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Finite Hilbert Systems for Weak Kleene Logics

Abstract. Multiple-conclusion Hilbert-style systems allow us to finitely axiomatize every logic defined by a finite matrix. Having obtained such axiomatizations for Paraconsistent Weak Kleene and Bochvar–Kleene logics, we modify them by replacing the multiple-conclusion rules with carefully selected single-conclusion ones. In this way we manage to introduce the first *finite* Hilbert-style single-conclusion axiomatizations for these logics.

Keywords: Hilbert-style systems, Bochvar–Kleene, Paraconsistent Weak Kleene, Containment logics, Multiple-conclusion logics.

1. Introduction

In his classic book [25], S.C. Kleene employs two different sets of three-valued truth tables to introduce the logical systems known, in today's parlance, as *Strong Kleene* and *Weak Kleene* logics. The latter, independently considered in 1937 by Bochvar [6,7], is also called *Bochvar–Kleene logic* (henceforth **BK**).

From a formal point of view, the main difference between the strong and the weak Kleene tables is that in the latter the third truth value ($\mathtt{u}$) exhibits an *infectious* behaviour: any interaction between $\mathtt{u}$ and either of the classical values ($\mathtt{t}$ and $\mathtt{f}$) delivers $\mathtt{u}$ itself. This feature, as we shall see, makes the resulting logics somewhat less tractable than most well-known three-valued logics, both from an algebraic and a proof-theoretic point of view.

From the Bochvar–Kleene tables two logics naturally arise. One (**BK**) is obtained by choosing the single truth value $\mathtt{t}$ as designated; the other, which we call *Paraconsistent Weak Kleene* (**PWK**), results from designating both $\mathtt{t}$ and $\mathtt{u}$. Concerning both these systems, a positive and a negative result are particularly worth mentioning in the present context.

The good news is that both logics are closely related, from a formal point of view, to the classical: more precisely, **PWK** and **BK** are, respectively,

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the *left* and *right variable inclusion companions* of classical logic (more on this below). The bad news, on the other hand, is that for neither of these logics a *finite* Hilbert-style axiomatization currently exists (this observation was made in [22], to which we also refer the reader for further background and examples of axiomatizations of three-valued logics). In other words, we do not know whether these logics admit a *finite basis* [35]. This is precisely the gap we wish to bridge in the present paper, thus solving a fundamental open problem concerning these logics.

According to Bochvar's original paper [6,7], the intended applications of the Bochvar–Kleene logic are in the formalization of paradoxes, future contingent statements and presuppositions (see e.g. [18] for a more recent computational interpretation of **BK**). The third value is therefore meant to represent nonsensical statements, or corrupted data in the interpretation given by Kleene. This explains the infectious behaviour of the third value, because any complex formula having a nonsensical or paradoxical subformula should be regarded as nonsensical/paradoxical too.

Paraconsistent Weak Kleene (**PWK**) seems to have been considered already by S. Halldén in his 1949 monograph [24], and two decades later by A. Prior [32], but has only recently been studied in more depth (see e.g. [11] and [17], the latter of which explores applications to the theory of truth).

The proof theory of **BK** and **PWK** has been intensively developed in the last years employing different formalisms and approaches, like sequent calculi [9,30], natural deduction [4,31] and tableaux [9,30].

In addition, a number of Hilbert-like systems for these logics exist in the literature [2,8,12]. However, as explained in Section 3, none of them are finite Hilbert-style systems in the usual sense (we shall call these Set-Fmla *H-systems*).

For **BK**, a finite but non-standard axiomatization may be obtained by taking any complete Set-Fmla H-system for classical logic (with *modus ponens* as its only rule) and, while keeping all the axioms, replacing *modus ponens* by a restricted version that satisfies the *containment condition* [11, Prop. 4], [12]. The finite Hilbert-style system for **BK** we introduce here will instead be standard, i.e. consisting of a finite number of axioms and unrestricted rule schemas.

For both **BK** and **PWK**, *infinite* Hilbert-style systems may be found in [8,12]; we note that the completeness proofs found in these papers are essentially algebraic, and rely on the above-mentioned observation that **BK** and **PWK** are, respectively, the right and the left variable inclusion companion of classical logic [34, Thm. 4, p. 258], [12,15].

In the present paper we follow a two-step strategy. Relying on the general observation that every finite logical matrix can be finitely axiomatized by means of a Hibert-style multiple-conclusion system (here called a SET–SET *H-system*), we first introduce finite SET–SET H-systems for **BK** and **PWK**, then show how from these SET-FMLA axiomatizations may be obtained preserving finiteness.

The paper is organized as follows. In Section 2 we formally introduce the language and semantics of **BK** and **PWK**. Section 3 contains as much theory of SET–SET and SET-FMLA H-systems as we shall need in order to introduce our axiomatic systems for **PWK** and **BK**. The former is then presented and shown to be complete in Section 4 (**PWK**), the latter in Section 5 (**BK**). The final Section 6 contains concluding remarks and suggestions for future research.

2. Language and Semantics of BK and PWK

Let $\wedge$, $\vee$ and $\to$ be binary connectives and $\neg$ be a unary connective. Call a collection Σ of these connectives a *propositional signature*. We may write $\Sigma_{©_1 \cdots ©_n}$ for the signature $\{©_1, \ldots, ©_n\} \subseteq \{\wedge, \vee, \to, \neg\}$.

A Σ-*algebra* is a structure $\mathbf{A} := \langle A, \cdot_{\mathbf{A}} \rangle$ such that A is a nonempty set called the *carrier* of $\mathbf{A}$ and, for each k-ary connective $© \in \Sigma$, the k-ary mapping $©_{\mathbf{A}} : A^k \to A$ is the *interpretation* (or *truth table*) *of* $©$ *in* $\mathbf{A}$.

Given a denumerable set P of *propositional variables*, we denote by $\mathbf{L}_\Sigma(P)$ the term algebra over Σ generated by P or, more briefly, the Σ-*language (generated by* P*)*, whose universe is denoted by $L_\Sigma(P)$. The elements of the latter are called Σ-*formulas*. Propositional variables will be denoted by lowercase letters p, q, r, s, and Σ-formulas will be denoted by Greek letters $\varphi, \psi, \gamma, \delta$, possibly subscripted with positive integers.

The endomorphisms on $\mathbf{L}_\Sigma(P)$ are called Σ-*substitutions*. By $\mathsf{subf}(\Phi)$ we denote the set of all subformulas of the formulas in $\Phi \subseteq L_\Sigma(P)$. Moreover, we will usually write Φ, Ψ to denote $\Phi \cup \Psi$ and we will omit curly braces when writing sets of formulas. Also, we write Φ^{c} for $L_\Sigma(P) \backslash \Phi$.

We take $\Sigma_{\wedge\vee\neg}$ to be the signature of classical logic as well as that of **PWK** and **BK** in the present work. We are going to define these logics in a moment via matrix semantics.

Let $\mathbf{B} := \langle \{\mathtt{f}, \mathtt{t}\}, \cdot_{\mathbf{B}} \rangle$ be the standard two-element Boolean $\Sigma_{\wedge\vee\neg}$-algebra. For $B_{\mathtt{u}} := \{\mathtt{f}, \mathtt{u}, \mathtt{t}\}$, define the $\Sigma_{\wedge\vee\neg}$-algebra $\mathbf{B}_{\mathtt{u}} := \langle B_{\mathtt{u}}, \cdot_{\mathbf{B}_{\mathtt{u}}} \rangle$ such that the connectives in $\Sigma_{\wedge\vee\neg}$ are interpreted according to the following truth tables:

$\wedge_{\mathbf{B}_{\mathtt{u}}}$	$\mathtt{f}$	$\mathtt{u}$	$\mathtt{t}$
$\mathtt{f}$	$\mathtt{f}$	$\mathtt{u}$	$\mathtt{f}$
$\mathtt{u}$	$\mathtt{u}$	$\mathtt{u}$	$\mathtt{u}$
$\mathtt{t}$	$\mathtt{f}$	$\mathtt{u}$	$\mathtt{t}$

$\vee_{\mathbf{B}_{\mathtt{u}}}$	$\mathtt{f}$	$\mathtt{u}$	$\mathtt{t}$
$\mathtt{f}$	$\mathtt{f}$	$\mathtt{u}$	$\mathtt{t}$
$\mathtt{u}$	$\mathtt{u}$	$\mathtt{u}$	$\mathtt{u}$
$\mathtt{t}$	$\mathtt{t}$	$\mathtt{u}$	$\mathtt{t}$

	$\neg_{\mathbf{B}_{\mathtt{u}}}$
$\mathtt{f}$	$\mathtt{t}$
$\mathtt{u}$	$\mathtt{u}$
$\mathtt{t}$	$\mathtt{f}$

As we will see in a moment, such $\Sigma_{\wedge\vee\neg}$-algebra provides the interpretation structure for the logical matrices that determine the logics **PWK** and **BK**. Note that we have, for all $© \in \Sigma_{\wedge\vee\neg}$ of arity k, $©_{\mathbf{B}_{\mathtt{u}}}(\vec{a}) = ©_{\mathbf{B}}(\vec{a})$ if $\vec{a} \in \{\mathtt{f},\mathtt{t}\}^k$ and $©_{\mathbf{B}_{\mathtt{u}}}(\vec{a}) = \mathtt{u}$ otherwise. In other words, the above truth tables result from extending the classical two-valued tables with an *infectious truth value* [15].

