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On Pretabular Extensions of Relevance Logic

Abstract. We exhibit infinitely many semisimple varieties of semilinear De Morgan monoids (and likewise relevant algebras) that are not tabular, but which have only tabular proper subvarieties. Thus, the extension of relevance logic by the axiom $(p \to q) \vee (q \to p)$ has infinitely many pretabular axiomatic extensions, regardless of the presence or absence of Ackermann constants.

Keywords: Pretabular variety, De Morgan monoid, Relevant algebra, Semilinearity, Relevance logic.

1. Introduction

When a logic **S** is algebraized, in the sense of [7], by a variety K of algebras, the axiomatic extensions of **S** and the subvarieties of K form anti-isomorphic lattices, so the former can be studied through the lens of the latter. The lower part of the subvariety lattice is more transparent when K is congruence distributive, as its *tabular* subvarieties—i.e., those generated by a finite algebra—have then only finitely many subvarieties of their own [3, Cor. 5.12]. Under these conditions it is of interest to determine, where possible, the *pretabular* subvarieties of K, i.e., the minimal non-tabular ones.

In the cases of intuitionistic propositional logic and **S4**, it is known that there are just three pretabular varieties of Heyting algebras [27], and just five of interior algebras [28]; also see [4,13,16,29]. These varieties are generated by their finite members. The situation in relevance logic is more complicated. According to the main result of [39], there are $2^{\aleph_0}$ pretabular varieties of relevant algebras, so it is natural to shift attention to strengthenings of the principal system **R** of Anderson and Belnap [2].

The extension **RM** of **R** by the *mingle* axiom $p \to (p \to p)$ is algebraized by the variety of Sugihara algebras, whose pretabularity was shown by Dunn [12]. This locally finite variety also models the *Gödel-Dummett* axiom $(p \to q) \vee (q \to p)$. Although Anderson and Belnap associated these two

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postulates with ‘paradoxes of material implication’, their effects have been studied extensively in the ambient literature, because of the mathematical intelligibility that they add to substructural logics like **R**.

A complicating factor is the presence or absence of the Ackermann constant **t** (or its negation **f**) in the signature. The system that results from adding the axioms **t** and $\mathbf{t} \to (p \to p)$ to a logic **S** is denoted as $\mathbf{S}^{\mathbf{t}}$. It is often a conservative extension (as in the case of **R** itself), but the move from **S** to $\mathbf{S}^{\mathbf{t}}$ may affect the subvariety lattice of the model class. For instance, the variety of Sugihara monoids (which algebraizes $\mathbf{RM}^{\mathbf{t}}$) is not pretabular; it has a pretabular proper subvariety, axiomatized by $e = \neg e$, where e is the algebraic incarnation of **t** (see [31, Thm. 4.4]).

The extension **RL** of **R** by the Gödel-Dummett axiom and its expansion $\mathbf{RL}^{\mathbf{t}}$ are algebraized, respectively, by the relevant algebras and the De Morgan monoids that are *semilinear*—i.e., subdirect products of chains. The business of isolating the pretabular varieties of such algebras is in its infancy. Apart from Sugihara algebras, two pretabular varieties of semilinear relevant algebras were identified in [17], and a pretabular variety of semilinear De Morgan monoids in [34, Ex. 9.5]. The first two are generated by their finite members; the last is not.

The present paper establishes that there are infinitely many pretabular varieties of semilinear De Morgan monoids, and of semilinear relevant algebras (Theorems 3.17 and 4.12). Our examples will be locally finite semisimple varieties—in fact, discriminator varieties. Each of them has a well-ordered subvariety lattice of order type $\omega + 1$, and is generated as a quasivariety by a single denumerable algebra. These results partially answer questions in [22] and [17, Sec. 5].

2. Conventions

Below, ω denotes the set of non-negative integers, and $\mathbb{N}$ the set of positive integers.

The universe of an algebra $\boldsymbol{A}$ is denoted as A, and is never empty. The smallest congruence of $\boldsymbol{A}$ containing a pair $\langle a, b\rangle$ is denoted as $\Theta^{\boldsymbol{A}}(a, b)$. We say that $\boldsymbol{A}$ has the *congruence extension property* (CEP) if every congruence on a subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ is the restriction $(B \times B) \cap \varphi$ of some congruence φ of $\boldsymbol{A}$. In this case, for all $a, b \in B$, we have $\Theta^{\boldsymbol{B}}(a, b) = (B \times B) \cap \Theta^{\boldsymbol{A}}(a, b)$. A class of similar algebras is said to have the CEP if all of its members have the CEP.

A variety K of algebras is said to have *equationally definable principal congruences* (EDPC) if there is a finite set Σ of pairs of 4-ary terms in its signature such that, whenever $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$, then

$$\langle c, d\rangle \in \Theta^{\boldsymbol{A}}(a, b) \text{ iff } \big(\alpha^{\boldsymbol{A}}(a, b, c, d) = \beta^{\boldsymbol{A}}(a, b, c, d) \text{ for all } \langle \alpha, \beta\rangle \in \Sigma\big).$$

The class operator symbols $\mathbb{S}$, $\mathbb{I}$, $\mathbb{H}$, $\mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$ stand, respectively, for closure under subalgebras, isomorphic and homomorphic images, direct products and ultraproducts, while $\mathbb{V}$ and $\mathbb{Q}$ denote varietal and quasivarietal generation, i.e., $\mathbb{V} = \mathbb{HSP}$ and $\mathbb{Q} = \mathbb{ISPP}_{\mathrm{U}}$. We write $\mathbb{V}(\{\boldsymbol{A}\})$ as $\mathbb{V}(\boldsymbol{A})$, etc.

Jónsson's Theorem [24, 25] states that, for any subclass L of a congruence distributive variety of algebras, each finitely subdirectly irreducible member of $\mathbb{V}(\mathsf{L})$ belongs to $\mathbb{HSP}_{\mathrm{U}}(\mathsf{L})$.

For a variety K, we use K_{SI} to denote the class of all subdirectly irreducible members of K.

3. De Morgan Monoids

Definition 3.1. A *De Morgan monoid* is an algebra $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprising a distributive lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, e\rangle$ and a function $\neg\colon A \longrightarrow A$, called an *involution*, where $\boldsymbol{A}$ satisfies $\neg\neg x = x$ and $x \leqslant x^2 := x \cdot x$ and

$$x \cdot y \leqslant \neg z \iff x \cdot z \leqslant \neg y. \tag{1}$$

Here, $\alpha \leqslant \beta$ abbreviates $\alpha = \alpha \wedge \beta$. We refer to $\cdot$ as *fusion*, and we define

$$f = \neg e \text{ and } x \rightarrow y = \neg(x \cdot \neg y) \text{ and } x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

We denote by DMM the class of all De Morgan monoids, which is a variety [21, Thm. 2.7]. De Morgan monoids were introduced by J.M. Dunn [11], who proved that DMM algebraizes the relevance logic $\mathbf{R^t}$; also see [2, 32]. A brief summary of the algebraization process can be found in [33, Sec. 4.1]. Our focus here will be on the algebras.

For $\boldsymbol{A}$ as in Definition 3.1, fusion distributes over $\vee$, while $\neg$ is an anti-automorphism of $\langle A; \wedge, \vee\rangle$, so De Morgan's laws hold. Every De Morgan monoid satisfies $\neg x = x \rightarrow f$ and

$$x \cdot y \leqslant z \iff x \leqslant y \rightarrow z \quad \text{(the law of residuation).} \tag{2}$$

The following facts about any De Morgan monoid $\boldsymbol{A}$ are known (see [33] for sourcing).

