Heinrich Wansing
Hitoshi Omori

Connexive Logic, Connexivity, and Connexivism: Remarks on Terminology

Abstract. Over the past ten years, the community researching connexive logics is rapidly growing and a number of papers have been published. However, when it comes to the terminology used in connexive logic, it seems to be not without problems. In this introduction, we aim at making a contribution towards both unifying and reducing the terminology. We hope that this can help making it easier to survey and access the field from outside the community of connexive logicians. Along the way, we will make clear the context to which the papers in this special issue on *Frontiers of Connexive Logic* belong and contribute.

Keywords: Connexive logic, Connexivity, Connexivism, Contra-classicality, Negation, Conditional, Negation inconsistency.

1. Introduction

Storrs McCall in his doctoral dissertation from 1963 [34] and his seminal paper [36] introduced the notion of a *connexive logic* (and adopted the expression 'connexive implication' from Bocheński [3]). Much later, in [38] he wrote a *history of connexivity*, and Routley [62] and some other later authors speaks of *connexivism*.

The distinction between connexive logic and connexivism might seem to echo the distinction between intuitionistic logic and intuitionism, but there are significant differences between these two distinctions. First of all, 'intuitionistic logic' is the name of one particular usually either zero- or first-order logic, whereas 'connexive logic' is the name of a sub-field of logic as a discipline, and connexive logics are formal systems that satisfy a number of

[1] One might want to distinguish between intuitionistic logic semantically defined by means of, for example, Kripke semantics or Beth semantics, but still the term 'intuitionistic logic' does not refer to a research area.

Special Issue: Frontiers of Connexive Logic
Edited by **Hitoshi Omori and Heinrich Wansing**.

Received September 20, 2023

conditions, similar to, say, normal modal logics being formal systems that satisfy certain other conditions.[1] Moreover, the expressions ‘intuitionism’ and ‘connexivism’ are not analogous in their meaning. Intuitionism is understood as Brouwer’s constructivist philosophy of mathematics, and whatever connexivism exactly is, it is different from one particular scholar’s view on some subject matter.

The notion of connexivity in fact is a rather loose concept that refers to certain informal ideas related to the meaning of conditionals and negation, or to certain properties making an implication connective or a logical system connexive, almost connexive, or connexive and satisfying additional constraints. The expression ‘connexivism’ refers to a family of views such as the idea that systems of connexive logic are respectable, that the notion of a connexive implication makes sense, that the ideas that have led to defining systems of connexive logic are interesting and substantial, that the characteristic principles of connexive logic reflect the correct understanding of conditionals (of a certain kind) in natural language, that systems of connexive logic are in this or that respect better than other systems of nonclassical logic, etc.

The long history of the discussion of ‘connexive implication’ notwithstanding, modern connexive logic (the field) is younger than, for example, intuitionistic logic (the particular system) or many-valued logic (the field), and it is also an area with a changing history. After its establishment in the 1960 s, on the threshold of the 21st century, connexive logic was a virtually dead research programme. The situation has changed after the inclusion of an entry on connexive logic in the *Stanford Encyclopedia of Philosophy* (SEP) [71] in 2006 and the beginning of a series of annual workshops on connexive logic in 2015.[2] The present double special issue of *Studia Logica* has emerged from the 21st *Trends in Logic* conference at Ruhr University Bochum, Germany, December 6–8, 2021 under the title “Frontiers of connexive logic”.

As connexive logic is still an unfolding and rapidly developing field, there is a risk that terminology will get out of hand. Such a proliferation may not be healthy for the area and could deter or discourage potentially interested outsiders. Therefore, we will introduce the contributions to the present special issue with a view on terminology.

[2] For the details, see https://sites.google.com/site/connexivelogic/events.

2. Established Terminology

2.1. Connexive Logics

In [36] Storrs McCall names the thesis $\sim(\sim p \to p)$ 'Aristotle's thesis' and refers to the formula $(q \to r) \to \sim(q \to \sim r)$ as 'Boethius' thesis' (using Łukasiewicz notation instead of infix notation). He explains that connexive logic "must not be equivalential logic. If it is to be of any interest, it must exclude the characteristic thesis $(p \to q) \to (q \to p)$ of equivalence" [36, p. 417]. Regarding the four-valued matrices introduced in Richard Angell's paper [1] he remarks that these truth tables

> satisfy the theses of Aristotle and Boethius, reject the formula $(p \to q) \to (q \to p)$ and hence define a system of connexive logic. [36, p. 418]

In [37, p. 350] he writes that "$(p \to q) \to \sim(p \to \sim q)$ is a connexive thesis, and together with $\sim(p \to \sim p)$ (since neither is a two-valued thesis) serves to distinguish connexive implication sharply from material, strict, intuitionist, 'rigorous', and other known types of implication." Following the definition given in [36], and assuming that $(p \to q) \to \sim(p \to \sim q)$ is as plausible as $(p \to \sim q) \to \sim(p \to q)$, systems of connexive logic are now standardly required to contain the following principles as theorems

AT $\sim(A \to \sim A)$ (Aristotle's thesis),

AT′ $\sim(\sim A \to A)$ (Aristotle's thesis′),

BT $(A \to B) \to \sim(A \to \sim B)$ (Boethius' thesis),

BT′ $(A \to \sim B) \to \sim(A \to B)$ (Boethius' thesis′),

and to satisfy the condition of non-symmetry of implication, saying that the schema $(A \to B) \to (B \to A)$ is not a theorem.

Although (i) in [37, p. 350] McCall remarks that "[f]urther characteristics of connexive implication include the rejection of the paradoxes of material and strict implication, and the avoidance of what have come to be known as the fallacies of relevance and necessity", (ii) in [37, p. 355] he excludes the laws of conjunctive simplification, i.e., $(A \wedge B \to A)$ and $(A \wedge B \to B)$, from connexive propositional logic, (iii) in [34, p. 109] with respect to Aristotle's and Boethius' theses he writes that "we may take the truth of one or both of these theses as a <u>mark</u> (necessary and sufficient condition) of connexive implication", and (iv) in [38] and elsewhere he discusses non-theorems of classical logic different from AT, AT′, BT, and BT′ as connexive theses, the definition based on the characterization in his 1966 *JSL*-paper [36] is the now widely agreed upon definition of connexive logic.

A connexive logic thus must contain at least a conditional and a unary connective of negation. In [34, p. 5] McCall explains that

> Certainly it is natural that any axiomatic basis for the theory of deduction should contain the central notion of implication, and for this reason implication/negation bases ('C-N bases') are in the writer's opinion to be preferred to A-N bases.

However, he does not discuss the notion of negation and just seems to presuppose a concept of negation that is not questionable as such. An in-depth discussion of what negation in a system of formal logic is started only much later, see [16,17,67].

For the benefit of the discussion of connexive logic and the development of the field, we strongly suggest to work with McCall's established definition, to keep the introduction of new terminology to a well justified minimum, and to avoid all too fancy terminology. Connexive logics according to McCall's definition have been called "minimally connexive" in [9] and "properly connexive" in [22]. We suggest to apply terminological parsimony and to do without adverbial qualifications in this case.

Note that Wolfgang Lenzen [30] proposes another language reform, namely to call defenders of connexive logics "hardcore connexivists". We advise not to follow this proposal. The suggestion seems to be motivated mainly by an opinion about what Lenzen considers to be non-hardcore conceptions of 'normal' conditionals.

We shall summarize our proposal of essential terminology towards the end of this introductory note.

2.2. Hyperconnexive Logics

Following Richard Sylvan (formerly Routley) [66], a connexive logic is sometimes called "hyperconnexive" if it validates the converses of Boethius' theses, i.e,

cBT $\sim(A\to\sim B)\to(A\to B)$,

cBT′ $\sim(A\to B)\to(A\to\sim B)$.

Sylvan actually demands in addition that conjunctive simplification fails. However, as there exist nontrivial connexive logics that validate cBT and cBT′ *and* conjunctive simplification, it makes sense to drop the latter requirement. The connexive logics **C** from [68], **C3** from [47], **LImp** from [41], and **CN** from [5] are hyperconnexive in this less restrictive sense.[3] As the negation connective in these logics can with good reasons be seen to express a notion of (support of) falsity, **C**, **C3**, **LImp**, and **CN** can be motivated as suggesting a certain falsity condition for implications $A \rightarrow B$, expressed by the following equivalential version of BT′:

eBT′ $\sim(A \rightarrow B) \leftrightarrow (A \rightarrow \sim B)$.[4]

This may well be part of the reason why according to Graham Priest [58, p. 178], the logic **C**, there called 'W', is "one of the simplest and most natural" connexive logics.

> *A note on the papers in the special issue.* In their "An algebraic investigation of the connexive logic **C**" (font-style adjusted), Davide Fazio and Sergei Odintsov start from the established definition of a connexive logic. They prove that axiomatic extensions of the paraconsistent connexive logic **C**, introduced in [68], and its extension by a falsum constant, $\mathbf{C}^{\perp}$, are algebraizable in the sense of Blok and Pigozzi with respect to sub-varieties of **C**-algebras, respectively $\mathbf{C}^{\perp}$-algebras. Fazio and Odintsov show the presentability of these algebras in terms of twist-like constructions over implicative lattices, clarify the relationship between the classes of algebras in question, and make some observations concerning the lattice of axiomatic extensions of **C** and $\mathbf{C}^{\perp}$. They also show the algebraizability of certain extensions of **C** and $\mathbf{C}^{\perp}$, namely the logics **C3** and $\mathbf{C3}^{\perp}$ obtained from **C** and $\mathbf{C}^{\perp}$ by adding the law of excluded middle (for the primitive strong negation) and the logics **MC** and $\mathbf{MC}^{\perp}$ obtained from **C** and $\mathbf{C}^{\perp}$ by adding the generalized law of excluded middle, $A \vee (A \rightarrow B)$.

Nevertheless, requiring the validity of cBT and cBT′ is not uncontroversial. In particular, McCall [38] challenged their intuitive plausibility. For a critical discussion of McCall's concerns see [75], and for an experimentally confirmed endorsement of natural language instances of cBT and cBT′, see [49].