We now extend the above observation to the derived operations of $\mathbf{B}_{\mathtt{u}}$. Let $\varphi(p_1,\ldots,p_k)$ indicate that $p_1,\ldots,p_k$ are the propositional variables occurring in φ (in which case φ is said to be k-ary—unary if $k=1$, binary if $k=2$), and let $\varphi(\psi_1,\ldots,\psi_k)$ refer to the formula resulting from replacing ψ_i for each occurrence of p_i in φ, for each $1 \leq i \leq k$. Given a Σ-algebra $\mathbf{A} := \langle A, \cdot_{\mathbf{A}}\rangle$ and a Σ-formula φ, we denote by $\varphi_{\mathbf{A}}$ the *derived operation* induced on $\mathbf{A}$ by φ. That is, for all $a_1,\ldots,a_k \in A$, provided a valuation v with $v(p_i) = a_i$, if $\varphi = ©(\psi_1,\ldots,\psi_k)$ we have $\varphi_{\mathbf{A}}(a_1,\ldots,a_k) = ©_{\mathbf{A}}(v(\psi_1),\ldots,v(\psi_k))$ and, if $\varphi = p_i$, then $\varphi_{\mathbf{A}}(p_i) = v(p_i)$. By induction on the structure of Σ-formulas, we then obtain that $\mathtt{u}$ is infectious also on the derived operations of $\mathbf{B}_{\mathtt{u}}$:

PROPOSITION 1. *For all $k \in \mathbb{N}$, $\varphi(p_1,\ldots,p_k) \in L_{\Sigma_{\wedge\vee\neg}}(P)$ and $\vec{a} \in B_{\mathtt{u}}^k$, $\varphi_{\mathbf{B}_{\mathtt{u}}}(\vec{a}) = \varphi_{\mathbf{B}}(\vec{a})$ if $\vec{a} \in \{\mathtt{f},\mathtt{t}\}^k$ and $\varphi_{\mathbf{B}_{\mathtt{u}}}(\vec{a}) = \mathtt{u}$ otherwise.*

In what follows, for every set X, let $\mathsf{Pow}(X)$ denote the power set of X. We now formally introduce the notion of *logic* considered in this work.

A *finitary* SET–SET *consequence relation* (or a SET–SET *logic*) over $L_\Sigma(P)$ is a binary relation $\rhd$ on $\mathsf{Pow}(L_\Sigma(P))$ satisfying **(O)**verlap, **(D)**ilution, **(C)**ut, **(S)**ubstitution-invariance and **(F)**initariness, for all $\Phi,\Psi,\Phi',\Psi' \subseteq L_\Sigma(P)$:

- **(O)** if $\Phi \cap \Psi \neq \varnothing$, then $\Phi \rhd \Psi$
- **(D)** if $\Phi \rhd \Psi$, then $\Phi,\Phi' \rhd \Psi,\Psi'$
- **(C)** if $\Pi,\Phi \rhd \Psi,\Pi^{\mathsf{c}}$ for all $\Pi \subseteq L_\Sigma(P)$, then $\Phi \rhd \Psi$
- **(S)** if $\Phi \rhd \Psi$, then $\sigma[\Phi] \rhd \sigma[\Psi]$, for every Σ-substitution σ
- **(F)** if $\Phi \rhd \Psi$, then $\Phi^{\mathrm{f}} \rhd \Psi^{\mathrm{f}}$ for some finite $\Phi^{\mathrm{f}} \subseteq \Phi$ and $\Psi^{\mathrm{f}} \subseteq \Psi$

Set–Set consequence relations have been thoroughly investigated by T. Shoesmith and T. Smiley in the book [33], to which we refer the reader for further background and details.

A *finitary* Set-Fmla *consequence relation* (or a Set-Fmla *logic*) over $L_\Sigma(P)$ is a relation $\vdash \subseteq \mathsf{Pow}(L_\Sigma(P)) \times L_\Sigma(P)$ satisfying the well-known Tarskian properties of reflexivity, monotonicity, transitivity, substitution-invariance and finitariness. Set-Fmla logics are a particular case of Set–Set logics. One may further check that each Set–Set logic $\rhd$ determines a Set-Fmla logic $\vdash_\rhd$ over $L_\Sigma(P)$ such that $\Phi \vdash_\rhd \psi$ if, and only if, $\Phi \rhd \{\psi\}$, which is called the Set-Fmla *companion of* $\rhd$. Pairs of the form (Φ, Ψ) or (Φ, ψ) are dubbed *statements*, and the statements belonging to a logic are called *consecutions* (of that logic).

A Σ-*matrix* is a structure $\mathbb{M} := \langle \mathbf{A}, D\rangle$, where $\mathbf{A}$ is a Σ-algebra and $D \subseteq A$. We write $\overline{D}$ for the set-theoretic complement $A \backslash D$. The homomorphisms from $\mathbf{L}_\Sigma(P)$ into $\mathbf{A}$ are called $\mathbb{M}$-*valuations*. Every Σ-matrix $\mathbb{M}$ determines a Set–Set consequence relation $\rhd_\mathbb{M}$ over $L_\Sigma(P)$ such that

$$\Phi \rhd_\mathbb{M} \Psi \text{ if, and only if, there is no } \mathbb{M}\text{-valuation } v \text{ satisfying}$$
$$v[\Phi] \subseteq D \text{ and } v[\Psi] \subseteq \overline{D}.$$

We denote by $\vdash_\mathbb{M}$ the Set-Fmla companion of $\rhd_\mathbb{M}$, which matches the canonical Set-Fmla consequence relation over $L_\Sigma(P)$ induced by $\mathbb{M}$, that is, $\Phi \vdash_\mathbb{M} \psi$ if, and only if, there is no $\mathbb{M}$-valuation v satisfying $v[\Phi] \subseteq D$ and $v(\psi) \in \overline{D}$. As expected, the $\Sigma_{\wedge\vee\neg}$-matrix $\mathbb{M}_{\mathbf{CL}} := \langle \mathbf{B}, \{\mathtt{t}\}\rangle$ determines the Set–Set and Set-Fmla consequence relations corresponding to classical logic, which we denote respectively by $\rhd_{\mathbf{CL}}$ and $\vdash_{\mathbf{CL}}$.

Consider the $\Sigma_{\wedge\vee\neg}$-matrices $\mathbb{M}_{\mathbf{PWK}} := \langle \mathbf{B}_\mathtt{u}, \{\mathtt{u}, \mathtt{t}\}\rangle$ and $\mathbb{M}_{\mathbf{BK}} := \langle \mathbf{B}_\mathtt{u}, \{\mathtt{t}\}\rangle$. Then Paraconsistent Weak Kleene (**PWK**) and Bochvar–Kleene (**BK**) logics are defined, respectively, as the Set-Fmla logics $\vdash_{\mathbb{M}_{\mathbf{PWK}}}$ and $\vdash_{\mathbb{M}_{\mathbf{BK}}}$, which we write $\vdash_{\mathbf{PWK}}$ and $\vdash_{\mathbf{BK}}$ for brevity. We will also be interested in the Set–Set logics determined by these matrices ($\rhd_{\mathbb{M}_{\mathbf{PWK}}}$ and $\rhd_{\mathbb{M}_{\mathbf{BK}}}$) which we denote simply by $\rhd_{\mathbf{PWK}}$ and $\rhd_{\mathbf{BK}}$, respectively. We may refer to them as the Set–Set *versions* of **PWK** and **BK**.

In what follows, given a Set-Fmla logic $\vdash$, we say that $\Phi \subseteq L_\Sigma(P)$ is $\vdash$-*explosive* in case $\Phi \vdash \varphi$ for all $\varphi \in L_\Sigma(P)$. As mentioned earlier, it is well-known that **PWK** and **BK** are, respectively, the *left variable inclusion companion* and the *right variable inclusion companion* of classical logic, in the sense expressed by the following facts (see [10,12,15] for general definitions and results concerning *inclusion logics*).

THEOREM 2. [12] *Let* $\Phi, \{\varphi, \psi\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$. *Then the following are equivalent:*

1. $\Phi \vdash_{\mathbf{PWK}} \psi$.
2. *There is* $\Phi' \subseteq \Phi$ *with* $\mathsf{props}[\Phi'] \subseteq \mathsf{props}(\psi)$ *such that* $\Phi' \vdash_{\mathbf{CL}} \psi$.

THEOREM 3. ([34], Theorem 2.3.1) *Let* $\Phi, \{\varphi, \psi\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$. *Then the following are equivalent:*

1. $\Phi \vdash_{\mathbf{BK}} \psi$.
2. $\Phi \vdash_{\mathbf{CL}} \psi$ *and* $\mathsf{props}(\psi) \subseteq \mathsf{props}[\Phi]$, *or else* Φ *is* $\vdash_{\mathbf{CL}}$*-explosive.*

3. Basics of Hilbert-Style Axiomatizations

Logical matrices are a semantical way to define SET-FMLA and SET–SET logics. Another popular way are *proof systems*, which manipulate syntactical objects envisaging the construction of derivations that bear witnesses to consecutions. Proof systems can be classified with respect to the *proof formalism* they belong to, based mainly on the objects they manipulate and the shape of their rules of inference and derivations. Each proof system induces a logic based on the derivations one may build via its rules of inference.

In this work, we are interested in *Hilbert-style proof systems*, or *H-systems* for short. As main characteristics, these have (a) rules of inference with the same shape of the consecutions of the induced logic; (b) derivations as trees labelled with sets of formulas; and (c) the fact that they represent a logical basis for the logics they induce, meaning that the latter is the least logic containing the rules of inference of the system [35].

Before the work of Shoesmith and Smiley [33], rules of inference in H-systems were constrained to be SET-FMLA statements, that is, pairs $(\Gamma, \delta) \in \mathsf{Pow}(L_\Sigma(P)) \times L_\Sigma(P)$, usually denoted by $\frac{\Gamma}{\delta}$, where Γ is called the *antecedent* and δ, the *succedent* of the rule. For this reason, we call them SET-FMLA *rules of inference* and sets thereof constitute SET-FMLA or *traditional H-systems*. They are also referred to as *single-conclusion* H-systems. In the above-mentioned work, H-systems were generalized to allow for multiple formulas in the succedent of rules of inference. In other words, rules of inference became SET–SET statements, that is, pairs of the form $(\Gamma, \Delta) \in \mathsf{Pow}(L_\Sigma(P)) \times \mathsf{Pow}(L_\Sigma(P))$, which we usually denote by $\frac{\Gamma}{\Delta}$. Collections of these so-called SET–SET *rules of inference* constitute what we refer to as SET–SET or *multiple-conclusion H-systems*.

In both formalisms, rules of inference are usually presented *schematically*, that is, as being induced by applying Σ-substitutions over representative rules called *rule schemas*. An H-system is *finite* when it is presented via a finite number of rule schemas.

