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Representability of Kleene Posets and Kleene Lattices

Abstract. A Kleene lattice is a distributive lattice equipped with an antitone involution and satisfying the so-called normality condition. These lattices were introduced by J. A. Kalman. We extended this concept also for posets with an antitone involution. In our recent paper (Chajda, Länger and Paseka, in: Proceeding of 2022 IEEE 52th International Symposium on Multiple-Valued Logic, Springer, 2022), we showed how to construct such Kleene lattices or Kleene posets from a given distributive lattice or poset and a fixed element of this lattice or poset by using the so-called twist product construction, respectively. We extend this construction of Kleene lattices and Kleene posets by considering a fixed subset instead of a fixed element. Moreover, we show that in some cases, this generating poset can be embedded into the resulting Kleene poset. We investigate the question when a Kleene poset can be represented by a Kleene poset obtained by the mentioned construction. We show that a direct product of representable Kleene posets is again representable and hence a direct product of finite chains is representable. This does not hold in general for subdirect products, but we show some examples where it holds. We present large classes of representable and non-representable Kleene posets. Finally, we investigate two kinds of extensions of a distributive poset $\mathbf{A}$, namely its Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{A})$ and a completion $G(\mathbf{A})$ which coincides with $\mathbf{DM}(\mathbf{A})$ provided $\mathbf{A}$ is finite. In particular we prove that if $\mathbf{A}$ is a Kleene poset then its extension $G(\mathbf{A})$ is also a Kleene lattice. If the subset X of principal order ideals of $\mathbf{A}$ is involution-closed and doubly dense in $G(\mathbf{A})$ then it generates $G(\mathbf{A})$ and it is isomorphic to $\mathbf{A}$ itself.

Keywords: Kleene lattice, Normality condition, Kleene poset, Pseudo-Kleene poset, Representable Kleene lattice, Embedding, Twist-product, Dedekind-MacNeille completion.

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1. Introduction

Kleene lattices are De Morgan algebras of a specific sort. These are distributive lattices equipped with a negation $'$ satisfying the double negation law $x'' = x$. In our case the unary operation $'$ is assumed to be antitone

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with respect to the induced order, but it need not be a complementation. In order to enrich the properties of such an operation, it is natural to add the so-called *normality condition*, i.e., the inequality

$$x \wedge x' \leq y \vee y'.$$

Of course, if $'$ is a complementation, then this inequality is satisfied automatically. But often, the negation in a De Morgan algebra has this property, and hence such a negation turns out to be close to complementation. Distributive lattices with an antitone involution satisfying the normality condition are called *Kleene lattices* and were introduced by Kalman [7] (under a different name). To emphasize the importance of this concept, let us note that every MV-algebra, i.e., the algebraic semantics of Łukasiewicz's many-valued logic, is a Kleene lattice. Moreover, MV-algebras are also crucial in the logic of quantum events because every lattice effect algebra is composed of blocks, which are MV-algebras. Due to this, the question how to construct Kleene lattices is of some interest and importance.

If instead of lattices, only posets are considered, one obtains *Kleene posets*. If we also forget distributivity, we get *pseudo-Kleene posets*. We will introduce both notions later.

Our previous paper [5] showed how to construct a Kleene lattice $\mathbf{K}$ from a given distributive lattice $\mathbf{L} = (L, \vee, \wedge)$ employing the so-called twist product and its reduction using a non-empty subset S of L. In such a case, we say that $\mathbf{K}$ is representable. However, a lot of problems mentioned in [5] remain open. Among them, we would like to try to solve the following ones:

- Determine classes of representable Kleene lattices as well as classes of not representable Kleene lattices.
- Can these constructions be extended to Kleene posets?
- Can every poset be embedded into a Kleene poset obtained by such a construction?
- Is the Dedekind-MacNeille completion of a representable poset a representable Kleene lattice?

The present paper aims to get at least partial answers to the mentioned questions. We show that direct products of chains can be considered as representable Kleene lattices and study certain ordinal sums of distributive lattices. We prove that if a pseudo-Kleene poset $\mathbf{K}$ of odd cardinality can be represented by a distributive poset $\mathbf{A}$ and a non-empty subset S of A, then S must be a singleton, i.e., $S = \{a\}$, and $\mathbf{A}$ can be embedded into $\mathbf{K}$

in such a way that a is mapped onto the unique fixed point of $'$. We prove further results on representable Kleene posets and Kleene lattices.

First, we recall or introduce several concepts on ordered sets (posets).

Let $(A, \leq)$ be a poset, $b, c \in A$ and $B, C \subseteq A$. We say

$$B \leq C \text{ if } x \leq y \text{ for all } x \in B \text{ and } y \in C.$$

Instead of $B \leq \{c\}$ and $\{b\} \leq C$ we simply write $B \leq c$ and $b \leq C$, respectively. Further, we define

$$\begin{aligned} L(B) &:= \{x \in A \mid x \leq B\}, \\ U(B) &:= \{x \in A \mid B \leq x\}. \end{aligned}$$

Instead of $L(B \cup C)$, $L(B \cup \{c\})$, $L(\{b\} \cup C)$, $L(\{b, c\})$ and $L\big(U(B)\big)$ we simply write $L(B, C)$, $L(B, c)$, $L(b, C)$, $L(b, c)$ and $LU(B)$, respectively. Similarly for finitely many subsets and/or elements of A. Moreover, for all $X, Y \subseteq A$ we have:

$$\begin{aligned} Y \subseteq L(X) &\text{ if and only if } X \subseteq U(Y) \\ LUL(X) = L(X) &\text{ and } ULU(X) = U(X). \end{aligned}$$

Also, $L(A) = \{0\}$ if A has a smallest element 0 and $L(A) = \emptyset$ otherwise. Similarly, $U(A) = \{1\}$ if A has a greatest element 1 and $U(A) = \emptyset$ otherwise.

A subset I of A is said to be a *Frink ideal* if $LU(M) \subseteq I$ for each finite subset $M \subseteq I$. Similarly, a subset F of A is said to be a *Frink filter* if $UL(N) \subseteq F$ for each finite subset $N \subseteq F$.

An order-preserving map f between posets $\mathbf{A}$ and $\mathbf{B}$ is said to be *an LU-morphism* if

$$L\big(f(X)\big) = L\big(f(UL(X)\big) \quad \text{and} \quad U\big(f(X)\big) = U\big(f(LU(X)\big) \tag{1}$$

for all non-empty finite subsets $X \subseteq A$.

We say that an LU-morphism f is an *LU-embedding*, respectively an *LU-isomorphism* if f is *order reflecting* (thus $f(x) \leq f(y)$ implies $x \leq y$ for any $x, y \in A$), respectively f is bijective and the inverse map to f is an LU-morphism.

An *antitone involution* on $\mathbf{A}$ is a unary operation $'$ on A, satisfying for any $x, y \in A$

$$\begin{aligned} &x \leq y \text{ implies } y' \leq x', \\ &x'' = x. \end{aligned}$$

For any $X \subseteq A$ we put $X' = \{x' \mid x \in X\}$. Hence, for any $X, Y \subseteq A$ we have the so-called *De Morgan laws*:

$$L(X,Y)' = U(X',Y') \text{ and } U(X,Y)' = L(X',Y').$$

The *poset* $\mathbf{A}$ is called *distributive* if one of the following equivalent LU-identities is satisfied:

$$\begin{aligned}
L\big(U(x,y),z\big) &\approx LU\big(L(x,z),L(y,z)\big),\\
U\big(L(x,z),L(y,z)\big) &\approx UL\big(U(x,y),z\big),\\
U\big(L(x,y),z\big) &\approx UL\big(U(x,z),U(y,z)\big),\\
L\big(U(x,z),U(y,z)\big) &\approx LU\big(L(x,y),z\big),\\
L\big(U(x_1,x_2,\dots,x_n),z\big) &\approx LU\big(L(x_1,z),L(x_2,z),\dots,L(x_n,z)\big),\\
U\big(L(x_1,x_2,\dots,x_n),z\big) &\approx UL\big(U(x_1,z),U(x_2,z),\dots,U(x_n,z)\big).
\end{aligned}$$

An element $y \in A$ is said to be a *complement* of $x \in A$ if $L(x,y) = L(A)$ and $U(x,y) = U(A)$. $\mathbf{A}$ is said to be *complemented* if each element of A has a complement in $\mathbf{A}$. $\mathbf{A}$ is said to be *Boolean* if it is distributive and complemented.

If $\mathbf{A}$ has a greatest element 1 and a smallest element 0, then an antitone involution $'$ on A is called an *orthocomplementation* if $L(x,x') = \{0\}$ and $U(x,x') = \{1\}$.

2. Constructions of Kleene Posets

Let $\mathbf{A} = (A, \leq)$ be a poset. We define the *twist product* of $\mathbf{A}$ as $(A^2, \sqsubseteq) = (A, \leq) \times (A, \geq)$ (the direct product of $\mathbf{A}$ with its dual), so that

$$(x,y) \sqsubseteq (z,u) \text{ if and only if } x \leq z \text{ and } u \leq y$$

for all $(x,y),(z,u) \in A^2$. So the twist product of $\mathbf{A}$ is a poset again. This construction was successfully applied in the study of the so-called Nelson-type algebras [2].

Note the following evident but useful fact. Let $x, y, z, u, v \in A$. If y covers x in A then (y,z) covers (x,z) in $(A^2, \sqsubseteq)$. This follows from the observation that "$(x,z) \sqsubset (u,v)$" *and* "$u < y$" $\sqsubset (y,z)$ implies $z = v$. Hence $x < u \leq y$ which is not possible.

Now we define the central concept of our paper, which is used to represent Kleene lattices and Kleene posets within twist products. Let S be a non-empty subset of A. Define

$$P_S(\mathbf{A}) := \{(x, y) \in A^2 \mid L(x, y) \leq S \leq U(x, y)\}.$$

Instead of $P_{\{a\}}(\mathbf{A})$, we simply write $P_a(\mathbf{A})$.

A poset with antitone involution $(A, \leq, ')$ is called a *pseudo-Kleene poset* if the *normality condition*

$$L(x, x') \leq U(y, y') \text{ for all } x, y \in A \tag{2}$$

holds. A *Kleene poset* is a distributive pseudo-Kleene poset.

Recall that Zhu in [9] introduced the notion of a Kleene poset as an ordered triple $(A, \leq, ')$ such that $'$ is an antitone involution on $\mathbf{A}$ and the *Zhu condition*

$$x \leq x' \text{ and } y' \leq y \text{ implies } x \leq y \text{ for all } x, y \in A \tag{3}$$

holds. In fact, his concept is precisely the pseudo-Kleene poset in our sense because he does not assume distributivity of $(A, \leq)$. Indeed, we have the following.

LEMMA 2.1. *Let $\mathbf{A} = (A, \leq, ')$ be a poset with an antitone involution $'$. Then the normality condition (2) is equivalent to the Zhu condition (3).*

PROOF. Assume first that the normality condition holds. Let $x, y \in A$ such that $x \leq x'$, and $y' \leq y$. Then $L(x) = L(x, x') \leq U(y, y') = U(y)$. Hence $x \leq y$.

Now assume that the Zhu condition holds. Let $x, y \in A$, $u \in L(x, x')$, and $v \in U(y, y')$. Since $u \leq x$ we obtain that $x' \leq u'$. It follows that $u \leq u'$. Similarly, $v' \leq v$. From the Zhu condition we conclude $u \leq v$, i.e., $L(x, x') \leq U(y, y')$. ∎

Since Kleene algebras are distributive lattices and Zhu does not assume any distributivity condition in his definition, we will use the notion of a Kleene poset in our sense.

For any poset $\mathbf{A} = (A, \leq)$ and any non-empty subset S of A we define $(x, y)' := (y, x)$ for all $(x, y) \in A^2$ and $\mathbf{P}_S(\mathbf{A}) := \big(P_S(\mathbf{A}), \sqsubseteq, '\big)$. Instead of $\mathbf{P}_{\{a\}}(\mathbf{A})$ we simply write $\mathbf{P}_a(\mathbf{A})$.

Let p_1 and p_2 denote the first and second projection from $P_S(\mathbf{A})$ to A, respectively.

The following lemma shows how to produce pseudo-Kleene posets from an arbitrarily given poset.

Lemma 2.2. *Let $\mathbf{A} = (A, \leq)$ be a poset. Then, in the twist product of $\mathbf{A}$:*

$$L(X) = L(p_1(X)) \times U(p_2(X)),$$
$$U(X) = U(p_1(X)) \times L(p_2(X))$$

for all $X \subseteq A^2$.

Proof. Let $X \subseteq A^2$. Then the following are equivalent for all $(a, b) \in A^2$:

$$(a, b) \sqsubseteq X,$$
$$a \leq p_1(X), b \geq p_2(X),$$
$$a \in L(p_1(X)), b \in U(p_2(X)),$$
$$(a, b) \in L(p_1(X)) \times U(p_2(X)).$$

Hence $L(X) = L(p_1(X)) \times U(p_2(X))$. Similarly, we obtain that $U(X) = U(p_1(X)) \times L(p_2(X))$. ■

Lemma 2.3. *Let $\mathbf{A} = (A, \leq)$ be a poset and S a non-empty subset of A. Then $\mathbf{P}_S(\mathbf{A})$ is a pseudo-Kleene poset in which*

$$L(X) = \big(L(p_1(X)) \times U(p_2(X))\big) \cap P_S(\mathbf{A}),$$
$$U(X) = \big(U(p_1(X)) \times L(p_2(X))\big) \cap P_S(\mathbf{A})$$

for all $X \subseteq P_S(\mathbf{A})$.