(I) $\boldsymbol{A}$ is simple iff e has just one strict lower bound in $\boldsymbol{A}$.

(II) $\boldsymbol{A}$ is finitely subdirectly irreducible iff e is join-irreducible (or equivalently, join-prime) in $\boldsymbol{A}$.

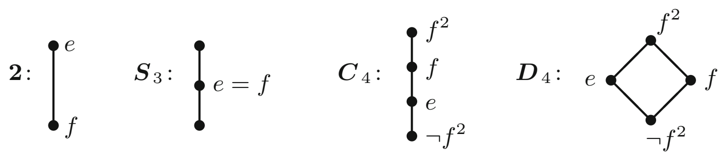
(III) $\boldsymbol{A}$ is subdirectly irreducible iff e is completely join-irreducible in $\boldsymbol{A}$.

(IV) If $\boldsymbol{A}$ has a least element $\bot$, then $a \cdot \bot = \bot$ for all $a \in A$. In this case we say that $\boldsymbol{A}$ is *bounded*, as $\top := \neg\bot$ is the greatest element of $\boldsymbol{A}$.

(V) $\boldsymbol{A}$ satisfies $f \leqslant e$ iff it is *idempotent* (i.e., $a^2 = a$ for all $a \in A$). In this case $\boldsymbol{A}$ is called a *Sugihara monoid*. The *odd* Sugihara monoids are the ones in which $f = e$.

(VI) $\boldsymbol{A}$ satisfies $x \leqslant e$ iff it is a Boolean algebra (in which $\cdot$ duplicates $\wedge$).

We depict below the two-element Boolean algebra $\mathbf{2}$, the three-element odd Sugihara monoid $\boldsymbol{S}_3$, and two four-element De Morgan monoids, $\boldsymbol{C}_4$ and $\boldsymbol{D}_4$. In each case, the labeled Hasse diagram determines the structure. Note that $\neg f^2$ abbreviates $\neg(f^2)$.



As it happens, the varieties generated, respectively, by these four algebras are exactly the minimal (nontrivial) subvarieties of DMM [33, Thm. 6.1]. Moreover, any simple De Morgan monoid that has no proper subalgebra is isomorphic to $\mathbf{2}$ or to $\boldsymbol{C}_4$ or to $\boldsymbol{D}_4$; this is implicit in [37,38] and explicit in [33, Thm. 5.20].

The variety DMM has EDPC [21, Thm. 3.55]. In fact, for every De Morgan monoid $\boldsymbol{A}$, with $a, b, c, d \in A$, we have

$$\langle c, d\rangle \in \Theta^{\boldsymbol{A}}(a, b) \text{ iff } (a \leftrightarrow b) \wedge e \leqslant c \leftrightarrow d. \tag{3}$$

Lemma 3.2. *In a variety with EDPC, let L be a class of simple algebras. Then every nontrivial finitely subdirectly irreducible member of $\mathbb{V}(\mathsf{L})$ belongs to $\mathbb{ISP}_{\mathrm{U}}(\mathsf{L})$ and is simple.*

Proof. Every variety with EDPC is congruence distributive and has the CEP [6, Thm. 1.2]. By Jónsson's Theorem, each finitely subdirectly irreducible member of $\mathbb{V}(\mathsf{L})$ belongs to $\mathbb{HSP}_{\mathrm{U}}(\mathsf{L})$. In a variety with EDPC, ultraproducts of simple algebras are simple [6, p. 211], and the CEP ensures that nontrivial subalgebras of simple algebras are simple, so the nontrivial members of $\mathbb{HSP}_{\mathrm{U}}(\mathsf{L})$ belong to $\mathbb{ISP}_{\mathrm{U}}(\mathsf{L})$ and are simple. ∎

Examples 3.3 and 3.4 below are not merely illustrative; they introduce algebras that will help us to construct pretabular varieties of De Morgan monoids.

EXAMPLE 3.3. ([34, Ex. 9.1]) For each positive integer p, let $\boldsymbol{A}_p$ be the De Morgan monoid on the chain

$$0 < 1 < 2 < 4 < 8 < 16 < \ldots < 2^{p-1} < 2^p < 2^{p+1},$$

where fusion is ordinary multiplication, truncated at 2^{p+1}. Thus, $|A_p| = p+3$ and e is the integer 1, while $\neg(2^k) = 2^{p-k}$ for all $k \in \{0, 1, \ldots, p\}$, with $\neg 0 = 2^{p+1}$ and $\neg 2^{p+1} = 0$. In particular, $f = 2^p$. Clearly, $\boldsymbol{A}_p$ is simple and generated by 2, and we may identify $\boldsymbol{C}_4$ with the subalgebra of $\boldsymbol{A}_p$ on $\{0, 1, 2^p, 2^{p+1}\}$. It is proved in [34] that, when p is prime, then $\boldsymbol{C}_4$ is the only proper subalgebra of $\boldsymbol{A}_p$, and $\mathbb{V}(\boldsymbol{A}_p)$ is a join-irreducible cover of $\mathbb{V}(\boldsymbol{C}_4)$ in the lattice of varieties of De Morgan monoids. Note that when p is an odd prime, then $\neg$ has no fixed point in $\boldsymbol{A}_p$. ∎

EXAMPLE 3.4. Given $p, i \in \mathbb{N}$, we now construct a totally ordered simple De Morgan monoid $\boldsymbol{B} = \boldsymbol{B}_{p,i}$, of which $\boldsymbol{A}_p$ is a subalgebra, as follows.

The universe $B = B_{p,i}$ and the total order $\leqslant$ of $\boldsymbol{B}$ arise by inserting into $\boldsymbol{A}_p$ the new elements x_j and $-x_j$ indicated and ordered below:

$$1 < x_1 < x_2 < \ldots < x_i < 2 \quad \text{and} \quad 2^{p-1} < -x_i < \ldots < -x_2 < -x_1 < 2^p.$$

This determines the (distributive) lattice operations of $\boldsymbol{B}$. Observe that $|B| = p + 3 + 2i$. We extend $\neg$ from $\boldsymbol{A}_p$ to $\boldsymbol{B}$ by defining

$$\neg x_j = -x_j \quad \text{and} \quad \neg(-x_j) = x_j \text{ for } j = 1, \ldots, i.$$

Thus, $\boldsymbol{B}$ satisfies $\neg\neg x = x$ and if p is an odd prime, then $\neg$ has no fixed point in $\boldsymbol{B}$. It is convenient to define $g\colon B \to A_p$ by

$$g(a) = \begin{cases} 2 & \text{if } a = x_j \text{ for some } j; \\ 2^{p-1} & \text{if } a = -x_j \text{ for some } j; \\ a & \text{otherwise.} \end{cases}$$

We then extend the fusion $\cdot$ of $\boldsymbol{A}_p$ to a commutative binary operation $\cdot^{\boldsymbol{B}}$ on B, with neutral element 1, by stipulating that for $1 \neq b, c \in B$,

$$b \cdot^{\boldsymbol{B}} c = \begin{cases} g(b) \cdot g(c) & \text{if } b \leqslant \neg c; \\ 2^{p+1} & \text{if } b > \neg c. \end{cases}$$

It follows that $0 \cdot^{\boldsymbol{B}} b = 0$ for all $b \in B$, and

$$b \cdot^{\boldsymbol{B}} c \in A_p, \text{ unless } \{b, c\} \text{ is } \{1, x_j\} \text{ or } \{1, -x_j\} \text{ for some } j. \tag{4}$$

It is easy to see that, for all $b, c, d \in B$, we have $b \leqslant b^2$ and

$$\text{if } c \leqslant d, \text{ then } \quad b \cdot^{\boldsymbol{B}} c \leqslant b \cdot^{\boldsymbol{B}} d,$$

whence $\leqslant$ is compatible with $\cdot^{\boldsymbol{B}}$.