Users of traditional H-systems are accustomed to derivations that are sequences of formulas, where each member is either a premise or results from the application of a rule of inference of the H-system on previous formulas in the sequence. A proof in a traditional H-system $\mathcal{H}$ of a statement (Φ, ψ) is then a derivation where the set of premises is Φ and the last formula is ψ. Equivalently, we could see these derivations as rooted labelled linear trees whose nodes are labelled with sets of formulas, where the root node is labelled with the set of premises and the child of each non-leaf node n is labelled with the label Γ of n plus the succedent of a rule of inference of $\mathcal{H}$ whose antecedent is contained in Γ. A proof of (Φ, ψ), then, is just a linear tree whose root node is labelled with Φ (or a subset thereof) and whose leaf node contains ψ.

Every Set-Fmla H-system $\mathcal{H}$ induces a Set-Fmla logic $\vdash_{\mathcal{H}}$ such that $\Phi \vdash_{\mathcal{H}} \psi$ if and only if there is a proof of (Φ, ψ) in $\mathcal{H}$. Given a Set-Fmla logic $\vdash$ and a Set-Fmla H-system $\mathcal{H}$, we say that $\mathcal{H}$ is *sound for* $\vdash$ when $\vdash_{\mathcal{H}} \subseteq \vdash$; that $\mathcal{H}$ is *complete for* $\vdash$ when $\vdash \subseteq \vdash_{\mathcal{H}}$; and that $\mathcal{H}$ *axiomatizes* $\vdash$ (or is an *axiomatization of*) $\vdash$ when it is both sound and complete for $\vdash$, that is, when $\vdash = \vdash_{\mathcal{H}}$.

Example 1. The following is a well-known Set-Fmla axiomatization of classical logic in the signature $\Sigma_{\to\neg}$, which we call $\mathcal{H}_{\mathbf{CL}}$ (note that it is presented by four rule schemas):

$$\frac{\varnothing}{q \to (p \to q)}\,\mathcal{H}_{\mathbf{CL}1} \qquad \frac{\varnothing}{(p \to (q \to r)) \to ((p \to q) \to (p \to r))}\,\mathcal{H}_{\mathbf{CL}2}$$

$$\frac{\varnothing}{(\neg r \to \neg q) \to ((\neg r \to q) \to r)}\,\mathcal{H}_{\mathbf{CL}3} \qquad \frac{p, p \to q}{q}\,\mathcal{H}_{\mathbf{CL}4}$$

Here is a proof in $\mathcal{H}_{\mathbf{CL}}$ bearing witness to $\varnothing \vdash_{\mathcal{H}_{\mathbf{CL}}} p \to p$:

1.	$(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$	$\mathcal{H}_{\mathbf{CL}2}$
2.	$p \to ((p \to p) \to p)$	$\mathcal{H}_{\mathbf{CL}1}$
3.	$(p \to (p \to p)) \to (p \to p)$	$1, 2, \mathcal{H}_{\mathbf{CL}4}$
4.	$p \to (p \to p)$	$\mathcal{H}_{\mathbf{CL}1}$
5.	$p \to p$	$3, 4, \mathcal{H}_{\mathbf{CL}4}$

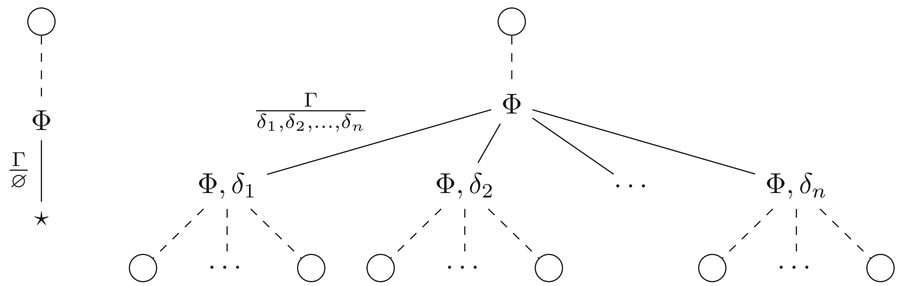

Figure 1. Graphical representation of R-derivations, where R is a SET–SET system. The dashed edges and blank circles represent other branches that may exist in the derivation. We usually omit the formulas inherited from the parent node, exhibiting only the ones introduced by the applied rule of inference. Recall that, in both cases, we must have $\Gamma \subseteq \Phi$

The passage from SET-FMLA H-systems to SET–SET H-systems demands an adaptation on the latter notions of derivations and proofs. Now a non-leaf node n may have a single child labelled with $\star$ (a *discontinuation symbol*) when there is a rule of inference in the H-system with empty succedent and whose antecedent Γ is contained in the label of n. This symbol indicates that the node does not need further development (see Example 2). It may alternatively have m child nodes $\mathsf{n}_1, \ldots, \mathsf{n}_m$ when there is a rule of inference $\frac{\Gamma}{\psi_1,\ldots,\psi_m}$ in the H-system whose antecedent Γ is, as in the previous case, contained in the label of n. The label of each n_i, in this situation, is the label of n union $\{\psi_i\}$, for all $1 \leq i \leq m$. See Figure 1 for a general scheme of these derivations. A proof of a statement (Φ, Ψ) in a SET–SET H-system, then, is a labelled rooted tree whose root node is labelled with Φ (or a subset thereof) and whose leaf nodes (now there may be more than one) are labelled either with $\star$ or with a set having a nonempty intersection with Ψ.

Note that SET–SET H-systems generalize SET-FMLA H-systems because when all rules of inference in a SET–SET H-system have a single formula in the conclusion (that is, they are SET-FMLA rules), the derivations in that system will always be rooted labelled *linear* trees, which matches our definition of SET-FMLA derivations.

Every SET–SET H-system R induces a SET–SET logic $\rhd_{\mathsf{R}}$ such that $\Phi \rhd_{\mathsf{R}} \Psi$ if and only if there is a proof of (Φ, Ψ) in R. Given a SET–SET logic $\rhd$ and a SET–SET H-system R, the notions of R being sound, complete or an axiomatization for $\rhd$ are defined analogously as in the SET-FMLA case.

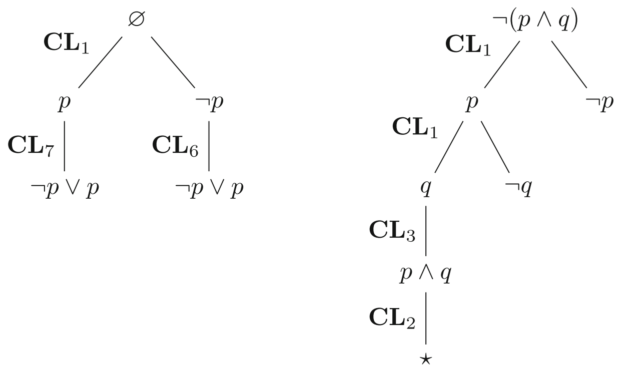

Figure 2. Proofs in $\mathsf{R}_{\mathbf{CL}}$ bearing witness, respectively, to $\varnothing \rhd_{\mathsf{R}_{\mathbf{CL}}} p \to p$ and $\neg(p \wedge q) \rhd_{\mathsf{R}_{\mathbf{CL}}} \neg p, \neg q$

Example 2. The following is a Set–Set axiomatization for classical logic in the signature $\Sigma_{\wedge\vee\neg}$ (in its Set–Set version). See Figure 2 for examples of derivations.

$$\frac{\varnothing}{p, \neg p}\mathbf{CL}_1 \quad \frac{p, \neg p}{\varnothing}\mathbf{CL}_2 \quad \frac{p, q}{p \wedge q}\mathbf{CL}_3 \quad \frac{p \wedge q}{p}\mathbf{CL}_4 \quad \frac{p \wedge q}{q}\mathbf{CL}_5$$
$$\frac{p}{p \vee q}\mathbf{CL}_6 \quad \frac{q}{p \vee q}\mathbf{CL}_7 \quad \frac{p \vee q}{p, q}\mathbf{CL}_8$$

The derivations shown in Figure 2 have an important property: only subformulas of the formulas in the respective statements (Φ, Ψ) being proved appear in the labels of the nodes. In fact, every statement that is provable in $\mathsf{R}_{\mathbf{CL}}$ has a proof with such feature. For this reason, we say that $\mathsf{R}_{\mathbf{CL}}$ is *analytic*. Traditional (Set-Fmla) H-systems have been historically avoided in tasks involving proof search, as they rarely satisfy the property of analyticity (note how the non-analyticity of $\mathcal{H}_{\mathbf{CL}}$ shows up in Example 1). The solution has usually been to employ another deductive formalism, usually one with more meta-linguistic resources, allowing one to prove meta-results that guarantee analyticity (a typical example being cut elimination in sequent-style systems [29]).

Recent work by C. Caleiro and S. Marcelino [13, 27] demonstrates that the much simpler passage to Set–Set H-systems is enough to obtain analytic proof systems (and thus bounded proof search) for a plethora of non-classical logics. This observation will be key to us, for we will be able to apply the techniques developed in the above-mentioned studies to provide finite H-systems for **PWK** and **BK**. This result, however, demands a slight generalization of the notion of analyticity in addition to the already mentioned modification of the proof formalism to Set–Set. In order to understand it, consider first a set Θ of formulas on a single propositional variable, and let R be a SET–SET system. The main idea is to allow for not only subformulas of a statement to appear in an analytic proof, but also formulas resulting from substitutions of those subformulas over the formulas in Θ. For example, if $\Theta = \{r, \neg r\}$, a Θ-analytic proof witnessing that $\neg p$ follows from $\neg(p \wedge q)$ would use only formulas in $\{p, q, \neg p, \neg q, \neg\neg p, p \wedge q, \neg(p \wedge q), \neg\neg(p \wedge q)\}$. Formally, we say that R is Θ-*analytic* whenever $\Phi \rhd_{\mathsf{R}} \Psi$ implies that there is a Θ-*analytic proof* of (Φ, Ψ) in R, that is, a proof whose nodes are labelled only with formulas in the set $\mathsf{subf}(\Phi \cup \Psi) \cup \{\varphi(\psi) \mid \varphi \in \Theta, \psi \in \mathsf{subf}(\Phi \cup \Psi\}$), i.e. the Θ-*subformulas* of (Φ, Ψ).