[3] A system equivalent to **LImp** in a different language can be found in [43].

[4] With eBT′ expressing the falsity condition of implications, **C**, **C3**, and **CN** emerge as logics on the 'Bochum Plan' [7], obtained by a tweaking of a more standard falsity condition, namely that of Almukdad and Nelson's paraconsistent constructive logic **N4**, see also the classification in [46] and [71]. A certain combination of **N4** and **C** has recently been suggested in [23].

A note on the papers in the special issue. Alexander Belikov opens his paper "A simple way to overcome hyperconnexivity" with the remark that "[i]t is hard to say that there exists a uniform and undisputed criterion of connexivity." This is a situation which we hope can be overcome by a general agreement to work with McCall's definition, so that there is no need to refer to connexive logics in McCall's sense as minimally or properly connexive.

Instead of entering a discussion of the intuitive plausibility of cBT and cBT′, referred to by Belikov as "Hyper-Boethius' theses", Belikov investigates how the falsity condition for implications in **C** can be modified so as to break the validity of cBT and cBT′ and at the same time keep the resulting logic connexive. This project prompts Belikov to introduce new terminology and to refer to connexive logics that fail to be hyperconnexive, as "mesoconnexive", thereby suggesting that they are middle way between connexive and hyperconnexive logics. The idea is to modify the falsity condition for implications in **C** by imposing an extra constraint. It is required that for a conditional $A \rightarrow B$ to be false at a state w, A constructively implies the falsity of B at w *or* B is not true at w. The resulting logic **MeC**, "mesoconnexive **C**", is shown to be connexive (and constructive and decidable), but the method for obtaining **MeC** does not work in the case of **C3**. Therefore another modification of the falsity condition for conditionals in **C** is called for to obtain the connexive but not hyperconnexive system **qMeC3**. Sound and complete axiomatic proof systems for the two semantically defined logics are presented, and their relationships with some other connexive logics are discussed.

We see the point of Belikov's introduction of new terminology, but believe that the guiding thought of terminological parsimony justifies talking about connexive though not hyperconnexive logics.

2.3. Strongly Connexive Logics

Andreas Kapsner [25] suggested to impose certain further semantical conditions on systems of connexive logic, namely

Unsat1 In no model, $(A \rightarrow \sim A)$ is satisfiable, and neither is $(\sim A \rightarrow A)$,

Unsat2 In no model $(A \rightarrow B)$ and $(A \rightarrow \sim B)$ are satisfiable simultaneously (for any A and B),

and called logics that satisfy Aristotle's and Boethius' theses as well as the Unsat1 and Unsat2 constraints "strongly connexive" and those that only satisfy the theses named after Aristotle and Boethius 'weakly connexive' (without imposing the non-symmetry of implication condition explicitly).[5] As Kapsner adds conditions to the list of properties defining a connexive logic, the resulting concept is in that sense stronger than the notion of a connexive logic. Note that the proposal of Unsat1 and Unsat2 is based on what Kapsner takes to be "robust pre-theoretical intuitions" [25, p. 2] about the validity of Aristotle's and Boethius' theses, but that there may well be conflicting intuitions. One's semantic intuitions could, for instance, agree with the understanding of negation and implication in the connexive logics **C** or **CN** [5], where both $A \rightarrow \sim A$ and $\sim A \rightarrow A$ are satisfiable and Unsat2 fails as well. The conditions Unsat1 and Unsat2 are problematic for any paraconsistent logic like **C** that admits of a trivial model that satisfies any formula whatsoever.

A note on the papers in the special issue. Davide Fazio, Antonio Ledda, and Francesco Paoli in their contribution also start from McCall's [36] definition and quote from the SEP entry on connexive logic, where this understanding is said to give "the now standard notion of connexive logic." The title of their paper sounds provocative: "Intuitionistic logic is a connexive logic". Connexive logics are contraclassical, how then can intuitionistic logic, being a subsystem of classical logic, possibly be a connexive logic? Fazio, Ledda, and Paoli show that a connexive implication 'lives within' intuitionistic logic. If we use $\neg A$ to stand for the intuitionistic negation of A defined as $A \rightarrow \bot$, and use $\rightarrow$ for the intuitionistic implication, then the binary connective $A \rightarrow_c B$ (this is our ad hoc notation) defined as $(A \rightarrow B) \wedge (\neg\neg B \rightarrow \neg\neg A)$, or equivalently as $(A \rightarrow B) \wedge (\neg A \rightarrow \neg B)$, satisfies non-symmetry of implication and the characteristic principles of connexive logic:

- $\neg(A \rightarrow_c \neg A)$, $\neg(\neg A \rightarrow_c A)$,
- $(A \rightarrow_c B) \rightarrow_c \neg(A \rightarrow_c \neg B)$, $(A \rightarrow_c \neg B) \rightarrow_c \neg(A \rightarrow_c B)$.

The logic with the connexive implication as primitive is called *Connexive Heyting Logic*, **CHL**, and satisfies the Unsat1 and Unsat2 conditions. Fazio, Ledda, and Paoli present an algebraic investigation of **CHL**, show

[5]We find that Kapsner's distinction between weakly connexive and strongly connexive logics is unfortunate. The dichotomy may amount to eliminating terminology that has been established since the beginning of modern connexive logic in the 1960s because there would then be no connexive logic *simpliciter* anymore. Of course, this was not his intention, but we would still like to highlight this as a cautious remark.

the term equivalence between structures they call *connexive Heyting algebras* and Heyting algebras, and provide Hilbert-style and Gentzen-style proof systems for **CHL**. They also suggest a computational interpretation of **CHL**'s connexive implication.

Before Fazio, Ledda, and Paoli studied **CHL**, only few logics were known to be strongly connexive, namely McCall's system **CC1** [34], which axiomatizes Angell's [1] truth tables for implication, negation and conjunction, and which according to McCall [38, p. 429] "is an awkward system in many ways" and the Boolean connexive logics from Tomasz Jarmużek and Jacek Malinowski [22].[6] Given that there exist continuum many logics intermediate between intuitionistic and classical propositional logic, there are uncountably many strongly connexive logics.

In [47] we noted that in the vicinity of the four-valued first-degree entailment logic, **FDE**, there are always at least two ways to formalize classical notions since truth and non-falsity (or, falsity and non-truth) come apart. If we consider valuation functions that assign values from the set $\{\varnothing, \{0\}, \{1\}, \{0,1\}\}$ to state-variable pairs and extend them to interpretation functions I in models for the connexive logic **C3**, then two notions of satisfiability can be distinguished, namely a positive one, defined in terms of the presence of the classical value 1 (*true*), and a negative one, defined in terms of the absence of the classical value 0 (*false*):

- A is positively satisfiable iff for some **C3**-model $\langle W, \leq, V\rangle$, $1 \in I(w, A)$ for some $w \in W$.
- A is negatively satisfiable iff for some **C3**-model $\langle W, \leq, V\rangle$, $0 \notin I(w, A)$ for some $w \in W$.

It then turns out that in **C3** the 'strong implication' defined by setting $(A{\Rightarrow}B) := (A{\rightarrow}B) \wedge ({\sim}B{\rightarrow}{\sim}A)$ satisfies Unsat1 and Unsat2. If negative satisfiability is defined with respect to models for **C**, then the strong implication in **C** satisfies Unsat1.

[6] Jarmużek and Malinowski [22] introduced the term "Boolean connexive logic". By definition, a Boolean connexive logic is an expansion of classical propositional logic with Boolean negation, conjunction and disjunction as primitive logical operations and material implication as a defined connective, by another connexive, 'relating' implication connective as a primitive conditional. As such, Boolean connexive logics follow a methodology that differs from McCall's original enterprise that was not meant to develop an expansion of classical logic but rather to deliberately break with classical logic.

> *A note on the papers in the special issue.* Hitoshi Omori and Andreas Kapsner in their contribution, "Angell and McCall meet Wansing", aim at offering a bridge between systems by Angell-McCall and Wansing. There seem to be more differences than similarities, but by focusing on a closely related three-valued system of Angell-McCall's **CC1**, some surprising similarities start to reveal themselves. Towards the end of the paper, they present a new system that expands the well-known **BCI** logic, a purely implicational logic that is given by the combinators B, C and I from combinatory logic, and some open problems are listed. The new system, called **AMW**, is both hyperconnexive and strongly connexive.

2.4. Superconnexive Logics

Towards the end of the paper [25], Kapsner ventilated the idea of expressing the Unsat1 and Unsat2 conditions in the object language through the following formulas.

- $(A{\rightarrow}{\sim}A){\rightarrow}B$ (Super-Aristotle),
- $({\sim}A{\rightarrow}A){\rightarrow}B$ (Super-Aristotle′),
- $(A{\rightarrow}B){\rightarrow}((A{\rightarrow}{\sim}B){\rightarrow}C)$ (Super-Boethius),
- $(A{\rightarrow}{\sim}B){\rightarrow}((A{\rightarrow}B){\rightarrow}C)$ (Super-Boethius′).

Moreover, he suggested to call strongly connexive logics that validate Super-Aristotle 'superconnexive'. It was already reported in [25] that the above formulas will be in conflict with the requirement of the Unsat principles, if substitution is assumed. Indeed, an instance of Super-Aristotle, namely $(A{\rightarrow}{\sim}A){\rightarrow}{\sim}(A{\rightarrow}{\sim}A)$ will be valid/derivable, but this goes against Unsat1. Therefore, being superconnexive is in tension with substitution. More recently, in [27], it is observed that if we assume Modus Ponens for $\rightarrow$, then we obtain *triviality*. The three line proof, presented in [27, Proposition 1], runs as follows.[7]

1	$(A{\rightarrow}{\sim}A){\rightarrow}{\sim}(A{\rightarrow}{\sim}A)$	[Super-Aristotle, substitution]
2	$((A{\rightarrow}{\sim}A){\rightarrow}{\sim}(A{\rightarrow}{\sim}A)){\rightarrow}B$	[Super-Aristotle, substitution]
3	B	[1, 2, MP]

As a corollary of this result, we can observe that Unsat 1 in combination with substitution, Modus Ponens and the deduction theorem will trivialize.