The associative law $\left(x \cdot^{\boldsymbol{B}} y\right) \cdot^{\boldsymbol{B}} z = x \cdot^{\boldsymbol{B}} \left(y \cdot^{\boldsymbol{B}} z\right)$ clearly holds when one of the factors is 0 or 1, so assume that $x, y, z > 1$. Then the law follows from the associativity of $\cdot$ in $\boldsymbol{A}_p$, except in one case: when $b > \neg c$, we have $b \cdot^{\boldsymbol{B}} c = 2^{p+1}$, whereas $g(b) \cdot g(c)$ may be 2^p. In this case, however, the presence of a third factor on either side of the law forces both sides to be 2^{p+1}, because $x, y, z > 1$. Thus, $\cdot^{\boldsymbol{B}}$ is associative.

Since $\leqslant$ well-orders B and is compatible with $\cdot^{\boldsymbol{B}}$, and as 0 is absorptive for $\cdot^{\boldsymbol{B}}$, it follows that for any $c, d \in B$, the element

$$c \to d := \max\nolimits_{\leqslant} \left\{b \in B : b \cdot^{\boldsymbol{B}} c \leqslant d\right\} \tag{5}$$

exists and that the law of residuation (2) holds in $\boldsymbol{B}$. Consequently, $\boldsymbol{B}$ satisfies $x \to (y \to z) = (x \cdot y) \to z = y \to (x \to z)$, as $\cdot^{\boldsymbol{B}}$ is commutative. The following property of $\to$ will be helpful later:

$$\text{whenever } b \in B \text{ and } 0 < b < 2^{p+1}, \text{ then } b \to b = 1 (= e). \tag{6}$$

Using (5), we see that $\neg b = b \to f\ (= b \to 2^p)$ for all $b \in B$, so because $b \to (c \to f) = c \to (b \to f)$, we have $b \to \neg c = c \to \neg b$ for all $b, c \in B$. This, with (2), establishes (1), so $\boldsymbol{B} = \langle B; \cdot^{\boldsymbol{B}}, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \neg, 1\rangle$ is a De Morgan monoid. And $\boldsymbol{B}$ is simple, by (I). ■

It is clear from the definitions of operations that $\boldsymbol{B}_{p,k}$ is a subalgebra of $\boldsymbol{B}_{p,i}$ whenever $k \leq i$.

Definition 3.5. For each $p \in \mathbb{N}$, let $\mathsf{K}_p = \mathbb{V}(\{\boldsymbol{B}_{p,i} : i \in \mathbb{N}\})$.

An algebra is said to be *n-generated*, where $n \in \omega$, if it has a generating subset of cardinality at most n. A variety K is said to be *locally finite* if each of its finitely generated members is a finite algebra. It has *finite type* if its set of operation symbols is finite. A variety K of finite type is locally finite iff there is a function $f \colon \omega \to \omega$ such that, for each $n \in \omega$, every n-generated member of K_{SI} has at most $f(n)$ elements [30, Thm. VI.3, p. 285].

A variety is said to be *tabular* (or *finitely generated*) if it has the form $\mathbb{V}(\boldsymbol{A})$ for some finite algebra $\boldsymbol{A}$. Such a variety is locally finite [3, Thm. 3.49].

Theorem 3.6. *For each prime $p \in \mathbb{N}$, the variety K_p is locally finite, but not tabular.*

Proof. For any finite algebra $\boldsymbol{F}$, we have $\mathbb{P}_{\mathrm{U}}(\boldsymbol{F}) \subseteq \mathbb{I}(\boldsymbol{F})$ [3, Thm. 5.5(2)]. If $\mathsf{K}_p = \mathbb{V}(\boldsymbol{F})$ for a finite algebra $\boldsymbol{F}$, then by Jónsson's Theorem, every

$\boldsymbol{F}' \in (\mathsf{K}_p)_{\mathrm{SI}}$ belongs to $\mathbb{HSP}_{\mathrm{U}}(\boldsymbol{F}) = \mathbb{HS}(\boldsymbol{F})$, whence $|F'| \leq |F|$, contradicting the unboundedness of the cardinalities of the simple algebras $\boldsymbol{B}_{p,i} \in \mathsf{K}_p$. Therefore, K_p is not tabular.

To prove that K_p is locally finite, let $\boldsymbol{G} \in \mathsf{K}_p$ be n-generated and subdirectly irreducible. It suffices to show that $|G| \leq p + 3 + 2n$. Suppose that some generating set Z for $\boldsymbol{G}$, with at most n elements, is given.

Because DMM has EDPC and since the algebras $\boldsymbol{B}_{p,i}$ are simple, it follows from Lemma 3.2 that we may identify $\boldsymbol{G}$ with a subalgebra of an ultraproduct $\boldsymbol{E} = \prod_{j \in J} \boldsymbol{C}_j / \mathcal{U}$, where each $\boldsymbol{C}_j$ is $\boldsymbol{B}_{p,i}$ for some $i \in \mathbb{N}$.

By Łos' Theorem [3, Thm. 5.21], a first order sentence Ψ that is true in 'almost all' of the $\boldsymbol{C}_j$ will be true in $\boldsymbol{E}$, and if it is a universal sentence, it will then be true in $\boldsymbol{G}$ also. Here, the meaning of 'almost all' is that

$$\{j \in J : \Psi \text{ is true in } \boldsymbol{C}_j\} \in \mathcal{U}.$$

In particular, $\boldsymbol{E}$ (and hence $\boldsymbol{G}$) is totally ordered.

Let $\boldsymbol{C} = \prod_{j \in J} \boldsymbol{C}_j$. We may write each $c \in C$ as $\langle c(j) : j \in J \rangle$. For each $m \in A_p$, define $\overline{m} \in C$ by

$$\overline{m}(j) = m \text{ for all } j \in J,$$

and let $\overline{A}_p = \{\overline{m}/\mathcal{U} : m \in A_p\}$, so $\overline{A}_p \subseteq E$. Now $\boldsymbol{A}_p$ is isomorphic to the subalgebra $\overline{\boldsymbol{A}}_p$ of $\boldsymbol{E}$ with universe $\overline{A}_p$, under the map $m \mapsto \overline{m}/\mathcal{U}$.