Proof. Clearly, $\mathbf{P}_S(\mathbf{A})$ is a poset with an antitone involution. Let (a, b), $(c, d) \in P_S(\mathbf{A})$, $(e, f) \in L\big((a, b), (b, a)\big)$ and $(g, h) \in U\big((c, d), (d, c)\big)$. Then $e \in L(a, b)$, $f \in U(a, b)$, $g \in U(c, d)$, $h \in L(c, d)$, $L(a, b) \leq S \leq U(a, b)$ and $L(c, d) \leq S \leq U(c, d)$ and hence $e \leq S \leq f$ and $h \leq S \leq g$ which implies $e \leq S \leq g$ and $h \leq S \leq f$, i.e. $(e, f) \sqsubseteq (g, h)$ showing $L\big((a, b), (b, a)\big) \leq U\big((c, d), (d, c)\big)$.

By Lemma 2.2, for any $X \subseteq P_S(\mathbf{A})$, we have:

$$L(X) = \big(L(p_1(X)) \times U(p_2(X))\big) \cap P_S(\mathbf{A}) \text{ and}$$
$$U(X) = \big(U(p_1(X)) \times L(p_2(X))\big) \cap P_S(\mathbf{A}).$$

■

Let $\mathbf{A} = (A, \leq)$ be a poset. Recall that $\mathbf{A}$ is said to satisfy the *Ascending Chain Condition* (ACC) or the *Descending Chain Condition* (DCC) if in $\mathbf{A}$, every strictly ascending chain or every strictly descending chain, respectively, is finite. Hence if $\mathbf{A} = (A, \leq)$ satisfies the ACC or the DCC then every $\emptyset \neq B \subseteq A$ contains maximal or minimal elements, respectively.

Let $B \subseteq A$. We denote by $\operatorname{Max} B$ and $\operatorname{Min} B$ the set of all maximal and minimal elements of B, respectively.

LEMMA 2.4. *Let $\mathbf{A} = (A, \leq)$ be a poset and S a non-empty subset of A.*

(i) *If $(S, \leq)$ satisfies the* ACC *and the* DCC, *then $P_S(\mathbf{A}) = P_{(\operatorname{Max} S)\cup(\operatorname{Min} S)}(\mathbf{A})$,*

(ii) *If $\bigwedge S$ and $\bigvee S$ exist, then $P_S(\mathbf{A}) = P_{\{\bigwedge S, \bigvee S\}}(\mathbf{A})$.*

PROOF.

(i) If $(S, \leq)$ satisfies the ACC, then every element of S lies under some maximal element of S, and if $(S, \leq)$ meets the DCC, then every element of S lies over some minimal element of S.

(ii) If $\bigwedge S$ and $\bigvee S$ exist then we have

$$P_S(\mathbf{A}) = \{(x, y) \in A^2 \mid L(x, y) \leq S \leq U(x, y)\} =$$
$$= \{(x, y) \in A^2 \mid L(x, y) \leq \bigwedge S \leq \bigvee S \leq U(x, y)\} = P_{\{\bigwedge S, \bigvee S\}}(\mathbf{A}).$$

■

A *subset* B of a poset $(A, \leq)$ is called *convex* if

$$x, z \in B, y \in A \text{ and } x \leq y \leq z \text{ imply } y \in B.$$

Let $S \subseteq A$. We put $co(S) := LU(S) \cap UL(S)$.

LEMMA 2.5. *Let $\mathbf{A} = (A, \leq)$ be a poset and $S \subseteq A$. Then $co(S)$ is a convex set including S, $L(S) = L(co(S))$, $U(S) = U(co(S))$, and $P_S(\mathbf{A}) = P_{co(S)}(\mathbf{A})$.*

PROOF. Let $x, z \in co(S), y \in A$ and $x \leq y \leq z$. Then $\{x, z\} \leq U(S)$. Since $y \leq z$, we obtain that $y \leq U(S)$, i.e., $y \in LU(S)$. Similarly, $y \in UL(S)$, hence $y \in co(S)$. Clearly, $S \subseteq LU(S) \cap UL(S) = co(S)$, thus $L(co(S)) \subseteq L(S)$ and $U(co(S)) \subseteq U(S)$. We have $co(S) \subseteq UL(S)$ and $co(S) \subseteq LU(S)$. We conclude $L(S) = LUL(S) \subseteq L(co(S))$ and $U(S) = ULU(S) \subseteq U(co(S))$, hence $L(S) = L(co(S))$ and $U(S) = U(co(S))$.

Finally, since $S \subseteq co(S)$ we obtain $P_{co(S)}(\mathbf{A}) \subseteq P_S(\mathbf{A})$. Assume now that $(x, y) \in P_S(\mathbf{A})$. Then $L(x, y) \leq S \leq U(x, y)$. Hence $L(x, y) \subseteq L(S) = L(co(S))$ and $U(x, y) \subseteq U(S) = U(co(S))$. We conclude $L(x, y) \leq co(S) \leq U(x, y)$, i.e., $(x, y) \in P_{co(S)}(\mathbf{A})$. ■

We are going to show that for a given poset $\mathbf{A} = (A, \leq)$ and an element a of A, the constructed pseudo-Kleene poset $\mathbf{P}_a(\mathbf{A})$ includes A as a convex subset.

LEMMA 2.6. *Let $\mathbf{A} = (A, \leq)$ be a poset, $a \in A$ and let f denote the mapping from A to $P_a(\mathbf{A})$ defined by*

$$f(x) := (x, a) \text{ for all } x \in A.$$

Then $f(A)$ is a convex subset of $\big(P_a(\mathbf{A}),\sqsubseteq\big)$, $\mathbf{A}$ can be LU-embedded into $\big(P_a(\mathbf{A}),\sqsubseteq\big)$, and f is an order isomorphism from $(A,\leq)$ to $\big(f(A),\sqsubseteq\big)$.

Proof. It is clear that f is a mapping from A to $P_a(\mathbf{A})$. If $b,c\in A$, $(d,e)\in P_a(\mathbf{A})$ and $(b,a)\sqsubseteq(d,e)\sqsubseteq(c,a)$, then $a\leq e\leq a$ and hence $e=a$, which implies $(d,e)\in f(A)$. This shows that $f(A)$ is a convex subset of $\big(P_a(\mathbf{A}),\sqsubseteq\big)$.

Assume now that $X\subseteq A$ is finite and non-empty. We compute:

$$\begin{aligned}U(f(X))&=\big(U(X)\times L(a)\big)\cap P_a(\mathbf{A})\\&=\big(ULU(X)\times L(a)\big)\cap P_a(\mathbf{A})=U(f(LU(X))),\\L(f(X))&=\big(L(X)\times U(a)\big)\cap P_a(\mathbf{A})\\&=\big(LUL(X)\times U(a)\big)\cap P_a(\mathbf{A})=L(f(UL(X))).\end{aligned}$$

Finally, for any $x,y\in A$, $x\leq y$ and $(x,a)\sqsubseteq(y,a)$ are equivalent. We conclude that f is an order isomorphism from $(A,\leq)$ to $\big(f(A),\sqsubseteq\big)$ and an LU-embedding from $\mathbf{A}$ into the poset $\big(P_a(\mathbf{A}),\sqsubseteq\big)$. ■

Corollary 2.7. *Let $\mathbf{A}=(A,\leq)$ be a poset, $a\in A$ and let f denote the mapping from A to $P_a(\mathbf{A})$ defined by*

$$f(x):=(x,a)\text{ for all }x\in A.$$

If $\big(P_a(\mathbf{A}),\sqsubseteq\big)$ is a lattice then $\mathbf{A}$ is a lattice.

3. Embeddings

In Lemma 2.6, we showed that for a poset $\mathbf{A}=(A,\leq)$ and an element a of A, there is an embedding of $\mathbf{A}$ into $\big(P_a(\mathbf{A}),\sqsubseteq\big)$. A similar result holds for a lattice $\mathbf{L}=(L,\vee,\wedge)$ and an element a of L. However, if the non-empty subset S of A is not a singleton, it is not so easy to find an embedding of $\mathbf{L}$ into the lattice $\big(P_S(\mathbf{L}),\sqcup,\sqcap\big)$ where

$$(a,b)\sqcup(c,d)=(a\vee c,b\wedge d)\text{ and }(a,b)\sqcap(c,d)=(a\wedge c,b\vee d)$$

for all $(a,b),(c,d)\in P_S(\mathbf{L})$. The following theorem provides a solution to this problem in a particular case.

Theorem 3.1. *Let $\mathbf{A}=(A,\leq)$ be a distributive poset and $a,b\in A$ with $a\leq b$ and assume that there exists an orthocomplementation $'$ on $([a,b],\leq)$. Further, assume that for every $x\in A$ satisfying $L(a)\subseteq L(U(x,a),b)\subseteq L(b)$ we have $L(U(x,a),b)=L(x)$. Then $\mathbf{A}$ can be LU-embedded into $\big(P_{\{a,b\}}(\mathbf{A}),\sqsubseteq\big)$ and the poset $([a,b],\leq,')$ is Boolean.*

PROOF. Put

$$I := \{x \in A \mid L(x,b) \leq a\},$$
$$F := \{x \in A \mid b \leq U(x,a)\}.$$

It is easy to see that $a \in I$ and $b \in F$. Let us show that I is a Frink ideal and F is a Frink filter. Assume that $X \subseteq I$, X finite. Let $X = \emptyset$. Then either $LU(X) = \emptyset$ or $LU(X) = \{0\}$ where 0 is the smallest element of $\mathbf{A}$. In both cases, $LU(X) \subseteq I$. Suppose now that $X \neq \emptyset$, $X = \{x_1, \ldots, x_n\}$ and $x \in LU(X)$. Then $L(x,b) \subseteq L(U(X),b) = LU\big(L(x_1,b), L(x_2,b), \ldots, L(x_n,b)\big) \subseteq LU\big(L(a)\big) = L(a)$. Hence $L(x,b) \leq a$. Similarly, F is a Frink filter.

Let $'$ be an orthocomplementation on $([a,b], \leq)$ and define $f\colon A \to P_{\{a,b\}}(\mathbf{A})$ as follows:

$$f(x) := \begin{cases} (x,b) & \text{if } x \in I, \\ (x,a) & \text{if } x \in F, \\ (x,x') & \text{otherwise} \end{cases}$$

$(x \in A)$. Of course, $a' = b$ and $b' = a$.

If $x \in I$, then $L(x,b) \leq a \leq b \leq U(x,b)$ and $(x,b) \in P_{\{a,b\}}(\mathbf{A})$. If $x \in F$, then $L(x,a) \leq a \leq b \leq U(x,a)$ and $(x,a) \in P_{\{a,b\}}(\mathbf{A})$.

Assume first that $I \cap F \neq \emptyset$. Let $z \in I \cap F$. Then $L(z,b) \leq a$ and $b \leq U(z,a)$. We conclude

$$b \in LU(z,a) \cap L(b) = LU\big(L(z,b), L(a,b)\big) \subseteq LU(L(a)) = L(a).$$

Hence $a = b$ and $I = F = A$. Evidently, f is an LU-embedding by Lemma 2.6.

From now on, we will assume that $I \cap F = \emptyset$. We have that $a < b$.

Let $c \in A$. If $c \notin I \cup F$ then $L(c,b) \nsubseteq L(a)$ and $U(c,a) \nsubseteq U(b)$, i.e., $L(b) = LU(b) \nsubseteq LU(c,a)$. From distributivity we compute:

$$L(a) \subseteq LU\big(L(c,b), a\big) = LU\big(L(c,b), L(a,b)\big) = L\big(U(c,a), b\big) \subseteq L(b).$$

Hence $L(U(c,a),b) = L(c) = LU\big(L(c,b), a\big)$. This shows $a \leq c \leq b$. Since $a \in I$ and $b \in F$ we obtain

$$A \setminus (I \cup F) \subseteq [a,b] \setminus \{a,b\}.$$

If $c \in I \cap [a,b]$, then $a \leq c \in L(c,b) \leq a$, which implies $c = a$. This implies

$$I \cap [a,b] = \{a\}.$$

Dually, we obtain

$$F \cap [a,b] = \{b\}.$$

Therefore, we have a partition

$$A = I \mathbin{\dot\cup} F \mathbin{\dot\cup} ([a,b] \setminus \{a,b\}).$$

We conclude

$$f(x) = (x, x') \text{ for all } x \in [a,b].$$

Since $'$ is an orthocomplementation on $([a,b], \leq)$ we obtain that $L(x,x') = L(a) \subset L(b) = LU(x,x')$, i.e., $(x,x') \in P_{\{a,b\}}(\mathbf{A})$.

We have proved that f is well-defined.

Let us verify that f is an LU-embedding. We first show that f is an LU-morphism. Clearly, f is order preserving.

Let us put $Z_I = Z \cap I$, $Z_{(a,b)} = Z \cap ([a,b]\setminus\{a,b\})$, and $Z_F = Z \cap F$ for any subset $Z \subseteq A$.

Assume now that $X \subseteq A$ is finite and non-empty. Suppose first that $X_F \neq \emptyset$. Then $LU(X) \cap F \neq \emptyset$ and $U(X) \subseteq F$.