As a response to the above result, there are at least three directions. First, we can try to stick to the original proposal made by Kapsner, and in this

[7] More related triviality/non-triviality proofs are reported in [27].

case, we need to do without substitution. Otherwise, there are two directions proposed in the literature. One is to follow the suggestion explored in [27] in which the theses characterizing superconnexive logics are formulated within a language with the falsum constant $\perp$ as follows (focusing only on Super-Aristotle):

- $(A{\rightarrow}{\sim}A){\rightarrow}\perp$.

The other option is to follow the suggestion made by Fazio, Ledda, and Paoli in their contribution to the present special issue, and reformulate the characteristic theses within a language with another conditional $\Rightarrow$ as follows (again focusing only on Super-Aristotle):

- $(A{\rightarrow}{\sim}A){\Rightarrow}B$.

Note that in this case, $\Rightarrow$ may obey both Modus Ponens and the deduction theorem, as noted by Fazio, Ledda, and Paoli in their contribution to the special issue.

In any case, further implications of superconnexive logics remain to be seen.

2.5. Totally Connexive Logics

Abelard's First Principle, the schema ${\sim}((A{\rightarrow}B){\wedge}(A{\rightarrow}{\sim}B))$, is among the non-theorems of classical logic discussed in the literature on connexive logic, and so is the schema ${\sim}((A{\rightarrow}B){\wedge}({\sim}A{\rightarrow}B))$ called 'Aristotle's Second Thesis' following [33], see [38]. Aristotle's Second Thesis is also known as the Principle of Subjunctive Contrariety after [1]. Jacek Malinowski and Rafał Palczewski [31] refer to Abelard's First Principle as 'Boethean Thesis' (p. 54) and as 'Third Boethean Thesis' (p. 55), and to Aristotle's Second Thesis as 'Third Aristotle's Thesis',[8] Luis Estrada-González and Elisángela Ramírez-Cámara [9] refer to Abelard's First Principle as 'Abelard's Principle'. This multiplicity of terminology can be confusing, so that it would be helpful to follow McCall's survey [38], and to speak of *Abelard's First Principle* and *Aristotle's Second Thesis*.

A note on the papers in the special issue. Niki Pfeifer and Giussepe Sanfilippo, in their contribution, "Probabilistic default reasoning, and compound conditionals", investigate various principles within two approaches,

[8] According to Bonevac and Dever [4, p. 192] Abelard's First Principle is the most famous thesis attributed to Boethius, but they also note that they fail to find it in Boethius.

namely the coherence-based probabilistic default reasoning and the coherence framework of compound and iterated conditionals. In the latter approach, the connexive principles are interpreted in terms of suitable conditional random quantities. In both approaches, the connexive principles are all valid, as well as Abelard's First Principle. Moreover, the additional conditions for strong connexivity are satisfied in both approaches. One of the differences between the two approaches is the status of cBT and cBT′, and these are not validated in the first approach, but are validated in the second approach.

We already pointed out that Abelard's First Principle is not required to be valid in a connexive logic as defined by McCall [36], and the same holds true of Aristotle's Second Thesis. Moreover, the connexive logic **C**, for example, fails to validate not only the former but also the latter.[9] Estrada-González and Ramírez-Cámara [9] refer to logics that validate both Abelard's First Principle and Aristotle's Second Thesis as 'Abelardian logics'.[10]

In fact, this is part of a longer list proposed by Estrada-González and Ramírez-Cámara. Indeed, they also considered the following properties on top of the validity of the theses discussed so far (namely Aristotle's theses, Boethius' theses, Abelard's First Principle, and Aristotle's Second Thesis).

Positive Paradox of Implication $\not\models A{\rightarrow}(B{\rightarrow}A)$,

Negative Paradox of Implication $\not\models A{\rightarrow}(\neg A{\rightarrow}B)$,

Paradox of Necessity $\not\models A{\rightarrow}(B{\rightarrow}C)$ where A is a contingent truth and $B{\rightarrow}C$ is a logical truth,

Simplification $\models (A \wedge B){\rightarrow}A$, $\models (A \wedge B){\rightarrow}B$,

Idempotence $\models (A \wedge A){\rightarrow}A$, $\models A{\rightarrow}(A \wedge A)$,

Kapsner-strong (i) $A{\rightarrow}{\sim}A$ is unsatisfiable and (ii) $A{\rightarrow}B$ and $A{\rightarrow}{\sim}B$ are not simultaneously satisfiable.

[9] Note, however, that the strong implication that can be defined in **C3** will satisfy both theses.

[10] A reason for being reluctant to introducing the term 'Abelardian connexive logics' could be that in the literature there is already the term 'Aristotelian-Abelardian predicate logic' which, according to Peter Seuren [64] "is a perfectly sound alternative to SMPL, from which it differs only in that, in the absence of any Fs, ALL F is G is considered false in the former but true in the latter", where 'SMPL' stands for 'Standard Modern Predicate Logic'.

Estrada-González and Ramírez-Cámara then introduced the notion of *totally connexive logics* as logics that satisfy all their desiderata, including the validity of the theses discussed so far. Moreover, they left as an open problem whether there are totally connexive logics, and if so then which is the minimal one (cf. [9, p. 348]). We only note that as we observed in [47] that **C3** can be seen as a totally connexive logic when examined via the strong implication. If there are other interesting totally connexive logics, remains to be an interesting open problem. There might be even some further discussions about the feasibility of the list suggested by Estrada-González and Ramírez-Cámara.

2.6. A Quick Summary

Before moving further, here is a table that summarizes the established terminology discussed in this section. This is terminology we propose to continue to use.

A logic is ...	if it validates		and satisfies	
connexive	AT, AT′	BT, BT′	non-symmetry of implication	
hyperconnexive	AT, AT′	BT, BT′ cBT, cBT′	non-symmetry of implication	
strongly connexive	AT, AT′	BT, BT′	non-symmetry of implication	Unsat1 Unsat2
superconnexive	$(A\to\sim A)\to B$ and is strongly connexive			
totally connexive	AT, AT′	BT, BT′	non-symmetry of implication	Unsat1 Unsat2
	and other properties listed in §2.5			

3. New Terminology

3.1. Partially Connexive Logics

As already highlighted, 'connexivity' is not a clearly defined expression and has been used to mean different things. We suggest to refer to logics that validate some but not all of AT, AT′, BT, and BT′ and satisfy non-symmetry of implication as *partially connexive logics*. Note that the 'subminimally connexive' logics of Estrada-González and Ramírez-Cámara [9] as well as the 'properly quasi-connexive' logics of Jarmużek and Malinowski [22] coincide with the partially connexive logics. For the 'demi-connexive' logics in [74], non-symmetry of implication was simply not mentioned, but not meant to be denied. Therefore, and in the spirit of terminological parsimony, the term 'demi-connexive' can be sacrificed in favour of the label 'partially connexive'. Likewise, as the non-symmetry constraint has been emphasized by McCall and seems to be both uncontroversial and convincing, we also suggest to do without the term 'quasi-connexive' used in [22] to refer to logics that validate at least one of AT, AT′, BT, and BT′ but not all, and do not satisfy non-symmetry of implication.

In what follows, we will revisit two main approaches in the literature to partially connexive logics, namely, logics of consequential implication and partially connexive logics inspired by the idea of negation as cancellation.

3.1.1. Logics of Consequential Implication Since his seminal paper [50] in 1977, Claudio Pizzi has been the only author that continues writing about logics of consequential implication to date, which are logics that validate Aristotle's theses but do not validate Boethius' theses.[11] At the beginning of his paper, Pizzi makes it explicit that he will focus on the following weak forms of Boethius' theses, where '$\supset$' denotes classical implication:

wBT $(A{\rightarrow}B){\supset}{\sim}(A{\rightarrow}{\sim}B)$.[12]

wBT′ $(A{\rightarrow}{\sim}B){\supset}{\sim}(A{\rightarrow}B)$.[13]

After offering three reasons why the weak BT should be included in the context of conditional logics, Pizzi introduced, in [50], two systems that include AT, AT′ and the weak BT. This is, therefore, a clear instance of partially connexive logics. Note also that there are some recent discussions related to conditional logics and connexive logics (e.g. [20, 60, 73]), but many of these are touched already in [50].

> *A note on the papers in the special issue.* Mateusz Klonowski and Luis Estrada-González in their paper, "Boolean connexive logic and content relationship", address the looseness of the concept of connexivity. They remark that "[o]ne of the main challenges for formal-philosophical research in connexive logic is to clarify the concept of connexivity" and

[11] For some of the later papers, two of them jointly with Timothy Williamson, see [52–56].

[12] Note that wBT stands for weak BT, and we are here following the terminology from [51]. However, in [50], it was referred to as the *conditional Boethius thesis.*

[13] Yale Weiss, in [78], refers to logics that validate wBT and wBT′ as *half-connexive.*

aim at presenting connexive logic as motivated by analyzing a content relationship between the antecedent and the succedent of conditionals.

A family of systems of Boolean connexive logic is introduced as counterparts of certain content relationship logics proposed by Richard Epstein. Motivation comes from the observation that the smallest Boolean connexive logic and some of its extensions lack the means to express content relationships in the object language, whereas Epstein's content relationship logics have such resources but fail to be connexive. The paper consists of two parts. In the first part axiomatic proof systems are introduced, starting from the smallest Boolean connexive logic and the smallest content relationship logic. The second part of the paper deals with relating- and set-assignment models. Sound and complete relating semantics are presented for all systems under consideration. Moreover, a set-assignment semantics is given for some Boolean connexive logics, thereby providing a formalization of content relationship understood either as content sharing or as content inclusion.

Klonowski and Estrada-González also consider what they call "weak Boolean connexive logics". Such logics are Boolean connexive logics with a relating implication that contain wBT and wBT′.