Let $\boldsymbol{D}$ be the subalgebra of $\boldsymbol{G}$ with universe $\overline{A}_p \cap G$. Then $\boldsymbol{D}$ is $\overline{\boldsymbol{A}}_p$ or $|D| = 4$, as $\boldsymbol{A}_p$ has no proper subalgebra other than $\boldsymbol{C}_4$. Thus,

$$|D| \leq |A_p| = p + 3.$$

Let $c = \langle c(j) : j \in J \rangle \in C$, and define:

$$\begin{aligned} c_m &= \{j \in J : c(j) = m\}, \text{ for each } m \in A_p; \\ c_X &= \{j \in J : 1 < c(j) < 2\}; \\ c_Y &= \{j \in J : 2^{p-1} < c(j) < 2^p\}. \end{aligned}$$

Then $c_X \cup c_Y \cup \left(\bigcup_{m \in A_p} c_m\right) = J \in \mathcal{U}$. Whenever the union of a finite (positive) number of subsets of a set belongs to an ultrafilter over that set, then one of the subsets belongs to the ultrafilter [3, p. 140]. So, $c_X \in \mathcal{U}$ or $c_Y \in \mathcal{U}$ or there exists $m \in A_p$ such that $c_m \in \mathcal{U}$. This means that $c/\mathcal{U}$ has one of the following forms:

$$\begin{aligned} &\overline{m}/\mathcal{U}, \text{ for some } m \in A_p; \\ &\overline{x}/\mathcal{U}, \text{ where } \overline{x} \in C \text{ and for each } j \in J, \text{ we have } \overline{x}(j) = x_r \text{ for some } r; \\ &\overline{y}/\mathcal{U}, \text{ where } \overline{y} \in C \text{ and for each } j \in J, \text{ we have } \overline{y}(j) = \neg x_r \text{ for some } r. \end{aligned}$$

Note that, in the second case, $\neg^{\boldsymbol{E}}(c/\mathcal{U})$ has the form of the third case, and vice versa, by definition of $\boldsymbol{E}$. Let us call $c/\mathcal{U}$ *extraneous* if it has the second or the third form shown above. So, the involutive image of an extraneous element of $\boldsymbol{E}$ is itself extraneous.

Remembering that $\boldsymbol{G}$ is a subalgebra of $\boldsymbol{E}$, let S consist of the extraneous elements of our generating set Z for $\boldsymbol{G}$, together with their involutive images. Then $|S| \leq 2n$ and $D \cup S$ includes $e^{\boldsymbol{E}}$ and all elements of Z. Also, $D \cup S$ is closed, in $\boldsymbol{E}$, under $\neg$ and the lattice operations (as $\boldsymbol{E}$ is a totally ordered De Morgan monoid and $\boldsymbol{D}$ is a subalgebra of $\boldsymbol{E}$).

If we can show that $D \cup S$ is also closed, in $\boldsymbol{E}$, under fusion, then it will follow that $G = D \cup S$, and hence that $|G| \leq p + 3 + 2n$, as required.

Because D is already closed under fusion, it suffices to note that the fusion of an extraneous element z of $\boldsymbol{E}$ with an element z' of $\overline{A}_p \cup S$ is always either z or an element of $\overline{A}_p$ (which will belong to D if $z \in S$ and $z' \in D$). This is a consequence of (4) (and the definition of $\boldsymbol{E}$). ∎

LEMMA 3.7. *For $p, i \in \mathbb{N}$, with p an odd prime, the subalgebras of $\boldsymbol{B}_{p,i}$ are, up to isomorphism, just $\boldsymbol{C}_4$, $\boldsymbol{A}_p$ and $\boldsymbol{B}_{p,k}$ for $k = 1, 2, \ldots, i$.*

PROOF. We have already noted that $\boldsymbol{A}_p$ has no proper subalgebra other than $\boldsymbol{C}_4$, because p is prime. Suppose $\boldsymbol{C}$ is a proper subalgebra of $\boldsymbol{B}_{p,i}$, other than $\boldsymbol{A}_p$ and $\boldsymbol{C}_4$. Then $Y := C \cap \{x_1, \ldots, x_i\} \neq \emptyset$, and $4 = x_j^2 \in C$ whenever $x_j \in Y$. But $4 \in A_p \backslash C_4$ (as $p > 2$), so $A_p \subseteq C$. As C is also closed under $\neg$, it follows that $\boldsymbol{C} \cong \boldsymbol{B}_{p,|Y|}$. ∎

REMARK 3.8. Lemma 3.7 becomes false when $p = 2$, as $\{0, 1, x_1, -x_1, 4, 8\}$ is a subuniverse of $\boldsymbol{B}_{2,1}$.

COROLLARY 3.9. *For each odd prime $p \in \mathbb{N}$, every finitely generated subdirectly irreducible member of K_p is isomorphic to $\boldsymbol{C}_4$ or to $\boldsymbol{A}_p$ or to $\boldsymbol{B}_{p,i}$ for some $i \in \mathbb{N}$.*

PROOF. Let $\boldsymbol{A} \in (\mathsf{K}_p)_{\mathrm{SI}}$ be finitely generated—hence finite, by Theorem 3.6. By Lemma 3.2, $\boldsymbol{A} \in \mathbb{ISP}_{\mathrm{U}}(\{\boldsymbol{B}_{p,i} : i \in \mathbb{N}\})$. But, since $\boldsymbol{A}$ is finite and of finite type, the attribute of not having a subalgebra isomorphic to $\boldsymbol{A}$ is first-order definable, and therefore persists in ultraproducts, so $\boldsymbol{A} \in \mathbb{IS}(\boldsymbol{B}_{p,i})$ for some $i \in \mathbb{N}$, and the result follows from Lemma 3.7. ∎

The following theorem originates in Blok and Pigozzi [6]. The version below is stated and proved in Jónsson [25, Thm. 6.6].

THEOREM 3.10. *Let K be a variety of finite type, with EDPC, and let $\boldsymbol{A} \in \mathsf{K}$ be finite and subdirectly irreducible. Then $\mathsf{K}^{\boldsymbol{A}} := \{\boldsymbol{B} \in \mathsf{K} : \boldsymbol{A} \notin \mathbb{SH}(\boldsymbol{B})\}$ is* *a subvariety of* K, *and for every subvariety* U *of* K, *we have*

$$\boldsymbol{A} \in \mathsf{U} \ \textit{ or } \mathsf{U} \subseteq \mathsf{K}^{\boldsymbol{A}}, \textit{ and not both.}$$

CONVENTION. For each prime $p \in \mathbb{N}$, it is now convenient to write $\boldsymbol{A}_p$ as $\boldsymbol{B}_{p,0}$ and $\boldsymbol{C}_4$ as $\boldsymbol{B}_{p,-1}$.

COROLLARY 3.11. *Let* p, k, i *be integers, where* $-1 \leq k < i$ *and* p *is a positive prime. Then* $\boldsymbol{B}_{p,k} \in \mathbb{V}(\boldsymbol{B}_{p,i})$ *and* $\boldsymbol{B}_{p,i} \notin \mathbb{V}(\boldsymbol{B}_{p,k})$.

PROOF. As $\boldsymbol{B}_{p,k} \in \mathbb{S}(\boldsymbol{B}_{p,i})$, the first claim holds. Also, $\boldsymbol{B}_{p,i} \notin \mathbb{SH}(\boldsymbol{B}_{p,k})$ (on cardinality grounds, as $k < i$), so $\boldsymbol{B}_{p,k} \in \mathsf{K}_p^{\boldsymbol{B}_{p,i}}$. Then Theorem 3.10 shows that $\mathbb{V}(\boldsymbol{B}_{p,k}) \subseteq \mathsf{K}_p^{\boldsymbol{B}_{p,i}}$, and hence that $\boldsymbol{B}_{p,i} \notin \mathbb{V}(\boldsymbol{B}_{p,k})$. ■

A variety K is said to be *pretabular* if it is not tabular but has only tabular proper subvarieties. (Any deductive system algebraized by K is then also said to be *pretabular*.)