We compute:

$$\begin{aligned} U(f(X)) &= \big(U(X) \times L(a)\big) \cap P_{\{a,b\}}(\mathbf{A}) \\ &= \big(ULU(X) \times L(a)\big) \cap P_{\{a,b\}}(\mathbf{A}) = U(f(LU(X))). \end{aligned}$$

From now on, we will assume that $X_F = \emptyset$.

Case 1 Let $X_{(a,b)} = \emptyset$. Then $X_I = X$ and $X \subseteq LU(X) \subseteq I$. We compute:

$$U\big(f(X)\big) = U(X \times \{b\}) = \big(U(X) \times L(b)\big) \cap P_{\{a,b\}}(\mathbf{A}) \text{ and } U\big(f(LU(X))\big) = U\big(LU(X) \times \{b\}\big) = \big(ULU(X) \times L(b)\big) \cap P_{\{a,b\}}(\mathbf{A}) = \big(U(X) \times L(b)\big) \cap P_{\{a,b\}}(\mathbf{A}).$$

Case 2 Let $X_{(a,b)} \neq \emptyset$. Then $f(X) = X_I \times \{b\} \cup \{(x,x') \mid x \in X_{(a,b)}\}$, $X'_{(a,b)} \subseteq [a,b] \setminus \{a,b\}$ and $L(a) \subseteq L(X'_{(a,b)}) \subseteq L(b)$. We compute: that

$$U\big(f(X)\big) = \big(U(X) \times L(X'_{(a,b)})\big) \cap P_{\{a,b\}}(\mathbf{A}) \supseteq U(f(LU(X))),$$

$$\begin{aligned} U\big(f(LU(X))\big) &= \begin{cases} \big(ULU(X) \times L(a)\big) \cap P_{\{a,b\}}(\mathbf{A}), & \text{if } LU(X) \cap F \neq \emptyset, \\ \big(ULU(X) \times L((LU(X)_{(a,b)})')\big) \cap P_{\{a,b\}}(\mathbf{A}), & \text{otherwise.} \end{cases} \\ &= \begin{cases} \big(U(X) \times L(a)\big) \cap P_{\{a,b\}}(\mathbf{A}), & \text{if } LU(X) \cap F \neq \emptyset, \\ \big(U(X) \times L((LU(X)_{(a,b)})')\big) \cap P_{\{a,b\}}(\mathbf{A}), & \text{otherwise.} \end{cases} \end{aligned}$$

Suppose first that $LU(X) \cap F \neq \emptyset$. Assume that $y \in L(X'_{(a,b)}) \setminus L(a)$. We have

$$L(a) \subset L(U(y,a), b) = L(U(y,a)) \subseteq L(x') \subset L(b)$$

for every $x \in X_{(a,b)}$. Since $X_{(a,b)}$ is non-empty, we therefore obtain

$$L(a) \subset L(U(y,a), b) = L(y) \subset L(b),$$

i.e., $a < y < b$ and $a < y' < b$. Moreover, $y \in L(X'_{(a,b)})$ yields $y' \in U(X_{(a,b)})$, i.e., $\emptyset \neq LU(X) \cap F \subseteq LU(X_I, y') \cap F$. Hence there is an element $z \in F$ such that $z \in LU(X_I, y')$ and $b \leq U(z, a)$. Let $X_I = \{x_{i_1}, \ldots, x_{i_k}\}$ and $X_{(a,b)} = \{x_{j_1}, \ldots, x_{j_l}\}$. We compute:

$$\begin{aligned}
b \in LU(z,a) \cap L(b) &\subseteq LU\big(LU(X_I, y'), a\big) \cap L(b) = LU(X_I, y') \cap L(b)\\
&= L\big(U(x_{i_1}, \ldots, x_{i_k}, y'), b\big) = LU\big(L(x_{i_1}, b), \ldots, L(x_{i_k}, b), L(y', b)\big)\\
&\subseteq LU\big(L(a), \ldots, L(a), L(y')\big) = L(y'),
\end{aligned}$$

i.e., $b \leq y' < b$, a contradiction. Hence $L(a) = L(X'_{(a,b)})$ and $U\big(f(LU(X))\big) = U\big(f(X)\big)$.

Suppose now that $LU(X) \cap F = \emptyset$.

Let us check that $L(X'_{(a,b)}) = L\big(\{u' \mid u \in LU(X) \cap ([a,b] \setminus \{a,b\})\}\big)$.

Clearly, $L(a) \subseteq L\big(\{u' \mid u \in LU(X) \cap ([a,b] \setminus \{a,b\})\}\big) \subseteq L(X'_{(a,b)}) \subseteq L(b)$. The first inclusion holds since $a < u < b$ and $a < u' < b$. The second one follows from the fact that $X_{(a,b)} \subseteq LU(X)$ and $X_{(a,b)} \subseteq ([a,b] \setminus \{a,b\})$. Hence also $X_{(a,b)} \subseteq LU(X) \cap ([a,b] \setminus \{a,b\}) \subseteq [a,b]$, i.e., $X'_{(a,b)} \subseteq \big(LU(X) \cap ([a,b] \setminus \{a,b\})\big)'$.

Assume that $y \in L(X'_{(a,b)}) \setminus L(a)$. We have

$$L(a) \subset L\big(U(y,a), b\big) = L\big(U(y,a)\big) \subseteq L(x') \subset L(b)$$

for every $x \in X_{(a,b)}$. Since $X_{(a,b)}$ is non-empty (i.e., at least one $x \in X_{(a,b)}$ exists) we obtain

$$L(a) \subset L\big(U(y,a), b\big) = L(y) \subset L(b),$$

i.e., $a < y < b$ and $a < y' < b$. Moreover, $y \in L\big(X'_{(a,b)}\big)$ yields $y' \in U\big(X_{(a,b)}\big)$, i.e., $LU\big(X_{(a,b)}\big) \leq y'$.

Let $u \in LU(X) \cap ([a,b] \setminus \{a,b\})$. We compute:

$$\begin{aligned}
u \in LU(X) \cap L(b) &= LU\big(L(x_{i_1}, b), \ldots, L(x_{i_k}, b), L(x_{j_1}, b), \ldots, L(x_{j_l}, b)\big)\\
&\subseteq LU\big(L(a), \ldots, L(a), L(x_{j_1}), \ldots, L(x_{j_l})\big) = LU\big(X_{(a,b)}\big) \leq y'.
\end{aligned}$$

This implies $u \leq y'$, i.e., $y \leq u'$ and $LU(X) \cap L(b) = LU\big(X_{(a,b)}\big)$. Therefore, $y \in L\big(\{u' \mid u \in LU(X) \cap ([a,b] \setminus \{a,b\})\}\big)$.

We compute:

$$\begin{aligned}
U\big(f(LU(X))\big) &= U\big(\big(LU(X) \cap I\big) \times \{b\} \cup \{(u, u') \mid u \in LU(X) \cap ([a,b] \setminus \{a,b\})\}\big)\\
&= \big(ULU(X) \times L\big(\{u' \mid u \in LU(X) \cap ([a,b] \setminus \{a,b\})\}\big)\big) \cap P_{\{a,b\}}(\mathbf{A})\\
&= \big(ULU(X) \times L(X'_{(a,b)})\big) \cap P_{\{a,b\}}(\mathbf{A})\big) = \big(U(X) \times L(X'_{(a,b)})\big) \cap P_{\{a,b\}}(\mathbf{A})\big)\\
&= U\big(f(X)\big).
\end{aligned}$$

Similarly, we obtain that $L\big(f(UL(X))\big) = L\big(f(X)\big)$ for every finite and non-empty $X \subseteq A$.

Assume now that $c, d \in A$, $f(c) = (c, u) \sqsubseteq (d, v) = f(d)$. Then $c \leq d$ and f is order-reflecting.

It remains to check that $([a, b], \leq, ')$ is a Boolean poset. Clearly, it is complemented. We have to verify that it is distributive.

Assume that $x, y, z \in [a, b]$. If $\{x, y, z\} \cap \{a, b\} \neq \emptyset$ then evidently $L_{[a,b]}\big(U_{[a,b]}(x, y), z\big) = L_{[a,b]}U_{[a,b]}\big(L_{[a,b]}(x, z), L_{[a,b]}(y, z)\big)$. Hence we may assume that $a < x, y, z < b$.

Let $d \in U_{[a,b]}\big(L_{[a,b]}(x, z), L_{[a,b]}(y, z)\big)$ and $e \in L_{[a,b]}\big(U_{[a,b]}(x, y), z\big)$. We will show that $d \in U\big(L(x, z), L(y, z)\big)$ and $e \in L\big(U(x, y), z\big)$.

Then $d \geq L(x, z)$. Namely, if $d \not\geq L(x, z)$ then there is $h \in A \setminus [a, b]$ such that $h \not\leq d$, $h < x$ and $h < z$. Hence also $h < b$ and $h \not\leq a$.

Clearly, $L(a) \subseteq L\big(U(h, a), b\big) = LU(h, a) \subseteq L(b)$. Assume first $L(a) = LU(h, a)$. Then $h \in LU(h, a) = L(a)$, a contradiction. We have $LU(h, a) \subseteq L(x) \subset L(b)$. Hence $L(a) \subset L\big(U(h, a), b\big) = L\big(U(h, a)\big) \subset L(b)$, i.e., $a < h < b$, a contradiction again. Therefore really $d \geq L(x, z)$ and similarly $d \geq L(y, z)$. Hence $d \in U\big(L(x, z), L(y, z)\big)$.

Further, $e \in L(z)$ since $e \in L_{[a,b]}(z)$. Let us check that $e \in LU(x, y)$. Assume that $e \notin LU(x, y)$. Since $e \in L_{[a,b]}\big(U_{[a,b]}(x, y), z\big)$ there exists $g \in U(x, y) \setminus U_{[a,b]}(x, y)$ such that $e \not\leq g$. We have $a < x \leq g \neq b$. Hence $L(a) \subseteq L\big(U(g, a), b\big) = L(g, b) \subseteq L(b)$. Suppose first that $L(g, b) = L(b)$. Then $e \leq b < g$, a contradiction with $e \not\leq g$. Also, $L(a) \subset L(x) \subseteq L(g, b) = L\big(U(g, a), b\big) \subset L(b)$. We conclude $a < g < b$, which is impossible since $g \notin [a, b]$. Therefore $e \in LU(x, y)$ and we obtain $e \in LU(x, y) \cap L(z) = L\big(U(x, y), z\big)$. Since $\mathbf{A}$ is distributive we see that $e \leq d$. Consequently $U_{[a,b]}L_{[a,b]}\big(U_{[a,b]}(x, y), z\big) \supseteq U_{[a,b]}\big(L_{[a,b]}(x, z), L_{[a,b]}(y, z)\big)$ which yields that $([a, b], \leq)$ is distributive. ■

Remark 3.2. First, note that Theorem 3.1 generalizes [5, Lemma 6] formulated for distributive lattices. We also prefer to use orthocomplementation instead of antitone complementation as in [5, Lemma 6].

Second, the conditions of Theorem 3.1 are equivalent with the requirement that $\mathbf{A}$ is a distributive poset such that $A \setminus (I \cup F) = [a, b] \setminus \{a, b\}$ and $[a, b]$ is a Boolean poset.

Example 3.3. Let $\mathbf{A}$ be the distributive poset (in fact a lattice) shown in Figure 1. Clearly, conditions of Theorem 3.1 for the choice of a and b are trivially satisfied. Then $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$ is visualized in Figure 2. The non-filled circles indicate the embedding of $\mathbf{A}$ into $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$.

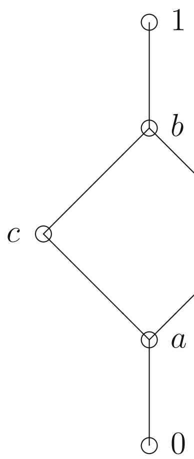


Figure 1. Distributive poset $\mathbf{A}$ that is LU-embeddable into a Kleene poset $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$

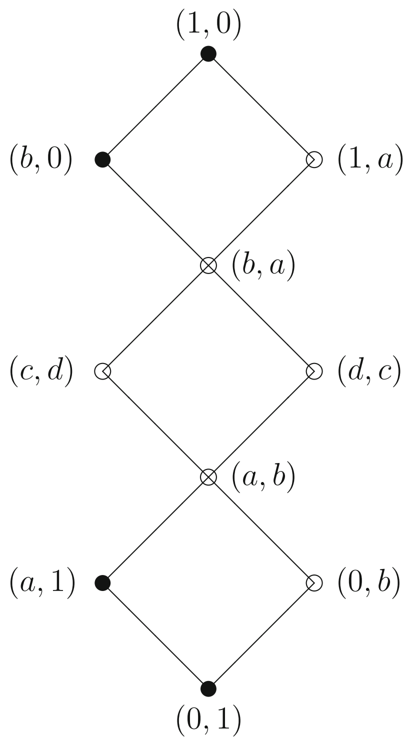


Figure 2. Distributive poset $\mathbf{A}$ that is LU-embeddable into a Kleene poset $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$

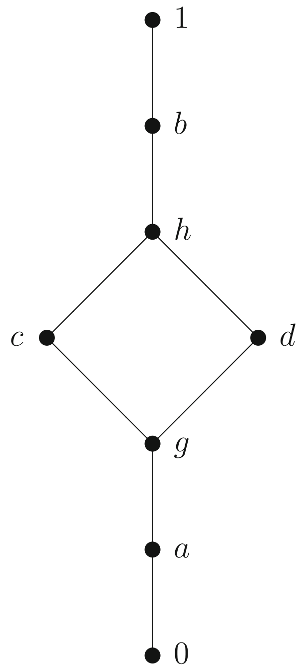


Figure 3. Distributive poset $\mathbf{A}$ that is not LU-embeddable into a Kleene poset $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$

Example 3.4. Let $\mathbf{A}$ be the distributive poset (in fact a lattice) shown in Figure 3. Since $[a,b]$ is a poset with every antitone involution on it being non-Boolean, conditions of Theorem 3.1 for the choice of a and b are not satisfied. The poset $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$ is visualized in Figure 4. One immediately sees that there is no LU-embedding of $\mathbf{A}$ into $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$.