3.1.2. Priest's Partially Connexive Logics Another instance of partially connexive logics can be found in [57] by Graham Priest. A distinctive feature of Priest's contribution is that he attempts to connect connexive logic with the cancellation account of negation, according to which "$\sim A$ deletes, neutralizes, erases, cancels A (and similarly, since the relation is symmetrical, A erases $\sim A$), so that $\sim A$ together with A leaves nothing, no content" [63, p. 205], for a critical discussion see [75]. For the purpose of modelling the cancellation behavior of negation, Priest, in brief, takes a strict conditional that builds on **S5**, adds a non-vacuity condition to the truth condition for the strict conditional, and finally requires that the premise has a model in defining the semantic consequence relation $\models$.

Then, the following holds for $\models$:

- $\models \sim(p\rightarrow\sim p)$,
- $\models \sim(\sim p\rightarrow p)$,
- $\models \sim((p\rightarrow q) \wedge (p\rightarrow\sim q))$,
- $\not\models (p\rightarrow q)\rightarrow\sim(p\rightarrow\sim q)$,
- $\not\models (p\rightarrow\sim q) \rightarrow \sim(p\rightarrow q)$,
- $p\rightarrow q \models \sim(p\rightarrow\sim q)$,

- $p \rightarrow \sim q \models \sim(p \rightarrow q)$.

That is, we have both AT and AT', and also weak BT in view of the third item given that the negation-conjunction fragment is classical. The fourth item shows that BT fails, and therefore the system is not connexive, but it is a partially connexive logic.[14] The final item shows that a *rule form* of BT holds, but we return to this shortly in the next approach.

Priest's idea was later detached from the cancellation account of negation by removing the requirement of the existence of a model for the premise in the definition of the semantic consequence relation, and explored systematically by Guido Gherardi and Eugenio Orlandelli in [18,19] under the label *super-strict implication*.

> *A note on the papers in the special issue.* Guido Gherardi, Eugenio Orlandelli, and Eric Raidl in their contribution, "Proof systems for super-strict implication", offer proof systems for the super-strict implication that are based on the modal logics **S2** to **S5**. The proof systems they work with are axiomatic systems as well as G3-style labelled sequent calculi.

3.1.3. Other Examples of Partially Connexive Logics In [44], a variant of **C**, called $\mathcal{N}$, is introduced for the purpose of analyzing some of the features of the system introduced in [13]. The formula $\sim(A \rightarrow B) \leftrightarrow (\sim A \rightarrow B)$ is called 'half-connexive' because in the system $\mathcal{N}$, in which it is provable, AT and AT' are provable, but BT and BT' are not. Therefore, the logic $\mathcal{N}$ is another example of a partially connexive logic.

Moreover, the logics **CR1** and **CR2** introduced by Weiss in [78] are both partially connexive. Indeed, **CR1** satisfies AT and AT', and **CR2** satisfies not only AT and AT', but also wBT and wBT', but neither of them satisfies BT nor BT'.

More recently, Hao Wu and Minghui Ma [80] construct an infinity of expansions of intuitionistic logic by a connective, $\sim$, for strong negation. In addition to the De Morgan laws and the biconditional $\sim(A \rightarrow B) \leftrightarrow (A \rightarrow \sim B)$ (notation adjusted) from the axiomatic presentation of the logic **C**, double negation axioms $\sim^{2m+n} A \leftrightarrow \sim^{n} A$ are assumed, and the resulting logics are called $\mathbf{C}_{m,n}$. For any $m > 0$ and $n \geq 0$, the lattice of extensions of $\mathbf{C}_{m,n}$ is isomorphic to the lattice of intermediate logics. For $m > 1$ or $n > 0$, the logics $\mathbf{C}_{m,n}$ contain AT$'$ and BT$'$ but not AT and BT. Wu and Ma refer

[14] Note, however, that the listed properties are stated for arbitrary propositional variables instead of arbitrary formulas. The reason is that substitution does not hold for Priest's systems.

to the systems $\mathbf{C}_{m,n}$ as intuitionistic connexive logics because they contain at least one of AT and AT′ and at least one of BT and BT′. With our convention, the logics $\mathbf{C}_{m,n}$ are partially connexive.

Finally, we have Alessandro Giordani's system introduced in his contribution to the special issue as another example of a partially connexive logic.

A note on the papers in the special issue. In his paper "Situation-based connexive logic", Alessandro Giordani charts a "landscape of connexive logics" and introduces an axiomatic system of situation-based modal connexive logic, **SC**, that expands classical propositional logic. In drawing a landscape of systems that are connexive in a broad sense, he makes use of a generic conditional, $\hookrightarrow$, and a generic negation connective *neg*, such that (notation adjusted) "*neg* is defined so that $neg(A)$ coincides with $\neg A$, if A is not a negation, and with the negated formula in A, if A is a negation", where $\neg A$ is the classical negation of A. Moreover, in **SC**, a **KT**-like necessity operator $\Box$ can be defined by setting: $\Box A := \top \hookrightarrow A$.

Giordani's classification scheme does not match with what we have seen so far because Aristotle's thesis, Boethius' thesis and the weak Boethius' thesis wBT are presented in a way that allows the presence of two different negation connectives, which is why we here use a font for the acronyms of the generalized versions of these principles different from the one used in §2.1:

`AT` $\neg(A \hookrightarrow neg\, A)$,

`BT` $(A \hookrightarrow B) \hookrightarrow \neg(A \hookrightarrow neg\, B)$,

`wBT` $(A \hookrightarrow B) \supset \neg(A \hookrightarrow neg\, B)$.

The model theory of **SC**, with respect to which **SC** is shown to be sound and complete, comes with three clauses in the truth definition of conditionals. A local inclusion condition captures a content relationship between the situation referred to in the consequent and the situation referred to in the antecedent, a modal inclusion condition makes the implication strict, and a modal compatibility condition requires the simultaneous satisfiability of the antecedent and the consequent. In **SC**, *neg* is instantiated by classical negation, and it turns out that while wBT is provable in **SC**, BT is not, and that, according to our new terminology, **SC** is a partially connexive logic, satisfying Unsat 1 and Unsat2.

Note also that Giordani speaks of "theses" to refer to both certain schematic formulas discussed in the literature on connexive logic and satisfiability constraints imposed on certain schematic formulas. While that is completely unproblematic in itself, for uniformity of terminology it might be recommendable to reserve the term 'theses' for schematic formulas from some object language.

3.2. Restrictedly Connexive Logics

Kapsner [26] introduced the notion of 'plain humble connexivity' which restricts Aristotle's theses, Boethius' theses, Unsat1, and Unsat2 to satisfiable antecedents. If the restriction to satisfiable antecedents is dropped for Unsat1 and Unsat2, Kapsner obtains plainly weakly humbly connexive logics. The qualification as 'plain' is used because a distinction is drawn between different kinds of 'humble' connexivity.

A fine-grained distinction between various types of 'humble' connexivity may be helpful in certain contexts, but in order to avoid extremely small-grained and, moreover, morally tinged distinctions (suggesting that unrestrictedly connexive logics are immodest), it might be reasonable to collectively refer to logics that satisfy non-symmetry of implication and validate AT, AT′, BT, and BT′ if certain syntactical or semantical restrictions are imposed on the antecedents or succedents of some of these principles as *restrictedly connexive* logics. As an example, we obtain that Robert Stalnaker's system, presented in [65], may be seen as a restrictedly connexive logic since AT is restricted. As another example, the system **WBK**, introduced in [45], is restrictedly connexive.[15] Moreover, in [39], Satoru Niki considers a variant of **C** by replacing eBT′ by $\sim(A\to B)\leftrightarrow(\neg\neg A\to\sim B)$, where $\neg$ is the intuitionistic negation. The resulting variant is restrictedly connexive for all four theses, restricted in terms of formulas related to the elimination of double intuitionistic negation.

> *A note on the papers in the special issue.* Andrea Iacona in his contribution, "Connexivity in the logic of reasons", explores the implications of reading the conditional $p\to q$ as 'p is a reason for q' in relation to connexive theses. To this end, Iacona presents a semantic framework that builds on his joint work [6] with Vincenzo Crupi on what they call the *evidential conditional*, and observes that all the connexive theses are included with some restrictions.

Another example of a restrictedly connexive logic is studied in Eric Raidl, Andrea Iacona, and Vincenzo Crupi's contribution to this special issue.

[15] Note also that once the modality is strengthened, then some of the extensions of **WBK** become connexive.

A note on the papers in the special issue. Eric Raidl, Andrea Iacona, and Vincenzo Crupi, in their paper "An axiomatic system for concessive conditionals", offer a proof system for what is called the *concessive conditional.* The system is restrictedly connexive since a restricted version of Aristotle's thesis is valid/derivable. The system also contains a restricted version of Aristotle's Second Thesis, as well as the weak Boethius' thesis wBT.

Note that we may also combine the two new pieces of terminology, as there exist logics that are partially and restrictedly connexive. For example, we already observed that $\mathbf{C}_{m,n}$ due to Wu and Ma are partially connexive, when $m > 1$ or $n > 0$, but we can also add that the systems are restrictedly connexive. Indeed, since eBT' is included in $\mathbf{C}_{m,n}$, we obtain, as a special case, that $\sim(A\rightarrow\sim A)\leftrightarrow(A\rightarrow\sim\sim A)$. In particular, we obtain that $(A\rightarrow\sim\sim A)\rightarrow\sim(A\rightarrow\sim A)$, and this shows that AT is restricted by the introduction of the double negation law.

3.3. A Quick Summary

Before moving further, here is again a table that summarizes the new terminology discussed in this section.

A logic is ...	if it validates		and satisfies
partially connexive	at least one but not all of AT, AT′,	BT, BT′	non-symmetry of implication
restrictedly connexive	at least one but not all of AT, AT′ with restrictions imposed on antecedents or consequents	BT, BT′	non-symmetry of implication

4. Variations

We now hope to have cleared and streamlined the exuberant and hence possibly irritating terminology, at least a little bit, if not completely. In this section, we wish to discuss three interesting topics that are closely related to the terminological issues in connexive logic discussed so far.

