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Boolean Connexive Logic and Content Relationship

Abstract. We present here some Boolean connexive logics (BCLs) that are intended to be connexive counterparts of selected Epstein's content relationship logics (CRLs). The main motivation for analyzing such logics is to explain the notion of connexivity by means of the notion of content relationship. The article consists of two parts. In the first one, we focus on the syntactic analysis by means of axiomatic systems. The starting point for our syntactic considerations will be the smallest BCL and the smallest CRL. In the first part, we also identify axioms of Epstein's logics that, together with the connexive principles, lead to contradiction. Moreover, we present some principles that will be equivalent to the connexive theses, but not to the content connexive theses we will propose. In the second part, we focus on the semantic analysis provided by relating- and set-assignment models. We define sound and complete relating semantics for all tested systems. We also indicate alternative relating models for the smallest BCL, which are not alternative models of the connexive counterparts of the considered CRLs. We provide a set-assignment semantics for some BCLs, giving thus a natural formalization of the content relationship understood either as content sharing or as content inclusion.

Keywords: Boolean connexive logic, Logic of content relationship, Relating semantics, Set-assignment semantics.

1. Introduction

A connexive logic is a contra-classical logic that standardly contains Aristotle's Theses $\neg(\varphi \to \neg\varphi), \neg(\neg\varphi \to \varphi)$ and Boethius' Theses $(\varphi \to \psi) \to \neg(\varphi \to \neg\psi), (\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$. One of the main challenges for formal-philosophical research in connexive logic is to clarify the concept of connexivity. There are various attempts to overcome such challenge, for example:

- connexivity explained by the notion of compatibility (see [22, 24, 27–29])
- connexivity explained by the notion of consistency (see [18, 21])
- connexivity explained by the notion of relevance (see [5, 22, 31])

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- connexivity explained by the notion of negation as cancellation (see [30, 32, 33])
- connexivity explained by pragmatics (see [6, 14]).

Our article aims to present connexive logic motivated by analyzing a content relationship. In general terms, connexivity is a non-extensional relationship between sentences that cannot be defined merely in terms of sequences of logical values. Through connexive theses, we try to grasp properties of sentences other than their having a logical value. By such theses, we state that sentences can be related to each other in some way, but the sentence and its negation are never related, and the relation of two sentences excludes the first being simultaneously related to the negation of the second sentence. In our article, we assume that this still unidentified relationship, outlined by determining which sentences cannot be related, can be explained through the concept of a content relationship. In this approach, we consider the content relationship, understood in one of the following ways:

(a) φ and ψ are related iff the contents of φ and ψ have something in common, formally: $s(\varphi) \cap s(\psi) \neq \emptyset$

(b) φ and ψ are related iff the content of φ is contained in the content of ψ, formally: $s(\varphi) \subseteq s(\psi)$

(c) φ and ψ are related iff the content of ψ is contained in the content of φ, formally: $s(\psi) \subseteq s(\varphi)$

(d) φ and ψ are related iff the contents of φ and ψ are equal, formally: $s(\varphi) = s(\psi)$,

where $s(\varphi)$ and $s(\psi)$ represent contents of the sentences φ and ψ, respectively. Such an approach was proposed by Richard Epstein [3] in the context of conditional sentence analysis. According to Epstein's analysis, conditional sentences should be interpreted by the following truth condition:

$\varphi \rightarrow \psi$ is true iff 1) φ is false or ψ is true, 2) φ and ψ are related. (†)

In other words, the Epstein implication is a special case of relating implication (see [7, 13, 16]).

From the syntactic point of view, the content relationships analyzed by Epstein can be expressed by the schema $\varphi \rightarrow (\psi \rightarrow \psi)$ (see [3, 16]). Indeed, taking a representation of content s for which we assume at least non-emptiness (i.e., $s(\varphi) \neq \emptyset$) and that the content of a sentence is the same as the content of its self-implication (i.e., $s(\varphi) = s(\varphi \rightarrow \varphi)$), we obtain that for any of the considered content relations (in the sense of (a), (b) (c) or (d) above), φ is related with φ, and φ is related with ψ iff φ is related with $\psi \to \psi$. Thus, we have:

$$\varphi \to (\psi \to \psi) \text{ is true iff } \varphi \text{ and } \psi \text{ are related.}$$

Notice that by (†) and since ψ is related with ψ, in the sense of (a), (b), (c) or (d), $\psi \to \psi$ is always true.

We will call an arbitrary relation over sentences a *content relation* iff there is a representation of a content s such that at least $s(\varphi) \neq \emptyset$ and $s(\varphi \to \varphi) = s(\varphi)$, and the relation is equal to a relation specified as in (a), (b), (c) or (d).

In the article, we consider the possibility of combining two logics. The first one is Boolean connexive logic proposed by Jarmużek and Malinowski [11] (cf. also [12,15,19,25]). Such logic is based on the idea of obtaining a connexive logic by modifying classical logic as less as possible. In this case, connexive implication is expressed by means of a relating one, i.e., by truth condition (†) (see [7,13,16]). However, in this case, being related is taken as a primitive notion, but it has to satisfy some additional, connexively justified restrictions like, for instance, that no sentence is related with its negation. In this way, Boolean connexive logic is a special case of relating logic based on Boolean logic. The second logic is the content relationship logic proposed by Epstein [3] (cf. also [8,16]). Such logic is an attempt to take into account the content relationship in the formal analysis of conditional sentences. His logic might also be considered a special case of relating logic based on Boolean logic. In this case, as noted above, an implication is also a relating one but of a different kind than a connexive one, and to combine them some substantial modifications of the systems are required.

Our main objective is to advance an explanation of the notion of connexivity by means of the notion of content relationship. Boolean connexive logic is a Boolean logic with a relating implication, just as Epstein's logics of content relationship are. Nonetheless, Boolean connexive logic, at least the smallest one and some of its extensions considered in the literature (see [11,12,15]), lacks the means to express Epstein's content relationships in the object language, and whereas Epstein's logics do clearly have those resources, they are not connexive. Thus, combining both approaches makes sense.

The structure of the paper is as follows. In "Preliminaries" and "Boolean Connexive Logic and Content Relationship Logic" sections we present a syntactic analysis of Boolean connexive logic and content relationship logic. And, we introduce exemplary connexive counterparts of content relationship logics. In "Relating Semantics and Set-Assignment Semantics" section, we present semantic analyses for the considered logics, including proofs of soundness and completeness.

2. Preliminaries

In what follows we consider a propositional language $\mathcal{L}$ consisting of propositional variables $p, q, r, p_1, q_1, r_1, \ldots$; Boolean connectives $\neg$, $\wedge$, $\vee$; relating implication $\rightarrow$ and brackets:), (. Thus, $\mathcal{L}$ is an expansion of the Boolean language obtained by adding relating implication $\rightarrow$. The set of formulas in $\mathcal{L}$ is defined in the standard way and denoted by For. We use the following abbreviations: $\varphi \supset \psi {:=} \neg\varphi \vee \psi$ and $\varphi \equiv \psi {:=} (\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi)$.

In the metalanguage, we will mainly use expressions of natural language. In some cases we will use the following symbols: $\sim, \forall, \Rightarrow$ to denote the following metalinguistic constants: negation, universal quantifier and implication, respectively. We give the considered metalinguistic expressions a classical meaning.

Let us also distinguish some sets of formulas, which we will refer to in the sequel. For any $\varphi \in \mathsf{For}$ we define the operation $\downarrow$ in the following way:

$$\downarrow\varphi = \begin{cases} \varphi, & \text{if } \varphi \neq \psi \rightarrow \psi, \text{for any } \psi \in \mathsf{For} \\ \downarrow\psi, & \text{if } \varphi = \psi \rightarrow \psi, \text{for some } \psi \in \mathsf{For}. \end{cases}$$

We define the $\downarrow$-complexity of formulas in the following way: the $\downarrow$-complexity of φ is equal to 1, if $\varphi \neq \psi \rightarrow \psi$, for any $\psi \in \mathsf{For}$; the $\downarrow$-complexity of φ is equal to the $\downarrow$-complexity of ψ plus 1, if $\varphi = \psi \rightarrow \psi$, for some $\psi \in \mathsf{For}$.

For any $\varphi \in \mathsf{For}$, by $\mathsf{For}_{\downarrow\varphi}$ we denote the least set $\Sigma \subseteq \mathsf{For}$ such that the following conditions hold:

- $\downarrow\varphi \in \Sigma$
- $\psi \in \Sigma \Rightarrow \psi \rightarrow \psi \in \Sigma$.

We have the following fact:

Fact 2.1. *1. For any $\varphi \in \mathsf{For}$, $\varphi \in \mathsf{For}_{\downarrow\varphi} = \mathsf{For}_{\downarrow\varphi\rightarrow\varphi}$.*

2. For any $\varphi, \psi \in \mathsf{For}$, $\varphi \in \mathsf{For}_{\downarrow\psi}$ iff $\varphi \rightarrow \varphi \in \mathsf{For}_{\downarrow\psi}$.

3. For any $\varphi, \psi \in \mathsf{For}$, $\downarrow\varphi = \downarrow\psi$ iff $\mathsf{For}_{\downarrow\varphi} \cap \mathsf{For}_{\downarrow\psi} \neq \emptyset$.

4. For any $\varphi, \psi, \chi \in \mathsf{For}$, if $\varphi, \psi \in \mathsf{For}_{\downarrow\chi}$, then $\mathsf{For}_{\downarrow\varphi} \cap \mathsf{For}_{\downarrow\psi} \neq \emptyset$.

Proof. For 1 and 2 the proof is straightforward, by means of the definition of $\downarrow$ and the definition of the considered sets of formulas.

For 3, the left-to-right implication holds by 1. We will prove the right-to-left implication. Let us assume for simplicity that $\varphi \neq \chi \to \chi$ and $\psi \neq \chi \to \chi$, for any $\chi \in \mathsf{For}$. We can show then that if $\chi \in \mathsf{For}_{\downarrow\varphi}$ and $\chi \in \mathsf{For}_{\downarrow\psi}$, then $\downarrow\varphi = \downarrow\chi = \downarrow\psi$. Let χ be of $\downarrow$-complexity equal to 1. Suppose $\chi \in \mathsf{For}_{\downarrow\varphi}$ and $\chi \in \mathsf{For}_{\downarrow\psi}$. Thus, $\chi = \varphi$ and $\chi = \psi$. Therefore, $\downarrow\varphi = \downarrow\chi = \downarrow\psi$. Let χ be of $\downarrow$-complexity equal to $n+1$, where $n \geq 1$. Suppose $\chi \in \mathsf{For}_{\downarrow\varphi}$ and $\chi \in \mathsf{For}_{\downarrow\psi}$. Let $\chi = \chi' \to \chi'$. Thus, $\chi' \in \mathsf{For}_{\downarrow\varphi}$ and $\chi' \in \mathsf{For}_{\downarrow\psi}$. Therefore, $\downarrow\varphi = \downarrow\chi = \downarrow\psi$, since $\downarrow\chi = \downarrow\chi'$.

