} := x^*_i \cup \{x_{i+1}\}$. By the closure properties of M, all of these sets are in M by induction. Thus $x^*_n = x \in M$. Contradiction! ■

[10] Cf. Footnote 6.

Proposition 19 excludes the five theories $\mathtt{SDU\cancel{P}\cancel{C}\cancel{R}}$, $\mathtt{SDU\cancel{P}C\cancel{R}}$, $\mathtt{SDU\cancel{P}CR}$, $\mathtt{SDUPC\cancel{R}}$, and $\mathtt{SDUP\cancel{C}\cancel{R}}$ as impossible in substructures of $\mathbf{HF}$.

Proposition 20. *Any non-empty transitive subset $M \subseteq \mathbf{HF}$ that is a model of* $\mathsf{Powerset} + \mathsf{Sep}$ *is equal to* $\mathbf{HF}$.

Proof. We prove by induction on n that $\mathbf{V}_n \in M$. Since M is transitive, this is sufficient. Because M is nonempty, we have that $\varnothing = \mathbf{V}_0 \in M$. Suppose $\mathbf{V}_n \in M$. The power set axiom in M gives us the M-power set of $\mathbf{V}_n$ in M. But since $\mathbf{V}_n$ is finite, every $\mathbf{HF}$-subset of $\mathbf{V}_n$ is definable by a quantifier-free formula with parameters, and thus by Sep in M, the M-power set of $\mathbf{V}_n$ and the $\mathbf{HF}$-power set of $\mathbf{V}_n$ coincide. But the latter is $\mathbf{V}_{n+1}$. ■

Proposition 20 excludes all remaining four theories, viz. $\mathtt{S\cancel{D}\cancel{U}PC\cancel{R}}$, $\mathtt{S\cancel{D}UPC\cancel{R}}$, $\mathtt{SD\cancel{U}PC\cancel{R}}$, and $\mathtt{S\cancel{D}\cancel{U}PCR}$.

4.5. Beyond Hereditarily Finite Sets

While the theories excluded in Section 4.1 are inconsistent, this is in general not the case with the nine theories excluded in Section 4.4: these theories cannot be obtained in well-founded locally finite graphs, but might be true in well-founded graphs that are not locally finite; well-known examples are

$$\begin{aligned} \mathbf{V}_{\omega+\omega} &\models \mathtt{SDUPC\cancel{R}} \text{ and} \\ \mathbf{HC} &\models \mathtt{SDU\cancel{P}CR}, \end{aligned}$$

where $\mathbf{HC}$ is the set of hereditarily countable sets. Applying the method of axiom closure to infinite graphs produces further examples; we do not know whether graph models for all nine theories can be produced by axiom closures or even whether all nine theories are consistent.

Acknowledgements. The authors would like to thank *The World Academy of Sciences* (TWAS) and the *Deutsche Forschungsgemeinschaft* (DFG) for their support of the project Independence phenomena in multi set theory (LO834/17-1) funding the first author's visit at the Universität Hamburg in the spring of 2019.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

[1] Enayat, A., J. H. Schmerl, and A. Visser, ω-models of finite set theory, in J. Kennedy, and R. Kossak, (eds.), *Set Theory, Arithmetic and Foundations of Mathematics: Theorems, Philosophies*, vol. 36 of *Lecture Notes in Logic*, Cambridge University Press, 2011, pp. 43–65.

[2] Gitman, V., J. D. Hamkins, and T. A. Johnstone, What is the theory ZFC without power set? *Mathematical Logic Quarterly* 62(4-5):391–406, 2016.

[3] Howard, P., and J. E. Rubin, *Consequences of the axiom of choice*, vol. 59 of *Mathematical Surveys and Monographs*, American Mathematical Society, 1998.

[4] Kunen, K., *Set Theory. An introduction to independence proofs*, vol. 102 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, 1980.

[5] Mathias, A. R. D., Slim models of Zermelo set theory, *Journal of Symbolic Logic* 66(2):487–496, 2001.

[6] Mathias, A. R. D., The strength of Mac Lane set theory, *Annals of Pure and Applied Logic* 110(1-3):107–234, 2001.

[7] Mathias, A. R. D., Weak systems of Gandy, Jensen and Devlin, in J. Bagaria, and S. Todorcevic, (eds.), *Set theory. Centre de Recerca Matemàtica Barcelona, 2003-2004*, Birkhäuser, Basel, 2006, pp. 149–224.

[8] Rubin, H., and J. E. Rubin, *Equivalents of the axiom of choice*, vol. 34 of *Studies in Logic and the Foundations of Mathematics, North-Holland*, 1970.

[9] Rubin, H., and J. E. Rubin, *Equivalents of the axiom of choice. II*, vol. 116 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, 1985.

[10] Strohm, M. A., *Unabhängigkeitsresultate schwacher Mengenlehren mit Unendlichkeitsaxiomen*, B.Sc. Thesis, Universität Hamburg, 2021.

[11] van Marle, M., and F. Dekker, *Simple models of set theory*, project report, B.Sc. Wiskunde, Universiteit van Amsterdam, 2021.

[12] Zarach, A., Unions of ZF^--models which are themselves ZF^--models. in D. van Dalen, D. Lascar, and T. J. Smiley, (eds.), *Logic Colloquium '80. Papers intended for the European Summer Meeting of the Association for Symbolic Logic to have been held in Prague, Aug 24–30, 1980*, vol. 108 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, 1982, pp. 315–342.

[13] Zarach, A., Replacement $\not\to$ collection, in P. Hájek, (ed.), *Gödel '96. Logical foundations of mathematics, computer science, and physics—Kurt Gödel's legacy. Proceedings of the conference held in Brno, 1996*, vol. 6 of *Lecture Notes in Logic*, Association for Symbolic Logic, 1996, pp. 307–322.

F. Balogun
Department of Mathematics
Federal University Dutsin-Ma
Kilometer 60, Katsina-Kankara Road
Dutsin-Ma
Nigeria
fbalogun@fudutsinma.edu.ng

B. Löwe
Fachbereich Mathematik
Universität Hamburg
Bundesstraße 55
20146 Hamburg
Germany
loewe@math.uni-hamburg.de

and

B. Löwe
Institute for Logic, Language and Computation
Universiteit van Amsterdam
Postbus 942421090 GE Amsterdam GE
The Netherlands

and

B. Löwe
Churchill College, Lucy Cavendish College, St Edmund's College,
& Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Storey's Way
Cambridge CB3 0DS
England