4.1. Weakly Connexive Logics

In [76] the phrase 'weakly connexive logics' is used to refer to logics that validate BT and BT′ only in the rule form, i.e.,

rBT $A{\to}B \vdash {\sim}(A{\to}{\sim}B)$,

rBT′ $A{\to}{\sim}B \vdash {\sim}(A{\to}B)$.

We suggest to use the term 'weakly connexive' in this latter sense and to require for weakly connexive logics in addition the non-interderivability of implication: $A{\to}B \not\vdash B{\to}A$. One may wonder why the additional requirement is in the form of non-interderivability of implication, rather than the non-symmetry of implication. One of the reasons can be that if we take the latter option, then we can interpret the conditional as conjunction, and that will satisfy all the conditions of weakly connexive logics. However, just as we do not want the conditional to be interpreted as a biconditional, we do not want the conditional to be interpreted as conjunction.

As an example of weakly connexive logics, beside the systems discussed in [76], we can observe that Pizzi's as well as Priest's systems are weakly connexive. However, these are systems that satisfy wBT and wBT', and therefore, these cases are rather trivial and not terribly interesting qua weakly connexive logics.

A more interesting case can be found in [69]. In the abstract of that paper, the sequent calculi introduced in the paper are classified as connexive logics, but they are weakly connexive logics in the present terminology. Here are a bit more details. In sequent calculi used in Categorial Grammar, a sequent of the form $A_1, \ldots, A_n \vdash A$ (notation adjusted) is read as "every sequence of expressions of syntactic types $A_1, \ldots, A_n$ is of syntactic type A". Therefore empty antecedents of sequents are not envisaged and the rules for introducing the directional implication connectives in succedent position of a sequent:

$$\frac{X \vdash B/A}{X, A \vdash B} \qquad \frac{X \vdash A\backslash B}{A, X \vdash B}$$

where X is a finite but non-empty sequence of formulas, simply cannot be applied to obtain provable conditionals B/A or $A\backslash B$, derivable from the empty sequence. (The formula B/A ($A\backslash B$) is the syntactic type of expressions that result in an expression of type B if they are combined from the right (the left) with an expression of type A.) There are no theorems, AT and AT′ are not provable, and Boethius' theses take the rule form:

$$A\backslash B \vdash {\sim}(A\backslash{\sim}B), \quad A\backslash{\sim}B \vdash {\sim}(A\backslash B),$$
$$B/A \vdash {\sim}({\sim}B/A), \quad {\sim}B/A \vdash {\sim}(B/A).$$

A more recent example of weakly connexive logics can be found in [11] in which Nicholas Ferenz expands the logic **FDE** by adding a conditional from the conditional logics literature, and one of his systems, called $\mathbf{CE_{FDE}}$, is an example of weakly connexive logics.

A note on the papers in the special issue. Xuefeng Wen in "Stalnakerian connexive logics" starts from the standard definition of a connexive logic and a presentation of various rules, principles, and conditions discussed in the literature, including Boethius' theses in rule form, Boethius' theses in equivalence form, Unsat1, and Unsat2. Wen then presents a strategy used to obtain weakly connexive logics that become connexive upon addition of the axiom $A \rightarrow A$. His semantics is a modification of Stalnaker's selection function semantics for conditionals. The idea is to allow the selection function to be a partial function, thereby combining Kleene's three-valued logic with Stalnaker's semantics. Whereas in [46] and [71] connexive logics have been classified according to whether they (i) adjust (support of) truth conditions or add semantical machinery to familiar truth conditions or (ii) tweak (support of) falsity conditions, Wen's strategy is "to put some precondition on both truth and falsity conditions. The precondition is only for the truth and falsity of conditionals to be defined, without changing the original semantics".

Wen defines two connexive and four weakly connexive conditional logics in terms of different classes of models and making use of two notions of validity. In addition to preservation of truth, he also considers preservation of "tolerant truth" understood as backward falsity preservation from the conclusion to the premises of an inference. Soundness and completeness results for axiomatic presentations of these logics are given. Moreover, there is a careful comparison between the logics introduced in the paper and other conditional logics, including the weakly connexive logic **cCL** and the connexive systems **CCL** from [76] and **CN**.[16]

[16] Wen points out that "[t]here seems to be inconsistency between [39] (the 2022 version of reference [71] in this note) and [42] (reference [76] in this note) in using 'weakly connexive'. In the former, logics validating both AT and BTr but not BT are called weakly connexive. In the latter, however, the authors call their logic **cCL** weakly connexive. But **cCL**, lacking the frame condition for $A \rightarrow A$ does not validate AT. Here we follow the usage in [42]." The observation is correct. In [76, p. 569] the logic **cCL** is called a "weakly connexive conditional logic" because it validates Boethius' theses in rule form but not the schemata BT and BT$'$, and it is pointed out that AT and AT$'$ fail for **cCL**. The remark in [71] has now been corrected.

4.2. Kapsner Strong Logics

As already mentioned, Estrada-González and Ramírez-Cámara introduced the term *Kapsner strong* in [9], by detaching the Unsat conditions from the basic principles included in the definition of a connexive logic.[17] On the one hand, this may seem to be a rather substantial deviation. On the other hand, as we shall see, it is a useful notion that may function as an interesting notion to group a number of systems. For this special issue, we have two papers that will benefit from this terminology.

> *A note on the papers in the special issue.* Hans Rott in his paper "Difference-making conditionals and connexivity" points out that the term 'connexive logic', as it seems, goes back to Storrs McCall's [34]. Nevertheless Rott does not work with McCall's definition of the notion of a connexive logic from [36]. Rott's contribution is based on belief revision theory. He proves a representation theorem for a certain difference-making conditional, i.e., a non-nesting conditional governed by what is called "the Relevant Ramsey Test", with respect to a basic AGM revision function, and remarks that his paper "does not aim at presenting a conditional logic in the sense of delineating a set of logical truths or theorems. It rather endorses the view that the task of logic lies in identifying what can be validly inferred from what. And this concept will be explicated semantically, not with the help of possible worlds and truth conditions, but with the help of rational belief states and conditions of belief or acceptance." Rott thus does not consider embeddings of conditionals and presents what he takes to be "three central principles of connexivity" in a metalinguistic way, namely (notation adjusted):
>
> (Arist1) Not $(A{\to}{\sim}A)$,
>
> (Arist2) Not both $A{\to}C$ and ${\sim}A{\to}C$,
>
> (Boet-Abel) Not both $A{\to}C$ and $A{\to}{\sim}C$.
>
> Readers should therefore pay attention when Rott explains that
>
> > The interpretation of difference-making conditionals in the spirit of connexive logic is already present in 'DMC' [i.e., the paper [61]]. But the account of conditionals presented there is not fully connexive,

[17] Although it is not made explicit in the literature, we suggest that we add the non-interderivability of implication, namely $A{\to}B \nvdash B{\to}A$, for the same reason we offered for weakly connexive logic.

because it restricts the Principle of Abaelard and Boethius to non-absurd antecedents.

Being a 'fully connexive' conditional here thus does not mean that the conditional is connexive in the sense of McCall [36]. Restricting the Principle of Abelard and Boethius to non-absurd antecedents is not a reason for falling short of giving rise to a connexive logic, because (i) (Boet-Abel) is not stated as an object language formula and (ii) even if it was stated as the formula $\sim((A \rightarrow B) \wedge (A \rightarrow \sim B))$, it is not required to be valid by McCall's [36] definition of a connexive logic.

A relation to connexive logic is discussed only very late in Thomas Ferguson's contribution to the present special issue.

A note on the papers in the special issue. Thomas Ferguson in "Executability and connexivity in an interpretation of Griss" presents a connexive logic as "a deductive system that contains one or more of" AT and BT and thus considers what we here suggest to call 'partially connexive logics'. Moreover, he takes the metalinguistic version of Aristotle's thesis considered also in Hans Rott's contribution as a "proto-connexive feature".

The main part of the paper deals with developing logics that satisfy a number of desiderata from G.F.C. Griss's philosophy of mathematics. Executability understood as the requirement that mental constructions are possible only if some corresponding mental activity can indeed be carried out is identified as a key component of Griss's philosophy. This perspective reveals that Griss's "negationless mathematics" is after all compatible with several types of negation, especially the strong negation in Ahmad Almukdad and David Nelson's constructive logic **N4** with strong negation. Like **N4**, the connexive logic **C** is faithfully embeddable into positive intuitionistic logic both at the propositional and the first-order level. It is therefore natural to think of both **N4**, **C**, and related systems in an enterprise to pursue negationless constructive mathematics.

Ferguson develops a sequence of propositional logics that culminates in the pair of logics $\mathbf{N_1PAI}$ and $\mathbf{N_2PAI}$, which are intended to formalize Griss's account of negationless constructive mathematics.

Another use of the expression 'proto-connexive' can be found in [10]. If $\langle A, \leq \rangle$ is a pre-order, $\langle A; \sim, 1 \rangle$ is an algebra of type $(1, 0)$, and $\Rightarrow$ is a binary function on A, the algebra $\langle A; \leq, \sim, 1, \Rightarrow\rangle$ is there said to be proto-connexive if the following equational versions of Aristotle's and Boethius' theses hold:

$1 \leq \sim(x \Rightarrow \sim x)$,

$1 \leq \sim(\sim x \Rightarrow x)$,

$1 \leq (x{\Rightarrow}y){\Rightarrow}\sim(x \Rightarrow \sim y)$,

$1 \leq (x{\Rightarrow}\sim y){\Rightarrow}\sim(x \Rightarrow y)$,

and $\Rightarrow$ is not non-symmetric, i.e, there are no $x, y \in A$ such that either $(x{\Rightarrow}y) \leq (y{\Rightarrow}x)$ or $(y{\Rightarrow}x) \leq (x{\Rightarrow}y)$ does not hold. The non-symmetry of implication is a fundamental requirement because it excludes logical operations such as the equivalence connective of classical logic from being connexive. Therefore, we do not include 'proto-connexive' in any of the tables related to terminology.

Another term that is not included in any of the tables is the expression 'subconnexive', introduced in [68], because the subsystem of **QC** considered there is still a connexive logic.