For 4. Let us assume for simplicity that $\chi \neq \chi' \to \chi'$. Let φ be of $\downarrow$-complexity equal to 1. Suppose $\varphi, \psi \in \mathsf{For}_{\downarrow\chi}$. Thus $\varphi = \chi$. Therefore, $\psi \in \mathsf{For}_{\downarrow\varphi}$. Let φ be of $\downarrow$-complexity equal to $n+1$, where $n \geq 1$. Suppose $\varphi, \psi \in \mathsf{For}_{\downarrow\chi}$. We have that, $\varphi = \varphi' \to \varphi' \in \mathsf{For}_{\downarrow\chi}$. Thus, $\varphi' \in \mathsf{For}_{\downarrow\chi}$. Therefore, $\mathsf{For}_{\downarrow\varphi} \cap \mathsf{For}_{\downarrow\psi} \neq \emptyset$, since $\downarrow\varphi = \downarrow\varphi' \to \varphi' = \downarrow\varphi'$. ■

By Fact 2.1 we obtain the following corollary:

COROLLARY 2.2. *1. For any* $\varphi, \psi \in \mathsf{For}$, $\mathsf{For}_{\downarrow\varphi} \cap \mathsf{For}_{\downarrow\psi} \neq \emptyset$ *iff* $\mathsf{For}_{\downarrow\varphi} = \mathsf{For}_{\downarrow\psi}$.

2. For any $\varphi, \psi \in \mathsf{For}$, *either* $\varphi \notin \mathsf{For}_{\downarrow\psi}$ *or* $\neg\varphi \notin \mathsf{For}_{\downarrow\psi}$.

3. For any $\varphi, \psi \in \mathsf{For}$, *if* $\downarrow\varphi \neq \neg\psi$, *then* $\neg\psi \notin \mathsf{For}_{\downarrow\varphi}$.

PROOF. For 1. By Fact 2.1.3, if $\mathsf{For}_{\downarrow\varphi} \cap \mathsf{For}_{\downarrow\psi} \neq \emptyset$, then $\downarrow\varphi = \downarrow\psi$. Thus, if $\mathsf{For}_{\downarrow\varphi} \cap \mathsf{For}_{\downarrow\psi} \neq \emptyset$, then $\mathsf{For}_{\downarrow\varphi} = \mathsf{For}_{\downarrow\psi}$. By Fact 2.1.1, $\mathsf{For}_{\downarrow\varphi} \neq \emptyset$. Thus, if $\mathsf{For}_{\downarrow\varphi} = \mathsf{For}_{\downarrow\psi}$, then $\mathsf{For}_{\downarrow\varphi} \cap \mathsf{For}_{\downarrow\psi} \neq \emptyset$.

For 2. We have that for any $\varphi \in \mathsf{For}$, $\downarrow\varphi \neq \downarrow\neg\varphi$. Since $\downarrow\neg\varphi = \neg\varphi \neq \downarrow\varphi$. Thus, by Fact 2.1 cases 3 and 4, for any $\varphi \in \mathsf{For}$, either $\varphi \notin \mathsf{For}_{\downarrow\psi}$ or $\neg\varphi \notin \mathsf{For}_{\downarrow\psi}$.

For 3. If $\downarrow\varphi \neq \neg\psi$, then $\downarrow\varphi \neq \neg\psi = \downarrow\neg\psi$, for any $\psi \in \mathsf{For}$. Thus, if $\downarrow\varphi \neq \neg\psi$, then, by Fact 2.1.3, $\neg\psi \notin \mathsf{For}_{\downarrow\varphi}$, for any $\psi \in \mathsf{For}$. ■

Let us also introduce some additional notation. For any schema $(\mathtt{X})$ of the form $\varphi \equiv \psi$, by $(\mathtt{X}^{\supset})$ we denote the schema $\varphi \supset \psi$, and by $(\mathtt{X}^{\subset})$ we denote the schema $\psi \supset \varphi$. We say that a set of formulas $\Sigma \subseteq \mathsf{For}$ *contains the schema* $(\mathtt{X})$ iff all formulas of the form of $(\mathtt{X})$ are elements of Σ.

Let $(\mathtt{BL})$ be a set of truth-functional tautologies defined with respect to $\neg, \wedge, \vee$ in $\mathcal{L}$. A *Boolean logic with relating implication* (*BLRI*) is any set of formulas $\Lambda \subseteq \mathsf{For}$ such that Λ contains all truth-functional tautologies, i.e.:

$$\mathsf{BL} \subseteq \Lambda \tag{BL}$$

Λ contains *Weakening of Relating Implication*, i.e., it contains the following schema:

$$(\varphi \to \psi) \supset (\varphi \supset \psi) \tag{$\mathtt{W}_{\to}$}$$

and Λ is closed under *Material Detachment*, i.e., for all $\varphi, \psi \in \mathsf{For}$:

$$\varphi, \varphi \supset \psi \in \Lambda \Rightarrow \psi \in \Lambda. \tag{MD}$$

The least BLRI is denoted by **BR**. For any BLRI Λ, $\Lambda \oplus \Sigma$ denotes the least BLRI that contains $\Lambda \cup \Sigma$ (for BLRI see [9]).

Notice that by ($\mathtt{W}_{\rightarrow}$) and (MD), if Λ is a BLRI, then Λ is closed under *Modus Ponens*, i.e., for all $\varphi, \psi \in \mathsf{For}$:

$$\varphi, \varphi \rightarrow \psi \in \Lambda \Rightarrow \psi \in \Lambda. \tag{MP}$$

Notice also that if Λ is a BLRI and contains *Strengthening to Relating Implication*, i.e., it contains the following schema:

$$(\varphi \supset \psi) \supset (\varphi \rightarrow \psi), \tag{$\mathtt{S}_{\rightarrow}$}$$

then $\rightarrow$ in Λ behaves exactly like material implication, and so $\mathbf{BR} \oplus (\mathtt{S}_{\rightarrow})$ is another formulation of classical logic.

Let Λ be a BLRI for the remainder of this section. We define a *relation of syntactic consequence of* Λ $\vdash_\Lambda \subseteq \mathcal{P}(\mathsf{For}) \times \mathsf{For}$ in the following way: for all $\Sigma \cup \{\varphi\} \subseteq \mathsf{For}$, $\Sigma \vdash_\Lambda \varphi$ iff there is $\{\psi_1, \ldots, \psi_n\} \subseteq \Sigma$ such that $(\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi \in \Lambda$.

We say that a formula φ is *derivable from* Σ *in* Λ iff $\Sigma \vdash_\Lambda \varphi$. Similarly, a schema (X) is *derivable from* Σ *in* Λ iff all formulas of the form of (X) are derivable from Σ. Obviously, φ is derivable from Λ in Λ iff Λ contains φ, and similarly for schemata. We also say that φ and ψ are *mutually derivable* in Λ if φ is derivable from ψ in Λ and ψ is derivable from φ in Λ, and similarly for schemata.

We define a notion of an inconsistent set with respect to Λ in a standard way. A set $\Sigma \subseteq \mathsf{For}$ is Λ-*inconsistent* iff $\Sigma \vdash_\Lambda p \wedge \neg p$. And $\Sigma \subseteq \mathsf{For}$ is Λ-*consistent* iff Σ is not inconsistent. We can easily prove that if $\Sigma \nvdash_\Lambda \varphi$, then $\Sigma \cup \{\neg\varphi\}$ is Λ-consistent. Moreover, we say Λ is an *inconsistent logic* iff Λ is Λ-inconsistent. Consequently, Λ is an inconsistent logic iff there is $\varphi \in \mathsf{For}$ such that $\varphi, \neg\varphi \in \Lambda$.[1]

3. Boolean Connexive Logic and Content Relationship Logic

Our goal is to combine the notion of content relationship proposed by Epstein and connexivity captured in Boolean connexive logic as presented by

[1] By (BL) and (MD), there is only one inconsistent BLRI, namely the set of formulas For. But we will use the plural 'inconsistent logics', having in mind different formulations of the only inconsistent logic in this setting.

Jarmużek and Malinowski. In other words, we want to define a Boolean connexive logic motivated by an analysis of content relationship. We start with a syntactic presentation of the logics we are interested in.

3.1. Boolean Connexive Logic

A *Boolean connexive logic* (*BCL*) is any set of formulas $\Lambda \subseteq \mathsf{For}$ such that Λ is a BLRI, contains *Aristotle's Theses*, i.e., it contains the following schemata:

$$\neg(\varphi \to \neg\varphi) \tag{A1}$$

$$\neg(\neg\varphi \to \varphi) \tag{A2}$$

and contains *Boethius's Theses*, i.e., it contains the following schemata:

$$(\varphi \to \psi) \to \neg(\varphi \to \neg\psi) \tag{B1}$$

$$(\varphi \to \neg\psi) \to \neg(\varphi \to \psi). \tag{B2}$$

Obviously, any BCL contains the so-called *weak Boethius' Theses*, i.e., it contains the following schemata:

$$(\varphi \to \psi) \supset \neg(\varphi \to \neg\psi) \tag{wB}$$

$$(\varphi \to \neg\psi) \supset \neg(\varphi \to \psi). \tag{wB$'$}$$

FACT 3.1. *1. (wB) is derivable from (B1) in any BLRI.*

2. (wB$'$) is derivable from (B2) in any BLRI.

3. (wB) and (wB$'$) are mutually derivable in any BLRI.

4. (wB$'$) is derivable from (B1) in any BLRI.

5. (wB) is derivable from (B2) in any BLRI.

PROOF. By ($\mathtt{W}_{\to}$), (MD) and (BL). ■

A *weak Boolean connexive logic* (*wBCL*) is any set of formulas $\Lambda \subseteq \mathsf{For}$ such that Λ is a BLRI and contains (A1), (A2), (wB).

The least BCL and the least wBCL are denoted by **BC** and **wBC**, respectively, i.e., $\mathbf{BC} = \mathbf{BR} \oplus \{\mathtt{A1}, \mathtt{A2}, \mathtt{B1}, \mathtt{B2}\}$ and $\mathbf{wBC} = \mathbf{BR} \oplus \{\mathtt{A1}, \mathtt{A2}, \mathtt{wB}\} = \mathbf{BR} \oplus \{\mathtt{A1}, \mathtt{A2}, \mathtt{wB}'\}$. It is known that neither **BC** nor **wBC** contains *Symmetry of Implication*, i.e., they do not contain the following schema (see [11]):

$$(\varphi \to \psi) \supset (\psi \to \varphi).$$

Let us note that in any BCL we can change (A1) and (A2) to the following weaker versions of Aristotle's Theses:

$$\neg(\varphi \to \neg\varphi) \vee \varphi \tag{A1$^\vee$}$$

$$\neg(\neg\varphi \to \varphi) \vee \neg\varphi. \tag{A2$^\vee$}$$

And (wB) can be changed to the following weaker version of weak Boethius' Thesis:

$$(\varphi \to \psi) \supset (\neg(\varphi \to \neg\psi) \vee \varphi). \tag{wB$^\vee$}$$

FACT 3.2. *1. (A1) and (A1$^\vee$) are mutually derivable in any BLRI.*

2. (A2) and (A2$^\vee$) are mutually derivable in any BLRI.

3. (wB) and (wB$^\vee$) are mutually derivable in any BLRI.

PROOF. By ($W_\to$), (MD) and (BL). ■

Thus $\mathbf{wBC} = \mathbf{BR} \oplus \{A1^\vee, A2^\vee, wB^\vee\}$ and $\mathbf{BC} = \mathbf{BR} \oplus \{A1^\vee, A2^\vee, B1, B2\}$.

These weaker connexive schemata formulated by means of disjunction do not affect the basic BCL but, as we will see, they are not enough to formulate some extensions of the basic BCL, and so neither Aristotle's Theses nor weak Boethius' Thesis are.