One can show that any finite logical matrix[1] satisfying a very mild expressiveness requirement is effectively axiomatized by a finite Θ-analytic SET–SET system, for some finite Θ. This requirement is called *monadicity* (or *sufficient expressivess*), and intuitively means that every truth value of the matrix can be described by formulas on a single variable (the set of these formulas will be precisely Θ). Let us make this notion precise and formally state the axiomatization result. We say that a matrix $\mathbb{M} := \langle \mathbf{A}, D \rangle$ is *monadic* whenever for every pair of distinct truth values $x, y \in A$ there is a formula φ in one propositional variable such that $\varphi_{\mathbf{A}}(x) \in D$ and $\varphi_{\mathbf{A}}(y) \in A \backslash D$ or vice-versa. These formulas are called *separators*. Then we have that:

THEOREM 4. ([27], Theorem 3.5) *For every finite monadic logical matrix $\mathbb{M}$, the logic $\rhd_{\mathbb{M}}$ is axiomatized by a finite Θ-analytic* SET–SET *system (which we call $\mathsf{R}^{\Theta}_{\mathbb{M}}$) where Θ is a finite set of separators for every pair of truth values of $\mathbb{M}$.*

The next lemma shows why this result is so important for us in the present context.

LEMMA 5. *The matrices $\mathbb{M}_{\mathbf{PWK}}$ and $\mathbb{M}_{\mathbf{BK}}$ are monadic, with set of separators $\Theta := \{p, \neg p\}$.*

PROOF. In both matrices, p is a separator for $(\mathtt{t}, \mathtt{f})$. In $\mathbb{M}_{\mathbf{PWK}}$, the same formula separates $(\mathtt{f}, \mathtt{u})$ and $\neg p$ separates $(\mathtt{u}, \mathtt{t})$. In $\mathbb{M}_{\mathbf{BK}}$, we have that p separates $(\mathtt{t}, \mathtt{u})$ and $\neg p$ separates $(\mathtt{f}, \mathtt{u})$. ■

The above fact anticipates that we will be able to provide finite and $\{p, \neg p\}$-analytic SET–SET systems for the SET–SET versions of **PWK** and **BK**. However, it is not obvious how to obtain traditional finite H-systems

[1] Actually, the result applies to a much more general scenario, which is not needed in the present work: the matrix can even be partial non-deterministic [13] in the sense of [1,3]. It may also be infinite, but then the generated system might be infinite as well.

for the original (and most studied) Set-Fmla versions of these logics. In the next couple of sections, we will not only exhibit the announced Set–Set systems, but also show how to use them to obtain finite Set-Fmla H-systems for **PWK** and **BK**, thus solving the question regarding their finite axiomatizability.

4. Finite H-Systems for PWK

Let us begin with the task of axiomatizing the Set–Set version of **PWK**. The following Set–Set system was generated from the matrix $\mathbb{M}_{\mathbf{PWK}}$ by the algorithm and simplification procedures described in [27] and implemented in [20, Appendix A], using $\{p, \neg p\}$ as a set of separators (in view of Lemma 5).

Definition 1. Let $\mathsf{R}_{\mathbf{PWK}}$ be the Set–Set system presented by the following rule schemas:

$$\frac{\varnothing}{p, \neg p}\mathbf{PWK}_1^{\rhd} \quad \frac{p}{\neg\neg p}\mathbf{PWK}_2^{\rhd} \quad \frac{\neg\neg p}{p}\mathbf{PWK}_3^{\rhd}$$

$$\frac{p, q}{p \wedge q}\mathbf{PWK}_4^{\rhd} \quad \frac{p \wedge q}{p, q}\mathbf{PWK}_5^{\rhd} \quad \frac{p \wedge q}{p, \neg q}\mathbf{PWK}_6^{\rhd} \quad \frac{p \wedge q}{\neg p, q}\mathbf{PWK}_7^{\rhd} \quad \frac{\neg(p \wedge q)}{\neg p, \neg q}\mathbf{PWK}_8^{\rhd}$$

$$\frac{p, \neg p}{p \wedge q}\mathbf{PWK}_9^{\rhd} \quad \frac{\neg p}{\neg(p \wedge q)}\mathbf{PWK}_{10}^{\rhd} \quad \frac{q, \neg q}{p \wedge q}\mathbf{PWK}_{11}^{\rhd} \quad \frac{\neg q}{\neg(p \wedge q)}\mathbf{PWK}_{12}^{\rhd}$$

$$\frac{p}{p \vee q}\mathbf{PWK}_{13}^{\rhd} \quad \frac{q}{p \vee q}\mathbf{PWK}_{14}^{\rhd} \quad \frac{p \vee q}{p, q}\mathbf{PWK}_{15}^{\rhd} \quad \frac{\neg(p \vee q)}{p, \neg q}\mathbf{PWK}_{16}^{\rhd}$$

$$\frac{\neg(p \vee q)}{\neg p, q}\mathbf{PWK}_{17}^{\rhd} \quad \frac{\neg(p \vee q)}{\neg p, \neg q}\mathbf{PWK}_{18}^{\rhd} \quad \frac{p, \neg p}{\neg(p \vee q)}\mathbf{PWK}_{19}^{\rhd} \quad \frac{q, \neg q}{\neg(p \vee q)}\mathbf{PWK}_{20}^{\rhd}$$

Since this system is equivalent to the system $\mathsf{R}_{\mathbb{M}_{\mathbf{PWK}}}^{\{p,\neg p\}}$ mentioned in Theorem 4 (when specialized to $\mathbb{M}_{\mathbf{PWK}}$), and since the mentioned simplification procedures preserve Θ-analyticity, we obtain:

Theorem 6. $\mathsf{R}_{\mathbf{PWK}}$ *is* $\{p, \neg p\}$*-analytic and* $\rhd_{\mathsf{R}_{\mathbf{PWK}}} = \rhd_{\mathbf{PWK}}$.

Our goal now is to find a finite Set-Fmla H-system for **PWK**. We will see that this task is easily solved because the disjunction connective in this logic allows us to convert $\mathsf{R}_{\mathbf{PWK}}$ into the desired finite Set-Fmla system. More generally, every Set-Fmla logic $\vdash$ is finitely axiomatized by a Set-Fmla H-system whenever it satisfies two conditions which we now describe [33, Theorem 5.37]. First, the logic is the Set-Fmla companion of a Set–Set logic finitely axiomatized by a Set–Set H-system, say R. Second, it satisfies the following property for some binary formula $C(p, q)$ (said to be a *definable binary connective* in this context):

$$\begin{aligned}&\text{for all } \Phi \cup \{\varphi, \psi, \gamma\} \subseteq L_\Sigma(P), \\ &\Phi, \varphi \vdash \gamma \text{ and } \Phi, \psi \vdash \gamma \text{ if, and only if, } \Phi, C(\varphi, \psi) \vdash \gamma.\end{aligned} \tag{disj}$$

The proof of this fact in [33] reveals how to effectively convert R into the desired Set-Fmla H-system. Let us see how to perform this conversion and then apply the transformation to $\mathsf{R}_{\mathbf{PWK}}$.[2] In what follows, when $\Phi{:=}\{\varphi_1, \ldots, \varphi_n\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$ $(n \geq 1)$, let $\bigvee \Phi{:=}(\ldots(\varphi_1 \vee \varphi_2) \vee \ldots) \vee \varphi_n$. Also, let $\Phi \vee \psi{:=}\{\varphi \vee \psi \mid \varphi \in \Phi\}$. Note that the latter set is empty when Φ is empty.

Definition 2. Let R be a Set–Set system and p_0 be a propositional variable not occurring in the rule schemas of R. Define the system $\mathsf{R}^\vee$as being presented by the rule schemas $\left\{\frac{p}{p\vee q}, \frac{p\vee q}{q\vee p}, \frac{p\vee(q\vee r)}{(p\vee q)\vee r}\right\} \cup \{\mathsf{r}^\vee \mid \mathsf{r} \text{ is a rule schema of } \mathsf{R}\}$ where $\mathsf{r}^\vee$ is $\frac{\varnothing}{\varphi}$ if $\mathsf{r} = \frac{\varnothing}{\varphi}$, $\frac{\Phi\vee p_0}{(\bigvee\Psi)\vee p_0}$ if $\mathsf{r} = \frac{\Phi}{\Psi}$, and $\frac{\Phi\vee p_0}{p_0}$ if $\mathsf{r} = \frac{\Phi}{\varnothing}$.

Below we present the result of this procedure when applied to $\mathsf{R}_{\mathbf{PWK}}$. Note that the conversion of rule $\mathbf{PWK}_{15}^{\triangleright}$ results in a rule of the form φ/φ, and thus can be discarded.

Definition 3. Let $\mathcal{H}_{\mathbf{PWK}}$ be the Set-Fmla system presented by the following rule schemas:

$$\frac{\varnothing}{p \vee \neg p}\mathbf{PWK}_1 \quad \frac{r \vee p}{r \vee \neg\neg p}\mathbf{PWK}_2 \quad \frac{r \vee \neg\neg p}{r \vee p}\mathbf{PWK}_3$$

$$\frac{p, q}{p \wedge q}\mathbf{PWK}_4 \quad \frac{r \vee (p \wedge q)}{r \vee (p \vee q)}\mathbf{PWK}_5 \quad \frac{r \vee (p \wedge q)}{r \vee (p \vee \neg q)}\mathbf{PWK}_6$$

$$\frac{r \vee (p \wedge q)}{r \vee (\neg p \vee q)}\mathbf{PWK}_7 \quad \frac{r \vee \neg(p \wedge q)}{r \vee (\neg p \vee \neg q)}\mathbf{PWK}_8 \quad \frac{p, \neg p}{p \wedge q}\mathbf{PWK}_9$$

$$\frac{\neg p}{\neg(p \wedge q)}\mathbf{PWK}_{10} \quad \frac{q, \neg q}{p \wedge q}\mathbf{PWK}_{11} \quad \frac{\neg q}{\neg(p \wedge q)}\mathbf{PWK}_{12}$$

$$\frac{p}{p \vee q}\mathbf{PWK}_{13} \quad \frac{q}{p \vee q}\mathbf{PWK}_{14} \quad \frac{r \vee \neg(p \vee q)}{r \vee (p \vee \neg q)}\mathbf{PWK}_{15}$$

$$\frac{r \vee \neg(p \vee q)}{r \vee (\neg p \vee q)}\mathbf{PWK}_{16} \quad \frac{r \vee \neg(p \vee q)}{r \vee (\neg p \vee \neg q)}\mathbf{PWK}_{17} \quad \frac{p, \neg p}{\neg(p \vee q)}\mathbf{PWK}_{18}$$

[2] Note that we use $\vee$ to simplify notation, but the same definition could be rephrased with the derived connective $C(p, q)$.

$$\frac{q, \neg q}{\neg(p \vee q)}\mathbf{PWK}_{19} \quad \frac{p \vee q}{q \vee p}\mathbf{PWK}_{20} \quad \frac{p \vee (q \vee r)}{(p \vee q) \vee r}\mathbf{PWK}_{21}$$

As anticipated in the previous discussion, we have that:

THEOREM 7. ([33], Theorem 5.37) *If* $\rhd_{\mathsf{R}} = \rhd$ *and* $\vdash_{\rhd}$ *satisfies* (disj), *then* $\vdash_{\mathsf{R}^{\vee}} = \vdash_{\rhd}$.