4.3. Connexive Logical Operations and 'Poly-connexivity'

The characteristic principles of connexive logic highlight a binary connective understood as a conditional and a unary sentential operator read as a negation connective. It is therefore maybe not surprising that the conditional in a system of connexive logic is called a 'connexive implication'. Although the characteristic principles of connexive logic do feature a negation connective, it is already less obvious that the negation in a system of connexive logic should be called a 'connexive negation', and pondering this terminology brings one easily to the notorious questions of what is a genuine conditional and what is a genuine negation. Moreover, one may ask if there are other kinds of logical operations that can be 'connexive' in a certain sense. It seems to us that it is pretty clear that if the history of connexive logic is traced back to discussions in ancient and medieval philosophy and McCall's seminal paper [36] titled "Connexive implication" is seen to mark the beginning of modern connexive logic, then connexive logic is primarily concerned with a notion of *if A then B*.

> *A note on the papers in the special issue.* An approach to a concept of a connexive negation could be to fix some recognized conditional and to investigate the range of unary connectives such that the characteristic theses of connexive logic are validated. This is the path followed in Luis

Estrada-González and Ricardo Nicolás-Francisco's contribution "Connexive negation". Their starting point is the four-valued Dunn semantics for first-degree entailment logic, **FDE**, for which they consider four conditionals, two of which are "clear cases of conditionals" according to their terminology, namely the extensional conditional $A \rightarrow B$ defined in **FDE** by $\sim A \vee B$, and a 'material' conditional' $A \rightarrow_m B$ obtained by a certain tweaking of the truth condition for $\rightarrow$. They address the vexing issue of what is a negation connective, fix some necessary conditions for a one-place connective to be a negation and observe that for $\rightarrow_m$, four out of 32 negation connectives under consideration give rise to a connexive logic.

Estrada-González and Nicolás-Francisco also compare various properties of the negations under consideration with those of 'negation as cancellation', and they look into whether binary connectives can be defined that meet a set of constraints suggested to characterize the notion of a compatibility connective.

There are other occurrences of the term 'connexive negation' in the literature we are aware of. Richard and Valerie Routley [63, p. 208] use the term to refer to negation as cancellation. Shahid Rahman [59, p.139], in a discussion of dialogue games, presents a "kind of negation-operator ... linked to negation by default (in non-monotonic reasoning) and [that] has also been deployed to render the semantics of connexive negation." Since negation as cancellation gives rise to non-monotonic reasoning, he probably has negation as cancellation in mind as well. Claudio Pizzi [52, p. 125] contrasts 'connexive negation' with 'relevant negation', and he thus distinguishes between negation in connexive logic and negation in relevance logic. In the very final paragraph of [32], João Marcos mentions "the case of MacColl & McCall's connexive negation" thereby referring to negation in connexive logic, pointing to the work of Hugh MacColl and Storrs McCall. Satoru Niki [39] mentions "the sibling notion of connexive negation" in contrast to the strong negation in **N4**. These uses seem not to postulate a particular 'connexive' nature of negation.

We do not deny that it may be convenient to call a unary connective a 'connexive negation' if it exhibits some negation-like behaviour and if its combination with a given conditional gives rise to a connexive implication. However, in [36, p. 415] McCall explains that "[t]he definition of connexive implication is transmitted to us by Sextus Empiricus:

> 'And those who introduce the notion of connexion say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedents.' "

This consideration leads McCall to postulating BT. It is the binary connexive conditional that in virtue of validating Aristotle's and Boethius' theses expresses a *connection* between the antecedent and the consequent of a 'sound' conditional and not the unary negation.

4.3.1. 'Poly-connexivity' Nissim Francez [14,15] has suggested the notion 'poly-connexivity' to talk not only about connexive conditionals but 'connexive' variants of other logical operations as well, especially conjunction and disjunction. The guiding line for this approach seems to be the Bochum Plan (cf. [7]), and there is nothing wrong with following that plan as a research programme. However, given the looseness of the term 'connexivity', it might make sense to view the logical operations obtained by executing the Bochum Plan as giving rise not to a multiplicity of connexivity but to a way of generating a plurality of contra-classical logics (cf. [21]). A critical assessment of Francez's strategy to define connexive conjunction and disjunction connectives can be found in [8].

4.3.2. Bi-connexive Logics In classical logic conjunction (disjunction) is the dual of disjunction (conjunction), and negation is self-dual. The notion of duality applied is that if $A \vdash B$, then $B^* \vdash A^*$, where * is the operation that replaces every occurrence of a connective in a formula by its dual. One can regard the latter condition as defining the notion of duality in terms of order-inversion at the level of derivability, or couple the latter with classical logic to leave open the possibility of defining other notions of duality, especially in systems of non-classical logic with a non-definable, primitive conditional and in a bilateralist setting with more than one derivability relation.

A "bi-intuitionistic connexive logic", **BCL**, has been studied in [24]. The system **BCL** is an expansion of $\mathbf{C}^{\perp}$ by a 'connexive' variant of the co-implication used in Heyting-Brouwer logic (also known as bi-intuitionistic logic). The co-implication, denoted by '$\prec$', is in a sense dual to implication, and a formula $B \prec A$ is read as "A co-implies B". The logic **BCL** is also called "connexive Heyting-Brouwer Logic". In its Kripke semantics, the co-implication of Heyting-Brouwer logic and **BCL** is defined semantically in terms of a backward looking existential quantifier over states and therefore hardly a conditional. It is a system with a connexive implication and a co-implication connective whose falsity condition has been modified to the effect that the following equivalence is provable in the sequent calculus for **BCL**:

$$\sim(B \prec A) \leftrightarrow (\sim B \prec A).$$

A 'bi-connexive' logic, **2C**, has been defined in [70]. In addition to a constructive implication, it contains in its language a constructive co-implication, also denoted by '$\prec$', that is dual to implication in another sense. The logic **2C** is bilateral in the sense that its natural deduction proof system, N2C, makes use of two derivability relations, namely a provability relation that captures preservation of support of truth from the premises to the conclusion of an inference, and a relation of dual provability (refutability) that captures the preservation of support of falsity from the premises to the conclusion of an inference. Whereas the constructive implication internalizes provability into the object language, the constructive co-implication of **2C** internalizes refutability into the object language. The encoding of derivations in **2C** by typed λ-terms makes use of a two-sorted typed λ-calculus that encodes both the introduction of a conditional and the introduction of a co-implication by λ-abstraction. In that sense, the co-implication of **2C** *is* a conditional, its connexive version may be seen as a connexive co-implication, and **2C** may be regarded as a bi-connexive logic.

In the natural deduction proof system for **2C**, the following dual versions of Aristotle's and Boethius' theses are *refutable*:

dAT $\sim(\sim A \prec A)$,

dAT′ $\sim(A \prec \sim A)$,

dBT $\sim(\sim B \prec A) \prec (B \prec A)$,

dBT′ $\sim(B \prec A) \prec (\sim B \prec A)$.

Moreover, $(A \prec B) \prec (B \prec A)$ is not refutable in N2C. We can then say that a logic is dually connexive if its co-implication connective satisfies non-symmetry of co-implication and if it validates dAT–dBT′ for the entailment relation coinciding with the derivability relation internalized by co-implication.

Due to the presence of the strong negation from **C**, implication and co-implication in **2C** are interdefinable, and due to the presence of the falsum constant $\perp$, **2C** is definitionally equivalent with $\mathbf{C}^{\perp}$.[18]

[18] Note that $\sim\neg A$ is provable in $\mathbf{C}^{\perp}$ and **2C** for any formula A, if $\neg A$ is defined as $A \to \perp$. The provability of $\sim(A \to \perp)$ in the logic **dLP**, (dialetheic **LP**), has been observed in [43, Remark 13] and its provability in **2C** has been pointed out in [70, Sect. 5].

4.3.3. Quantifiers Another approach to extending the range of 'connexive' logical operations opens up with the occasionally observed parallelism between the constructive conditional and the universal quantifier in first-order intuitionistic logic. In terms of Kripke models for intuitionistic first-order logic, the analogy is eye-catching:[19]

- $A \rightarrow B$ is verified at state w iff for all $v \geq w$, if A is verified at v then B is verified at v,
- $\forall x A x$ is verified at state w iff for all $v \geq w$, if a is in the domain of v then $A(a)$ is verified at v,

and

- $A \rightarrow B$ is not verified at w iff for some $v \geq w$, A is verified at v but B is not,
- $\forall x A x$ is not verified at w iff for some $v \geq w$ and some a is in the domain of v, $A(a)$ is not verified at v.

If the support of falsity condition for implications is taken from the Kripke semantics of **C**, the parallelism is lost:

> $A \rightarrow B$ receives support of falsity at w iff for all $v \geq w$, if A receives support of truth at v then B receives support of falsity at v.

It is therefore not unnatural to mirror the support of falsity condition for implications in **C** by the following clause:

> $\forall x A x$ receives support of falsity at w iff for all $v \geq w$, if a is in the domain of v then $A(a)$ receives support of falsity at v.

This move has as a remarkable consequence that the support of falsity condition for formulas $\forall x A$ is now the same as the support of falsity condition for $\exists x A$ in **QC**, quantified **C**, see [42, 47, 68]. Grigory Olkhovikov [42] introduces the symbol Æ for this 'hybrid' quantifier, because both $Æ x A x \leftrightarrow \forall x A x$ and $\sim Æ x A x \leftrightarrow \sim \exists x A x$ are validated, given the validity of $\forall x \sim A x \leftrightarrow \sim \exists x A x$. He critically remarks that as a result $\sim Æ x A x \leftrightarrow Æ \sim A x$ is validated. In [77], Olkhovikov's quantifier Æ is taken as a universal quantifier, $\mathbb{A}$, and supplemented by a corresponding existential quantifier, $\mathbb{E}$. The universal quantifier is suggested to be used in formalizations of bare plurals in natural language, and it is observed that the quantifiers $\mathbb{A}$ and $\mathbb{E}$ have already been considered

[19] We talk of verification here because in intuitionistic logic no distinction is drawn between support of truth and support of falsity as in the case of logics with strong negation.

in the literature on bilattice logics as the infinitary meet and join operations of the information order in bilattices, see for instance [2, 12].