In the sequel, we use the following notation: If $\mathbf{A}_1$ and $\mathbf{A}_2$ are posets with top and bottom elements, respectively, then by $\mathbf{A}_1 +_a \mathbf{A}_2$ we denote the ordinal sum of $\mathbf{A}_1$ and $\mathbf{A}_2$ where the top element a of $\mathbf{A}_1$ is identified with the bottom element of $\mathbf{A}_2$. If $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are posets with top element, bottom and top element, and bottom element, respectively, then by $\mathbf{A}_1 +_a \mathbf{A}_2 +_b \mathbf{A}_3$ we denote the ordinal sum of $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ where the top element a of $\mathbf{A}_1$ is identified with the bottom element of $\mathbf{A}_2$ and the top element b of $\mathbf{A}_2$ is identified with the bottom element of $\mathbf{A}_3$. For every poset $\mathbf{A}$, let $\mathbf{A}^d$ denote its dual.

Lemma 3.5. *Let $\mathbf{A}_1 = (A_1, \le)$ and $\mathbf{A}_2 = (A_2, \le)$ be distributive posets with top element a and bottom element a, respectively. Then $\mathbf{A}_1 +_a \mathbf{A}_2$ is a distributive poset.*

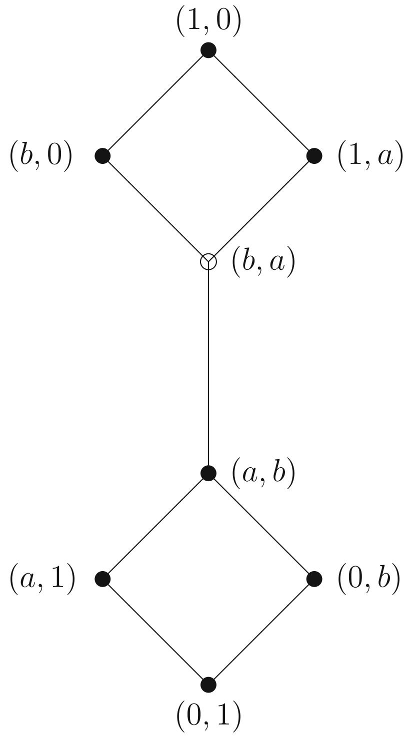


Figure 4. Distributive poset $\mathbf{A}$ that is not LU-embeddable into a Kleene poset $\big(P_{\{a,b\}}(\mathbf{A}), \sqsubseteq\big)$

PROOF. Let $x \in A_1$ and $y \in A_2$. Then $L_{\mathbf{A}_1 +_a \mathbf{A}_2}(x) = L_{\mathbf{A}_1}(x)$, $L_{\mathbf{A}_1 +_a \mathbf{A}_2}(y) = L_{\mathbf{A}_2}(y) \cup A_1$, $U_{\mathbf{A}_1 +_a \mathbf{A}_2}(y) = U_{\mathbf{A}_2}(y)$, $U_{\mathbf{A}_1 +_a \mathbf{A}_2}(x) = U_{\mathbf{A}_1}(x) \cup A_2$.

Assume that $x, y, z \in \mathbf{A}_1 +_a \mathbf{A}_2$. If some pair of elements x, y, z is comparable, then evidently

$$\begin{aligned} &L_{\mathbf{A}_1 +_a \mathbf{A}_2}\big(U_{\mathbf{A}_1 +_a \mathbf{A}_2}(x, y), z\big) \\ &\quad = L_{\mathbf{A}_1 +_a \mathbf{A}_2} U_{\mathbf{A}_1 +_a \mathbf{A}_2}\big(L_{\mathbf{A}_1 +_a \mathbf{A}_2}(x, z), L_{\mathbf{A}_1 +_a \mathbf{A}_2}(y, z)\big). \end{aligned}$$

Suppose now that there is no pair of comparable elements from $\{x, y, z\}$. Then either $\{x, y, z\} \subseteq A_1$ or $\{x, y, z\} \subseteq A_2$. Assume first that $\{x, y, z\} \subseteq A_1$. We compute:

$$\begin{aligned} &L_{\mathbf{A}_1 +_a \mathbf{A}_2}\big(U_{\mathbf{A}_1 +_a \mathbf{A}_2}(x, y), z\big) \\ &\quad = L_{\mathbf{A}_1 +_a \mathbf{A}_2}\big(U_{\mathbf{A}_1}(x, y) \cup A_2, z\big) \\ &\quad = L_{\mathbf{A}_1}\big(U_{\mathbf{A}_1}(x, y), z\big) = L_{\mathbf{A}_1} U_{\mathbf{A}_1}\big(L_{\mathbf{A}_1}(x, z), L_{\mathbf{A}_1}(y, z)\big) \\ &\quad = L_{\mathbf{A}_1} U_{\mathbf{A}_1}\big(L_{\mathbf{A}_1 +_a \mathbf{A}_2}(x, z), L_{\mathbf{A}_1 +_a \mathbf{A}_2}(y, z)\big) \end{aligned}$$

$$= L_{\mathbf{A}_1+_a\mathbf{A}_2} U_{\mathbf{A}_1+_a\mathbf{A}_2}\big(L_{\mathbf{A}_1+_a\mathbf{A}_2}(x,z), L_{\mathbf{A}_1+_a\mathbf{A}_2}(y,z)\big).$$

The case $\{x,y,z\} \subseteq A_2$ can be verified by the same procedure. ■

The following proposition enables us to determine a broad class of representable Kleene posets by using ordinal sums of distributive posets.

Proposition 3.6. *Let $\mathbf{A}_1 = (A_1,\leq)$ and $\mathbf{A}_2 = (A_2,\leq)$ be distributive posets with top element a and bottom element b, respectively, and $\mathbf{B} = (B,\leq,')$ a Boolean poset with bottom element a and top element b. If $a \neq b$ or a is join-irreducible in $\mathbf{A}_1$ and meet-irreducible in $\mathbf{A}_2$ then $\big(P_{ab}(\mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2), \sqsubseteq, '\big)$ is a Kleene poset and*

$$\big(P_{ab}(\mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2), \sqsubseteq\,\big) \cong (\mathbf{A}_1 \times \mathbf{A}_2^d) +_{(a,b)} \mathbf{B} +_{(b,a)} (\mathbf{A}_2 \times \mathbf{A}_1^d).$$

Proof. We have

$$P_{ab}(\mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2) = (A_1 \times A_2) \cup \{(x,x') \mid x \in B\} \cup (A_2 \times A_1)$$

and the order relations on both sides coincide. The equality follows from the following facts:

1. if $x \in B \setminus \{a,b\}$ then the only element $y \in A_1 \cup B \cup A_2$ satisfying $L_{\mathbf{A}_1+_a\mathbf{B}+_b\mathbf{A}_2}(x,y) \leq a$ and $U_{\mathbf{A}_1+_a\mathbf{B}+_b\mathbf{A}_2}(x,y) \geq b$ is the element $x' \in B$,
2. if $x \in A_1$ and $(x,y) \in P_{ab}(\mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2)$ then $y \in A_2$,
3. if $x \in A_2$ and $(x,y) \in P_{ab}(\mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2)$ then $y \in A_1$,
4. $(A_1 \times A_2) \cup \{(x,x') \mid x \in B\} \cup (A_2 \times A_1) \subseteq P_{ab}(\mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2)$.

Since the dual of a distributive poset is again distributive and the cartesian product of distributive posets is distributive, we obtain by Lemma 3.5 that the ordinal sum $(\mathbf{A}_1 \times \mathbf{A}_2^d) +_{(a,b)} \mathbf{B} +_{(b,a)} (\mathbf{A}_2 \times \mathbf{A}_1^d)$ is also distributive. Hence $\big(P_{ab}(\mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2), \sqsubseteq, '\big)$ is a Kleene poset. ■

Corollary 3.7. *Let $\mathbf{A}_1 = (A_1,\leq)$ and $\mathbf{A}_2 = (A_2,\leq)$ be distributive posets with top element a and bottom element b, respectively, and $\mathbf{B} = (B,\leq,')$ a Boolean poset with bottom element a and top element b. If $a \neq b$ or a is join-irreducible in $\mathbf{A}_1$ and meet-irreducible in $\mathbf{A}_2$ then $\big(P_{ab}(\mathbf{A}_1 +_a \mathbf{B}), \sqsubseteq, '\big)$ and $\big(P_{ab}(\mathbf{B} +_b \mathbf{A}_2), \sqsubseteq, '\big)$ are Kleene posets and*

$$\big(P_{ab}(\mathbf{A}_1 +_a \mathbf{B}), \sqsubseteq\,\big) \cong \mathbf{A}_1 +_{(a,b)} \mathbf{B} +_{(b,a)} \mathbf{A}_1^d,$$
$$\big(P_{ab}(\mathbf{B} +_b \mathbf{A}_2), \sqsubseteq\,\big) \cong \mathbf{A}_2^d +_{(a,b)} \mathbf{B} +_{(b,a)} \mathbf{A}_2.$$

Proof. It is enough to put $\mathbf{A}_2 := \mathbf{1}$ or $\mathbf{A}_1 := \mathbf{1}$ and use Proposition 3.6. ■

Corollary 3.8. *Let* $\mathbf{A}_1 = (A_1, \leq)$ *and* $\mathbf{A}_2 = (A_2, \leq)$ *be distributive posets with top and bottom element* a*, respectively. If* a *is join-irreducible in* $\mathbf{A}_1$ *and meet-irreducible in* $\mathbf{A}_2$ *then*

$$\big(P_a(\mathbf{A}_1 +_a \mathbf{A}_2), \sqsubseteq\big) \cong (\mathbf{A}_1 \times \mathbf{A}_2^d) +_{(a,a)} (\mathbf{A}_2 \times \mathbf{A}_1^d),$$
$$\big(P_a(\mathbf{A}_1), \sqsubseteq\big) \cong \mathbf{A}_1 +_{(a,a)} \mathbf{A}_1^d,$$
$$\big(P_a(\mathbf{A}_2), \sqsubseteq\big) \cong \mathbf{A}_2^d +_{(a,a)} \mathbf{A}_2.$$

Proof. It is enough to put $\mathbf{B} := \mathbf{1}$ and use Proposition 3.6 and Corollary 3.7. ∎

For some of the Kleene posets described in Proposition 3.6, we can construct the embeddings as follows:

Corollary 3.9. *Let* $\mathbf{A}_1 = (A_1, \leq)$ *and* $\mathbf{A}_2 = (A_2, \leq)$ *be distributive posets with top element* a *and bottom element* b*, respectively, and* $\mathbf{B} = (B, \leq, ', a, b)$ *a non-trivial bounded Boolean poset, put* $\mathbf{A} := \mathbf{A}_1 +_a \mathbf{B} +_b \mathbf{A}_2$ *and define* $f\colon A \to P_{ab}(\mathbf{L})$ *as follows:*

$$f(x) := \begin{cases} (x, b) & \text{if } x \leq a, \\ (x, x') & \text{if } a \leq x \leq b, \\ (x, a) & \text{if } b \leq x \end{cases}$$

$(x, y \in A)$*. Then* f *is an embedding from* $\mathbf{A}$ *into* $\big(P_{ab}(\mathbf{A}), \sqsubseteq\big)$*, and* $f(A)$ *is a convex subset of* $\big(P_{ab}(\mathbf{L}), \sqsubseteq\big)$*.*

Proof. The first assertion is a special case of Theorem 3.1. We have

$$\mathbf{A} = \mathbf{A}_1 +_a \mathbf{A} +_2 \mathbf{A}_2,$$
$$P_{ab}(\mathbf{A}) = (A_1 \times A_2) \cup \{(x, x') \mid x \in B\} \cup (A_2 \times A_1),$$
$$f(A) = (A_1 \times \{b\}) \cup \{(x, x') \mid x \in B\} \cup (A_2 \times \{a\}).$$

Now assume $(c, d), (h, i) \in f(A)$, $(e, g) \in P_{ab}(\mathbf{A})$ and $(c, d) \sqsubseteq (e, g) \sqsubseteq (h, i)$. Then $c \leq e \leq h$ and $a \leq i \leq g \leq d \leq b$, and hence $a \leq g \leq b$. If $e \in B$ then $f(e) = (e, e') = (e, g)$. Assume now that $e < a$. Then $g \geq b$, i.e., $g = b$. We conclude that $f(e) = (e, b) = (e, g)$. Finally, suppose that $b < e$. Hence $g \leq a$, i.e., $g = a$ and $f(e) = (e, a) = (e, g)$.