REMARK 1. The authors of [33] also show that a similar conversion between SET–SET and SET-FMLA is possible when the logic has a definable binary connective $C(p, q)$ that satisfies the so-called *deduction theorem*:

$$\begin{aligned} &\text{for all } \Phi \cup \{\varphi, \psi\} \subseteq L_{\Sigma}(P), \\ &\Phi, \varphi \vdash \psi \text{ if, and only if, } \Phi \vdash C(\varphi, \psi). \end{aligned} \tag{ded}$$

Theorem 7 can then be applied to **PWK** because $\vdash_{\mathbf{PWK}} = \vdash_{\rhd_{\mathbf{PWK}}}$ and it satisfies (disj), as we establish below.

PROPOSITION 8. *For all* $\Phi \cup \{\varphi, \psi, \gamma\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$,

$$\Phi, \varphi \vdash_{\mathbf{PWK}} \gamma \textit{ and } \Phi, \psi \vdash_{\mathbf{PWK}} \gamma \textit{ if, and only if, } \Phi, \varphi \vee \psi \vdash_{\mathbf{PWK}} \gamma.$$

PROOF. The reader can easily check that the presence of rules $\mathbf{PWK}^{\rhd}_{13}$, $\mathbf{PWK}^{\rhd}_{14}$ and $\mathbf{PWK}^{\rhd}_{15}$ in $\mathsf{R}_{\mathbf{PWK}}$ is enough to prove this statement. ■

In other words,

THEOREM 9. $\vdash_{\mathcal{H}_{\mathbf{PWK}}} = \vdash_{\mathbf{PWK}}$.

5. Finite H-Systems for BK

We shall proceed as in the previous case, starting with the axiomatization of the SET–SET version of **BK**. In view of Lemma 5, we can apply the same reasoning as the one applied to axiomatize **PWK** in SET–SET, that is, we can automatically generate a $\{p, \neg p\}$-analytic axiomatization for **BK**:

DEFINITION 4. Let $\mathsf{R}_{\mathbf{BK}}$ be the SET–SET system presented by the following rule schemas:

$$\frac{p, \neg p}{\varnothing}\mathbf{BK}^{\rhd}_{1} \quad \frac{p}{\neg\neg p}\mathbf{BK}^{\rhd}_{2} \quad \frac{\neg\neg p}{p}\mathbf{BK}^{\rhd}_{3}$$

$$\frac{p, q}{p \wedge q}\mathbf{BK}^{\rhd}_{4} \quad \frac{\neg p, \neg q}{\neg(p \wedge q)}\mathbf{BK}^{\rhd}_{5} \quad \frac{\neg p, q}{\neg(p \wedge q)}\mathbf{BK}^{\rhd}_{6} \quad \frac{p, \neg q}{\neg(p \wedge q)}\mathbf{BK}^{\rhd}_{7}$$

$$\frac{\neg(p \wedge q)}{\neg p, p}\mathbf{BK}^{\rhd}_{8} \quad \frac{\neg(p \wedge q)}{\neg q, q}\mathbf{BK}^{\rhd}_{9} \quad \frac{p \wedge q}{p}\mathbf{BK}^{\rhd}_{10} \quad \frac{p \wedge q}{q}\mathbf{BK}^{\rhd}_{11}$$

$$\frac{\neg p, \neg q}{\neg(p \vee q)}\mathbf{BK}^{\triangleright}_{12} \quad \frac{\neg(p \vee q)}{\neg p}\mathbf{BK}^{\triangleright}_{13} \quad \frac{\neg(p \vee q)}{\neg q}\mathbf{BK}^{\triangleright}_{14} \quad \frac{p \vee q}{p, \neg p}\mathbf{BK}^{\triangleright}_{15}$$

$$\frac{p \vee q}{q, \neg q}\mathbf{BK}^{\triangleright}_{16} \quad \frac{\neg p, q}{p \vee q}\mathbf{BK}^{\triangleright}_{17} \quad \frac{p, \neg q}{p \vee q}\mathbf{BK}^{\triangleright}_{18} \quad \frac{p, q}{p \vee q}\mathbf{BK}^{\triangleright}_{19} \quad \frac{p \vee q}{p, q}\mathbf{BK}^{\triangleright}_{20}$$

As in the case of **PWK**, since $\mathsf{R}_{\mathbf{BK}}$ is equivalent to the system $\mathsf{R}^{\{p,\neg p\}}_{\mathbb{M}_{\mathbf{BK}}}$ mentioned in Theorem 4 (when particularized to $\mathbb{M}_{\mathbf{BK}}$), and the employed simplification procedures preserve Θ-analyticity, we have:

Theorem 10. $\mathsf{R}_{\mathbf{BK}}$ *is* $\{p, \neg p\}$*-analytic and* $\triangleright_{\mathsf{R}_{\mathbf{BK}}} = \triangleright_{\mathbf{BK}}$.

Remark 2. It is not hard to see that $\mathbb{M}_{\mathbf{BK}}$ results from a renaming of the truth-values of the logical matrix $\mathbb{M}' := \langle \mathbf{B}'_{\mathtt{u}}, \{\mathtt{f}\}\rangle$, where $\mathbf{B}'_{\mathtt{u}}$ has the same set $B_{\mathtt{u}}$ of truth values and its truth tables are such that $\vee_{\mathbf{B}'_{\mathtt{u}}} := \wedge_{\mathbf{B}_{\mathtt{u}}}$, $\wedge_{\mathbf{B}'_{\mathtt{u}}} := \vee_{\mathbf{B}_{\mathtt{u}}}$ and $\neg_{\mathbf{B}'_{\mathtt{u}}} := \neg_{\mathbf{B}_{\mathtt{u}}}$ (just swap $\mathtt{t}$ and $\mathtt{f}$ in the interpretations and in the designated set). Note also that, if we take $\mathbb{M}_{\mathbf{PWK}}$ and replace its designated set $\{\mathtt{t}, \mathtt{u}\}$ by $\{\mathtt{f}\}$ and swap the truth tables of $\wedge$ and $\vee$, we obtain $\mathbb{M}'$. The axiomatization procedure of [27] implies in this situation that $\mathbb{M}'$ is axiomatized simply by taking $\mathsf{R}_{\mathbf{PWK}}$ and turning its rules of inference upside down (antecedents become succedents, and vice-versa), in addition to replacing $\wedge$ by $\vee$ and vice-versa in the rules. We call the resulting system the *dualization* of $\mathsf{R}_{\mathbf{PWK}}$. Because $\mathbb{M}_{\mathbf{BK}}$ results from $\mathbb{M}'$ by this simple renaming of truth values, we have that it is axiomatized by this same Set–Set system. The reader can easily check that, indeed, $\mathsf{R}_{\mathbf{BK}}$ is just the dualization of $\mathsf{R}_{\mathbf{PWK}}$.

Finding a finite Set-Fmla axiomatization for **BK** turns out to be harder than in the case of **PWK**. The reason, as we prove in the next proposition, is that in **BK** it is impossible to define a binary connective $C(p, q)$ satisfying (disj) or (ded).

Proposition 11. *The following holds for* **BK***:*

1. *For no binary formula* $C(p, q) \in L_{\Sigma_{\wedge\vee\neg}}(P)$ *we have* $\Phi, \varphi \vdash_{\mathbf{BK}} \gamma$ *and* $\Phi, \psi \vdash_{\mathbf{BK}} \gamma$ *whenever* $\Phi, C(\varphi, \psi) \vdash_{\mathbf{BK}} \gamma$*, for all* $\Phi \cup \{\varphi, \psi, \gamma\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$.
2. *For no binary formula* $C(p, q) \in L_{\Sigma_{\wedge\vee\neg}}(P)$ *we have* $\Phi \vdash_{\mathbf{BK}} C(\varphi, \psi)$ *whenever* $\Phi, \varphi \vdash_{\mathbf{BK}} \psi$*, for all* $\Phi \cup \{\varphi, \psi\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$.

Proof. For item 1, note that $C(p, q) \vdash_{\mathbf{BK}} C(p, q) \vee \neg C(p, q)$, however $q \nvdash_{\mathbf{BK}} C(p, q) \vee \neg C(p, q)$, as a **BK**-valuation assigning $\mathtt{u}$ to p and $\mathtt{t}$ to q would be a countermodel for the latter consecution (see Theorem 1). Similarly, for item 2, note that $\neg p, p \vdash_{\mathbf{BK}} q$, but $\neg p \nvdash_{\mathbf{BK}} C(p, q)$, what can be seen by considering a **BK**-valuation assigning $\mathtt{f}$ to p and $\mathtt{u}$ to q. ■

Therefore, up to this point, the mere existence of a finite SET–SET system for **BK** does not guarantee that this logic is finitely axiomatizable in SET-FMLA. It does not mean, however, that such system cannot help us in an *ad hoc* effort to finitely axiomatize **BK**.