The relationship between the conditional in **C** and the universal quantifier $\mathbb{A}$ might be seen as a justification for understanding $\mathbb{A}$ as a 'connexive quantifier', but the relationship is thin, and the coincidence with infinitary meet and join operations reveals a sense in which the non-standard quantifiers are conjunction, respectively disjunction operations after all. Therefore, connexive logic would nevertheless in the first place be a theory of the meaning of "[a] new kind of implication" [35].

4.4. A Quick Summary

Before moving further, here is a table, for the third and final time, that summarizes the terminology related to variations of connexive logics discussed in this section.

A logic is ...	if it validates		and satisfies	
weakly connexive		BT, BT$'$ in rule form	non-interderivability of implication	
Kapsner strong			non-interderivability of implication	Unsat1 Unsat2
bi-connexive	if it is connexive and dually connexive			

5. Concluding Remarks: More Dividing Lines?

A number of properties could in principle be used to introduce further distinctions between connexive logics and logics in their vicinity, covered by the established and new terminology. We suggest being very reserved in this regard and to use modifiers that express the respective properties without introducing new brands of 'connexivity'. Still, there are several perspectives that can be helpful to understand varieties of systems of connexive logic. Here are some of them that come to us as useful and important.

Perspective 1: What kind of conditional?

Since the birth of modern connexive logics, one of the main connectives, namely the conditional, has been considered in a number of different ways. For example, Angell thought of the conditional as representing a subjunctive conditional, while McCall's [37] discussion within the context of syllogisms is not following that path. Pizzi's works are closely related to conditional logic, while there were a few contributions in connection to relevance logic. John

Cantwell introduced his hyperconnexive system **CN** with the intention to model indicative conditionals in English, and McCall [38] thinks of connexive implication as capturing a concept of causal implication. Another type of conditionals that has attracted special attention is the concessive *even if* conditional addressed in the contribution by Raidl, Iacona, and Crupi to this special issue.[20] Then there are the evidential and the difference-making conditionals treated in the contributions by Iacona and by Rott to the present special issue. The typology of conditionals is certainly relevant to the motivation and to applications of systems of connexive or restrictedly connexive logic, but note also that some distinctions here may be contentious.

Perspective 2: What kind of semantics, if any?
There has also been a variety of different semantics for systems of connexive logic. One of the first connexive logics, namely **CC1**, is a four-valued logic, whereas Pizzi's systems enjoy a modal semantics, like many others in the literature of connexive logic. Connexive logics related to relevance logics have been rather complicated in the semantic treatment, but there is also Mortensen's three-valued logic, known as **M3V**, which turned out to be equivalent to Cantwell's three-valued logic **CN**, even though Cantwell's system was not designed to be a system of connexive logic. Finally, there are some algebraic treatments as well.

Perspective 3: Consistency versus negation inconsistency
There is a more unorthodox dividing line that can be drawn between systems of connexive logic because some of these contra-classical logics turn out to be negation consistent, such as McCall's **CC1**, while others are non-trivial but negation inconsistent, such as **C**, **MC**, **C3**, **CN**, and **M3V**, and the negation inconsistent pure theories of connexive implication in [79].[21] However, there seems to be no pressing need to stipulate different kinds of connexivity coming with the distinction between negation consistent and negation inconsistent connexive logics. Non-trivial negation inconsistent logics may appear to be unusual, but there are actually various kinds of well motivated such logics that have been arrived at independently of any desire to fabricate non-trivial contradictory logics (cf. [48, 72]).

---

[20] We cannot enter into linguistic considerations here. According to [29, p. 220] "in the majority of languages, as in English, concessive conditionals are morphologically similar, if not identical, to causal conditionals." For a discussion of concessive conditionals in linguistics see [28].

[21] For a study of the negation inconsistency of **C** and **C3** see [40].

There can be more and more perspectives, such as the choice of the language, connections to experiments, as well as historical figures, but let us emphasize again that we should not keep on introducing more and more terminology. We strongly believe that instead of diversifying the terminology it is more interesting and important to discuss the reasons for studying various systems of connexive logic, both old and new, and deepen our understanding of connexive logic. We hope that this special issue will contribute towards reaching more deep and surprising results in connexive logics. A table summarizing our suggestion for terminology can be found in the Appendix.

Acknowledgements. We would like to thank the members of the programme and organization committees (different from ourselves) of the *Trends in Logic* conference at Ruhr University Bochum, Germany, December 6–8, 2021, and all the referees who helped us to edit the present special issue of *Studia Logica*. Heinrich Wansing has been supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant agreement ERC-2020-ADG, 101018280, ConLog, and Hitoshi Omori has been supported by a Sofja Kovalevskaja Award of the Alexander von Humboldt-Foundation, funded by the German Ministry for Education and Research. Moreover, we are grateful to Sara Ayhan, Norihiro Kamide, Nils Kürbis, Satoru Niki, Grigory Olkhovikov, Yaroslav Shramko and those contributors to this special issue (different from ourselves) who gave us feedback on this introductory note, especially Hans Rott and Luis Estrada-González. Needless to say that we take sole responsibility for the views presented and the suggestions made in this paper.

Funding Open Access funding enabled and organized by Projekt DEAL.

Appendix

Suggested terminology

Established terminology

A logic is ...	AT	AT′	BT	BT′	NSI	Unsat 1,2	cBT	cBT′	More
connexive	✓	✓	✓	✓	✓				
hyperconnexive	✓	✓	✓	✓	✓		✓	✓	
strongly connexive	✓	✓	✓	✓	✓	✓			
superconnexive	✓	✓	✓	✓	✓	✓			SA
totally connexive	✓	✓	✓	✓	✓	✓			cf. §2.5

where NSI stands for non-symmetry of implication, and SA for $(A \rightarrow \sim A) \rightarrow B$.

New terminology

- A logic is **partially connexive** iff
  - at least one but not all of AT, AT′, BT, BT′ is valid, and
  - non-symmetry of implication is satisfied.
- A logic is **restrictedly connexive** iff
  - at least one but not all of AT, AT′, BT, BT′ is valid with restrictions imposed on antecedents or consequents, and
  - non-symmetry of implication is satisfied.

Variations

- A logic is **weakly connexive** iff
  - BT and BT′ holds in the rule form, and
  - non-interderivability of implication.
- A logic is **Kapsner strong** iff
  - Unsat 1 and Unsat 2 holds, and
  - non-interderivability of implication.
- A logic is **bi-connexive** iff it is connexive and dually connexive.

References

[1] Angell, R. B., A propositional logic with subjunctive conditionals, *Journal of Symbolic Logic* 27(3):327–343, 1962.

[2] Arieli, O., and A. Avron, Reasoning with logical bilattices, *Journal of Logic, Language and Information* 5:25–63, 1996.

[3] Bocheński, J. M., *Formale Logik*, Alber, Freiburg, 1956 (Revised translation *A History of Formal Logic*, University of Notre Dame Press, Indiana 1961).

[4] Bonevac, D., and J. Dever., A history of the connectives, in D. Gabbay, F.J. Pelletier, and J. Woods, (eds.), *Handbook of the History of Logic*, vol. 11, Elsevier, 2012, pp. 175–233.

[5] Cantwell, J., The Logic of Conditional Negation, *Notre Dame Journal of Formal Logic* 49(3):245–260, 2008.

[6] Crupi, V., and A. Iacona, The evidential conditional, *Erkenntnis* 87(6):2897–2921, 2022.

[7] Estrada-González, L., The Bochum Plan and the foundations of contra-classical logics, *CLE e-Prints*, 19(1), 2020.

[8] Estrada-González, L., An analysis of poly-connexivity, *Studia Logica* 110:925–947, 2022.

[9] Estrada-González, L., and E. Ramírez-Cámara, A comparison of connexive logics, *IfCoLog Journal of Logics and their Applications* 3(3):341—355, 2016.

[10] Fazio, D., and G. St. John, Connexive implications in substructural logics. arXiv:2209.02830, 2022.

[11] Ferenz, N., Conditional **FDE** logics, in K. Bimbó, (ed.), *Relevance Logics and other Tools for Reasoning. Essays in Honour of Michael Dunn*, College Publications, 2022, pp. 182–214.

[12] Fitting, M., Bilattices in logic programming, in G. Epstein, (ed.), *Proceedings of The International Symposium on Multiple-Valued Logic*, IEEE, New York, 1990, pp. 238–246.

[13] Francez, N., Natural-deduction for two connexive logics, *IfCoLog Journal of Logics and their Applications* 3:479–504, 2016.

[14] Francez, N., A poly-connexive logic, *Logic and Logical Philosophy* 29:143–57, 2020.

[15] Francez, N., *A View of Connexive Logics*, College Publications, London, 2021.

[16] Gabbay, D., What is negation in a system?, in F. Drake, and J. Truss, (eds.), *Logic Colloquium' 86*, Elsevier, Amsterdam, 1988, pp. 95–112.

[17] Gabbay, D., and H. Wansing, (eds.), *What is Negation?*, Kluwer, Dordrecht, 1999.

[18] Gherardi, G., and E. Orlandelli, Super-strict implications, *Bulletin of the Section of Logic* 50(1):1–34, 2021.

[19] Gherardi, G., and E. Orlandelli, Non-normal super-strict implications, *Electronic Proceedings in Theoretical Computer Science* 358:1–11, 2022.

[20] Günther, M., A connexive conditional, *Logos & Episteme* 13(1):55–63, 2022.

[21] Humberstone, L., Contra-classical logics, *Australasian Journal of Philosophy* 78(4):438–474, 2000.

[22] Jamrużek, O., and J. Malinowski, Boolean connexive logics: Semantics and tableau approach, *Logic and Logical Philosophy* 28(3):427–448, 2019.

[23] Kamide, N., Kripke-Completeness and cut-elimination theorems for intuitionistic paradefinite logics with and without quasi-explosion, *Journal of Philosophical Logic* 49:1185–1212, 2020.