3.2. Content Relationship Logics

Let us now consider content relationship logic. For any of Epstein's logics, the following schema plays the role of expressing syntactically that two sentences φ and ψ are related with respect to content (see [3, pp. 77–78]):

$$\varphi \to (\psi \to \psi). \tag{E}$$

However, it is not the case for every BLRI that the schema (E) allows to express that two sentences are related. Any BLRI for which that is possible must contain the following schemata:

$$\varphi \to \varphi \tag{$R_\to$}$$

$$(\varphi \to \psi) \supset (\varphi \to (\psi \to \psi)) \tag{$W^+_\to$}$$

$$((\varphi \supset \psi) \wedge (\varphi \to (\psi \to \psi)) \supset (\varphi \to \psi). \tag{$S^+_\to$}$$

The schema ($R_\to$) says that $\to$ is reflexive and might be called *Reflexivity of Implication Thesis*. By means of ($W^+_\to$) and ($S^+_\to$) we get Epstein's definition of relating implication:

$$(\varphi \to \psi) \equiv ((\varphi \supset \psi) \wedge (\varphi \to (\psi \to \psi))).$$

The schema ($\mathtt{W}^+_\to$) is a kind of weakening of relating implication and ($\mathtt{S}^+_\to$) a kind of strengthening to relating implication. By means of additional schemata formulated with the help of the schema ($\mathtt{E}$) we can specify what content relationship we want to consider.

In order to define basic content relationship logic we need two more schemata:

$$((\varphi \to \varphi) \to (\psi \to \psi)) \equiv (\varphi \to (\psi \to \psi)) \tag{CR0}$$

$$(((\varphi \to (\psi \to \psi)) \wedge (\psi \to (\chi \to \chi))) \supset$$
$$\supset (\varphi \to (\chi \to \chi))) \vee ((\varphi' \to (\psi' \to \psi')) \supset (\psi' \to (\varphi' \to \varphi'))). \tag{DS}$$

A *content relationship logic* (*CRL*) is any set of formulas $\Lambda \subseteq \mathsf{For}$ such that Λ is a BLRI and contains the schemata ($\mathtt{R}_\to$), ($\mathtt{W}^+_\to$), ($\mathtt{S}^+_\to$), ($\mathtt{CR0}$) and ($\mathtt{DS}$). The least content relationship logic is denoted by **CR**, i.e., $\mathbf{CR} = \mathbf{BR} \oplus \{\mathtt{R}_\to, \mathtt{S}^+_\to, \mathtt{W}^+_\to, \mathtt{CR0}, \mathtt{DS}\}$.

There are two more schemata expressed by ($\mathtt{E}$) that capture important properties of any content relationship considered by Epstein:

$$\varphi \to (\varphi \to \varphi) \tag{CR1}$$

$$(\varphi \to (\psi \to \psi)) \equiv (\varphi \to ((\psi \to \psi) \to (\psi \to \psi))). \tag{CR2}$$

Let us now comment on ($\mathtt{DS}$), ($\mathtt{CR0}$), ($\mathtt{CR1}$) and ($\mathtt{CR2}$). According to Epstein's analysis, all content relationships are either transitive or symmetric, and ($\mathtt{DS}$) allows us to say that we only consider relations between sentences that are either transitive or symmetric.[2] ($\mathtt{CR1}$) expresses that any sentence is related to itself, so any sentence shares content with itself and its content contains its content. ($\mathtt{CR2}$) with ($\mathtt{CR0}$) express that a sentence is exchangeable with a conditional sentence whose antecedent and consequent are that very sentence. Thus, those two schemata allow saying that content is insensitive to the repetition of sentences within conditional sentences. In other words, that the repetition of a sentence in the conditional does not influence the content. We have the following fact:

FACT 3.3. *1. (CR1) is derivable in any CRL.*

2. (CR2) is derivable in any CRL.

PROOF. For 1, by ($\mathtt{R}_\to$), ($\mathtt{W}^+_\to$) and ($\mathtt{MD}$). For 2, by ($\mathtt{R}_\to$), ($\mathtt{W}^+_\to$), ($\mathtt{S}^+_\to$), ($\mathtt{MD}$) and ($\mathtt{BL}$). ■

[2] The schemata of this kind were first introduced and analyzed in [10] in the context of finding a meet of two relating logics.

Epstein's logic, which captured some specific notions of content relationship, might be defined by means of some additional schemata that enable us to express a particular understanding of content relationship. We will focus on two systems: the logic **S** (for Symmetric Relatedness Logic) and the logic **DD** (for Dual Dependence Logic).

S is the least set $\Lambda \subseteq \mathsf{For}$ such that Λ is a CRL and contains the following schemata:

$$(\varphi \to (\psi \to \psi)) \supset (\psi \to (\varphi \to \varphi)) \tag{S0}$$

$$(\neg\varphi \to (\psi \to \psi)) \equiv (\varphi \to (\psi \to \psi)) \tag{S1}$$

$$((\varphi \wedge \psi) \to (\chi \to \chi)) \equiv ((\varphi \to (\chi \to \chi)) \vee (\psi \to (\chi \to \chi))) \tag{S2}$$

$$((\varphi \vee \psi) \to (\chi \to \chi)) \equiv ((\varphi \to (\chi \to \chi)) \vee (\psi \to (\chi \to \chi))) \tag{S3}$$

$$((\varphi \to \psi) \to (\chi \to \chi)) \equiv ((\varphi \to (\chi \to \chi)) \vee (\psi \to (\chi \to \chi))). \tag{S4}$$

On the other hand, **DD** is the least set $\Lambda \subseteq \mathsf{For}$ such that Λ is a CRL and contains (S1) plus the following schemata:

$$((\varphi \to (\psi \to \psi)) \wedge (\psi \to (\chi \to \chi))) \supset (\varphi \to (\chi \to \chi)) \tag{DD0}$$

$$(\varphi \to (\neg\psi \to \neg\psi)) \equiv (\varphi \to (\psi \to \psi)) \tag{DD1}$$

$$((\varphi \wedge \psi) \to (\chi \to \chi)) \equiv ((\varphi \to (\chi \to \chi)) \wedge (\psi \to (\chi \to \chi))) \tag{DD2}$$

$$((\varphi \vee \psi) \to (\chi \to \chi)) \equiv ((\varphi \to (\chi \to \chi)) \wedge (\psi \to (\chi \to \chi))) \tag{DD3}$$

$$((\varphi \to \psi) \to (\chi \to \chi)) \equiv ((\varphi \to (\chi \to \chi)) \wedge (\psi \to (\chi \to \chi))). \tag{DD4}$$

Therefore we have $\mathbf{S} = \mathbf{CR} \oplus \{\mathtt{S0}, \mathtt{S1}, \mathtt{S2}, \mathtt{S3}, \mathtt{S4}\}$ and $\mathbf{DD} = \mathbf{CR} \oplus \{\mathtt{DD0}, \mathtt{DD1}, \mathtt{S1}, \mathtt{DD2}, \mathtt{DD3}, \mathtt{DD4}\}$.[3]

Note that, instead of (S2)–(S3), we could use the following schemata in the definition of **S**:

$$(\varphi \to ((\psi \wedge \chi) \to (\psi \wedge \chi))) \equiv ((\varphi \to (\psi \to \psi)) \vee (\varphi \to (\chi \to \chi))) \tag{S2$'$}$$

$$(\varphi \to ((\psi \vee \chi) \to (\psi \wedge \chi))) \equiv ((\varphi \to (\psi \to \psi)) \vee (\varphi \to (\chi \to \chi))) \tag{S3$'$}$$

$$(\varphi \to ((\psi \vee \chi) \to (\psi \to \chi))) \equiv ((\varphi \to (\psi \to \psi)) \vee (\varphi \to (\chi \to \chi))), \tag{S4$'$}$$

which can be observed by the following fact:

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[3] In the article we propose slightly different axiom schemata than those proposed by Epstein. Such an approach, formulated in an extended but not more expressible language, was discussed in [10].

FACT 3.4. *1. (S2) and (S2′) are mutually derivable from (S1) in any CRL.*

2. (S3) and (S3′) are mutually derivable from (S1) in any CRL.

3. (S4) and (S4′) are mutually derivable from (S1) in any CRL.

PROOF. By (S0), (BL) and (MD). ■

Moreover, in the definition of **DD** we could use the following schemata instead of (S1) and (DD1):

$$\neg\varphi \to (\varphi \to \varphi) \tag{S1$'$}$$

$$\varphi \to (\neg\varphi \to \neg\varphi). \tag{DD1$'$}$$

We have the following fact:

FACT 3.5. *1. (S1′) is derivable from ($S1^{\subset}$) in any CRL.*

2. (S1′) is derivable from ($DD1^{\supset}$) in any CRL.

3. (DD1′) is derivable from ($S1^{\supset}$) in any CRL.

4. (DD1′) is derivable from ($DD1^{\subset}$) in any CRL.

5. (S1) is derivable from {DD0,DD1′,S1′} in any CRL.

6. (DD1) is derivable from {DD0,S1′,DD1′} in any CRL.

PROOF. By (CR1), (MD) and (BL). ■

3.3. Boolean Connexive Content Relationship Logics

Before we define some connexive counterparts of Epstein's logics, let us consider connexive theses in the context of CRL and the application of the schema (E). We have the following fact:

FACT 3.6. *1. (A1), (A2), (wB′) are derivable from (wB) in any CRL.*

2. (A1), (A2), (wB) are derivable from (wB′) in any CRL.

PROOF. For 1. For (wB), by ($R_{\to}$) and (MD). For (wB), by Fact 3.1.3. For 2, by ($R_{\to}$) and (MD). ■

Let us note that by means of (E) we can express a connexive dependence expressed by weak Boethius' Theses (wB) in a different way. Consider the following schema:

$$(\varphi \to (\psi \to \psi)) \supset \neg(\varphi \to (\neg\psi \to \neg\psi)). \tag{C}$$

The schema (C) says that if the contents of φ and ψ are related, then the contents of φ and $\neg\psi$ are not related. We call (C) *Connexive Content Connection Thesis.*

In a similar way, we can obatin the following counterparts of (A1) and (A2):

$$\neg(\varphi \to (\neg\varphi \to \neg\varphi)) \tag{AC1}$$

$$\neg(\neg\varphi \to (\varphi \to \varphi)). \tag{AC2}$$

(AC1) and (AC2) express a lack of connection between a sentence and its negation with respect to content. We call (AC1) and (AC2) *Aristotle's Content Connection Theses*.

Since in a CRL $\to$ expresses some connection between antecedent and consequent, (B1) and (B2) also express that $\varphi \to \psi$ is somehow related with $\neg(\varphi \to \neg\psi)$ and $\varphi \to \neg\psi$ is somehow related with $\neg(\varphi \to \psi)$. The following schemata will express these connections with respect to content by means of the schema (E):

$$(\varphi \to \psi) \to (\neg(\varphi \to \neg\psi) \to \neg(\varphi \to \neg\psi)) \tag{BC1}$$

$$(\varphi \to \neg\psi) \to (\neg(\varphi \to \psi) \to \neg(\varphi \to \psi)). \tag{BC2}$$

We call (BC1) and (BC2) *Boethius' Content Connection Theses*.

We have the following fact:

FACT 3.7. *1. (wB) and (wB′) are derivable from (C) in any CRL.*

2. (AC1) and (AC2) are derivable from (C) in any CRL.

3. (B1) is derivable from {C, BC1} in any CRL.

4. (B2) is derivable from {C, BC2} in any CRL.

5. (BC1) is derivable from (B1) in any CRL.

6. (BC2) is derivable from (B2) in any CRL.