THEOREM 3.12. *For each odd prime* $p \in \mathbb{N}$, *the variety* K_p *is pretabular.*

PROOF. We already noted in Theorem 3.6 that K_p is not tabular. Let L be a proper subvariety of K_p. We must show that L is tabular, so we may assume that L is nontrivial, whence $\boldsymbol{C}_4 = \boldsymbol{B}_{p,-1} \in \mathsf{L}$.

Let m be the least non-negative integer such that $\boldsymbol{B}_{p,m} \notin \mathsf{L}$. The existence of m follows from the propriety of L. By Corollary 3.11,

$$\boldsymbol{B}_{p,-1}, \ \ldots, \ \boldsymbol{B}_{p,m-1} \in \mathsf{L} \ \text{ and } \boldsymbol{B}_{p,k} \notin \mathsf{L} \quad \text{for all integers} \quad k \geq m. \tag{7}$$

Clearly, $\mathbb{V}(\boldsymbol{B}_{p,m-1}) \subseteq \mathsf{L}$, and it suffices to show that $\mathsf{L} \subseteq \mathbb{V}(\boldsymbol{B}_{p,m-1})$.

Because every variety is generated by its finitely generated subdirectly irreducible members, we need only show that every finitely generated algebra $\boldsymbol{D} \in \mathsf{L}_{\mathrm{SI}}$ belongs to $\mathbb{V}(\boldsymbol{B}_{p,m-1})$. But, since every such $\boldsymbol{D}$ belongs to K_p, it is isomorphic to some $\boldsymbol{B}_{p,j}$ (by Corollary 3.9), and we must have $j < m$ (by (7)). Then $\boldsymbol{D} \in \mathbb{V}(\boldsymbol{B}_{p,m-1})$, by Corollary 3.11. ■

COROLLARY 3.13. *For each odd prime* $p \in \mathbb{N}$, *the subvariety lattice of* K_p *is the following chain of order type* $\omega + 1$, *where* T *is the trivial variety of De Morgan monoids:*

$$\mathsf{T} \subsetneq \mathbb{V}(\boldsymbol{C}_4) \subsetneq \mathbb{V}(\boldsymbol{A}_p) \subsetneq \mathbb{V}(\boldsymbol{B}_{p,1}) \subsetneq \mathbb{V}(\boldsymbol{B}_{p,2}) \subsetneq \mathbb{V}(\boldsymbol{B}_{p,3}) \subsetneq \ \ldots \ \subsetneq \mathsf{K}_p.$$

LEMMA 3.14. *For distinct primes* $p, q \in \mathbb{N}$, *we have* $\mathsf{K}_p \neq \mathsf{K}_q$.

PROOF. Let $p < q$, so q is odd. As $\boldsymbol{A}_p \in \mathsf{K}_p$, it suffices to show that $\boldsymbol{A}_p \notin \mathsf{K}_q$. If $\boldsymbol{A}_p \in \mathsf{K}_q$, then since $\boldsymbol{A}_p$ is finite and simple, Corollary 3.9 yields $\boldsymbol{A}_p \cong \boldsymbol{B}_{q,i}$ for some integer $i \geq -1$, and $|B_{q,i}|$ is $q + 3 + 2i$ ($i \in \mathbb{N}$) or $q + 3$ or 4. But this contradicts the fact that $|A_p| = p + 3$, with $p < q$. ■

Remark 3.15. An alternative explanation of Lemma 3.14 can be got by considering the equation $x^{p-1} \leftrightarrow \neg x = \neg f^2$, which we abbreviate as E_p. In algebras of the form $\boldsymbol{B}_{n,i}$, the value of $\neg f^2$ is 0, so E_p is a quantifier-free formulation of $\forall x\,(x^{p-1} \neq \neg x)$. For distinct primes p, q, with q odd, the algebra $\boldsymbol{A}_q$ satisfies E_p, because $\neg(2^k) = 2^{q-k}$ (in $\boldsymbol{A}_q$) for non-negative integers $k \leq q$, and because p does not divide q. One can then check that E_p remains valid in the algebras $\boldsymbol{B}_{q,i}$, by definition of the operations. Thus, E_p is valid in K_q, but it is not valid in K_p, because $2^{p-1} = \neg 2$ in $\boldsymbol{A}_p$. Note that E_p could be expressed without use of the symbol f as $x^{p-1} \leftrightarrow \neg x \leqslant y$.

Remark 3.16. The variety K_2 is not pretabular, as the equation E_2 (equivalently, $\forall x\,(x \neq \neg x)$) defines a subvariety of K_2 that is proper (as it excludes $\boldsymbol{A}_2$, in which $2 = \neg 2$), but not tabular. Tabularity fails as in the proof of Theorem 3.6, because $(\mathsf{K}_2)_{\mathrm{SI}}$ includes unboundedly large simple algebras with universes $B_{2,i}\backslash\{2\}$ $(i \in \mathbb{N})$, which satisfy E_2; cf. Remark 3.8.

A variety is said to be *semisimple* if its subdirectly irreducible members are simple. In view of Theorem 3.12 and Lemmas 3.2 and 3.14, we have proved:

Theorem 3.17. *There are infinitely many semisimple pretabular varieties of semilinear De Morgan monoids.*

Theorem 3.18. *For each odd prime $p \in \mathbb{N}$, there is a denumerable algebra $\boldsymbol{B}_{p,\infty}$ such that $\mathsf{K}_p = \mathbb{Q}(\boldsymbol{B}_{p,\infty})$.*

Proof. Let $\mathcal{U}$ be a nonprincipal ultrafilter over $\mathbb{N}$, and consider the ultraproduct $\boldsymbol{E} = \prod_{i\in\mathbb{N}} \boldsymbol{B}_{p,i}/\mathcal{U} \in \mathsf{K}_p$. By Łos' Theorem, $\boldsymbol{E}$ is totally ordered and simple, and contains an isomorphic copy of $\boldsymbol{A}_p$ (which we shall identify with $\boldsymbol{A}_p$). Also, $\boldsymbol{E}$ is bounded by 0 and 2^{p+1}, and each of its elements, other than 2^{p+1}, has an immediate successor, while every element other than 0 has an immediate predecessor. Therefore, the interval $(1, 2)$ of $\boldsymbol{E}$ has an initial section $x_1 < x_2 < x_3 < \ldots$ of order type ω (ascending from 1) and a final section $\ldots < x_3' < x_2' < x_1'$ (descending from 2); the interval $(2^{p-1}, 2^p)$ has the same features and terminates with $\ldots < \neg x_3 < \neg x_2 < \neg x_1$.

The denumerable subalgebra $\boldsymbol{B}_{p,\infty}$ of $\boldsymbol{E}$ generated by $\{x_j : j \in \mathbb{N}\}$ is structurally independent of $\mathcal{U}$. It has universe

$$B_{p,\infty} = A_p \cup \{x_j : j \in \mathbb{N}\} \cup \{\neg x_j : j \in \mathbb{N}\},$$

where $1 < x_j < 2$ and $2^{p-1} < \neg x_k < 2^p$ for all $j, k \in \mathbb{N}$. Our descriptions of the function g and the operation $\cdot^{\boldsymbol{B}_{p,i}}$ in Example 3.4 did not depend on the value of i, and they extend without amendment to $\boldsymbol{B}_{p,\infty}$.