[24] Kamide, N., and H. Wansing, Completeness of connexive Heyting–Brouwer logic, *The IfColog Journal of Logics and their Applications* 3:441–466, 2016.

[25] Kapsner, A., Strong connexivity, *Thought* 1(2):141–145, 2012.

[26] Kapsner, A., Humble connexivity, *Logic and Logical Philosophy* 28:513–536, 2019.

[27] Kapsner, A., and H. Omori, Superconnexivity reconsidered, in *Proceedings of NCL 2022*, EPTCS 358, 2022, pp. 160–173.

[28] König, E., Conditionals, concessive conditionals and concessives: Areas of contrast overlap and neutralization, in E.C. Traugott, A. ter Meulen, J.S. Reilly, and Ch.A. Ferguson, (eds.), *On Conditionals*, Cambridge University Press, 1986, pp. 229–246.

[29] König, E., Constraints on the form and meaning of the protasis, in E.C. Traugott, A. ter Meulen, J.S. Reilly, and Ch.A. Ferguson, (eds.), *On Conditionals*, Cambridge University Press, 1986, pp. 215–227.

[30] Lenzen, W., Rewriting the history of connexive logic, *Journal of Philosophical Logic* 51(1):525–553, 2022.

[31] Malinowski, J., and R. Palczewski, Connexive Modal Logic, in A. Giordani, and J. Malinowski, (eds.), *Logic in High Definition*, Springer, Cham, 2021, pp. 49–65.

[32] Marcos, J., On negation: Pure local rules, *Journal of Applied Logic* 3:185–219, 2005.

[33] Martin, C., Logic, in *The Cambridge Companion to Abelard*, Cambridge University Press, 2004, pp. 158–199.

[34] McCall, S., *Non-classical Propositional Calculi*, Ph.D. thesis, Oxford University, 1963.

[35] McCall, S., A new variety of implication (abstract), *Journal of Symbolic Logic* 29:151–152, 1964.

[36] McCall, S., Connexive implication, *Journal of Symbolic Logic* 31(3):415–433, 1966.

[37] McCall, S., Connexive implication and the syllogism, *Mind* 76(303):346–356, 1967.

[38] McCall, S., A history of connexivity, in D. Gabbay, F.J. Pelletier, and J. Woods, (eds.), *Handbook of the History of Logic*, vol. 11, Elsevier, 2012, pp. 415–449.

[39] Niki, S., Improving strong negation, *The Review of Symbolic Logic* 16:951–977, 2023.

[40] Niki, S., and H. Wansing, On the provable contradictions of the connexive logics **C** and **C3**, *Journal of Philosophical Logic* published online 29 July 2023.

[41] Olkhovikov, G., On a new three-valued paraconsistent logic (in Russian), in *Logic of Law and Tolerance*, Ural State University Press, Yekaterinburg, 2001, pp. 96–113.

[42] Olkhovikov, G. K., On the completeness of some first-order extensions of **C**, *Journal of Applied Logics - IfCoLog Journal* 10(1):57–114, 2023.

[43] Omori, H., From paraconsistent logic to dialetheic logic, in H. Andreas, and P. Verdée, (eds.), *Logical Studies of Paraconsistent Reasoning in Science and Mathematics*, Springer, 2016, pp. 111–134.

[44] Omori, H., A note on Francezc' half-connexive formula, *IfCoLog Journal of Logics and their Applications* 3:505–512, 2016.

[45] Omori, H., Towards a bridge over two approaches in connexive logic, *Logic and Logical Philosophy* 28:553–566, 2019.

[46] Omori, H., and H. Wansing, Connexive logics. An overview and current trends, *Logic and Logical Philosophy* 28:371–387, 2019.

[47] Omori, H., and H. Wansing, An extension of connexive logic C, in N. Olivetti, R. Verbrugge, S. Negri, and G. Sandu, (eds.), *Advances in Modal Logic*, vol. 13, College Publications, London, 2020, pp. 503–522.

[48] Omori, H., and H. Wansing, Varieties of negation and contra-classicality in view of Dunn semantics, in K. Bimbó, (ed.), *Relevance Logics and other Tools for Reasoning. Essays in Honour of Michael Dunn*, College Publications, 2022, pp. 309–337.

[49] Pfeifer, N., and L. Schöppl, Experimental philosophy of connexivity, in H. Wansing, and H. Omori, (eds.), *60 Years of Connexive Logic*, Springer, 202+. (22 pp.).

[50] Pizzi, C., Boethius' thesis and conditional logic, *Journal of Philosophical Logic* 6(1):283–302, 1977.

[51] Pizzi, C., Weak vs. Strong Boethius' Thesis: A Problem in the Analysis of Consequential Implication, in A. Ursini, and P. Aglinano, (eds.), *Logic and Algebra*, New York: Marcel Dekker, 1996, pp. 647–654.

[52] Pizzi, P., Aristotle's Thesis between paraconsistency and modalization, *Journal of Applied Logic* 3(1):119–131, 2005.

[53] Pizzi, C., Aristotle' Cubes and Consequential Implication, *Logica Universalis* 2:143–153, 2008.

[54] Pizzi, C., Two Kinds of Consequential Implication, *Studia Logica* 106:453–480, 2018.

[55] Pizzi, C., and T. Williamson, Strong Boethius' Thesis and Consequential Implication, *Journal of Philosophical Logic* 26(5):569–588, 1997.

[56] Pizzi, C., and T. Williamson, Conditional Excluded Middle in Systems of Consequential Implication, *Journal of Philosophical Logic* 34(5):333–362, 2005.

[57] Priest, G., Negation as cancellation and connexive logic, *Topoi* 18(2):141–148, 1999.

[58] Priest, G., *An Introduction to Non-Classical Logic: From If to Is*, Cambridge University Press, 2 edition, 2008.

[59] Rahman, S., NO-ARGUMENTS: Denials, refutations, negations and the constitution of arguments, *Characteristica Universalis Journal* 1:135–160, 2020.

[60] Raidl, E., Neutralization, Lewis' doctored conditional, or another note on "A connexive conditional", *Logos & Episteme* 14(1):101–118, 2023.

[61] Rott, H., Difference-making conditionals and the relevant Ramsey test, *Review of Symbolic Logic* 15:133–164, 2022.

[62] Routley, R., Semantics for connexive logics. I, *Studia Logica* 37(4):393–412, 1978.

[63] Routley, R., and V. Routley, Negation and contradiction, *Revista Colombiana de Matemáticas* 19(4):201–230, 1985.

[64] Seuren, P., A logic-based approach to problems in pragmatics, *Poznań Studies in Contemporary Linguistics* 46:519–532, 2010.

[65] Stalnaker, R. C., A theory of conditionals, in N. Rescher, (ed.), *Studies in Logical Theory*, Oxford, 1968, pp. 98–112.

[66] Sylvan, R., A preliminary western history of sociative logics, chapter 4 in *Bystanders' Guide to Sociative Logics, Research Series in Logic and Metaphysics* #9, Australian National University, Canberra, 1989 (Reprinted in: D. Hyde, and G. Priest, (eds.), chapter 5 of *Sociative Logics and their Applications. Essays by the Late Richard Sylvan*, Aldershot, Ashgate Publishing, 2000).

[67] Wansing, H., (ed.), *Negation. A Notion in Focus*, de Gryuter, Berlin, 1996.

[68] Wansing, H., Connexive Modal Logic, in R. Schmidt, I. Pratt-Hartmann, M. Reynolds, and H. Wansing, (eds.), *Advances in Modal Logic*, vol. 5, King's College Publications, London, 2005, pp. 367–383.

[69] Wansing, H., A Note on Negation in Categorial Grammar, *Logic Journal of the IGPL* 15:271–286, 2007.

[70] Wansing, H., Natural deduction for bi-connexive logic and a two-sorted typed λ-calculus, *IfCoLog Journal of Logics and their Applications* 3:413–439, 2016.

[71] Wansing, H., Connexive logic, in E.N. Zalta, (ed.), *The Stanford Encyclopedia of Philosophy*, http://plato.stanford.edu/archives/sum2023/entries/logic-connexive/, summer 2023 edition, 2023.

[72] Wansing, H., Beyond paraconsistency. A plea for a radical breach with the Aristotelean orthodoxy logic, in A. Rodrigues, H. Antunes, and A. Freire, (eds.), *Outstanding Contributions to Logic volume for Walter Carnielli*, Springer, 2023, forthcoming.

[73] Wansing, H., and H. Omori, A note on "a connexive conditional", *Logos & Episteme* 13(3):325–328, 2022.

[74] Wansing, H., H. Omori, and T. Ferguson, The tenacity of connexive logic: Preface to the special issue, *IfCoLog Journal of Logics and their Applications* 28(3):279–296, 2016.

[75] Wansing, H., and D. Skurt, Negation as cancellation, connexive logic, and *qLPm*, *Australasian Journal of Logic* 15:476–488, 2018.

[76] Wansing, H., and M. Unterhuber, Connexive conditional logic. Part I, *Logic and Logical Philosophy* 28(3):567–610, 2019.

[77] Wansing, H., and Z. Weber, Quantifiers in connexive logic (in general and in particular), Submitted for publication, 2023.

[78] Weiss, Y., Connexive extensions of regular conditional logic, *Logic and Logical Philosophy* 28(3):611–627, 2019.

[79] Weiss, Y., Semantics for pure theories of connexive implication, *Review of Symbolic Logic* 15(3):611–627, 2022.

[80] Wu, H., and M. Ma, An Infinity of Intuitionistic Connexive Logics, in M. Banerjee, and A.V. Sreejith, (eds.), *Logic and Its Applications. 10th Indian Conference, ICLA 2023*, vol 13963 of *Lecture Notes in Computer Science*, Springer, Cham, 2023, pp. 189–206.

H. Wansing
Department of Philosophy I
Ruhr University Bochum
Universitätsstraße 150
44780 Bochum
Germany
Heinrich.Wansing@rub.de

H. Omori
Graduate School of Information Sciences
Tohoku University
6-3-09 Aoba, Aramaki-aza Aoba-ku
Sendai 980-8579
Japan
hitoshiomori@gmail.com