7. (B1) is derivable from {BC1,wB$^\vee$} in any CRL.

8. (B2) is derivable from {BC2,wB$^\vee$} in any CRL.

PROOF. For 1. (wB) follows from (C) by ($\mathtt{W}^+_\to$), (BL) and (MD). (wB′) follows from (C) by (wB) and Fact 3.1.3. For 2: by (CR1) and (MD). For 3 and 4: by Fact 3.7.1, ($\mathtt{S}^+_\to$), (BL) and (MD). For 5 and 6, by ($\mathtt{W}^+_\to$), (BL) and (MD). For 7 and 8, by Fact 3.2, ($\mathtt{S}^+_\to$), (BL) and (MD). ■

Thus, in any CRL we can derive Aristotle's Theses and weak Boethius' Theses by means of (C). Moreover, Epstein's axiom schema (S0) together with Aristotle's Theses allow us to derive Aristotle's Content Connection Theses:

FACT 3.8. *(AC1) and (AC2) are derivable from {A1, A2, S0} in any CRL.*

PROOF. For (AC1).

1. $\neg(\varphi \to (\neg\varphi \to \neg\varphi)) \vee \neg(\varphi \supset \neg\varphi)$ — ($\mathtt{S}^+_\to$), (A1), (BL), (MD)
2. $\neg(\varphi \to (\neg\varphi \to \neg\varphi)) \vee \varphi$ — 1, (BL), (MD)
3. $\neg(\neg\varphi \to (\varphi \to \varphi)) \vee \neg(\neg\varphi \supset \varphi)$ — ($\mathtt{S}^+_\to$), (A2), (BL), (MD)
4. $\neg(\neg\varphi \to (\varphi \to \varphi)) \vee \neg\varphi$ — 3, (BL), (MD)
5. $\neg(\varphi \to (\neg\varphi \to \neg\varphi)) \vee \neg\varphi$ — 4, (S0), (BL), (MD)
6. $\neg(\varphi \to (\neg\varphi \to \neg\varphi))$ — 2, 5, (BL), (MD)

For (AC2) we reason similarly as for (AC1). ■

We will show in the sequel that (C) does not follow from (wB) in **CR**; likewise, (AC1) and (AC2) do not follow from (A1) and (A2), respectively. As we will find out, this is related to schemata ($\mathtt{A1}^\vee$), ($\mathtt{A2}^\vee$), ($\mathtt{wB}^\vee$) and the possibility of defining **wBC** and **BC** by models different from the one defined by Jarmużek and Malinowski. Such alternative models seem to define weaker logics than the connexive counterparts of Epstein's logics we focused on. Briefly, we can say that not always some connections between sentences matter for the truth of ($\mathtt{A1}^\vee$), ($\mathtt{A2}^\vee$) and ($\mathtt{wB}^\vee$), while for the truth of (AC1), (AC2) and (C) sentences of certain forms must always be related and of some other forms cannot be related.

Let us analyze a bit further the weaker versions of the connexive theses. Consider the counterpart of ($\mathtt{wB}^\vee$) formulated by (E) in the following way:

$$(\varphi \to (\psi \to \psi)) \supset (\neg(\varphi \to (\neg\psi \to \neg\psi)) \vee \varphi). \qquad (\mathtt{wBC}^\vee)$$

Schemata ($\mathtt{wB}^\vee$) and ($\mathtt{wBC}^\vee$) are mutually derivable. We have:

FACT 3.9. *($\mathtt{wB}^\vee$) and ($\mathtt{wBC}^\vee$) are mutually derivable in any CRL.*

PROOF. By (BL), (MD), ($\mathtt{S}^+_\to$) and ($\mathtt{W}^+_\to$). ■

We also have that the weaker Aristotle's Theses formulated by (E) will be mutually derivable with weaker Aristotle's Theses ($\mathtt{A1}^\vee$) and ($\mathtt{A2}^\vee$). We have the following counterparts of ($\mathtt{A1}^\vee$) and ($\mathtt{A2}^\vee$) formulated by (E):

$$\neg(\varphi \to (\neg\varphi \to \neg\varphi)) \vee \varphi \qquad (\mathtt{AC1}^\vee)$$

$$\neg(\neg\varphi \to (\varphi \to \varphi)) \vee \neg\varphi. \qquad (\mathtt{AC2}^\vee)$$

We have the following fact:

FACT 3.10. *1. ($\mathtt{AC1}^\vee$) and ($\mathtt{A1}^\vee$) are mutually derivable in any CRL.*

2. ($\mathtt{AC2}^\vee$) and ($\mathtt{A2}^\vee$) are mutually derivable in any CRL.

PROOF. By (BL), (MD), ($\mathtt{S}^+_\to$) and ($\mathtt{W}^+_\to$). ■

As a corollary of Facts 3.10, 3.2 and 3.6, we obtain:

Corollary 3.11. *1. ($\mathtt{AC1}^{\vee}$) and ($\mathtt{AC2}^{\vee}$) are derivable from ($\mathtt{wB}^{\vee}$) in any CRL.*

2. ($\mathtt{AC1}^{\vee}$) and ($\mathtt{AC2}^{\vee}$) are derivable from ($\mathtt{wBC}^{\vee}$) in any CRL.

In the definition of Boolean connexive logic of content relationship, we will focus on ($\mathtt{C}$), ($\mathtt{BC1}$) and ($\mathtt{BC2}$), as we are interested in combining Epstein's concept of content relationship, on the one hand, and connexivity dependencies captured by Jarmużek and Malinowski in BCL, on the other.

A *weak Boolean connexive content relationship logic* (wBCCRL) is any set of formulas $\Lambda \subseteq \mathsf{For}$ such that Λ is a CRL and it contains the schema ($\mathtt{C}$). A *Boolean connexive content relationship logic* (BCCRL) is any set of formulas $\Lambda \subseteq \mathsf{For}$ such that Λ is a wBCCRL and contains the schemata ($\mathtt{BC1}$), ($\mathtt{BC2}$). The least wBCCRL and the least BCCRL are denoted by **wBCCR** and **BCCR**, respectively. Thus, $\mathbf{wBCCR} = \mathbf{CR} \oplus (\mathtt{C})$ and $\mathbf{BCCR} = \mathbf{wBCCR} \oplus \{\mathtt{BC1}, \mathtt{BC2}\}$.

Some remarks are in order here. The schemata ($\mathtt{E}$), ($\mathtt{C}$), ($\mathtt{AC1}$), ($\mathtt{AC2}$), ($\mathtt{BC1}$) and ($\mathtt{BC2}$) are all invalid in Angell-McCall's **CC1** (and a fortiori in Angell's **PA1**, since the former is an extension of the latter; see [2], [22]). All those schemata but ($\mathtt{C}$) are valid in Wansing's **C**, and understandably so: ($\mathtt{C}$) expresses that if φ is related to the content of ψ, then φ cannot be related to content of the negation of ψ. But **C** makes room for such inconsistencies (see [34], [26]). And, like in our BCCRLs, all the schemata but ($\mathtt{E}$) are valid in Mortensen's [23] **M3V**. Again, this is understandably so: the implication of **M3V** is structurally the same as the one used by Anderson and Belnap in [1] to show the consistency of the relevance logic **E**, and ($\mathtt{E}$) is one of the fallacies of relevance that the implication was meant to avoid. But, unlike our BCCRLs, **M3V** is inconsistent (yet not trivial). To the best of our knowledge, none of the connexive content theses have been explicitly dealt with in the literature on connexive logic, though.[4]

Before we introduce connexive counterparts of Epstein's logics, let us identify the schemata, among the considered ones, that we must reject to avoid defining a trivial logic, i.e., the inconsistent logic. Obvious examples are the schemata specifying the content relation between a sentence and its negation.

Fact 3.12. *If Λ is a CRL, then the following logics are inconsistent:*

1. $\Lambda \oplus \{\mathtt{S1}^{\supset}, \mathtt{A1}\}$ and $\Lambda \oplus \{\mathtt{S1}^{\subset}, \mathtt{A2}\}$,

[4] There has been no substantial new work on **CC1** since the 1960's. For recent work on **M3V** see [4], [20]; for **C** — but also **M3V** — see [26].

2. $\Lambda \oplus \{$*DD1*$^{\subset}$,*A1*$\}$ *and* $\Lambda \oplus \{$*DD1*$^{\supset}$,*A2*$\}$.

PROOF. For 1. (†) $(\varphi \wedge \neg\varphi) \to \neg(\varphi \wedge \neg\varphi)$ is derivable from ($\mathtt{S1}^{\supset}$) in any CRL.

1. $(\varphi \wedge \neg\varphi) \to (\neg(\varphi \wedge \neg\varphi) \to \neg(\varphi \wedge \neg\varphi))$ Fact 3.5.3
2. $(((\varphi \wedge \neg\varphi) \supset \neg(\varphi \wedge \neg\varphi)) \wedge ((\varphi \wedge \neg\varphi) \to (\neg(\varphi \wedge \neg\varphi) \to \neg(\varphi \wedge \neg\varphi)))) \supset$ $\supset ((\varphi \wedge \neg\varphi) \to \neg(\varphi \wedge \neg\varphi))$ ($\mathtt{W}^{+}_{\to}$)
3. $((\varphi \wedge \neg\varphi) \supset \neg(\varphi \wedge \neg\varphi)) \supset ((\varphi \wedge \neg\varphi) \to \neg(\varphi \wedge \neg\varphi))$ 1, 2, (BL), (MD)
4. $(\varphi \wedge \neg\varphi) \supset \neg(\varphi \wedge \neg\varphi)$ (BL)
5. $(\varphi \wedge \neg\varphi) \to \neg(\varphi \wedge \neg\varphi)$ 3, 4, (MD)

Therefore, the logic $\Lambda \oplus \{\mathtt{S1}^{\supset},\mathtt{A1}\}$ is inconsistent.

(‡) $\neg(\varphi \wedge \neg\varphi) \to (\varphi \wedge \neg\varphi)$ is derivable from ($\mathtt{S1}^{\subset}$) in any CRL

1. $\neg(\varphi \wedge \neg\varphi) \to ((\varphi \wedge \neg\varphi) \to (\varphi \wedge \neg\varphi))$ Fact 3.5.1
2. $((\neg(\varphi \wedge \neg\varphi) \supset (\varphi \wedge \neg\varphi)) \wedge (\neg(\varphi \wedge \neg\varphi) \to ((\varphi \wedge \neg\varphi) \to (\varphi \wedge \neg\varphi)))) \supset$ $\supset (\neg(\varphi \wedge \neg\varphi) \to (\varphi \wedge \neg\varphi))$ ($\mathtt{W}^{+}_{\to}$)
3. $(\neg(\varphi \wedge \neg\varphi) \supset (\varphi \wedge \neg\varphi)) \supset (\neg(\varphi \wedge \neg\varphi) \to (\varphi \wedge \neg\varphi))$ 1, 2, (BL), (MD)
4. $\neg(\varphi \wedge \neg\varphi) \supset (\varphi \wedge \neg\varphi)$ (BL)
5. $\neg(\varphi \wedge \neg\varphi) \to (\varphi \wedge \neg\varphi)$ 3, 4, (MD)

Therefore, the logic $\Lambda \oplus \{\mathtt{S1}^{\subset},\mathtt{A2}\}$ is inconsistent.

For 2. To prove that $\Lambda \oplus \{\mathtt{DD1}^{\subset},\mathtt{A1}\}$ is inconsistent logic we reason as for (†). To prove that $\Lambda \oplus \{\mathtt{DD1}^{\supset},\mathtt{A2}\}$ is inconsistent logic we reason as for (‡).