We begin by noting that only the rules $\mathbf{BK}_i^{\triangleright}$, with $i \in \{8, 9, 15, 16, 20\}$, have multiple formulas in the succedent. We will replace the first four of these by the following SET-FMLA rules:

$$\frac{\neg(p \land q)}{\neg p \lor p}\mathbf{BK}_{8\star} \qquad \frac{\neg(p \land q)}{\neg q \lor q}\mathbf{BK}_{9\star} \qquad \frac{p \lor q}{p \lor \neg p}\mathbf{BK}_{15\star} \qquad \frac{p \lor q}{q \lor \neg q}\mathbf{BK}_{16\star}$$

DEFINITION 5. Let $\mathsf{R}_{\mathbf{BK}\star}$ be $\mathsf{R}_{\mathbf{BK}}$ but with $\mathbf{BK}_i^{\triangleright}$ replaced by $\mathbf{BK}_{i\star}$, for each $i \in \{8, 9, 15, 16\}$.

Then we have that this transformation preserves the induced logic:

PROPOSITION 12. *$\mathsf{R}_{\mathbf{BK}}$ and $\mathsf{R}_{\mathbf{BK}\star}$ induce the same* SET–SET *logic.*

PROOF. We just need to show that $\triangleright_{\mathsf{R}_{\mathbf{BK}}} = \triangleright_{\mathsf{R}_{\mathbf{BK}\star}}$. The right-to-left inclusion is easy, and the converse follows thanks to the presence of $\mathbf{BK}_{20}^{\triangleright}$. ∎

EXAMPLE 3. The following derivation bears witness to $\neg(p \land q) \triangleright_{\mathsf{R}_{\mathbf{BK}\star}} \neg p \lor \neg q$:

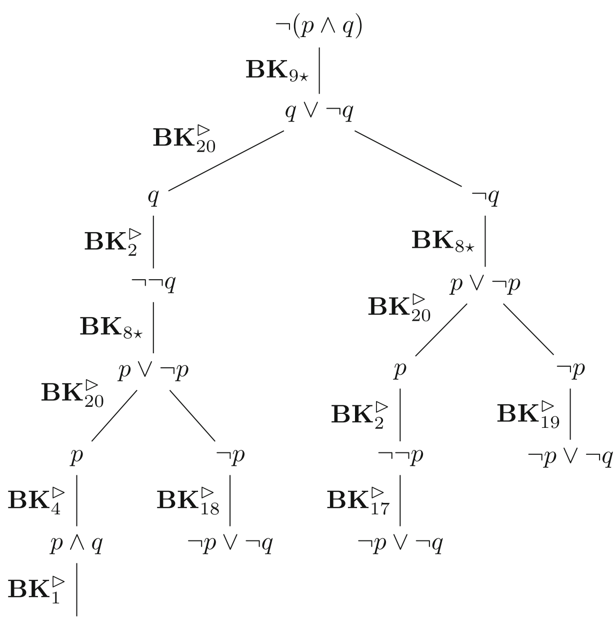

Figure 3. A derivation in $\mathsf{R}_{\mathbf{BK}\star}$ showing that $\neg(p \land q) \triangleright_{\mathsf{R}_{\mathbf{BK}\star}} \neg p \lor \neg q$

REMARK 3. The modifications in $\mathsf{R}_{\mathbf{BK}}$ that resulted in $\mathsf{R}_{\mathbf{BK}\star}$, despite preserving the induced logic, are not guaranteed to preserve $\{p, \neg p\}$-analyticity. The previous example may be seen as an illustration of this fact.

The fact that the only rule of $\mathsf{R}_{\mathbf{BK}\star}$ with more than one formula in the succedent is $\frac{p \vee q}{p, q}\mathbf{BK}^{\triangleright}_{20}$ will help us in providing a finite SET-FMLA system for **BK**, thus answering positively the question of its finite axiomatizability. Before showing why and how, let us introduce some transformations over SET-FMLA rules that will be useful in our endeavour:

DEFINITION 6. Let $\frac{\varphi_1, \ldots, \varphi_m}{\psi}\mathsf{r}$ be a SET-FMLA inference rule and r be a propositional variable not occurring in any of the formulas $\varphi_1, \ldots, \varphi_m$ and ψ. For simplicity, we define the binary connective $\rightarrow$ by abbreviation: for all $\varphi, \psi \in L_{\Sigma_{\wedge\vee\neg}}(P)$, let $\varphi \rightarrow \psi {:=} \neg\varphi \vee \psi$. Then:

1. The $\vee$-*lifted version* of r is the rule
$$\frac{r \vee \varphi_1, \ldots, r \vee \varphi_m}{r \vee \psi}\mathsf{r}^{\vee}$$
2. The $\rightarrow$-*lifted version* of r is the rule
$$\frac{r \rightarrow \varphi_1, \ldots, r \rightarrow \varphi_m}{r \rightarrow \psi}\mathsf{r}^{\rightarrow}$$

The following characterization of rules of inference will also be useful to us, in view of Theorem 3:

DEFINITION 7. A SET-FMLA inference rule $\frac{\Phi}{\psi}\mathsf{r}$ is said to satisfy the *containment condition* whenever $\mathsf{props}(\psi) \subseteq \mathsf{props}[\Phi]$.

We will provide a SET-FMLA H-system resulting from $\mathsf{R}_{\mathbf{BK}\star}$ essentially by the following modifications:

(a) removing the rule $\mathbf{BK}^{\triangleright}_{20}$;

(b) replacing $\mathbf{BK}^{\triangleright}_{1}$, a rule with empty succedent, with a new rule called $\mathbf{BK}_{1\star}$ having a fresh variable in the succedent;

(c) adding some rules concerning $\vee$;

(d) adding all $\vee$-lifted versions (see Definition 6) of all rules but $\mathbf{BK}_{1\star}$.

Having the lifted rules for all rules satisfying the containment condition will be important for completeness, as we will see. Our task, then, boils down to showing that applications of $\mathbf{BK}^{\triangleright}_{1}$ and $\mathbf{BK}^{\triangleright}_{20}$ in derivations in $\mathsf{R}_{\mathbf{BK}\star}$ of SET-FMLA statements may be replaced by applications of rules of the proposed SET-FMLA system. We display this system below for clarity and ease of reference.

DEFINITION 8. Let $\mathcal{H}_{\mathbf{BK}}$ be the SET-FMLA system given by the rule schemas

$$\frac{p, \neg p}{q}\mathbf{BK}_{1\star} \quad \frac{p}{\neg\neg p}\mathbf{BK}_{2} \quad \frac{\neg\neg p}{p}\mathbf{BK}_{3}$$

$$\frac{p, q}{p \wedge q}\mathbf{BK}_{4} \quad \frac{\neg p, \neg q}{\neg(p \wedge q)}\mathbf{BK}_{5} \quad \frac{\neg p, q}{\neg(p \wedge q)}\mathbf{BK}_{6} \quad \frac{p, \neg q}{\neg(p \wedge q)}\mathbf{BK}_{7}$$

$$\frac{\neg(p \wedge q)}{\neg p \vee p}\mathbf{BK}_{8\star} \quad \frac{\neg(p \wedge q)}{\neg q \vee q}\mathbf{BK}_{9\star} \quad \frac{p \wedge q}{p}\mathbf{BK}_{10} \quad \frac{p \wedge q}{q}\mathbf{BK}_{11}$$

$$\frac{\neg p, \neg q}{\neg(p \vee q)}\mathbf{BK}_{12} \quad \frac{\neg(p \vee q)}{\neg p}\mathbf{BK}_{13} \quad \frac{\neg(p \vee q)}{\neg q}\mathbf{BK}_{14} \quad \frac{p \vee q}{p \vee \neg p}\mathbf{BK}_{15\star}$$

$$\frac{p \vee q}{q \vee \neg q}\mathbf{BK}_{16\star} \quad \frac{\neg p, q}{p \vee q}\mathbf{BK}_{17} \quad \frac{p, \neg q}{p \vee q}\mathbf{BK}_{18} \quad \frac{p, q}{p \vee q}\mathbf{BK}_{19}$$

$$\frac{p \vee q, \neg p}{q}\mathbf{BK}_{20} \quad \frac{p \vee (q \vee r)}{(p \vee q) \vee r}\mathbf{BK}_{21} \quad \frac{p \vee p}{p}\mathbf{BK}_{22} \quad \frac{p \vee q}{q \vee p}\mathbf{BK}_{23} \quad \frac{p \vee q, r}{\neg p \vee r}\mathbf{BK}_{24}$$

plus the $\vee$-lifted versions of the above rules that satisfy the containment condition (see Definition 7)—that is, all but $\mathbf{BK}_{1\star}$.

REMARK 4. Adding the $\vee$-lifted versions of the rules displayed above substantially increases the size of the proposed system. In classical logic, this can be readily avoided due to the presence of the rule $\frac{p}{p \vee q}$, which is not sound in **BK**.

Our first goal is to verify that the SET-FMLA system just defined is sound for **BK**. This can be proved by showing that each rule $\frac{\Gamma}{\delta}$ of the system is *sound for* $\vdash_{\mathbb{M}_{\mathbf{BK}}}$, i.e. that $\Gamma \vdash_{\mathbb{M}_{\mathbf{BK}}} \delta$. In this direction, we take advantage of the close relationship between $\mathbb{M}_{\mathbf{BK}}$ and classical logic described in Theorem 3.

LEMMA 13. $\vdash_{\mathcal{H}_{\mathbf{BK}}} \subseteq \vdash_{\mathbb{M}_{\mathbf{BK}}}$.

PROOF. Note that $\mathbf{BK}_{1\star}$ is the only rule that does not satisfy the containment condition. Since it is impossible for an $\mathbb{M}_{\mathbf{BK}}$-valuation to satisfy both p and $\neg p$, this rule is sound with respect to $\mathbb{M}_{\mathbf{BK}}$. Because the other rules satisfy the containment condition and are all sound in classical logic, by Theorem 3 we have that they are also sound with respect to $\mathbb{M}_{\mathbf{BK}}$. ∎

In what follows, we will abbreviate some SET-FMLA derivations by composing rules of inference: we write $\mathsf{r}_1, \mathsf{r}_2, \ldots, \mathsf{r}_n$ to mean that we apply first rule r_1, then r_2 considering the formula derived in the previous step, then r_3 and so on.