For each $i \in \mathbb{N}$, the subalgebra of $\boldsymbol{B}_{p,\infty}$ with universe

$$A_p \cup \{x_1, \dots, x_i\} \cup \{\neg x_i, \dots, \neg x_1\}$$

is isomorphic to $\boldsymbol{B}_{p,i}$, so $\boldsymbol{B}_{p,i} \in \mathbb{Q}(\boldsymbol{B}_{p,\infty})$. Like any variety, K_p is generated as a quasivariety by the finitely generated members of $(\mathsf{K}_p)_{\mathrm{SI}}$, so $\mathsf{K}_p \subseteq \mathbb{Q}(\boldsymbol{B}_{p,\infty})$, by Corollary 3.9. Thus, $\mathsf{K}_p = \mathbb{Q}(\boldsymbol{B}_{p,\infty})$. ■

Theorem 3.18 is significant because a variety K is generated as a quasivariety by a single algebra iff it satisfies a certain kind of 'relevance principle', namely: for any finite set $\Delta \cup \Gamma \cup \{\mu = \nu\}$ of equations, where Δ is not explosive (i.e., there are terms α, β such that $\mathsf{K} \not\models (\& \, \Delta) \Longrightarrow \alpha = \beta$) and no variable occurs both in a member of Δ and in a member of $\Gamma \cup \{\mu = \nu\}$,

$$\text{if } \mathsf{K} \models (\& \, (\Delta \cup \Gamma)) \Longrightarrow \mu = \nu, \text{ then } \mathsf{K} \models (\& \, \Gamma) \Longrightarrow \mu = \nu.$$

This is an algebraic analogue of the *Łoś-Suszko Theorem* ([26, p. 182], corrected in [41]); a proof of the analogue can be found in [15, Thm. 3].

4. Relevant Algebras

Reducts of an algebra arise by just discarding some of its basic operations (without altering its universe). *Subreducts* are subalgebras of indicated reducts. In particular, when a discarded operation of an algebra $\boldsymbol{A}$ is nullary, the corresponding distinguished element of $\boldsymbol{A}$ remains an element of the reduct, where it is no longer distinguished. Consequently, it need not belong to all of the resulting subreducts of $\boldsymbol{A}$.

Definition 4.1. A *relevant algebra* is an e-free subreduct of a De Morgan monoid (i.e., a subalgebra of the reduct $\langle A; \cdot, \wedge, \vee, \neg \rangle$ of some $\boldsymbol{A} \in \mathsf{DMM}$).

These algebras form a variety RA, algebraizing the logic $\mathbf{R}$ of [2] (i.e., the fragment of $\mathbf{R^t}$ that lacks the Ackermann constants). A finite equational basis for RA is given in [18] (also see [14], [23, Cor. 4.11] and [33, Sec. 7]). In a relevant algebra, $x \to y = \neg(x \cdot \neg y)$ is still defined, and the law of residuation (2) persists.

A De Morgan monoid and its relevant algebra reduct have the same congruences, but not necessarily the same subuniverses. For example, the e-free reduct of the two-element Boolean algebra $\mathbf{2}$ embeds into every nontrivial relevant algebra (see [33, Cor. 7.5], for instance), but $\mathbf{2}$ itself does not embed into the Sugihara monoid $\boldsymbol{S}_3$.

The following lemma is a specialization of [36, Thm. 5.3], but it has logical antecedents in [1, Lem. 2] and [2, p. 343].

LEMMA 4.2. *If a relevant algebra is finitely generated, then it is the e–free* reduct $\langle A; \cdot, \wedge, \vee, \neg \rangle$ *of a De Morgan monoid $\boldsymbol{A}$. In this case, the unique neutral element of $\boldsymbol{A}$ is the greatest lower bound of all $a \to a$, where a ranges over any finite generating set for $\langle A; \cdot, \wedge, \vee, \neg \rangle$.*

The variety RA is congruence distributive (as its members have lattice reducts), but RA lacks EDPC. In fact, the CEP fails for finite relevant algebras [9, p. 289]. This would seem at first to block transference of the arguments in Sect. 3 from the setting of DMM to that of RA (i.e., from $\mathbf{R^t}$ to $\mathbf{R}$). We shall show, however, that EDPC holds for a subvariety of RA within which we can imitate the arguments of Sect. 3.

We start by abstracting features of $\boldsymbol{B}_{p,i}$ (p an odd prime), including (6), that can be expressed without using the nullary symbol e (or f). There is clearly a first-order sentence Φ in the language of relevant algebras such that a relevant algebra $\boldsymbol{C}$ satisfies Φ iff

> $\boldsymbol{C}$ is totally ordered and bounded, and $\neg$ has no fixed point in $\boldsymbol{C}$ (whence $\boldsymbol{C}$ is nontrivial), and for every element c of C, other than the two bounds, $c \to c$ is a neutral element for fusion in $\boldsymbol{C}$ and is an atom of $\langle C; \wedge, \vee \rangle$, and $(\neg(c \to c))^2$ is the greatest element of $\boldsymbol{C}$.

LEMMA 4.3. *Let M be a class of relevant algebras satisfying Φ. Then M consists of simple algebras and $\mathbb{ISP}_{\mathrm{U}}(\mathsf{M}) \models \Phi$.*

PROOF. Like any first-order sentence, Φ persists under $\mathbb{P}_{\mathrm{U}}$. Let $\boldsymbol{C} \in \mathsf{M}$ and $\boldsymbol{D} \in \mathbb{S}(\boldsymbol{C})$, so $\boldsymbol{D}$ is totally ordered. It suffices to show that $\boldsymbol{D}$ is simple and satisfies Φ.

As $\boldsymbol{C} \models \Phi$, the operation $\neg^{\boldsymbol{C}}$ has no fixed point, so $\boldsymbol{D}$ is nontrivial and $\neg^{\boldsymbol{D}}$ has no fixed point. If $|D| = 2$, then $\boldsymbol{D}$ is simple and $\boldsymbol{D} \models \Phi$, so assume that $|D| > 2$.

Now there exists $c \in D$ such that c is neither of the bounds of $\boldsymbol{C}$. As $\boldsymbol{C} \models \Phi$, we have $e := c \to c \in D$, so $f := \neg e \in D$ and $\top := f^2 \in D$ and $\bot := \neg f^2 \in D$, and $\bot \leqslant d = e \cdot d \leqslant \top$ for all $d \in D$, and e is an atom of $\boldsymbol{D}$, and $d \to d = e$ for all $d \in D \backslash \{\bot, \top\}$ (as neutral elements are unique). Thus, $\boldsymbol{D} \models \Phi$.