■

We obtain the following corollary:

COROLLARY 3.13. *If* Λ *is a CRL, then the following logics are inconsistent:*

1. $\Lambda \oplus \{$*S1*$^{\supset}$,*wB*$\}$, $\Lambda \oplus \{$*S1*$^{\supset}$,*wB′*$\}$, $\Lambda \oplus \{$*S1*$^{\subset}$,*wB*$\}$ *and* $\Lambda \oplus \{$*S1*$^{\subset}$,*wB′*$\}$;

2. $\Lambda \oplus \{$*DD1*$^{\supset}$,*wB*$\}$, $\Lambda \oplus \{$*DD1*$^{\supset}$,*wB′*$\}$, $\Lambda \oplus \{$*DD1*$^{\subset}$,*wB*$\}$ *and* $\Lambda \oplus \{$*DD1*$^{\subset}$,*wB′*$\}$;

3. $\Lambda \oplus \{$*S1*$^{\supset}$,*B1*$\}$, $\Lambda \oplus \{$*S1*$^{\supset}$,*B2*$\}$, $\Lambda \oplus \{$*S1*$^{\subset}$,*B1*$\}$ *and* $\Lambda \oplus \{$*S1*$^{\subset}$,*B2*$\}$;

4. $\Lambda \oplus \{$*DD1*$^{\supset}$,*B1*$\}$, $\Lambda \oplus \{$*DD1*$^{\supset}$,*B2*$\}$, $\Lambda \oplus \{$*DD1*$^{\subset}$,*B1*$\}$ *and* $\Lambda \oplus \{$*DD1*$^{\subset}$,*B2*$\}$.

PROOF. For 1 and 2, by Facts 3.12 and 3.6. For 3 and 4, by 1 and Fact 3.1.

■

But not only (S1) and (DD1) are problematic if we want to add content connexive theses to Epstein's logics. We have the following fact:

FACT 3.14. *If* Λ *is a CRL, then* $\Lambda \oplus \{$*S2*$^{\subset}$,*C*$\}$,$\Lambda \oplus \{$*S3*$^{\subset}$,*C*$\}$ *and* $\Lambda \oplus \{$*S4*$^{\subset}$,*C*$\}$ *are inconsistent logics.*

PROOF. $(\varphi \wedge \neg\varphi) \to (\varphi \to \varphi)$ is derivable from ($\mathtt{S2}^{\subset}$) in any CRL.

1. $((\varphi \to (\varphi \to \varphi)) \vee (\neg\varphi \to (\varphi \to \varphi))) \supset ((\varphi \wedge \neg\varphi) \to (\varphi \to \varphi))$ Hyp.

2. $\varphi \to (\varphi \to \varphi)$ (CR1)
3. $(\varphi \wedge \neg\varphi) \to (\varphi \to \varphi)$ 1, 2, (BL), (MD)

$(\varphi \wedge \neg\varphi) \to (\neg\varphi \to \neg\varphi)$ is derivable from ($\mathtt{S2}^{\subset}$) in any CRL.

1. $((\varphi \to (\neg\varphi \to \neg\varphi)) \vee (\neg\varphi \to (\neg\varphi \to \neg\varphi))) \supset ((\varphi \wedge \neg\varphi) \to (\neg\varphi \to \neg\varphi))$ Hyp.
2. $\neg\varphi \to (\neg\varphi \to \neg\varphi)$ (CR1)
3. $(\varphi \wedge \neg\varphi) \to (\neg\varphi \to \neg\varphi)$ 1, 2, (BL), (MD)

To prove that $(\varphi \vee \neg\varphi) \to (\varphi \to \varphi)$ and $(\varphi \vee \neg\varphi) \to (\neg\varphi \to \neg\varphi)$ are derivable from ($\mathtt{S3}^{\subset}$) in any CRL, $(\varphi \to \neg\varphi) \to (\varphi \to \varphi)$ and $(\varphi \vee \neg\varphi) \to (\neg\varphi \to \neg\varphi)$ are derivable from ($\mathtt{S4}^{\subset}$) in any CRL, we reason similarly as above. Therefore, $\Lambda \oplus \{\mathtt{S2}^{\subset},\mathtt{C}\}$,$\Lambda \oplus \{\mathtt{S3}^{\subset},\mathtt{C}\}$ and $\Lambda \oplus \{\mathtt{S4}^{\subset},\mathtt{C}\}$ are inconsistent logics. ∎

Note that in case of ($\mathtt{S2}^{\subset}$) we can also show inconsistency with (wB).

Fact 3.15. *If Λ is CRL, then $\Lambda \oplus \{S2^{\subset}, wB\}$ is an inconsistent logic.*

Proof. By proof of Fact 3.14, we have that $(\varphi \wedge \neg\varphi) \to (\varphi \to \varphi)$ and $(\varphi \wedge \neg\varphi) \to (\neg\varphi \to \neg\varphi)$ are derivable from ($\mathtt{S2}^{\subset}$) in any CRL.

$(\varphi \wedge \neg\varphi) \to \varphi$ is derivable from ($\mathtt{S2}^{\subset}$) in any CRL.

1. $((\varphi \to (\varphi \to \varphi)) \vee (\neg\varphi \to (\varphi \to \varphi))) \supset ((\varphi \wedge \neg\varphi) \to (\varphi \to \varphi))$ Hyp.
2. $\varphi \to (\varphi \to \varphi)$ (CR1)
3. $(\varphi \wedge \neg\varphi) \to (\varphi \to \varphi)$ 1, 2, (MD)
4. $(\varphi \wedge \neg\varphi) \supset \varphi$ (MD)
5. $(((\varphi \wedge \neg\varphi) \to (\varphi \to \varphi)) \wedge ((\varphi \wedge \neg\varphi) \supset \varphi)) \supset ((\varphi \wedge \neg\varphi) \to \varphi)$ ($\mathtt{W}^{+}_{\to}$)
6. $(\varphi \wedge \neg\varphi) \to \varphi$ 4, 5, (MD)

To prove that $(\varphi \wedge \neg\varphi) \to \neg\varphi$ is derivable from ($\mathtt{S2}^{\subset}$) in any CRL, we reason similarly as above. Therefore, $\Lambda \oplus \{\mathtt{S2}^{\subset},\mathtt{wB}\}$ is an inconsistent logic. ∎

Let us also observe that, in the considered context, schemata ($\mathtt{DD2}^{\supset}$), ($\mathtt{DD3}^{\supset}$) and ($\mathtt{DD4}^{\supset}$) are also problematic.

Fact 3.16. *If Λ is a CRL, then $\Lambda \oplus \{DD2^{\supset}, C\}$, $\Lambda \oplus \{DD3^{\supset}, C\}$ and $\Lambda \oplus \{DD4^{\supset}, C\}$ are inconsistent logics.*

Proof. ($\mathtt{S2}^{\subset}$) is derivable from ($\mathtt{DD2}^{\supset}$) in any CRL

1. $((\psi \wedge \chi) \to ((\psi \wedge \chi) \to (\psi \wedge \chi))) \supset ((\psi \to ((\psi \wedge \chi) \to (\psi \wedge \chi))) \wedge (\chi \to$
   $\to ((\psi \wedge \chi) \to (\psi \wedge \chi))))$ Hyp.
2. $(\psi \wedge \chi) \to ((\psi \wedge \chi) \to (\psi \wedge \chi))$ (CR1)
3. $\psi \to ((\psi \wedge \chi) \to (\psi \wedge \chi))$ 1, 2, (BL), (MD)
4. $((\varphi \to (\psi \to \psi)) \wedge (\psi \to ((\psi \wedge \chi) \to (\psi \wedge \chi)))) \supset (\varphi \to ((\psi \wedge \chi) \to (\psi \wedge \chi)))$ (DD0)

5. $(\varphi \to (\psi \to \psi)) \supset (\varphi \to ((\psi \wedge \chi) \to (\psi \wedge \chi)))$ 3, 4, (BL), (MD)

To prove that ($\mathtt{S3}^{\subset}$) is derivable from ($\mathtt{DD3}^{\supset}$) in any CRL and ($\mathtt{S4}^{\subset}$) is derivable from ($\mathtt{DD4}^{\supset}$) in any CRL, we reason similarly as above. By Fact 3.14, $\Lambda \oplus \{\mathtt{DD2}^{\supset}, \mathtt{C}\}$, $\Lambda \oplus \{\mathtt{DD3}^{\supset}, \mathtt{C}\}$ and $\Lambda \oplus \{\mathtt{DD4}^{\supset}, \mathtt{C}\}$ are inconsistent logics. ■

Let us now propose some definitions of weak connexive and connexive versions of two of the Epstein's systems considered. The logic **wBCS** (**BCS**) is the least set $\Lambda \subseteq \mathsf{For}$ such that Λ is wBCCRL (BCCRL) and contains the schemata (S0), ($\mathtt{S2}^{\supset}$), ($\mathtt{S3}^{\supset}$), ($\mathtt{S4}^{\supset}$). The logic **wBCDD** (**BCDD**) is the least set $\Lambda \subseteq \mathsf{For}$ such that Λ is wBCCRL (BCCRL) and contains schemata (DD0), ($\mathtt{DD2}^{\subset}$), $\mathtt{DD3}^{\subset}$) ($\mathtt{DD4}^{\subset}$).

Moreover, we will consider some extensions of the considered counterparts of **S** and **DD**. Logic $\mathbf{wBCDD}^{+}$ ($\mathbf{BCDD}^{+}$) is the least set $\Lambda \subseteq \mathsf{For}$ such that Λ is wBCCRL (BCCRL) and contains schemata (DD0), ($\mathtt{DD2}^{\subset}$), ($\mathtt{DD3}^{\subset}$), ($\mathtt{DD4}^{\subset}$), ($\mathtt{S2'}^{\supset}$), ($\mathtt{S3'}^{\supset}$) ($\mathtt{S4'}^{\supset}$). The logic $\mathbf{wBCS}^{+}$ ($\mathbf{BCS}^{+}$) is the least set $\Lambda \subseteq \mathsf{For}$ such that Λ is wBCCRL (BCCRL) and contains schemata (S0), (DD0), ($\mathtt{S2}^{\supset}$), ($\mathtt{S3}^{\supset}$) ($\mathtt{S4}^{\supset}$). Note that $\mathbf{wBCDD}^{+}$ is a sub-logic of $\mathbf{wBCS}^{+}$ and $\mathbf{BCDD}^{+}$ is a sub-logic of $\mathbf{BCS}^{+}$. We have the following fact:

FACT 3.17. *1. ($\mathtt{S2'}^{\subset}$), ($\mathtt{S3'}^{\subset}$), ($\mathtt{S4'}^{\subset}$) are derivable in* $\mathbf{wBCS}^{+}$ *and* $\mathbf{BCS}^{+}$*.*

2. ($\mathtt{DD2}^{\subset}$), ($\mathtt{DD3}^{\subset}$), ($\mathtt{DD4}^{\subset}$) are derivable in $\mathbf{wBCS}^{+}$ *and* $\mathbf{BCS}^{+}$*.*

PROOF. For 1, by Fact 3.4.

For 2. ($\mathtt{DD2}^{\subset}$) is derivable in $\mathbf{wBCS}^{+}$ and $\mathbf{BCS}^{+}$.