Summing up, $f(A)$ is a convex subset of $\big(P_{ab}(\mathbf{A}), \sqsubseteq\big)$. ∎

4. Representable Kleene Posets

The following result shows how to construct representable Kleene posets using the direct product of known representable Kleene posets.

THEOREM 4.1. *Let $\mathbf{A}_i = (A_i, \leq)$ be a poset and S_i a non-empty subset of A_i for every $i \in I$. Put*

$$\mathbf{A} := \prod_{i\in I} \mathbf{A}_i \text{ and } S := \prod_{i\in I} S_i.$$

Then

$$\mathbf{P}_S(\mathbf{A}) \cong \prod_{i\in I} \mathbf{P}_{S_i}(\mathbf{A}_i).$$

Moreover, if $\mathbf{A}_i$ is a distributive poset for every $i \in I$ then $\mathbf{A}$ is a distributive poset.

PROOF. Let us denote, for every $j \in I$, by $p_j \colon \prod_{i\in I} \mathbf{A}_i \to \mathbf{A}_j$ the j-th projection defined by $p_j((x_i)_{i\in I}) = x_j$ for every $(x_i)_{i\in I} \in \prod_{i\in I} A_i$.

Recall that, for a non-empty subset $X \subseteq \prod_{i\in I} A_i$, $L_{\mathbf{A}}(X) = \prod_{i\in I} L_{\mathbf{A}_i}\big(p_i(X)\big)$ and $U_{\mathbf{A}}(X) = \prod_{i\in I} U_{\mathbf{A}_i}\big(p_i(X)\big)$.

Let us define a mapping f from the poset $(A^2, \sqsubseteq) = (\prod\limits_{i\in I} A_i, \prod\limits_{i\in I} \leq_i) \times (\prod\limits_{i\in I} A_i, \prod\limits_{i\in I} \geq_i)$ to the poset $\prod\limits_{i\in I} (A_i^2, \sqsubseteq_i) = \prod\limits_{i\in I} \big((A_i, \leq_i) \times (A_i, \geq_i)\big)$ by

$$f\big((x_i)_{i\in I}, (y_i)_{i\in I}\big) := (x_i, y_i)_{i\in I}$$

for all $(x_i, y_i)_{i\in I} \in (\prod\limits_{i\in I} A_i, \prod\limits_{i\in I} \leq_i) \times (\prod\limits_{i\in I} A_i, \prod\limits_{i\in I} \geq_i)$.

The mapping f is clearly an order isomorphism preserving $'$ since the order and involution on $\prod\limits_{i\in I} (A_i^2, \sqsubseteq_i)$ are defined componentwise. Moreover, $f(P_S(\mathbf{A})) = \prod\limits_{i\in I} P_{S_i}(\mathbf{A}_i)$ since, for any $x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \in A$, we have:

$$\begin{aligned}
&(x, y) \in P_S(\mathbf{A}) \text{ if and only if} \\
&\prod_{i\in I} L_{\mathbf{A}_i}((x_i, y_i)_{i\in I}) = L_{\mathbf{A}}(x, y) \leq S \leq U_{\mathbf{A}}(x, y) \\
&= \prod_{i\in I} U_{\mathbf{A}_i}((x_i, y_i)_{i\in I}) \text{ if and only if} \\
&(x_i, y_i) \in P_{S_i}(\mathbf{A}_i) \text{ for each } i \in I \text{ if and only if} \\
&f(x, y) \in \prod_{i\in I} P_{S_i}(\mathbf{A}_i).
\end{aligned}$$

Suppose now that $\mathbf{A}_i$ is a distributive poset for every $i \in I$. Since the distributive law for $\mathbf{A}$ can be checked componentwise, $\mathbf{A}$ is a distributive poset. ■

Lemma 4.2. *Let $\mathbf{C}$ be a bounded chain with involution and $a \in C$ such that $a \leq a'$ and $x \in [a, a']$ implies $x \in \{a, a'\}$. Then $\mathbf{C}$ is a representable Kleene poset and $\mathbf{C} \cong \mathbf{P}_{[a,a']}([a, 1])$.*

Proof. Assume first that $a = a'$. From [5, Lemma 18] we know that $P_a([a, 1])$
$= \big(\{a\} \times [a, 1]\big) \cup \big([a, 1] \times \{a\}\big) \cong \mathbf{C}$. The isomorphism f from $\mathbf{C}$ to $\mathbf{P}_a([a, 1])$ is given by

$$f(x) = \begin{cases} (a, x') & \text{if } x \leq a, \\ (x, a) & \text{if } a < x. \end{cases}$$

Moreover $f(a) = (a, a)$.

Suppose now that $a < a'$. Let us show that $\mathbf{C} \cong \mathbf{P}_{\{a,a'\}}([a, 1])$. We define an isomorphism g from $\mathbf{C}$ to $\mathbf{P}_{\{a,a'\}}([a, 1])$ as follows:

$$g(x) = \begin{cases} (a, x') & \text{if } x \leq a, \\ (x, a) & \text{if } a < x. \end{cases}$$

Moreover $g(a) = (a, a')$ and $g(a') = (a', a)$. ∎

Remark 4.3. Let us denote by $\mathcal{RC}$ the class of bounded chains with involution having elements a as in Lemma 4.2. Clearly, all finite chains are in $\mathcal{RC}$, which was proved already in [5, Corollary 21]. Hence due to Theorem 4.1 and Lemma 4.2 direct products of chains from $\mathcal{RC}$ form a class of representable Kleene lattices.

By Theorem 4.1, a direct product of representable Kleene posets $\mathbf{K}_i$ is again representable, and the set S for this product is just the direct product of the sets S_i for $\mathbf{K}_i$. The natural question arises if a similar result also holds for a subdirect product of representable Kleene lattices. We can show that, in particular cases, this is true. Let us consider the following example.

Example 4.4. Let $\mathbf{K}_1$ be the Kleene lattice depicted in Figure 5 and $\mathbf{K}_2$ the two-element chain considered as a Kleene lattice:

Then $\mathbf{K}_1$ is representable by means of $\mathbf{L}_1$ and S_1 and, similarly, $\mathbf{K}_2$ is representable by means of $\mathbf{L}_2$ and S_2 as shown in Figure 6.

Hence $\mathbf{K}_1 \cong \mathbf{P}_{S_1}(\mathbf{L}_1)$ and $\mathbf{K}_2 \cong \mathbf{P}_{S_2}(\mathbf{L}_2)$.

Consider now the Kleene lattice $\mathbf{K} = \mathbf{K}_1 \times \mathbf{K}_2$. By Theorem 4.1 it is representable by means of $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$ and $S = S_1 \times S_2$, see Figure 7.

Consider now two subdirect products of $\mathbf{K}$ which are Kleene lattices.

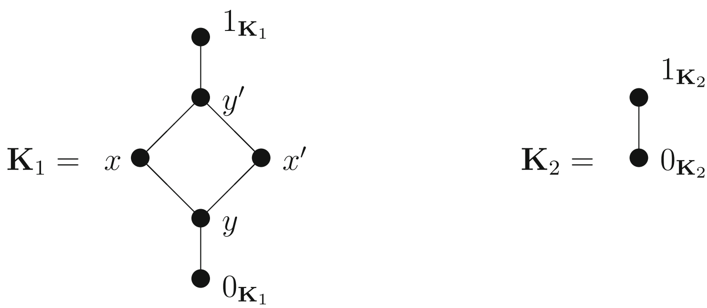
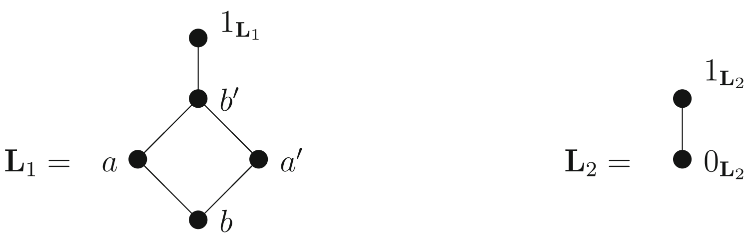


Figure 5. Representable Kleene lattices $\mathbf{K}_1$ and $\mathbf{K}_2$



$$S_1 = \{a, a', b, b'\} \qquad S_2 = \{0_{\mathbf{L}_2}, 1_{\mathbf{L}_2}\}$$

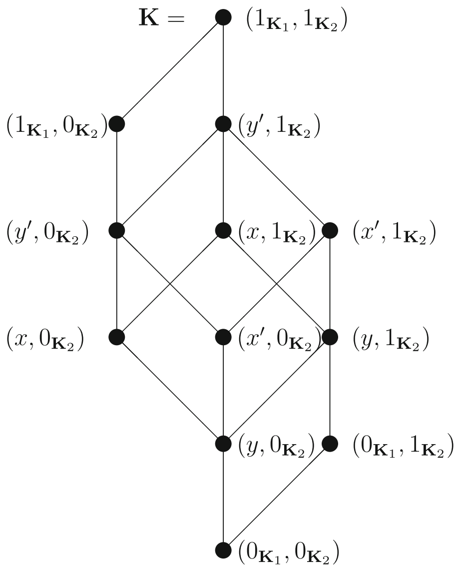
$$\begin{array}{rr} 1_{\mathbf{K}_1} \mapsto (1_{\mathbf{L}_1}, b), & y' \mapsto (b', b) \\ x \mapsto (a, a'), & x' \mapsto (a', a) \\ 0_{\mathbf{K}_1} \mapsto (b, 1_{\mathbf{L}_1}), & y \mapsto (b, b') \end{array} \qquad 1_{\mathbf{K}_2} \mapsto (1_{\mathbf{L}_2}, 0_{\mathbf{L}_2}), \quad 0_{\mathbf{K}_2} \mapsto (0_{\mathbf{L}_2}, 1_{\mathbf{L}_2})$$

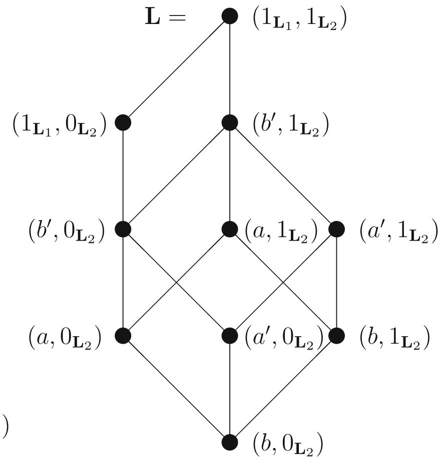
Figure 6. Representations of Kleene lattices $\mathbf{K}_1$ and $\mathbf{K}_2$

(1) We start with $\mathbf{K}^s$, see Figure 8.
Take $L^s := (L_1 \times L_2) \cap K^s$ and $S' := S \cap L^s$, see Figure 9. Then $L^s = (L_1 \times L_2) \setminus \{(1_{\mathbf{L}_1}, 0_{\mathbf{L}_2})\}$ and $S' = S$.
It is elementary to show that $\mathbf{K}^s \cong \mathbf{P}_{S'}(\mathbf{L}^s)$. Thus it is representable.

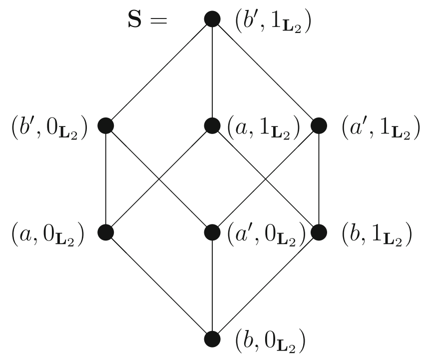
(2) Now, let us consider the subdirect product $\mathbf{K}^0$ and put $L^0 := (L_1 \times L_2) \cap K^0$ and $S^0 := S \cap L^0$ as depicted in Figure 10.
Then $\mathbf{K}^0 \cong \mathbf{P}_{S^0}(\mathbf{L}^0)$, and $\mathbf{K}^0$ is representable.