PROPOSITION 14. *The following rules are derivable in $\mathcal{H}_{\mathbf{BK}}$:*

$$\frac{p \vee q}{p \to (p \vee q)}\mathbf{BK}_{25} \quad \frac{p \to q, p}{q}\mathbf{BK}_{26} \quad \frac{(p \vee q) \vee r}{p \vee (q \vee r)}\mathbf{BK}_{27} \quad \frac{p \to r, q \to r}{(p \vee q) \to r}\mathbf{BK}_{28}$$

PROOF. Below we present the derivations of the above rules:

- $\frac{p\vee q}{p\to(p\vee q)}\mathbf{BK}_{25}$:

1.	$p \vee q$	Assumption
2.	$p \vee \neg p$	$1, \mathbf{BK}_{15\star}$
3.	$(p \vee q) \vee (p \vee \neg p)$	$2, \mathbf{BK}_{19}$
4.	$\neg p \vee ((p \vee q) \vee p)$	$3, \mathbf{BK}_{21}, \mathbf{BK}_{23}$
5.	$\neg p \vee ((p \vee p) \vee q)$	$4, \mathbf{BK}^{\vee}_{23}, \mathbf{BK}^{\vee}_{21}$
6.	$\neg p \vee (p \vee q)$	$5, \mathbf{BK}^{\vee}_{23}, \mathbf{BK}^{\vee}_{22}, \mathbf{BK}^{\vee}_{23}$

- $\frac{p\to q, p}{q}\mathbf{BK}_{26}$: clearly from $\mathbf{BK}_2$ and $\mathbf{BK}_{20}$.
- $\frac{(p\vee q)\vee r}{p\vee(q\vee r)}\mathbf{BK}_{27}$: clearly from $\mathbf{BK}_{21}$ and $\mathbf{BK}_{23}$.
- $\frac{p\to r, q\to r}{(p\vee q)\to r}\mathbf{BK}_{28}$:

1.	$\neg p \vee r$	Assumption
2.	$\neg q \vee r$	Assumption
3.	$r \vee \neg p$	$1, \mathbf{BK}_{23}$
4.	$r \vee \neg q$	$2, \mathbf{BK}_{23}$
5.	$r \vee \neg(p \vee q)$	$3, 4, \mathbf{BK}^{\vee}_{12}$
6.	$\neg(p \vee q) \vee r$	$5, \mathbf{BK}_{23}$

■

Now, should we also add as primitive rules the $\vee$-lifted versions of the primitive $\vee$-lifted rules (and continue this *ad infinitum*)? The following result shows that this is not necessary.

LEMMA 15. *For every primitive rule* r *of* $\mathcal{H}_{\mathbf{BK}}$ *but* $\mathbf{BK}_{1\star}$*, the* $\vee$*-lifted version of* r *is derivable.*

PROOF. Note that the $\vee$-lifted version of the rules depicted in Definition 8, with the exception of $\mathbf{BK}_{1\star}$, are primitive in $\mathcal{H}_{\mathbf{BK}}$. Thus it remains to show that the $\vee$-lifted versions thereof are derivable in this system. Let $\frac{r\vee\varphi_1,\ldots,r\vee\varphi_m}{r\vee\psi}\mathsf{r}^\vee$ be the $\vee$-lifted version of $\frac{\varphi_1,\ldots,\varphi_m}{\psi}\mathsf{r}$, this one being any of the primitive rules of $\mathcal{H}_{\mathbf{BK}}$ but $\mathbf{BK}_{1\star}$. Below we show that $\frac{s\vee(r\vee\varphi_1),\ldots,s\vee(r\vee\varphi_m)}{s\vee(r\vee\psi)}\mathsf{r}^{\vee\vee}$ is derivable in $\mathcal{H}_{\mathbf{BK}}$:

$$
\begin{array}{rll}
1. & s\vee(r\vee\varphi_1) & \text{Assumption}\\
 & \vdots & \\
m. & s\vee(r\vee\varphi_m) & \text{Assumption}\\
m+1. & (s\vee r)\vee\varphi_1 & 1,\mathbf{BK}_{21}\\
 & \vdots & \\
2m. & (s\vee r)\vee\varphi_m & m,\mathbf{BK}_{21}\\
2m+1. & (s\vee r)\vee\psi & m+1,\ldots,2m,\mathsf{r}^\vee\\
2m+2. & s\vee(r\vee\psi) & 2m+1,\mathbf{BK}_{27}
\end{array}
$$

■

With the above, we also obtain the following result, which will be useful to abbreviate some of the upcoming proofs.

Corollary 16. *For every primitive rule* r *of* $\mathcal{H}_{\mathbf{BK}}$ *but* $\mathbf{BK}_{1\star}$*, the* $\to$*-lifted version of* r *is derivable.*

Proof. Let $\frac{\varphi_1,\ldots,\varphi_m}{\psi}\mathsf{r}$ be a primitive rule of $\mathcal{H}_{\mathbf{BK}}$ but $\mathbf{BK}_{1\star}$. Then, from $\neg r\vee\varphi_1,\ldots,\neg r\vee\varphi_m$, we derive, in view of Lemma 15, $\neg r\vee\psi$, and we are done. ■

These two results extend easily to rules that can be proved derivable in $\mathcal{H}_{\mathbf{BK}}$ without the use $\mathbf{BK}_{1\star}$.

Corollary 17. *Let* r *be a derivable rule of* $\mathcal{H}_{\mathbf{BK}}$ *having a proof that does not use* $\mathbf{BK}_{1\star}$*. Then* $\mathsf{r}^\vee$ *and* $\mathsf{r}^\to$ *are derivable as well.*

Proof. By induction on the length of the proof of r in $\mathcal{H}_{\mathbf{BK}}$ (one that does not employ $\mathbf{BK}_{1\star}$), applying essentially Lemma 15 and Corollary 16. ■

Even though $\mathbf{BK}$ does not admit a deduction theorem in the usual sense (see Theorem 11), the following result provides analogous deduction theorems that will be enough for our purposes.

Proposition 18. *Let* $\delta\in\{\varphi,\psi\}\subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$ *and let* t *be a proof in* $\mathcal{H}_{\mathbf{BK}}$ *witnessing that* $\Phi,\varphi\vee\psi,\delta\vdash_{\mathcal{H}_{\mathbf{BK}}}\gamma$*.*

If the rule $\mathbf{BK}\star_1$ *was not applied in* t*, then* $\Phi,\varphi\vee\psi\vdash_{\mathcal{H}_{\mathbf{BK}}}\delta\to\gamma$*.*

PROOF. Let us first consider the case $\delta = \varphi$. Suppose that t is $\gamma_1, \ldots, \gamma_n = \gamma$. We will prove that $P(j) {:=} \Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \gamma_j$ for all $1 \leq j \leq n$, using strong induction on j. For the base case $j = 1$, we have that $\gamma_1 \in \Phi \cup \{\varphi \vee \psi, \varphi\}$, leading to the following cases:

1. if $\gamma_1 \in \Phi$, use $\mathbf{BK}_{24}$.
2. if $\gamma_1 = \varphi \vee \psi$, use $\mathbf{BK}_{25}$.
3. if $\gamma_1 = \varphi$, use $\mathbf{BK}_{15\star}$.

Suppose now that (IH): $P(j)$ holds for all $j < k$. We want to prove $P(k)$. The cases when $\gamma_k \in \Phi \cup \{\varphi \vee \psi, \varphi\}$ are as in the base case. We have to consider then γ_k resulting from applications of the rules of the system, except for $\mathbf{BK}_{1\star}$. Assume that γ_k resulted from an application of an m-ary rule r using formulas $\gamma_{k_1}, \ldots, \gamma_{k_m}$ as premises, which must have appeared previously in the proof. By (IH), then, we have $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \gamma_{k_i}$ for each $1 \leq i \leq m$. By Corollary 16, then, we have $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \gamma_k$. In particular, for $k = n$, we obtain $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \gamma$, as desired. The case $\delta = \psi$ follows easily by commutativity of $\vee$ and the case $\delta = \varphi$ just proved. ■

With this deduction theorem, we can derive some rules more easily, as the next result shows.

PROPOSITION 19. *The following rules are derivable in $\mathcal{H}_{\mathbf{BK}}$:*

$$\frac{\neg p \vee \neg q}{\neg(p \wedge q)}\mathbf{BK}_{29} \qquad \frac{p \to q}{\neg(q \wedge \neg q)}\mathbf{BK}_{30}$$

PROOF. We present below the derivations.

- $\frac{\neg p \vee \neg q}{\neg(p \wedge q)}\mathbf{BK}_{29}$: first of all, we prove that $\neg p \vee \neg q, \neg q \vee \neg\neg q, \neg p \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg(p \wedge q)$:

1.	$\neg p$	Assumption
2.	$\neg q \vee \neg\neg q$	Assumption
3.	$\neg\neg q \vee \neg p$	$1, 2, \mathbf{BK}_{24}$
4.	$\neg\neg q \vee \neg q$	$2, \mathbf{BK}_{23}$
5.	$\neg\neg q \vee \neg(p \wedge q)$	$3, 4, \mathbf{BK}^{\vee}_{5}$
6.	$\neg\neg q \vee \neg\neg\neg q$	$2, \mathbf{BK}_{16\star}$
7.	$\neg\neg\neg q \vee \neg\neg q$	$6, \mathbf{BK}_{23}$
8.	$\neg\neg\neg q \vee q$	$7, \mathbf{BK}^{\vee}_{3}$
9.	$\neg\neg\neg q \vee \neg p$	$1, 3, \mathbf{BK}_{24}$
10.	$\neg\neg\neg q \vee \neg(p \wedge q)$	$8, 9, \mathbf{BK}^{\vee}_{6}$

11.	$\neg q \to \neg(p \wedge q)$	5, Def. of $\to$
12.	$\neg\neg q \to \neg(p \wedge q)$	10, Def. of $\to$
13.	$(\neg q \vee \neg\neg q) \to \neg(p \wedge q)$	$11, 12, \mathbf{BK}_{28}$
14.	$\neg(p \wedge q)$	$2, 13, \mathbf{BK}_{26}$

Similarly, we can show that $\neg p \vee \neg q, \neg p \vee \neg\neg p, \neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg(p \wedge q)$, also without using $\mathbf{BK}_{1\star}$. Since $\mathbf{BK}_{1\star}$ was not employed in such derivations, we have $\neg p \vee \neg q, \neg q \vee \neg\neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg p \to \neg(p \wedge q)$ and $\neg p \vee \neg q, \neg p \vee \neg\neg p \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg q \to \neg(p \wedge q)$, by Proposition 18. Since $\neg p \vee \neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg p \vee \neg\neg p$ (by $\mathbf{BK}_{15\star}$) and $\neg p \vee \neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg q \vee \neg\neg q$ (by $\mathbf{BK}_{16\star}$), by transitivity of $\vdash_{\mathcal{H}_{\mathbf{BK}}}$ (the Set-Fmla version of **(C)**), we have $\neg p \vee \neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg p \to \neg(p \wedge q)$ and $\neg p \vee \neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg q \to \neg(p \wedge q)$. By $\mathbf{BK}_{28}$, then, $\neg p \vee \neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} (\neg p \vee \neg q) \to \neg(p \wedge q)$. Finally, by $\mathbf{BK}_{26}$ (modus ponens), we obtain $\neg p \vee \neg q \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg(p \wedge q)$.