1. $(\varphi \wedge \psi) \to ((\varphi \wedge \psi) \to (\varphi \wedge \psi))$ (CR1)
2. $((\varphi \wedge \psi) \to ((\varphi \wedge \psi) \to (\varphi \wedge \psi))) \supset (((\varphi \wedge \psi) \to (\varphi \to \varphi)) \vee ((\varphi \wedge \psi) \to$ $\to (\psi \to \psi)))$ ($\mathtt{S2}^{\supset}$)
3. $((\varphi \wedge \psi) \to (\varphi \to \varphi)) \vee ((\varphi \wedge \psi) \to (\psi \to \psi))$ 1, 2, (MD)
4. $(((\varphi \wedge \psi) \to (\varphi \to \varphi)) \wedge (\varphi \to (\chi \to \chi))) \supset ((\varphi \wedge \psi) \to (\chi \to \chi))$ (DD0)
5. $(((\varphi \wedge \psi) \to (\psi \to \psi)) \wedge (\psi \to (\chi \to \chi))) \supset ((\varphi \wedge \psi) \to (\chi \to \chi))$ (DD0)
6. $((((\varphi \wedge \psi) \to (\varphi \to \varphi)) \wedge (\varphi \to (\chi \to \chi))) \vee (((\varphi \wedge \psi) \to (\psi \to \psi)) \wedge (\psi \to$ $\to (\chi \to \chi)))) \supset ((\varphi \wedge \psi) \to (\chi \to \chi))$ 4, 5, (BL), (MD)
7. $((\varphi \to (\chi \to \chi)) \wedge (\psi \to (\chi \to \chi))) \supset ((\varphi \wedge \psi) \to (\chi \to \chi))$ 3, 6, (BL), (MD)

The reasoning to prove that ($\mathtt{DD3}^{\subset}$) and ($\mathtt{DD4}^{\subset}$) are derivable in $\mathbf{wBCS}^{+}$ is similar to the above. ■

In Figure 1, we present an ordering of all of logics discussed in the article. Note that in the figure, the dashed lines run to the connexive counterpart, not to the extension of the given logic.

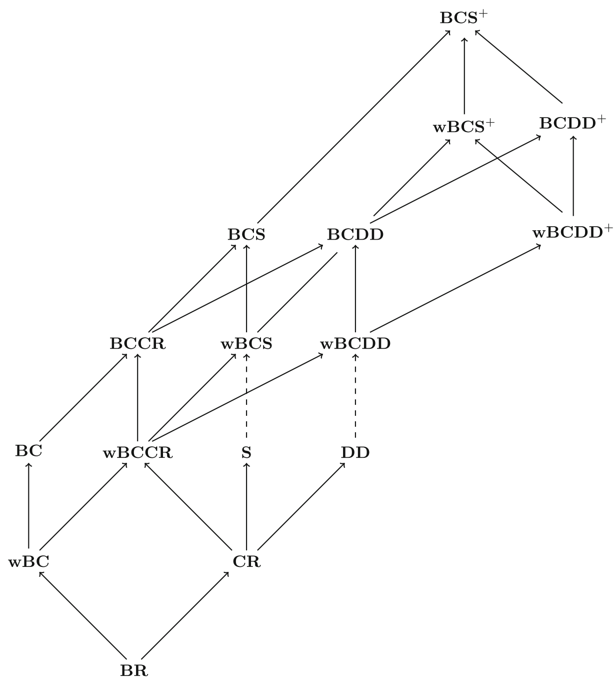

Figure 1. Order of the considered logics

4. Relating Semantics and Set-Assignment Semantics

In what follows, we consider two kinds of structures. The first is a relating model using which we can determine the family of relating logic (see [7,13,16]), particularly to define Jarmużek and Malinowski's BCL (see [11], cf. [12,19]) and Epstein's CRL (see [3, pp. 61–84, 115–143], cf. [16]). Epstein's systems were originally defined using a special case of relating semantics. Still, he also used another kind of structure, the so-called set-assignment model, the second kind of structure we consider. A set-assignment model is based on a function by means of which Epstein represented sentential content.

4.1. Relating Semantics

A *relating model* (an *r-model*) is an ordered pair $\langle v, R\rangle$ such that $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a classical valuation and $R \subseteq \mathsf{For} \times \mathsf{For}$ is a binary relation on the set of formulas. R is called *connection relation* or *relating relation*.

For any r-model $\langle v, R\rangle$, we assume the following truth conditions:

$$\begin{aligned}
\langle v, R\rangle \models \varphi &\text{ iff } \ v(\varphi) = 1, \ \text{if } \varphi \in \mathsf{Var} \\
\langle v, R\rangle \models \neg\varphi &\text{ iff } \ \text{not } \langle v, R\rangle \models \varphi \text{ (i.e. iff} \langle v, R\rangle \not\models \varphi) \\
\langle v, R\rangle \models \varphi \wedge \psi &\text{ iff } \ \langle v, R\rangle \models \varphi \text{ and } \langle v, R\rangle \models \psi \\
\langle v, R\rangle \models \varphi \vee \psi &\text{ iff } \ \langle v, R\rangle \models \varphi \text{ or } \langle v, R\rangle \models \psi \\
\langle v, R\rangle \models \varphi \to \psi &\text{ iff } \ \text{either } \langle v, R\rangle \not\models \varphi \text{ or } \langle v, R\rangle \models \psi, \ \text{and } R(\varphi, \psi).
\end{aligned}$$

Let M be a class of r-models. We define a *relation of semantic consequence* $\models_{\mathsf{M}} \subseteq \mathcal{P}(\mathsf{For}) \times \mathsf{For}$ in the following way: for all $\Sigma \cup \{\varphi\} \subseteq \mathsf{For}$, $X \models_{\mathsf{M}} \varphi$ iff for all $\mathfrak{M} \in \mathsf{M}$, if for all $\psi \in \Sigma$, $\mathfrak{M} \models \psi$, then $\mathfrak{M} \models \varphi$. We say that a formula φ is *valid in* M iff $\emptyset \models_{\mathsf{M}} \varphi$. Similarly, a schema $(\mathtt{X})$ is *valid in* M iff all formulas of the form of $(\mathtt{X})$ are valid in M.

Having defined semantic consequence, we describe the notions of soundness and completeness in the standard way. Let Λ be BLRI and M be a class of r-models. We say that:

- Λ is *sound with respect to* M iff for all $\Sigma \cup \{\varphi\} \subseteq \mathsf{For}$, if $\Sigma \vdash_{\Lambda} \varphi$, then $\Sigma \models_{\mathsf{M}} \varphi$
- Λ is *complete with respect to* M iff for all $\Sigma \cup \{\varphi\} \subseteq \mathsf{For}$, if $\Sigma \models_{\mathsf{M}} \varphi$, then $\Sigma \vdash_{\Lambda} \varphi$.

We say that Λ is *determined by a class of models* M iff Λ is sound and complete with respect to M.

4.1.1. Soundness and Completeness of Logic BR In order to prove soundness and completeness of **BR**, we adapt methods presented in [9, 10, 15]. The soundness proof for **BR** is straightforward:

THEOREM 4.1. **BR** *is sound with respect to the class of all r-models.*

PROOF. By truth conditions. ■

We use the notion of maximally consistent set and the Lindenbaum construction to prove completeness. Let Λ be a BLRI. A set $\Sigma \subseteq \mathsf{For}$ is a *maximally Λ-consistent set* iff Σ is Λ-consistent and there is no $\Gamma \subseteq \mathsf{For}$ such that Γ is Λ-consistent and $\Sigma \subset \Gamma$. The set of all maximally Λ-consistent sets is denoted by Max_{Λ}. We have the following fact, well-known as Lindenbaum's Lemma:

FACT 4.2. *Let Λ be a BLRI and $\Sigma \subseteq \mathsf{For}$. If Σ is a Λ-consistent set, then there is $\Gamma \in \mathsf{Max}_{\Lambda}$ such that $\Sigma \subseteq \Gamma$.*

We define a canonical model with respect to a maximally consistent set. Let Λ be a BLRI and $\Sigma \in \mathsf{Max}_\Lambda$. A *BLRI canonical Σ-model* (*BLRI-Σ-model*) is an ordered pair $\langle v_\Sigma, R_\Sigma \rangle$ such that for all $\varphi \in \mathsf{Var}$, $v_\Sigma(\varphi) = 1$ iff $\varphi \in \Sigma$; for all $\varphi, \psi \in \mathsf{For}$, $R_\Sigma(\varphi, \psi)$ iff $\varphi \to \psi \in \Sigma$.

By the properties of the maximally consistent set we obtain the following fact about the canonical model:

Lemma 4.3. (See [15, Lemma 4.8, p. 529], cf. [10, Lemma 5.13], [9, Lemma 6.6].) *Let Λ be a BLRI, $\Sigma \in \mathsf{Max}_\Lambda$ and $\mathfrak{M}_\Sigma$ be a BLRI-Σ-model. Then, for all $\varphi \in \mathsf{For}$, $\mathfrak{M}_\Sigma \models \varphi$ iff $\varphi \in \Sigma$.*

Using the standard argument, by Fact 4.2 and Lemma 4.3 we obtain completeness:

Theorem 4.4. **BR** *is complete with respect to the class of all r-models.*

In order to present determination results for the other logics considered here, we will use some relational conditions. For any condition $(X_{\mathtt{Y}})$ of the form $\forall_{\varphi_1,\ldots,\varphi_n \in \mathsf{For}}(F \text{ iff } G)$, by $(X_{\mathtt{Y}\supset})$ we denote condition $\forall_{\varphi_1,\ldots,\varphi_n \in \mathsf{For}}(F \Rightarrow G)$, and by $(X_{\mathtt{Y}\subset})$ we denote condition $\forall_{\varphi_1,\ldots,\varphi_n \in \mathsf{For}}(G \Rightarrow F)$.

4.1.2. Soundness and Completeness of Logics wBC and BC We may determine BCLs by means of relating models using the relational conditions introduced by Jarmużek and Malinowski [11,12]:

$$\forall_{\varphi \in \mathsf{For}} {\sim} R(\varphi, \neg\varphi) \tag{$\mathsf{R}_{\mathtt{AC1}}$}$$

$$\forall_{\varphi \in \mathsf{For}} {\sim} R(\neg\varphi, \varphi) \tag{$\mathsf{R}_{\mathtt{AC2}}$}$$

$$\forall_{\varphi,\psi \in \mathsf{For}}(R(\varphi, \psi) \Rightarrow {\sim} R(\varphi, \neg\psi)) \tag{$\mathsf{R}_{\mathtt{C}}$}$$

$$\forall_{\varphi,\psi \in \mathsf{For}} R(\varphi \to \psi, \neg(\varphi \to \neg\psi)) \tag{$\mathsf{R}_{\mathtt{BC1}}$}$$

$$\forall_{\varphi,\psi \in \mathsf{For}} R(\varphi \to \neg\psi, \neg(\varphi \to \psi)). \tag{$\mathsf{R}_{\mathtt{BC2}}$}$$

Treating a binary relation between formulas as a formal way to express that some formulas are somehow connected, we would like to say something more about the kind of connexivity we can represent by such relations. Even though the conditions ($\mathsf{R}_{\mathtt{AC1}}$), ($\mathsf{R}_{\mathtt{AC2}}$), ($\mathsf{R}_{\mathtt{C}}$), ($\mathsf{R}_{\mathtt{BC1}}$), ($\mathsf{R}_{\mathtt{BC2}}$) express some properties of connexivity, they do not make precise any specific meaning of connexivity. They are rather some general properties that all relations used for representation of connexivity in relating semantics should satisfy.