Note that not every subdirect product of two representable Kleene posets need be representable. On the one hand, if $\mathbf{K} = (K, \leq, ')$ is a Kleene poset that is isomorphic to some $\mathbf{P}_S(\mathbf{A})$, then $\mathbf{A}$ may not be embeddable into $\mathbf{K}$ (see Example 3.4) and thus S may not be considered as a subset of K. On the other hand, every finite distributive lattice is a subdirect product of





$$
\begin{aligned}
(1_{\mathbf{K}_1}, 1_{\mathbf{K}_2}) &\mapsto \big((1_{\mathbf{L}_1}, 1_{\mathbf{L}_2}), (b, 0_{\mathbf{L}_2})\big),\\
(0_{\mathbf{K}_1}, 0_{\mathbf{K}_2}) &\mapsto \big((b, 0_{\mathbf{L}_2}), (1_{\mathbf{L}_1}, 1_{\mathbf{L}_2})\big),\\
(1_{\mathbf{K}_1}, 0_{\mathbf{K}_2}) &\mapsto \big((1_{\mathbf{L}_1}, 0_{\mathbf{L}_2}), (b, 1_{\mathbf{L}_2})\big),\\
(0_{\mathbf{K}_1}, 1_{\mathbf{K}_2}) &\mapsto \big((b, 1_{\mathbf{L}_2}), (1_{\mathbf{L}_1}, 0_{\mathbf{L}_2})\big),\\
(y', 1_{\mathbf{K}_2}) &\mapsto \big((b', 1_{\mathbf{L}_2}), (b, 0_{\mathbf{L}_2})\big),\\
(y, 0_{\mathbf{K}_2}) &\mapsto \big((b, 0_{\mathbf{L}_2}), (b', 1_{\mathbf{L}_2})\big),\\
(y', 0_{\mathbf{K}_2}) &\mapsto \big((b', 0_{\mathbf{L}_2}), (b, 1_{\mathbf{L}_2})\big),\\
(y, 1_{\mathbf{K}_2}) &\mapsto \big((b, 1_{\mathbf{L}_2}), (b', 0_{\mathbf{L}_2})\big),\\
(x, 1_{\mathbf{K}_2}) &\mapsto \big((a, 1_{\mathbf{L}_2}), (a', 0_{\mathbf{L}_2})\big),\\
(x', 0_{\mathbf{K}_2}) &\mapsto \big((a', 0_{\mathbf{L}_2}), (a, 1_{\mathbf{L}_2})\big),\\
(x', 1_{\mathbf{K}_2}) &\mapsto \big((a', 1_{\mathbf{L}_2}), (a, 0_{\mathbf{L}_2})\big),\\
(x, 0_{\mathbf{K}_2}) &\mapsto \big((a, 0_{\mathbf{L}_2}), (a', 1_{\mathbf{L}_2})\big).
\end{aligned}
$$



$$
\begin{aligned}
S = \{&(a, 0_{\mathbf{L}_2}), (a', 0_{\mathbf{L}_2}), (b, 0_{\mathbf{L}_2}), (b', 0_{\mathbf{L}_2}),\\
&(a, 1_{\mathbf{L}_2}), (a', 1_{\mathbf{L}_2}), (b, 1_{\mathbf{L}_2}), (b', 1_{\mathbf{L}_2})\}
\end{aligned}
$$

Figure 7. Representation of a direct product of Kleene lattices $\mathbf{K}_1$ and $\mathbf{K}_2$

finite chains and every finite chain is representable, but non-representable Kleene lattices exist (see Theorem 4.10).

Lemma 4.5. *Let $\mathbf{A} = (A, \leq)$ be a finite poset and S a non-empty subset of A. Then $|P_S(\mathbf{A})|$ is odd if and only if $|S| = 1$.*

Proof. Let $a, b, c \in A$. Then $(b, c) \in P_S(\mathbf{A})$ if and only if $(c, b) \in P_S(\mathbf{A})$. Hence the number of elements $(b, c) \in P_S(\mathbf{A})$ such that $b \neq c$ is even. Assume that $(b, b) \in P_S(\mathbf{A})$. Then, for every $s \in S$, $b \leq s \leq b$. Hence

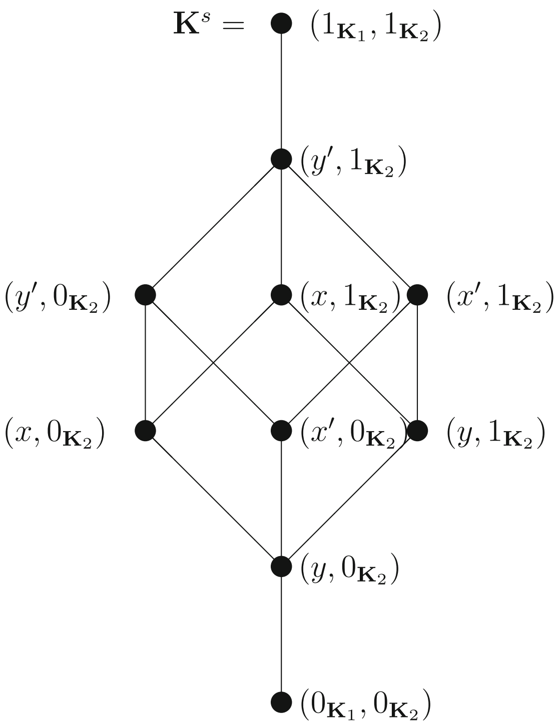


Figure 8. Subdirect product $\mathbf{K}^s$ of Kleene lattices $\mathbf{K}_1$ and $\mathbf{K}_2$

$S = \{b\}$ independently of b, i.e., $|S| = 1$ and (b, b) is unique. Conversely, if $|S| = 1$, say $S = \{a\}$, then $(b, b) \in P_S(\mathbf{A})$ if and only if $b = a$. ■

Lemma 4.6. *Let $\mathbf{A} = (A, \leq)$ be a poset and $a \in A$ and assume $\big(P_a(\mathbf{A}), \sqsubseteq\big)$ to have no three-element antichain. Then a is comparable with every element of A and join- and meet-irreducible.*

Proof. If there would exist some element b of A with $b \parallel a$ then $\{(a, a), (a, b), (b, a)\}$ would be a three-element antichain of $\big(P_a(\mathbf{A}), \sqsubseteq\big)$. If a would not be join-irreducible then there would exist $c, d \in A \setminus \{a\}$ with $c \vee d = a$ and $\{(a, a), (c, d), (d, c)\}$ would be a three-element antichain of $\big(P_a(\mathbf{A}), \sqsubseteq\big)$. If, finally, a would not be meet-irreducible then there would exist $e, f \in A \setminus \{a\}$ with $e \wedge f = a$ and $\{(a, a), (e, f), (f, e)\}$ would be a three-element antichain of $\big(P_a(\mathbf{A}), \sqsubseteq\big)$. ■

The following two lemmas will be helpful to determine a large class of representable, respectively non-representable, Kleene posets.

Lemma 4.7. *Let $\mathbf{K} = (K, \leq, ')$ be a pseudo-Kleene poset, $\mathbf{A} = (A, \leq)$ a poset and S a non-empty subset of A. Then*

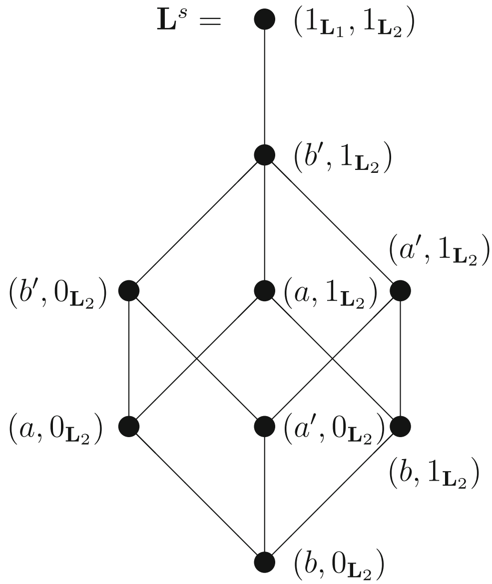


Figure 9. Representation of the subdirect product $\mathbf{K}^s$ of Kleene lattices $\mathbf{K}_1$ and $\mathbf{K}_2$

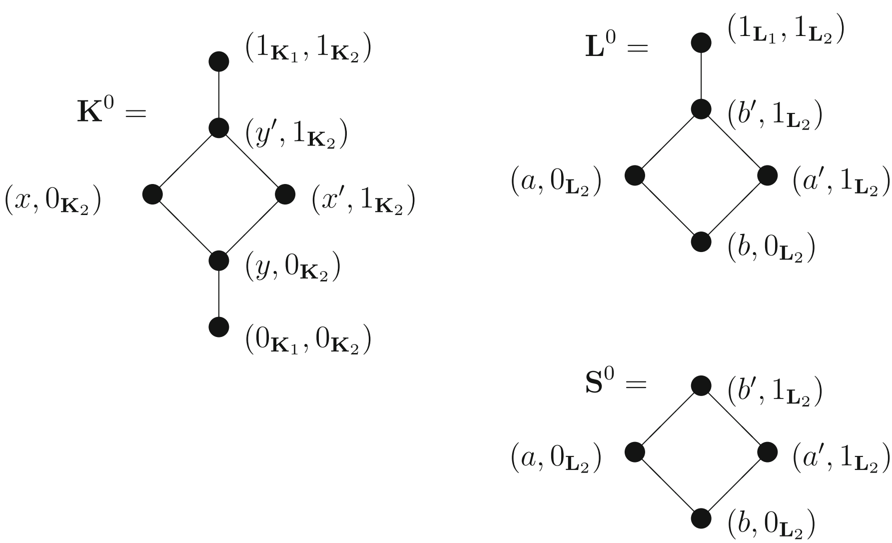


Figure 10. Representation of a subdirect product $\mathbf{K}^0$ of Kleene lattices

(i) *The antitone involution* $'$ *on* K *has at most one fixed point,*

(ii) *If* $\mathbf{K}$ *is finite then the antitone involution* $'$ *on* K *has a fixed point if and only if* $|K|$ *is odd,*

(iii) *The antitone involution* $'$ *on* $P_S(\mathbf{A})$ *has a fixed point if and only if* $|S| = 1$.

Proof. Let $a, b \in K$ and $c, d, e \in A$.

(i) If $a' = a$ and $b' = b$ then $L(a) = L(a, a') \leq U(b, b') = U(b)$ and $L(b) = L(b, b') \leq U(a, a') = U(a)$ and hence $a = b$.

(ii) The set

$$\bigcup_{x \in K} \{x, x'\}^2$$

is an equivalence relation on K having only two-element classes if $'$ on K has no fixed point and having precisely one one-element class otherwise.

(iii) We have $(c, c) \in P_c(\mathbf{A})$ and $(c, c)' = (c, c)$ in $P_c(\mathbf{A})$. Now assume $|S| > 1$. If (d, e) would be a fixed point of $'$ in $P_S(\mathbf{A})$ then $d = e$ and we would have $L(d) = L(d, d) \leq S \leq U(d, d) = U(d)$ and hence $d \leq s \leq d$, i.e., $d = s$ for all $s \in S$ contradicting $|S| > 1$. Hence $'$ has no fixed point in $P_S(\mathbf{A})$.

■

In the following lemma, we derive an upper bound for $|A|$ provided $P_a(\mathbf{A})$ is finite. This result will be used in the following theorem describing representable Kleene posets of odd cardinality.

Lemma 4.8. *Let* $\mathbf{A} = (A, \leq)$ *be a poset and* $a \in A$ *and assume* $P_a(\mathbf{A})$ *to be finite. Then* A *is finite and* $|A| < |P_a(\mathbf{A})|/2 + 1$.

Proof. Since $P_a(\mathbf{A}) \supseteq (\{a\} \times A) \cup (A \times \{a\})$ we have that A is finite and $|P_a(\mathbf{A})| \geq 2|A| - 1$ whence $|A| \leq (|P_a(\mathbf{A})| + 1)/2 < |P_a(\mathbf{A})|/2 + 1$. ■

Theorem 4.9. *Let* $\mathbf{K} = (K, \leq, ')$ *be a finite representable Kleene poset with an odd number of elements. Then* $'$ *has exactly one fixed point* a*, and there exists some subposet* $\mathbf{A}$ *of* $(K, \leq)$ *of cardinality less than* $|K|/2+1$ *containing* a *such that* $\mathbf{P}_a(\mathbf{A}) \cong \mathbf{K}$.

Proof. Since $\mathbf{K}$ is representable, there exists some poset $\mathbf{A}^* = (A^*, \leq)$ and some non-empty subset S of A^* such that $\mathbf{P}_S(\mathbf{A}^*) \cong \mathbf{K}$. According to Lemma 4.7, $'$ has a fixed point in K and hence $|S| = 1$, say $S = \{b\}$, again because of Lemma 4.7. According to Lemma 4.8, A^* is finite, and $|A^*| <$

$|K|/2+1$. Let f denote an isomorphism from $P_b(\mathbf{A}^*)$ to $\mathbf{K}$. Obviously, (b,b) is the unique fixed point of $'$ in $P_b(\mathbf{A}^*)$, and hence $f(b,b)=a$. Let g denote the embedding $x\mapsto(x,b)$ of $\mathbf{A}^*$ into $(P_b(\mathbf{A}^*),\sqsubseteq)$. Then $f\circ g$ is an embedding of $\mathbf{A}^*$ into $(K,\leq)$ mapping b onto a. Hence, if $A:=f\big(g(A^*)\big)$ then $\mathbf{A}:=(A,\leq)$ is a subposet of $(K,\leq)$ isomorphic to $\mathbf{A}^*$ and $\mathbf{P}_a(\mathbf{A})\cong\mathbf{P}_b(\mathbf{A}^*)\cong\mathbf{K}$. ■

The following theorem shows a class of non-representable Kleene lattices.

Theorem 4.10. *Let $\mathbf{C}$ be a finite chain containing more than one element, X a set such that $|X|\geq 2$, and $\mathbf{B}=\mathcal{P}(X)$ the powerset Boolean algebra. Let $\mathbf{K}=(K,\vee,\wedge,')$ denote the Kleene lattice $\mathbf{C}+_b\mathbf{B}+_c\mathbf{B}+_d\mathbf{C}$. Then $\mathbf{K}$ is not representable.*

Proof. Using the method of indirect proof, let us suppose $\mathbf{K}$ to be representable. By Lemma 4.7 there exists some poset $\mathbf{A}=(A,\leq)$ containing the element $e\in A$ such that $\mathbf{P}_e(\mathbf{A})\cong\mathbf{K}$ and $(e,e)\mapsto c$.