As e has just one strict lower bound in $\boldsymbol{D}$, the De Morgan monoid $\langle \boldsymbol{D}, e \rangle$ is a simple algebra, by (I). Then $\boldsymbol{D}$ is simple, because every congruence of $\boldsymbol{D}$ is a congruence of $\langle \boldsymbol{D}, e \rangle$. ■

DEFINITION 4.4. For a De Morgan monoid $\boldsymbol{A}$, let $\boldsymbol{A}^-$ denote the relevant algebra reduct of $\boldsymbol{A}$. For a class X of De Morgan monoids, let

$$\mathsf{X}^- = \{\boldsymbol{A}^- : \boldsymbol{A} \in \mathsf{X}\}.$$

Note that Φ fails in $\boldsymbol{A}_2^-$, where $\neg 2 = 2$. But $\boldsymbol{A}_p^-$ and $\boldsymbol{B}_{p,i}^-$ satisfy Φ when p is an *odd* prime. In each of them, $\{0, 2^{p+1}\}$ is the universe of a subalgebra isomorphic to $\mathbf{2}^-$, which shows that e is not defined by any term in these algebras. Otherwise, the non-empty subuniverses are the same as those of $\boldsymbol{A}_p$ and $\boldsymbol{B}_{p,i}$.

A class of similar algebras is said to be *elementary* if it is the model class of some set of first-order sentences.

LEMMA 4.5. *Let* K *be a variety of relevant algebras, such that* $\mathsf{K}_{\mathrm{SI}} \models \Phi$. *Then* K_{SI} *is an elementary class and* K *is semisimple.*

PROOF. By Birkhoff's Theorem [3, Thm. 4.41], there is a set of equations that axiomatizes K. The universal closures of those equations constitute a set $\mathfrak{S}$ of first-order sentences. Evidently, $\mathsf{K}_{\mathrm{SI}} \models \mathfrak{S} \cup \{\Phi\}$. Conversely, a model $\boldsymbol{C}$ of $\mathfrak{S} \cup \{\Phi\}$ belongs to K and, by Lemma 4.3, $\boldsymbol{C}$ is simple, so $\boldsymbol{C} \in \mathsf{K}_{\mathrm{SI}}$. Thus, $\mathfrak{S} \cup \{\Phi\}$ axiomatizes K_{SI}. And K is semisimple, again by Lemma 4.3. ■

THEOREM 4.6. ([10, Thm. 3.3]) *Let* K *be a congruence distributive variety such that* K_{SI} *is an elementary class. If* K_{SI} *has the CEP, then so does* K.

COROLLARY 4.7. *Let* K *be a variety of relevant algebras, such that* $\mathsf{K}_{\mathrm{SI}} \models \Phi$. *Then* K *has the CEP.*

PROOF. It suffices, by Lemma 4.5 and Theorem 4.6, to show that K_{SI} has the CEP. Let $\boldsymbol{A} \in \mathsf{K}_{\mathrm{SI}}$ and $\boldsymbol{B} \in \mathbb{S}(\boldsymbol{A})$. Then $\boldsymbol{B}$ is simple, by Lemma 4.3. Obviously, the only two congruences of $\boldsymbol{B}$ extend to $\boldsymbol{A}$, so $\boldsymbol{A}$ has the CEP, as required. ■

In relevant algebras and De Morgan monoids, we now abbreviate $x \to x$ as $|x|$. For any elements a, b of a De Morgan monoid $\boldsymbol{A}$, we have

(i) $e \leqslant |a|$;

(ii) $|a| \to a = a = e \to a$;

(iii) $e \leqslant a$ iff $|a| \leqslant a$;

(iv) $a \leqslant b$ iff $e \leqslant a \to b$.

LEMMA 4.8. *Let* $\boldsymbol{A}$ *be a De Morgan monoid satisfying* Φ, *with* $a, b \in A$. *Then*

$$(a \wedge e) \to b = (a \wedge |a| \wedge |b|) \to b. \tag{8}$$

PROOF. Let $\bot, \top$ be the least and greatest elements of $\boldsymbol{A}$, respectively. Now (8) is true if $|a| = e$ (by (i)), so assume that $|a| \neq e$. Then $a \in \{\bot, \top\}$, because $\boldsymbol{A} \models \Phi$. If $a = \bot$, then (8) is true, because bounded De Morgan monoids satisfy $\bot \to x = \top$. So, we may assume that $a = \top$, whence $|a| = \top$, as bounded De Morgan monoids satisfy $x \to \top = \top$. Then (8) is true, by (ii). ∎

Corollary 4.9. *Let* K *be a variety of relevant algebras, such that* $\mathsf{K}_{\mathrm{SI}} \models \Phi$. *Let* $\boldsymbol{D}$ *be a De Morgan monoid, with* $\boldsymbol{D}^- \in \mathsf{K}$. *Then*

$$\boldsymbol{D} \models (x \wedge e) \to y = (x \wedge |x| \wedge |y|) \to y.$$

Proof. As K is a variety and $\boldsymbol{D}^- \in \mathsf{K}$, the Subdirect Representation Theorem [3, Thm. 3.24] shows that there are congruences θ_i of $\boldsymbol{D}^-$ $(i \in I)$ and a subdirect embedding $h\colon d \mapsto \langle d/\theta_i : i \in I\rangle$ of $\boldsymbol{D}^-$ into $\prod_{i \in I} \boldsymbol{D}^-/\theta_i$, where $\boldsymbol{D}^-/\theta_i \in \mathsf{K}_{\mathrm{SI}}$ for all i. So, each $\boldsymbol{D}^-/\theta_i$ satisfies Φ. For each $i \in I$, since $\boldsymbol{D}$ and $\boldsymbol{D}^-$ have the same congruences, each θ_i is a congruence of $\boldsymbol{D}$, and $\boldsymbol{D}/\theta_i$ is a De Morgan monoid satisfying Φ, whence

$$\boldsymbol{D}/\theta_i \models (x \wedge e) \to y = (x \wedge |x| \wedge |y|) \to y$$

(by Lemma 4.8), and h is a subdirect embedding $\boldsymbol{D} \to \prod_{i \in I} \boldsymbol{D}/\theta_i$. Therefore, $\boldsymbol{D} \models (x \wedge e) \to y = (x \wedge |x| \wedge |y|) \to y$. ∎

Theorem 4.10. *Let* K *be a variety of relevant algebras, such that* $\mathsf{K}_{\mathrm{SI}} \models \Phi$. *Then* K *has EDPC.*

Proof. Let $\boldsymbol{A} \in \mathsf{K}$ and $a, b, c, d \in A$. It suffices to show that

$$\langle c, d\rangle \in \Theta^{\boldsymbol{A}}(a, b) \text{ iff } (a \leftrightarrow b) \wedge |a \leftrightarrow b| \wedge |c \leftrightarrow d| \leqslant c \leftrightarrow d. \tag{9}$$