The soundness and completeness of **BC** was proved in [15]. We reason similarly to prove the soundness and completeness of **wBC**.

Theorem 4.5. (See [15, Theorem 5.1 1, p. 534].)

1. **wBC** *is determined by the class of all such r-models that* (R_{AC1}), (R_{AC2}) *and* (R_C) *are satisfied.*
2. **BC** *is determined by the class of all such r-models that* (R_{AC1}), (R_{AC2}), (R_C), (R_{BC1}) *and* (R_{BC2}) *are satisfied.*

4.1.3. Alternative Relating Models for Logics wBC and BC As we indicated in the previous section, logics **wBC** and **BC** might be also determined by different kinds of r-models than the ones we considered above. Let us introduce the following conditions imposed on r-models:

$$\forall_{\varphi \in \mathsf{For}}(\sim R(\varphi, \neg\varphi) \text{ or } \langle v, R\rangle \models \varphi) \qquad (\mathsf{R}_{\mathtt{AC1}^{\vee}})$$

$$\forall_{\varphi \in \mathsf{For}}(\sim R(\neg\varphi, \varphi) \text{ or } \langle v, R\rangle \not\models \varphi) \qquad (\mathsf{R}_{\mathtt{AC2}^{\vee}})$$

$$\forall_{\varphi, \psi \in \mathsf{For}}(R(\varphi, \psi) \Rightarrow (\sim R(\varphi, \neg\psi) \text{ or } \langle v, R\rangle \models \varphi)). \qquad (\mathsf{R}_{\mathtt{C}^{\vee}})$$

We have the following correspondence between models and connexive theses:

Fact 4.6. *Let* $\mathfrak{M} = \langle v, R\rangle$ *be a r-model. Then:*

1. $\mathfrak{M} \models$(A1$^{\vee}$) *iff* $\mathfrak{M}$ *satisfies* ($R_{AC1^{\vee}}$)
2. $\mathfrak{M} \models$(A2$^{\vee}$) *iff* $\mathfrak{M}$ *satisfies* ($R_{AC2^{\vee}}$)
3. $\mathfrak{M} \models$(wB$^{\vee}$) *iff* $\mathfrak{M}$ *satisfies* ($R_{C^{\vee}}$)
4. $\mathfrak{M} \models$(B1) *iff* $\mathfrak{M}$ *satisfies* ($R_{C^{\vee}}$) *and* R *satisfies* (R_{BC1})
5. $\mathfrak{M} \models$(B2) *iff* $\mathfrak{M}$ *satisfies* ($R_{C^{\vee}}$) *and* R *satisfies* (R_{BC2}).

Proof. For 1. $\mathfrak{M} \models$(A1$^{\vee}$), by truth conditions, iff $\sim R(\varphi, \neg\varphi)$ or $\mathfrak{M} \models \varphi$.

For 2 we reason similarly as for 1.

For 3. $\mathfrak{M} \models$(wB$^{\vee}$), by truth conditions, iff $\mathfrak{M} \not\models \varphi \rightarrow \psi$ or $\mathfrak{M} \not\models \varphi \rightarrow \neg\psi$ or $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \not\models \varphi \rightarrow \psi$ or $\mathfrak{M} \not\models \varphi \rightarrow \neg\psi$ or $\mathfrak{M} \models \varphi$, by truth conditions, iff $\sim R(\varphi, \psi)$ or $\sim R(\varphi, \neg\psi)$ or $\mathfrak{M} \models \varphi$.

For 4. $\mathfrak{M} \models$ (B1), by truth conditions, iff $\mathfrak{M} \models$(wB$^{\vee}$) and R satisfies ($\mathsf{R}_{\mathtt{BC1}}$).

For 5 we reason similarly as for 4. ■

By Fact 4.6 we get another soundness result:

Theorem 4.7. *1.* **wBC** *is sound with respect to the class of all such r-models that* ($R_{AC1^{\vee}}$), ($R_{AC2^{\vee}}$) *and* ($R_{C^{\vee}}$) *are satisfied.*

2. **BC** *is sound with respect to the class of all such r-models that* ($R_{AC1^{\vee}}$), ($R_{AC2^{\vee}}$), ($R_{C^{\vee}}$), (R_{BC1}) *and* (R_{BC2}) *are satisfied.*

Proof. We can reason as in the proof for Theorem 4.1. By Fact 4.6, we obtain the validity of specific axiom schemata in the proper classes of models. ■

We obtain the following fact with respect to canonical models:

FACT 4.8. *1. If* $\Sigma \in \mathsf{Max}_{\mathbf{BR}\oplus(A1)}$ $(\Sigma \in \mathsf{Max}_{\mathbf{BR}\oplus(A2)})$*, then* $\langle v_\Sigma, R_\Sigma\rangle$ *satisfies* ($\mathsf{R}_{AC1^\vee}$) *((*$\mathsf{R}_{AC2^\vee}$*)).*

2. If $\Sigma \in \mathsf{Max}_{\mathbf{BR}\oplus(wB)}$ $(\Sigma \in \mathsf{Max}_{\mathbf{BR}\oplus(B1)}; \Sigma \in \mathsf{Max}_{\mathbf{BR}\oplus(B2)})$*, then* $\langle v_\Sigma, R_\Sigma\rangle$ *satisfies* ($\mathsf{R}_{C^\vee}$) *((*$\mathsf{R}_{C^\vee}$*) and (*R_{BC1}*); (*$\mathsf{R}_{C^\vee}$*) and (*R_{BC2}*)).*

PROOF. For 1. Suppose $\Sigma \in \mathsf{Max}_{\mathbf{BR}\oplus(\mathtt{A1})}$ $(\Sigma \in \mathsf{Max}_{\mathbf{BR}\oplus(\mathtt{A2})})$. By Fact 3.2, ($\mathtt{A1}^\vee$)$\in \Sigma$ (($\mathtt{A2}^\vee$)$\in \Sigma$). By Lemma 4.3, $\langle v_\Sigma, R_\Sigma\rangle \models$($\mathtt{A1}^\vee$) ($\langle v_\Sigma, R_\Sigma\rangle \models$($\mathtt{A2}^\vee$)). By Fact 4.6.1 (Fact 4.6.2), $\langle v_\Sigma, R_\Sigma\rangle$ satisfies ($\mathsf{R}_{\mathtt{AC1}^\vee}$) (($\mathsf{R}_{\mathtt{AC2}^\vee}$)).

For 2 we reason similarly as for 1. ■

We get another completeness result:

THEOREM 4.9. *1.* **wBC** *is complete with respect to the class of all such r-models that (*$\mathsf{R}_{AC1^\vee}$*), (*$\mathsf{R}_{AC2^\vee}$*) and (*$\mathsf{R}_{C^\vee}$*) are satisfied.*

2. **BC** *is complete with respect to the class of all such r-models that (*$\mathsf{R}_{AC1^\vee}$*), (*$\mathsf{R}_{AC2^\vee}$*), (*$\mathsf{R}_{C^\vee}$*), (*R_{BC1}*) and (*R_{BC2}*) are satisfied.*

PROOF. We can reason as in the case of proof for Theorem 4.4. We use Fact 4.8 to show that the considered BLRI-canonical-model belongs to suitable classes of models. ■

4.1.4. Soundness and Completeness of Logics CR, S and DD Let us now consider some CRLs. To determine any CRL at least the following conditions must be imposed on connection relations:

$$\forall_{\varphi,\psi\in\mathsf{For}}(R(\varphi\to\varphi,\psi) \text{ iff } R(\varphi,\psi)) \tag{$\mathsf{R}_{\mathtt{CR0}}$}$$

$$\forall_{\varphi\in\mathsf{For}}R(\varphi,\varphi) \tag{$\mathsf{R}_{\mathtt{CR1}}$}$$

$$\forall_{\varphi,\psi\in\mathsf{For}}(R(\varphi,\psi\to\psi) \text{ iff } R(\varphi,\psi)). \tag{$\mathsf{R}_{\mathtt{CR2}}$}$$

For any relation that satisfies conditions ($\mathsf{R}_{\mathtt{CR1}}$) and ($\mathsf{R}_{\mathtt{CR2}}$), schema ($\mathtt{E}$) enables us to express in the object language that such relation holds between given sentences.

FACT 4.10. *(Cf. [3, pp. 77–78]) Let* $\langle v, R\rangle$ *be a r-model such that (*R_{CR1}*) and (*R_{CR2}*) are satisfied. Then, for all* $\varphi, \psi \in \mathsf{For}$*,* $R(\varphi,\psi)$ *iff* $\langle v, R\rangle \models$*(E).*

Proof. Suppose $R(\varphi, \psi)$. By ($\mathsf{R}_{\mathtt{CR1}}$) and truth conditions, $\langle v, R\rangle \models \psi \to \psi$. By ($\mathsf{R}_{\mathtt{CR2}}$), $R(\varphi, \psi \to \psi)$. Thus, by truth conditions, $\langle v, R\rangle \models \varphi \to (\psi \to \psi)$.

Suppose $\langle v, R\rangle \models \varphi \to (\psi \to \psi)$. By truth conditions, $R(\varphi, \psi \to \psi)$. By ($\mathsf{R}_{\mathtt{CR2}}$), $R(\varphi, \psi)$. ■

Let us now consider adequate r-models for the logics **S** and **DD** (cf. [3, pp. 65–68, 72–73, 120–123, 133], [16, pp. 587–596, 603–604]). In order to define r-models for logic **S** we consider condition ($\mathsf{R}_{\mathtt{CR1}}$) and the following conditions:

$$\forall_{\varphi,\psi\in\mathsf{For}}(R(\varphi,\psi) \Rightarrow R(\psi,\varphi)) \tag{R_{S0}}$$

$$\forall_{\varphi,\psi\in\mathsf{For}}(R(\neg\varphi,\psi) \text{ iff } R(\varphi,\psi)) \tag{R_{S1}}$$

$$\forall_{\varphi,\psi\in\mathsf{For}}(R(\varphi,\neg\psi) \text{ iff } R(\varphi,\psi)) \tag{$\mathsf{R}_{\mathtt{DD1}}$}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi\wedge\psi,\chi) \text{ iff } (R(\varphi,\chi) \text{ or } R(\psi,\chi)) \tag{R_{S2}}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi,\psi\wedge\chi) \text{ iff } (R(\varphi,\psi) \text{ or } R(\varphi,\chi)) \tag{$\mathsf{R}_{\mathsf{S2'}}$}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi\vee\psi,\chi) \text{ iff } (R(\varphi,\chi) \text{ or } R(\psi,\chi)) \tag{R_{S3}}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi,\psi\vee\chi) \text{ iff } (R(\varphi,\psi) \text{ or } R(\varphi,\chi)) \tag{$\mathsf{R}_{\mathsf{S3'}}$}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi\to\psi,\chi) \text{ iff } (R(\varphi,\chi) \text{ or } R(\psi,\chi)) \tag{R_{S4}}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi,\psi\to\chi) \text{ iff } (R(\varphi,\psi) \text{ or } R(\varphi,\chi)). \tag{$\mathsf{R}_{\mathsf{S4'}}$}$$

Obviously, for any symmetric relation, i.e., that satisfies (R_{S0}), (R_{S1})–(R_{S4}) are equivalent with ($\mathsf{R}_{\mathtt{DD1}}$), ($\mathsf{R}_{\mathsf{S2'}}$)–($\mathsf{R}_{\mathsf{S4'}}$). And if a relation satisfies ($\mathsf{R}_{\mathsf{S4'}}$), then it satisfies ($\mathsf{R}_{\mathtt{CR2}}$).