Assume first that $e=1_A$. Then $\mathbf{A}\cong{\downarrow}c=\mathbf{C}+_b\mathbf{B}$ since every second coordinate of $\Downarrow(1_A,1_A)$ is 1_A. Similarly, $\mathbf{A}^d\cong{\uparrow}c=\mathbf{B}+_d\mathbf{C}$. Therefore $\mathbf{P}_e(\mathbf{A})=(A\times\{1_A\})\cup(\{1_A\}\times A)$. Clearly, there are elements $u,v\in A\setminus\{1_A\}$ such that $u\vee v=1_A$ and $u\wedge v\leq 1_A$ (e.g. the coatoms of $\mathbf{A}$). Hence $(u,v)\in P_e(\mathbf{A})$, a contradiction. We conclude that $e\neq 1_A$. By a dual argument we obtain that $e\neq 0_A$.

Hence there are elements $x,y\in A$ such that $e>x$ and $y>e$. This yields $(e,e)\sqsupset(x,e)$ and $(e,e)\sqsupset(e,y)$. We conclude (since $\Downarrow(e,e)$ is dually atomic) that there are maximal elements (u,p) and (q,v) of $\Downarrow(e,e)$ such that $(e,e)\sqsupset(u,p)\sqsupseteq(x,e)$ and $(e,e)\sqsupset(q,v)\sqsupseteq(e,y)$. We obtain $p=e$ and $q=e$. Therefore (e,e) covers (u,e) and (e,v) in $P_e(\mathbf{A})$, and e covers u and v covers e in A.

Suppose that there are $p,q\in A$ such that $p\wedge q\leq e\leq p\vee q$, (e,e) covers (p,q) and $(p,q)\neq(u,e)$, $(p,q)\neq(e,v)$. We conclude that $p\leq e\leq q$. Assume first that $p<e<q$. Then $(p,q)\sqsubset(p,e)\sqsubset(e,e)$, a contradiction. Therefore either $p=e$ or $q=e$.

So the set M of maximal elements of the principal ideal $\Downarrow(e,e)$ is of the form $M=\big(U\times\{e\}\big)\cup\big(\{e\}\times V\big)$, $u\in U$ and $v\in V$.

Since U corresponds to a subset $U\times\{e\}$ of a complete lattice $P_e(\mathbf{A})$ its meet in $\mathbf{A}$ exists (in fact $\bigsqcap\big(U\times\{e\}\big)=(\overline{u},e)$ for some $\overline{u}\in A$). Similarly, the join of V in $\mathbf{A}$ exists since V corresponds to a subset $\{e\}\times V$ ($\bigsqcap\big(\{e\}\times V\big)=(e,\overline{v})$ for some $\overline{v}\in A$).

Hence $\overline{u}=\bigwedge U$ and $\overline{v}=\bigvee V$. Evidently, $b=(\overline{u},e)\sqcap(e,\overline{v})=(\overline{u},\overline{v})\in P_e(\mathbf{A})$ and $d=(\overline{v},\overline{u})$.

Since $A\times\{e\}\subseteq\mathbf{P}_e(\mathbf{A})$ we conclude that either $\overline{u}=0_{\mathbf{A}}$ or there is $w\in A$ such that $(\overline{u},e)$ covers $(w,e)\in P_e(\mathbf{A})$.

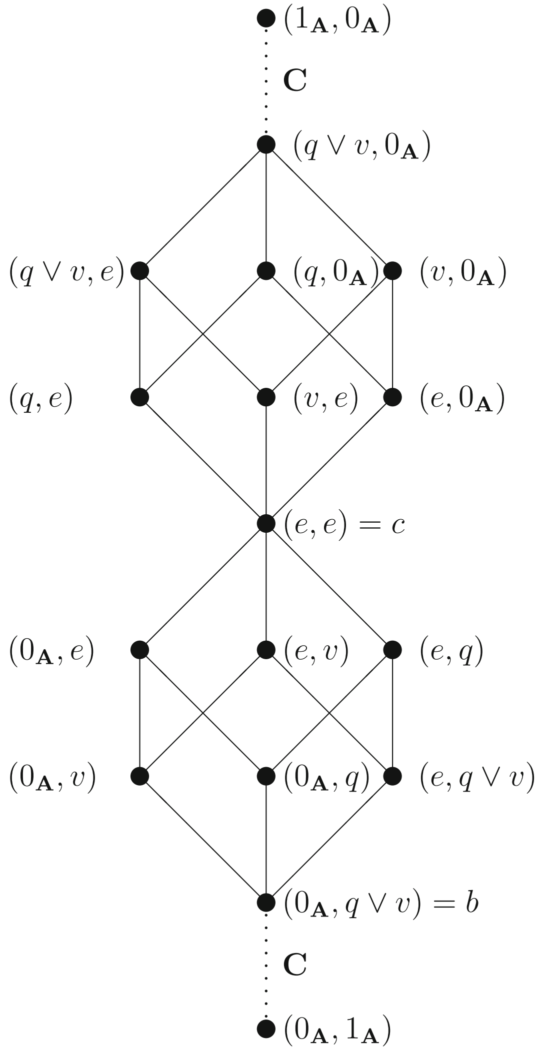


Figure 11. Ordinal sum of chains and Boolean algebras

Case $\overline{u} = 0_A$: We have that $b = (0_{\mathbf{A}}, \overline{v})$ (see Figure 11 for illustration if $|X| = 3$) and $d = (\overline{v}, 0_{\mathbf{A}})$. Moreover, the element $(1_{\mathbf{A}}, e) \sqsupseteq (\overline{u}, e)$ is contained in the interval $[c, d]$ since all elements above d are of the form $(z, 0_{\mathbf{A}}), z \geq e$. We conclude that $1_{\mathbf{A}} \leq \overline{v} \leq 1_{\mathbf{A}}$. But $(0_{\mathbf{A}}, \overline{v}) \neq (0_{\mathbf{A}}, 1_{\mathbf{A}})$ because $\mathbf{C}$ is a finite chain containing more than one element, a contradiction.

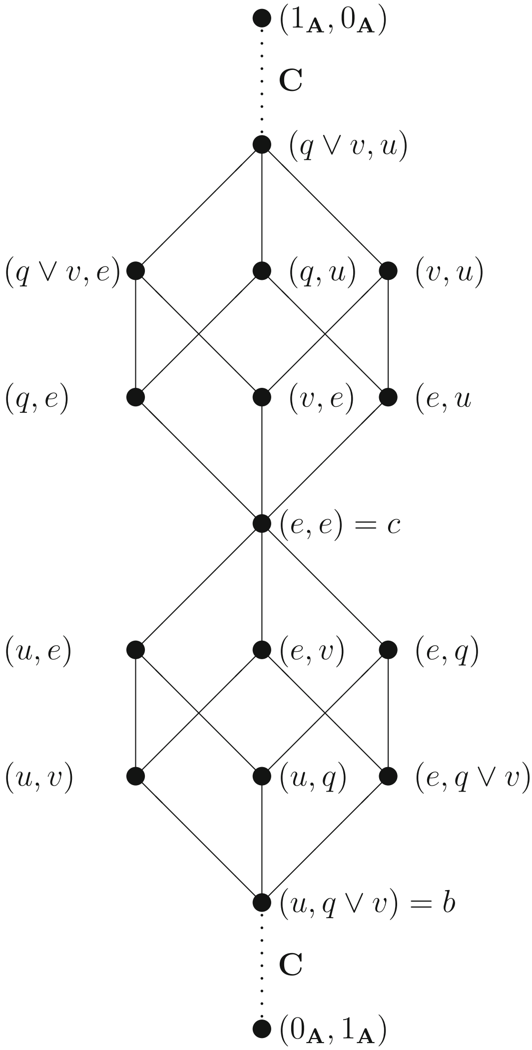


Figure 12. Ordinal sum of chains and Boolean algebras

Case $\overline{u} \neq 0_\mathbf{A}$: Then there is an element $w \in A$ such that $\overline{u}$ covers w. Hence also $(\overline{u}, e)$ covers (w, e). Since $(\overline{u}, e) > b = (\overline{u}, \overline{v})$ we have $(w, e) \geq b$. Therefore $w = \overline{u}$, a contradiction again (see Figure 12 for illustration if $|X| = 3$). ■

5. Dedekind-MacNeille Completion of Kleene Posets

Recall that the *Dedekind-MacNeille completion* $\mathbf{DM}(\mathbf{A})$ of a poset $\mathbf{A} = (A, \leq)$ is the complete lattice $\big(\mathrm{DM}(\mathbf{A}), \subseteq\big)$ where

$$\mathrm{DM}(\mathbf{A}) := \{L(B) \mid B \subseteq A\} = \{C \subseteq A \mid LU(C) = C\}.$$

For Kleene posets, we can show the following:

Example 5.1. Consider the Kleene poset $\mathbf{A}$ depicted in Figure 13 such that $a' = d$ and $b' = c$:

Put $S := \{a, b\}$. Then the poset $\big(P_S(\mathbf{A}), \sqsubseteq\big)$ is visualized in Figure 14.

The Dedekind-MacNeille completion $\mathbf{DM}\big(P_S(\mathbf{A}), \sqsubseteq\big)$ of this poset is depicted in Figure 15, where

$$\begin{aligned}
L\big((0,1)\big) &= \{(0,1)\},\\
L\big((0,c)\big) &= \{(0,1),(0,c)\},\\
L\big((0,d)\big) &= \{(0,1),(0,d)\},\\
L\big((a,b),(b,a)\big) &= \{(0,1),(0,c),(0,d)\},\\
L\big((a,b)\big) &= \{(0,1),(0,c),(0,d),(a,b)\},\\
L\big((b,a)\big) &= \{(0,1),(0,c),(0,d),(b,a)\},\\
L\big((d,0),(c,0)\big) &= \{(0,1),(0,c),(0,d),(a,b),(b,a)\},\\
L\big((d,0)\big) &= \{(0,1),(0,c),(0,d),(a,b),(b,a),(d,0)\},
\end{aligned}$$

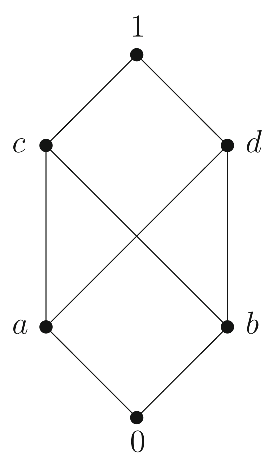


Figure 13. Kleene poset $\mathbf{A}$

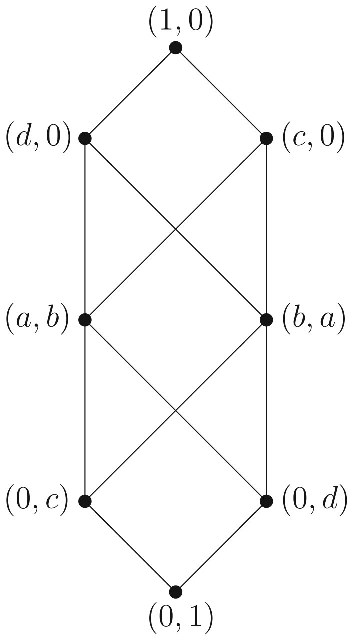


Figure 14. Kleene poset induced by **A**

$$L\big((c,0)\big) = \{(0,1),(0,c),(0,d),(a,b),(b,a),(c,0)\},$$
$$L\big((1,0)\big) = \{(0,1),(0,c),(0,d),(a,b),(b,a),(d,0),(c,0),(1,0)\}.$$

The Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{A})$ of the given poset $\mathbf{A}$ is visualized in Figure 16; here the involution is given by $L(a)' = L(d)$, $L(b)' = L(c)$: Finally, the lattice $\Big(P_{\{L(s)\mid s\in S\}}\big(\mathbf{DM}(\mathbf{A})\big), \sqcup, \sqcap\Big)$ is depicted in Figure 17, where

$$\begin{aligned}
\big(L(0), L(1)\big) &= (\{0\}, \{0,a,b,c,d,1\}),\\
\big(L(0), L(c)\big) &= (\{0\}, \{0,a,b,c\}),\\
\big(L(0), L(d)\big) &= (\{0\}, \{0,a,b,d\}),\\
\big(L(0), L(c,d)\big) &= (\{0\}, \{0,a,b\}),\\
\big(L(a), L(b)\big) &= (\{0,a\}, \{0,b\}),\\
\big(L(b), L(a)\big) &= (\{0,b\}, \{0,a\}),\\
\big(L(c,d), L(0)\big) &= (\{0,a,b\}, \{0\}),\\
\big(L(d), L(0)\big) &= (\{0,a,b,d\}, \{0\}),
\end{aligned}$$