Let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}$ generated by $\{a, b, c, d\}$, so $\boldsymbol{B} \in \mathsf{K}$. As K has the CEP (Corollary 4.7), we have $\Theta^{\boldsymbol{B}}(a, b) = (B \times B) \cap \Theta^{\boldsymbol{A}}(a, b)$. As $\boldsymbol{B}$ is a finitely generated relevant algebra, its fusion has a neutral element, e say (by Lemma 4.2), but e need not be a distinguished element of $\boldsymbol{B}$. Let $\boldsymbol{B}^+$ be the De Morgan monoid $\langle \boldsymbol{B}, e\rangle$. As $\boldsymbol{B}$ and $\boldsymbol{B}^+$ have the same congruences, $\Theta^{\boldsymbol{B}^+}(a, b) = \Theta^{\boldsymbol{B}}(a, b) = (B \times B) \cap \Theta^{\boldsymbol{A}}(a, b)$. Now

$$\begin{aligned}
&\langle c, d\rangle \in \Theta^{\boldsymbol{A}}(a, b) \text{ iff } \langle c, d\rangle \in \Theta^{\boldsymbol{B}^+}(a, b) \quad (\text{as } c, d \in B)\\
&\text{iff, in } \boldsymbol{B}^+, \text{ we have } (a \leftrightarrow b) \wedge e \leqslant c \leftrightarrow d \quad (\text{by (3), as } \boldsymbol{B}^+ \in \mathsf{DMM})\\
&\text{iff, in } \boldsymbol{B}^+, \text{ we have } e \leqslant ((a \leftrightarrow b) \wedge e) \to (c \leftrightarrow d) \quad (\text{by (iv) above})\\
&\text{iff, in } \boldsymbol{B}^+, \text{ we have } e \leqslant ((a \leftrightarrow b) \wedge |a \leftrightarrow b| \wedge |c \leftrightarrow d|) \to (c \leftrightarrow d)\\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{by Corollary 4.9, as } \boldsymbol{B} \in \mathsf{K})\\
&\text{iff, in } \boldsymbol{B}^+, \text{ we have } (a \leftrightarrow b) \wedge |a \leftrightarrow b| \wedge |c \leftrightarrow d| \leqslant c \leftrightarrow d \quad (\text{by (iv)})\\
&\text{iff, in } \boldsymbol{B}, \text{ we have } (a \leftrightarrow b) \wedge |a \leftrightarrow b| \wedge |c \leftrightarrow d| \leqslant c \leftrightarrow d\\
&\text{iff, in } \boldsymbol{A}, \text{ we have } (a \leftrightarrow b) \wedge |a \leftrightarrow b| \wedge |c \leftrightarrow d| \leqslant c \leftrightarrow d.
\end{aligned}$$

∎

Corollary 4.11. *$\mathbb{V}(\{\boldsymbol{B}_{p,i}^- : p, i \in \mathbb{N},\ p$ an odd prime$\})$ has EDPC.*

Proof. Let $\mathsf{S} = \{\boldsymbol{B}_{p,i} : p, i \in \mathbb{N},\ p$ an odd prime$\}$. We have noted that $\mathsf{S}^- \models \Phi$, so by Lemma 4.3, $\mathbb{ISP}_{\mathrm{U}}(\mathsf{S}^-)$ consists of simple algebras satisfying Φ. Thus, the nontrivial members of $\mathbb{HSP}_{\mathrm{U}}(\mathsf{S}^-)$ satisfy Φ, and they include all members of $\mathbb{V}(\mathsf{S}^-)_{\mathrm{SI}}$, by Jónsson's Theorem. The result therefore follows from Theorem 4.10. ■

Because of Corollary 4.11, the arguments from Sect. 3 can be repeated, with $\mathbb{V}(\mathsf{S}^-)$ (rather than DMM) as the ambient variety with EDPC, and they yield the following result.

Theorem 4.12. *The variety $\mathsf{L}_p := \mathbb{V}(\{\boldsymbol{B}_{p,i}^- : i \in \mathbb{N}\})$ is pretabular for each odd prime $p \in \mathbb{N}$, and these varieties are distinct for different primes p. There are therefore infinitely many semisimple pretabular varieties of semilinear relevant algebras. Also, each L_p coincides with $\mathbb{Q}(\boldsymbol{B}_{p,\infty}^-)$.*

For odd primes p, the subvariety lattice of L_p is as follows (cf. Corollary 3.13):

$$\mathsf{T}^- \subsetneq \mathbb{V}(\mathbf{2}^-) \subseteq \mathbb{V}(\boldsymbol{C}_4^-) \subsetneq \mathbb{V}(\boldsymbol{A}_p^-) \subsetneq \mathbb{V}(\boldsymbol{B}_{p,1}^-) \subsetneq \mathbb{V}(\boldsymbol{B}_{p,2}^-) \subsetneq \dots \subsetneq \mathsf{L}_p.$$

Here, $\mathbb{V}(\mathbf{2}^-)$ is the class of Boolean algebras, with undistinguished bounds.

Remark 4.13. A variety is *filtral* iff it is semisimple and has EDPC [19, 20]. The congruence permutable filtral varieties are exactly the *discriminator varieties* [5, 20]. For any such variety K, there is a term $\delta(x, y, z)$ such that, in any simple member $\boldsymbol{A}$ of K, with elements a, b, c, we have

$$\delta^{\boldsymbol{A}}(a, b, c) = a \ \text{ if } \ a \neq b, \ \text{ and } \ \delta^{\boldsymbol{A}}(a, a, c) = c.$$

As relevant algebras are congruence permutable [40, Prop. 8.3], the pretabular varieties in Theorems 3.17 and 4.12 are in fact discriminator varieties.

Remark 4.14. We leave open the question of whether there are uncountably many pretabular varieties of semilinear relevant algebras/De Morgan monoids. Note that, while some 'neighbouring' varieties of semilinear residuated structures have denumerable subvariety lattices (e.g., [35]), there are already $2^{\aleph_0}$ semisimple varieties of semilinear relevant algebras. Indeed, for $\mathsf{K} = \mathbb{V}(\{\boldsymbol{A}_p^- : p$ an odd prime$\})$ and any subset X of $\{\boldsymbol{A}_q^- : q$ an odd prime$\}$, if $\boldsymbol{A}_p^- \notin \mathsf{X}$, then $\boldsymbol{A}_p^- \notin \mathbb{SH}(\boldsymbol{A}_q^-)$ for each $\boldsymbol{A}_q^- \in \mathsf{X}$ (as $\boldsymbol{A}_q^-$ is simple and has no proper subalgebra other than $\mathbf{2}^-$ and $\boldsymbol{C}_4^-$), so $\mathsf{X} \subseteq \mathsf{K}^{\boldsymbol{A}_p^-}$, whence $\boldsymbol{A}_p^- \notin \mathbb{V}(\mathsf{X})$ (by Theorem 3.10 and Corollary 4.11). The varieties $\mathbb{V}(\mathsf{X})$ of this kind are therefore distinct for the $2^{\aleph_0}$ different choices of X.

REMARK 4.15. Blok and Pigozzi [8] proved that, when a deductive system $\mathbf{S}$ is algebraized by a variety K, then K has EDPC iff $\mathbf{S}$ has a deduction-detachment theorem (DDT). Thus, $\mathbf{R}$ has no DDT. It is well known that $\mathbf{R}^{\mathbf{t}}$ has the following DDT:

$$\Gamma \cup \{\alpha\} \vdash \beta \text{ iff } \Gamma \vdash (\alpha \wedge \mathbf{t}) \to \beta.$$

A DDT persists in axiomatic extensions. Using (9) and the tools of algebraization, one can show that the axiomatic extension $\mathbf{S}$ of $\mathbf{R}$ that is algebraized by $\mathbb{V}(\{\boldsymbol{B}_{p,i}^{-} : p, i \in \mathbb{N},\, p \text{ an odd prime}\})$ has the following DDT:

$$\Gamma \cup \{\alpha\} \vdash_{\mathbf{S}} \beta \text{ iff } \Gamma \vdash_{\mathbf{S}} (\alpha \wedge |\alpha| \wedge |\beta|) \to \beta$$

(for any set $\Gamma \cup \{\alpha, \beta\}$ of formulas in the language of relevant algebras).

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On Pretabular Extensions of Relevance Logic

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