- $\frac{p \to q}{\neg(q \wedge \neg q)}\mathbf{BK}_{30}$: clearly from $\mathbf{BK}_{16\star}$ and $\mathbf{BK}_{29}$.

■

In the negation fragment of classical logic (call it $\mathbf{CL}_{\neg}$) we have the following deduction theorem: if $\Phi, \varphi \vdash_{\mathbf{CL}_{\neg}} \neg\varphi$, then $\Phi \vdash_{\mathbf{CL}_{\neg}} \neg\varphi$. Similarly to what we did in Proposition 18, we show now that this result also holds for **BK** provided $\varphi \vee \psi$ is present in the context Φ. In this case, however, we do not need to impose any restriction on the rules applied in the derivations witnessing the consecution in the assumption. This result will also be useful for proving the desired completeness result for $\mathcal{H}_{\mathbf{BK}}$.

Proposition 20. *Let $\delta \in \{\varphi, \psi\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$. If $\Phi, \varphi \vee \psi, \delta \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\delta$, then $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\delta$.*

Proof. We begin with the case $\delta = \varphi$. Let $\mathtt{t} = \gamma_1, \ldots, \gamma_n$ be a proof witnessing that $\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\varphi$. In case no application of $\mathbf{BK}_{1\star}$ is used in $\mathtt{t}$, we have $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \neg\varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\varphi \vee \neg\varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\varphi$ by Proposition 18 and $\mathbf{BK}_{22}$, as desired. On the other hand, suppose that γ_k was the formula produced by the first application of $\mathbf{BK}_{1\star}$. Then $k > 2$ and there are γ_{m_1} and $\gamma_{m_2} = \neg\gamma_{m_1}$, with $m_1, m_2 < k$, such that $\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma_{m_1}$ and $\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\gamma_{m_1}$. By Proposition 18, then, we have (a): $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \gamma_{m_1}$, and $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \neg\gamma_{m_1}$, and so, by Corollary 16, $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to (\gamma_{m_1} \wedge \neg\gamma_{m_1})$, that is, $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\varphi \vee (\gamma_{m_1} \wedge \neg\gamma_{m_1}) \vdash_{\mathcal{H}_{\mathbf{BK}}} (\gamma_{m_1} \wedge \neg\gamma_{m_1}) \vee \neg\varphi$. But $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg(\gamma_{m_1} \wedge \neg\gamma_{m_1})$ by (a) and $\mathbf{BK}_{30}$, and thus $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\varphi$ by $\mathbf{BK}_{20}$. Now, for $\delta = \psi$, we have that from $\Phi, \varphi \vee \psi, \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\psi$ and the rule $\mathbf{BK}_{23}$, we get $\Phi, \psi \vee \varphi, \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\psi$. By Proposition 20, we get $\Phi, \psi \vee \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\psi$ and, again by $\mathbf{BK}_{23}$, we have $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\psi$. ■

A consequence of the previous result is the following.

PROPOSITION 21. *Let $\delta_1, \delta_2 \in \{\varphi, \psi\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$ with $\delta_1 \neq \delta_2$, and $\gamma_1, \ldots, \gamma_n$ be a proof in $\mathcal{H}_{\mathbf{BK}}$ witnessing that $\Phi, \varphi \vee \psi, \delta_1 \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$. If the rule $\mathbf{BK}_{1\star}$ was applied in such proof, then $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \delta_2$.*

PROOF. We will prove the case $\delta_1 = \varphi$ and the other will be analogous. Suppose that γ_k was the formula produced by the first application of $\mathbf{BK}_{1\star}$. Then $k > 2$ and there are γ_{m_1} and $\gamma_{m_2} = \neg\gamma_{m_1}$, with $m_1, m_2 < k$, such that $\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma_{m_1}$ and $\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\gamma_{m_1}$. But then, by $\mathbf{BK}_{1\star}$, we have $\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\varphi$. By Proposition 20, then, we have $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \neg\varphi$, and then $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \psi$ by $\mathbf{BK}_{20}$. ■

In Proposition 11, we proved that $\mathbf{BK}$ does not allow to express a connective satisfying (disj). Nevertheless, we now show that having $\varphi \vee \psi$ in the context is also enough to recover this result.

LEMMA 22. *For all $\Phi, \{\varphi, \psi, \gamma\} \subseteq L_{\Sigma_{\wedge\vee\neg}}(P)$, we have*
$\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$ and $\Phi, \varphi \vee \psi, \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$,
if, and only if, $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$.

PROOF. The right-to-left direction is obvious by reflexivity of the consequence relation. For the left-to-right direction, suppose that t_1 and t_2 are witnesses of $\Phi, \varphi \vee \psi, \varphi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$ and $\Phi, \varphi \vee \psi, \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$, respectively. Consider the following cases:

1. In both there are no applications of $\mathbf{BK}_{1\star}$: then, by Proposition 18, we have $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \varphi \to \gamma$ and $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \psi \to \gamma$. Thus $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} (\varphi \vee \psi) \to \gamma$, by $\mathbf{BK}_{28}$, and $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$, by $\mathbf{BK}_{26}$.
2. If there is an application of $\mathbf{BK}_{1\star}$ in t_1: then, by Proposition 21, we have $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \psi$. Then, by transitivity considering $\Phi, \varphi \vee \psi, \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$, we obtain $\Phi, \varphi \vee \psi \vdash_{\mathcal{H}_{\mathbf{BK}}} \gamma$.
3. If there is an application of $\mathbf{BK}_{1\star}$ in t_2: similar to the previous case.

■

Finally we get to the desired axiomatization result.

THEOREM 23. $\vdash_{\mathcal{H}_{\mathbf{BK}}} = \vdash_{\mathbf{BK}}$.

PROOF. We will show that $\Phi \vdash_{\mathcal{H}_{\mathbf{BK}}} \psi$ if, and only if, $\Phi \rhd_{\mathsf{R}_{\mathbf{BK}\star}} \{\psi\}$. The left-to-right direction easily follows, since every rule of $\mathcal{H}_{\mathbf{BK}}$ is sound with respect to the matrix of **BK** by Lemma 13, and thus derivable in $\mathsf{R}_{\mathbf{BK}\star}$. From the right to the left, we will show by induction on the structure of derivations in $\mathsf{R}_{\mathbf{BK}\star}$ that $P(\mathsf{t})$: if t witnesses that $\Phi \rhd_{\mathsf{R}_{\mathbf{BK}\star}} \{\psi\}$, then there is a proof in $\mathcal{H}_{\mathbf{BK}}$ bearing witness to $\Phi \vdash_{\mathcal{H}_{\mathbf{BK}}} \psi$. In the base case, t has a single node, meaning that $\psi \in \Phi$, and we are done by reflexivity of $\vdash_{\mathcal{H}_{\mathbf{BK}}}$. In the inductive step, we assume $P(\mathsf{t}')$ for each subtree t' of t and consider t resulting from an application of the rules of $\mathsf{R}_{\mathbf{BK}\star}$. Let us consider three cases:

1. t results from a rule that is derivable in $\mathcal{H}_{\mathbf{BK}}$: here, there is nothing to do, as the same rule may be applied to produce the desired derivation.
2. t results from an application of $\mathbf{BK}_1^{\rhd}$: use $\mathbf{BK}_{1\star}$ instead.
3. t results from an application of $\mathbf{BK}_{20}^{\rhd}$: if the root of t is labelled with Γ, then $\gamma \vee \delta \in \Gamma$, and we have, by the induction hypothesis, (a): $\Gamma, \gamma \vee \delta, \gamma \vdash_{\mathcal{H}_{\mathbf{BK}}} \psi$ and (b): $\Gamma, \gamma \vee \delta, \delta \vdash_{\mathcal{H}_{\mathbf{BK}}} \psi$. By Lemma 22, then, we obtain the desired result. ∎

6. Concluding Remarks

Taking stock of what we achieved in the previous sections, we highlight that we have settled fundamental questions regarding **BK** and **PWK**, two logics that are among the main subjects of this Special Issue. We also wish to mention an interesting corollary of our results, namely that some finite subset of the axioms employed in the papers [8, 12] must already suffice to axiomatize each of the two logics. We leave this observation as a suggestion for future developments.

Besides the intrinsic interest in the results established above, the present paper may also be seen as another illustration of the differences in expressive power among the various available proof-theoretic formalisms in logic, and in particular between Set–Set over Set-Fmla H-systems. The latter are obviously less expressive than the former—even weaker if compared to sequent systems—even though they afford more fine-grained tools for comparing and also for combining logics (in particular when one wishes to introduce the least possible interactions), as recent results amply demonstrate [14, 28].

Another direction for future research worth mentioning is the study of these and other logics associated to the algebra $\mathbf{B_u}$ (and other three-valued algebras) in the setting of different kinds of H-systems. In particular, a two-dimensional version of Set–Set H-systems [21, 23], whose induced logics are the so-called B-*consequence relations* [5], may be employed as a uniform setting for investigating *pure consequence relations* (like **BK** and **PWK**), their intersection (*order-theoretic consequence relations*) and *mixed consequence relations* (we use here the terminology of [16]), the latter being non-Tarskian consequence relations (lacking either reflexivity [26] or transitivity [19]).

Not only can a two-dimensional logic express all of these very different notions of logics in the same logical environment: we also have that it has a neat analytic two-dimensional axiomatization. That is, this two-dimensional logic has not only great theoretical value due to its expressiveness, but also constitutes an important tool for using the above-mentioned logics and studying their properties.

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