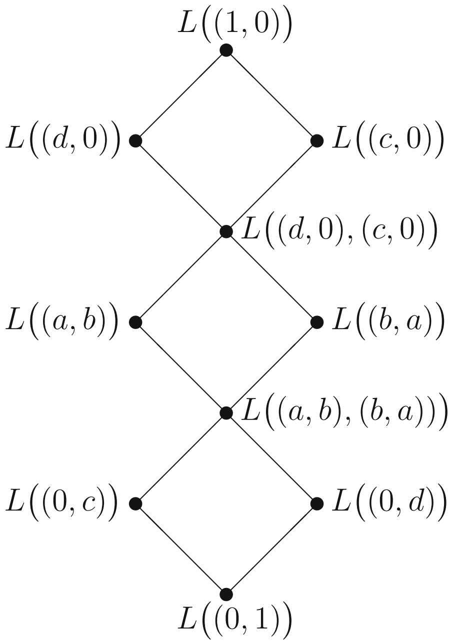


Figure 15. Dedekind-MacNeille completion of the Kleene poset induced by **A**

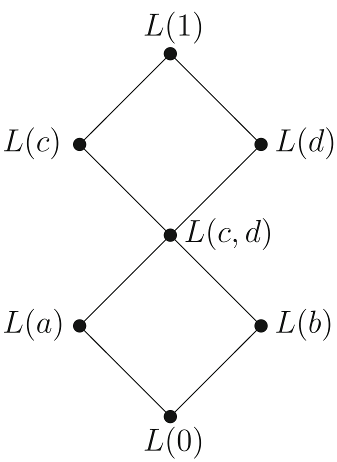


Figure 16. Dedekind-MacNeille completion of the Kleene poset **A**

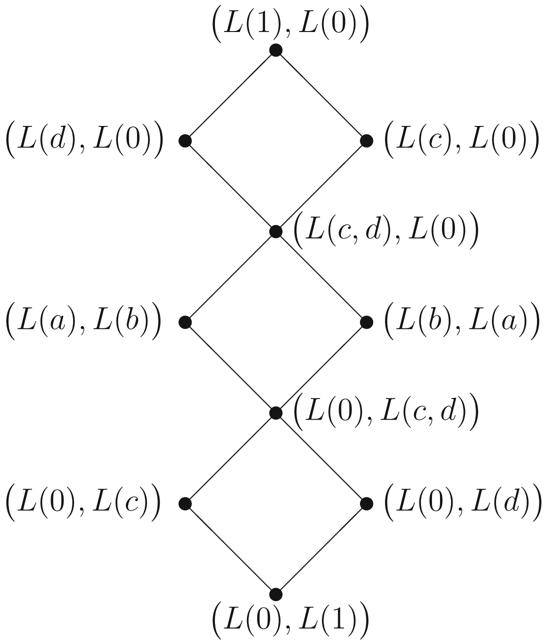


Figure 17. Kleene poset induced by the Dedekind-Mac Neille completion of **A**

$$\big(L(c), L(0)\big) = (\{0, a, b, c\}, \{0\}),$$
$$\big(L(1), L(0)\big) = (\{0, a, b, c, d, 1\}, \{0\}).$$

Hence, in this case, the lattices $\Big(P_{\{L(s)\mid s\in S\}}\big(\mathbf{DM}(\mathbf{A})\big), \sqcup, \sqcap\Big)$ and $\mathbf{DM}\,\big(P_S(\mathbf{A}), \sqsubseteq\,\big)$ are isomorphic.

The lattices mentioned above need not be isomorphic for distributive posets **A**, which are not Kleene posets.

Example 5.2. Consider the distributive poset **A** which is not a Kleene poset depicted in Figure 18:

Put $S := \{c, d\}$. Then the poset $\big(P_S(\mathbf{A}), \sqsubseteq\,\big)$ is visualized in Figure 19.

The Dedekind-MacNeille completion $\mathbf{DM}\,\big(P_S(\mathbf{A}), \sqsubseteq\,\big)$ of this poset is depicted in Figure 20 and the Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{A})$ of the given poset **A** is visualized in Figure 21:

Finally, the lattice $\Big(P_{\{L(s)\mid s\in S\}}\big(\mathbf{DM}(\mathbf{A})\big), \sqcup, \sqcap\Big)$ is depicted in Figure 22.

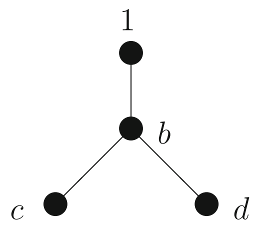


Figure 18. Distributive poset **A**

Figure 19. Kleene poset induced by the distributive poset **A**

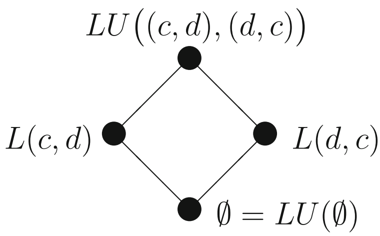


Figure 20. Dedekind-MacNeille completion of the Kleene poset induced by the distributive poset **A**

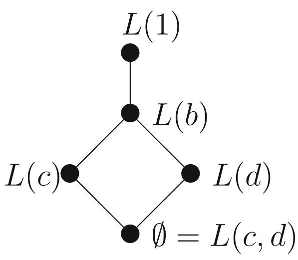


Figure 21. Dedekind-MacNeille completion of the distributive poset **A**

Hence, in this case, the lattices $\big(P_{\{L(s) \mid s \in S\}}\big(\mathbf{DM}(\mathbf{A})\big), \sqcup, \sqcap\big)$ and $\mathbf{DM}\big(P_S(\mathbf{A}), \sqsubseteq\big)$ are not isomorphic.

Let $\mathbf{A} = (A, \leq)$ be a distributive poset and let $Fin(A)$ denote the set of all finite subsets of A. We put (see [8])

$$G(\mathbf{A}) := \{L\big(U(A_1), \dots, U(A_n)\big) \mid n \in \mathbb{N}_+ \& \forall i, 1 \leq i \leq n, \emptyset \neq A_i \in Fin(\mathbf{A})\}.$$

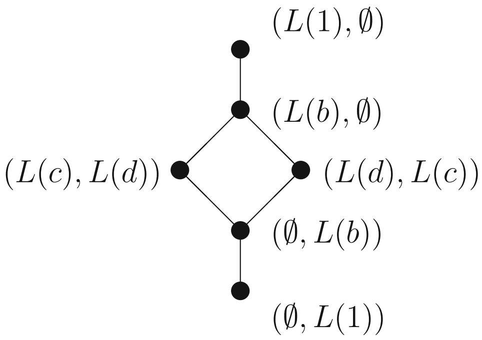


Figure 22. Kleene poset induced by the Dedekind-Mac Neille completion of the distributive poset $\mathbf{A}$

Note that $G(\mathbf{A})$ is a subset of $\mathbf{DM}(\mathbf{A})$, containing all principal ideals. It is worth noticing that $G(\mathbf{A}) = \mathbf{DM}(\mathbf{A})$ provided A is finite. Moreover, from ([8, Proposition 31]) we immediately obtain that any element of $G(\mathbf{A})$ is also of the form $LU\big(L(B_1), \ldots, L(B_n)\big)$ where B_i are finite non-empty subsets of A.

The following definition and theorem are motivated by a similar result of Niederle for Boolean posets ([8, Theorem 17]).

Definition 5.3.

(1) Let $\mathbf{A} = (A, \leq)$ be a poset. A subset X of A is called *doubly dense in* $\mathbf{A}$ if $a = \bigvee_{\mathbf{A}} \big(L(a) \cap X\big) = \bigwedge_{\mathbf{A}} \big(U(a) \cap X\big)$ for all $a \in A$.

(2) Let $\mathbf{A} = (A, \leq, ')$ be a poset with an antitone involution $'$. A subset X of A is called *involution-closed and doubly dense in* $\mathbf{A}$ if $X' \subseteq X$ and X is doubly dense in $\mathbf{A}$.

We will need the following

Proposition 5.4. *([8], Proposition 33) Let* $\mathbf{A} = (A, \leq)$ *be a distributive poset. Then* $\big(G(\mathbf{A}), \subseteq\big)$ *is a distributive lattice and* $X = \{L(a) \mid a \in A\}$ *is doubly dense in* $G(\mathbf{A})$*, generates* $G(\mathbf{A})$ *and* $(X, \subseteq)$ *is isomorphic to* $\mathbf{A}$*.*

In what follows, if $\mathbf{A} = (A, \leq, ')$ is a poset with an antitone involution $'$ and $X \subseteq A$, we define:

- $X' := \{x' \in A \mid x \in X\}$,
- $X^{\perp} := \{a \in A \mid a \leq x' \text{ for all } x \in X\} = L(X')$.

Remark 5.5. Recall that any involution-closed and doubly dense subset X in $\mathbf{A}$ is a poset with induced order and involution. Moreover, if $\mathbf{A} = (A, \leq, ')$

is a poset with an antitone involution $'$ then A is an involution-closed and doubly dense subset in its Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{A})$ with involution $^{\perp}$. This can be shown by the same arguments as in ([8, Theorem 16]), so we omit it.

By the preceding remark, Proposition 5.4 and [8, Theorem 34] we have the following

Corollary 5.6. *Let $\mathbf{A} = (A, \leq, ')$ be a distributive poset with an antitone involution $'$. Then $\big(G(\mathbf{A}), \subseteq, {}^{\perp}\big)$ is a distributive lattice with an antitone involution $^{\perp}$ and $X = \{L(a) \mid a \in A\}$ is involution-closed and doubly dense in $G(\mathbf{A})$, generates $G(\mathbf{A})$ and $(X, \subseteq, {}^{\perp})$ is isomorphic to $\mathbf{A}$.*

Corollary 5.7. *Embedding theorem for distributive posets with an antitone involution. The following conditions are equivalent for a poset $\mathbf{A}$:*

(i) *$\mathbf{A}$ is a distributive poset with an antitone involution;*

(ii) *$\mathbf{A}$ is an involution-closed and doubly dense subset of a distributive lattice with an antitone involution.*

But we can prove more.

Proposition 5.8. *Let $\mathbf{A} = (A, \leq, ')$ be a Kleene poset. Then $(G\big(\mathbf{A}), \subseteq, {}^{\perp}\big)$ is a Kleene lattice and $X = \{L(a) \mid a \in A\}$ is involution-closed and doubly dense in $G(\mathbf{A})$, generates $G(\mathbf{A})$ and $(X, \subseteq, {}^{\perp})$ is isomorphic to $\mathbf{A}$.*

Proof. It is enough to check that for all $C, D \in G(\mathbf{A})$, we have

$$C \cap C^{\perp} \subseteq L\big(U(D \cup D^{\perp})\big).$$

Assume first that $C = LU(E)$ and $D = LU(F)$ where E, F are non-empty finite subsets of A. Then $C = \bigvee\{L(e) \mid e \in E\}$ and $D = \bigvee\{L(f) \mid f \in F\}$. We compute:

$$\begin{aligned}
&\big(\bigvee\{L(e) \mid e \in E\}\big) \wedge \big(\bigwedge\{L(g)^{\perp} \mid g \in E\}\big) = \bigvee_{e\in E} \Big(L(e) \wedge \big(\bigwedge\{L(g)^{\perp} \mid g \in E\}\big)\Big)\\
&\quad \leq \bigvee_{e\in E} \big(L(e) \wedge L(e)^{\perp}\big) = \bigvee_{e\in E} \big(L(e) \wedge L(e')\big) = \bigvee_{e\in E} L(e, e') \leq \bigwedge_{h\in F} LU(h, h')\\
&\quad = \bigwedge_{h\in F} \big(L(h) \vee L(h')\big) \leq \bigwedge_{h\in F} \Big(\big(\bigvee\{L(f) \mid f \in F\}\big) \vee L(h')\Big)\\
&\quad = \big(\bigvee\{L(f) \mid f \in F\}\big) \wedge \big(\bigwedge\{L(h)^{\perp} \mid h \in F\}\big).
\end{aligned}$$

Now, assume that $C = \bigwedge_{i=1}^{n} C_i$, $D = \bigwedge_{j=1}^{m} D_i$ where $C_i = LU(E_i)$, $D_j = LU(F_j)$, E_i and F_j are non-empty finite subsets of A, $1 \leq i \leq n$ and $1 \leq j \leq m$. We compute:

$$\begin{aligned}(\textstyle\bigwedge_{i=1}^{n} C_i) \wedge (\bigvee_{k=1}^{n} C_k^{\perp}) &= \textstyle\bigvee_{k=1}^{n} \left(C_k^{\perp} \wedge (\bigwedge_{i=1}^{n} C_i)\right) \leq \bigvee_{k=1}^{n}(C_k^{\perp} \wedge C_k) \\ &\leq \textstyle\bigwedge_{l=1}^{m}(D_l^{\perp} \vee D_l) \\ &\leq \textstyle\bigwedge_{l=1}^{m} \left((\bigvee_{j=1}^{m} D_j^{\perp}) \vee D_l\right) = (\bigvee_{j=1}^{m} D_j^{\perp}) \vee (\bigwedge_{l=1}^{m} D_l).\end{aligned}$$

■

THEOREM 5.9. **Embedding theorem for Kleene posets.** *The following conditions are equivalent for a poset* $\mathbf{A}$*:*

(i) $\mathbf{A}$ *is a Kleene poset;*

(ii) $\mathbf{A}$ *is an involution-closed and doubly dense subset of a Kleene lattice.*

PROOF. (i) ⇒ (ii) has been proved in Proposition 5.8.

(ii) ⇒ (i): From Corollary 5.7, we know that $\mathbf{A}$ is a distributive poset with an antitone involution $'$. But the involution reflects the Kleene condition. Namely, let $x, y \in A$. Assume that $a \in L(x, x')$ and $b \in U(y, y')$. Then $a \leq x \wedge x' \leq y \vee y' \leq b$ in the Kleene lattice. Hence $a \leq b$ in $\mathbf{A}$, i.e., $\mathbf{A}$ is a Kleene poset. ■

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Representability of Kleene Posets and Kleene Lattices

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