In order to define r-models for the logic **DD** we consider condition ($\mathsf{R}_{\mathtt{CR1}}$) and the following conditions:

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}((R(\varphi,\psi) \text{ and } R(\psi,\chi)) \Rightarrow R(\varphi,\chi)) \tag{$\mathsf{R}_{\mathtt{DD0}}$}$$

$$\forall_{\varphi\in\mathsf{For}}R(\neg\varphi,\varphi) \tag{$\mathsf{R}_{\mathsf{S1'}}$}$$

$$\forall_{\varphi\in\mathsf{For}}R(\varphi,\neg\varphi) \tag{$\mathsf{R}_{\mathtt{DD1'}}$}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi\wedge\psi,\chi) \text{ iff } (R(\varphi,\chi) \text{ and } R(\psi,\chi))) \tag{$\mathsf{R}_{\mathtt{DD2}}$}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi\vee\psi,\chi) \text{ iff } (R(\varphi,\chi) \text{ and } R(\psi,\chi))) \tag{$\mathsf{R}_{\mathtt{DD3}}$}$$

$$\forall_{\varphi,\psi,\chi\in\mathsf{For}}(R(\varphi\to\psi,\chi) \text{ iff } (R(\varphi,\chi) \text{ and } R(\psi,\chi))). \tag{$\mathsf{R}_{\mathtt{DD4}}$}$$

Obviously, for any reflexive and transitive relation, i.e., that satisfies ($\mathsf{R}_{\mathtt{CR1}}$) and ($\mathsf{R}_{\mathtt{DD0}}$), the conditions ($\mathsf{R}_{\mathtt{S1'}}$), ($\mathsf{R}_{\mathtt{DD1'}}$) are equivalent with ($\mathsf{R}_{\mathtt{S1}}$), ($\mathsf{R}_{\mathtt{DD1}}$); furthermore, if it satisfies ($\mathsf{R}_{\mathtt{DD4}}$), then it satisfies ($\mathsf{R}_{\mathtt{CR2}}$).

By Fact 4.10 we obtain the following correspondence between relational conditions and axiom schemata:

Corollary 4.11. *Let $\langle v, R\rangle$ be a r-model such that ($\mathsf{R}_{\mathtt{CR1}}$) and ($\mathsf{R}_{\mathtt{CR2}}$) are satisfied. Then:*

1. *$\langle v, R\rangle \models (X)$ iff R satisfies (R_X), where X is any of the following CR0,DS, S0, DD0,S1′,DD1′, AC1,AC2, C,BC1, BC2*
2. *$\langle v, R\rangle \models (X^*)$ iff R satisfies (R_{X^*}), where X is any of the following S1–S4, S2′–S4′, DD1–DD4 and $* \in \{\supset, \subset\}$*
3. *$\langle v, R\rangle \models$(DS) iff R satisfies ($\mathsf{R}_{\mathtt{S0}}$) or ($\mathsf{R}_{\mathtt{DD0}}$).*

Let us now prove determination theorems for the considered CRLs.

Theorem 4.12. *1.* **CR** *is sound with respect to the class of all r-models such that ($\mathsf{R}_{\mathtt{CR0}}$), ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{CR2}}$) are satisfied and either ($\mathsf{R}_{\mathtt{S0}}$) or ($\mathsf{R}_{\mathtt{DD0}}$) is satisfied.*

2. **S** *is sound with respect to the class of all r-models such that ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{S0}}$), ($\mathsf{R}_{\mathtt{S1}}$), ($\mathsf{R}_{\mathtt{S2}}$), ($\mathsf{R}_{\mathtt{S3}}$) and ($\mathsf{R}_{\mathtt{S4}}$) are satisfied.*
3. **DD** *is sound with respect to the class of all r-models such that ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{DD0}}$), ($\mathsf{R}_{\mathtt{S1'}}$), ($\mathsf{R}_{\mathtt{DD1'}}$), ($\mathsf{R}_{\mathtt{DD2}}$) ($\mathsf{R}_{\mathtt{DD3}}$) and ($\mathsf{R}_{\mathtt{DD4}}$) are satisfied.*

Proof. Let $\langle v, R\rangle$ be a r-model such that ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{CR2}}$) and ($\mathsf{R}_{\mathtt{CR0}}$) are satisfied.

For ($\mathtt{R}_{\rightarrow}$). By ($\mathsf{R}_{\mathtt{CR1}}$) and truth conditions, $\langle v, R\rangle \models \varphi \rightarrow \varphi$.

For ($\mathtt{S}^{+}_{\rightarrow}$). Suppose $\langle v, R\rangle \models \varphi \rightarrow \psi$. Then, by truth conditions, $R(\varphi, \psi)$. By ($\mathsf{R}_{\mathtt{CR2}}$), $R(\varphi, \psi \rightarrow \psi)$. By Fact 4.10, $\langle v, R\rangle \models \varphi \rightarrow (\psi \rightarrow \psi)$.

For ($\mathtt{W}^{+}_{\rightarrow}$). Suppose $\langle v, R\rangle \models \varphi \supset \psi$ and $\langle v, R\rangle \models \varphi \rightarrow (\psi \rightarrow \psi)$. By Fact 4.10, $R(\varphi, \psi)$. Then, by truth conditions, $\langle v, R\rangle \models \varphi \rightarrow \psi$.

For (DS). Let $\langle v, R\rangle$ be a r-model such that ($\mathsf{R}_{\mathtt{DD0}}$) is satisfied. By Corollary 4.11, $\langle v, R\rangle \models ((\varphi \rightarrow (\psi \rightarrow \psi)) \wedge (\psi \rightarrow (\chi \rightarrow \chi))) \supset (\varphi \rightarrow (\chi \rightarrow \chi))$. Thus, $\langle v, R\rangle \models$(DS). Similarly if ($\mathsf{R}_{\mathtt{S0}}$) is satisfied.

For (CR0) and the specific axiom schemata of **S** and **DD** we just use Corollary 4.11. ∎

To prove completeness for CRLs, we need a different notion of canonical model. We will continue to use the method presented in previous sections, which is slightly different from the one applied by Epstein in [3]. Let Λ be a CRL and $\Sigma \in \mathsf{Max}_{\Lambda}$. A *CRL canonical Σ-model* (*CRL-Σ-model*) is an ordered pair $\langle v_\Sigma, R_\Sigma\rangle$ such that for all $\varphi \in \mathsf{Var}$, $v_\Sigma(\varphi) = 1$ iff $\varphi \in \Sigma$; for all $\varphi, \psi \in \mathsf{For}$, $R_\Sigma(\varphi, \psi)$ iff $\varphi \to (\psi \to \psi) \in \Sigma$.

For such a notion of canonical model we obtain again a completeness lemma which is a counterpart of Lemma 4.3:

LEMMA 4.13. (Cf. [10, Lemma 5.13, p.], [15, Lemma 4.8, p. 529]) *Let Λ be a CRL, $\Sigma \in \mathsf{Max}_\Lambda$ and $\mathfrak{M}_\Sigma$ be a CRL-Σ-model. Then, for all $\varphi \in \mathsf{For}$, $\mathfrak{M}_\Sigma \models \varphi$ iff $\varphi \in \Sigma$.*

PROOF. We use induction on the complexity of formulas. Let us focus on the case $\varphi = \psi \to \chi$.

Suppose $\mathfrak{M}_\Sigma \models \psi \to \chi$. By truth conditions, $\mathfrak{M}_\Sigma \not\models \psi$ or $\mathfrak{M}_\Sigma \models \chi$, and $R_\Sigma(\psi, \chi)$. Thus, by the inductive hypothesis, $\psi \notin \Sigma$ or $\chi \in \Sigma$, and $R_\Sigma(\psi, \chi)$. By the definition of CRL-Σ-model, $\psi \notin \Sigma$ or $\chi \in \Sigma$, and $\psi \to (\chi \to \chi) \in \Sigma$. Since $\Sigma \in \mathsf{Max}_{\mathbf{CR}}$, $(\psi \supset \chi) \wedge (\psi \to (\chi \to \chi)) \in \Sigma$. By ($\mathtt{S}^+_\to$), $\psi \to \chi \in \Sigma$.

Suppose $\psi \to \chi \in \Sigma$. Since $\Sigma \in \mathsf{Max}_{\mathbf{CR}}$ and by ($\mathtt{W}_\to$), ($\mathtt{W}^+_\to$), $\psi \supset \chi \in \Sigma$ and $\psi \to (\chi \to \chi) \in \Sigma$. Thus, $\psi \notin \Sigma$ or $\chi \in \Sigma$, and $\psi \to (\chi \to \chi) \in \Sigma$. By the inductive hypothesis and Fact 4.10, $\mathfrak{M}_\Sigma \not\models \psi$ or $\mathfrak{M}_\Sigma \models \chi$, and $R_\Sigma(\psi, \chi)$. By truth conditions, $\mathfrak{M}_\Sigma \models \psi \to \chi$. ■

As before, to prove completeness we need to show that the relation from the canonical model meets the required conditions.

FACT 4.14. *1. If $\Sigma \in \mathsf{Max}_{\mathbf{CR}}$, then R_Σ satisfies (R_{CR0}), (R_{CR1}), (R_{CR2}), and either (R_{S0}) or (R_{DD0}).*

2. *If $\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus(X)}$, then R_Σ satisfies* (R_X), *where X is any of the following CR0,DS,S0 ,DD0,S1′,DD1′.*

3. *If $\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus(X^*)}$, then R_Σ satisfies* ($\mathsf{R}_{X\supset}$), *where X is any of the following S1–S4,S2′ –S4′,DD1–DD4 and $* \in \{\supset, \subset\}$.*

4. *If $\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus(S1\subset)}$ ($\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus(DD1\subset)}$) R_Σ satisfies ($R_{S1'}$) (($R_{DD1'}$)).*

But this fact holds by Corollary 4.11 and Lemma 4.13.

We can now prove completeness:

THEOREM 4.15. *1. **CR** is complete with respect to the class of all such r-models that (R_{CR0}), (R_{CR1}), (R_{CR2}) are satisfied and either (R_{S0}) or (R_{DD0}) is satisfied.*

2. ***S** is complete with respect to the class of all such r-models that (R_{CR1}), (R_{S0}), (R_{S1}), (R_{S2}), (R_{S3}) and (R_{S4}) are satisfied.*

3. ***DD** is complete with respect to the class of all such r-models that (R_{CR1}), (R_{DD0}), ($R_{S1'}$), ($R_{DD1'}$), (R_{DD2}) (R_{DD3}) and (R_{DD4}) are satisfied.*

Proof. We can reason as in the proof for Theorem 4.4. We use Fact 4.14 to show that the considered CRL-canonical-model belongs to the suitable class of models. ■

4.1.5. Non-derivability of Content Connexive Theses We go back now to the problem of non-derivability of (AC1) from ($A1^{\vee}$); of (AC2) from ($A2^{\vee}$), and of (C) from ($wB^{\vee}$). To this end we will use some sets of formulas defined with respect to $\downarrow$.

We define a valuation $v\colon \mathsf{Var} \longrightarrow \{1,0\}$ and we set $v(\varphi) = 1$ for all $\varphi \in \mathsf{Var}$. We define two relations. For all $\varphi, \psi \in \mathsf{For}$, $R_1(\varphi, \psi)$ holds iff at least one of the following conditions hold:

(1) $\varphi \in \mathsf{For}_{\downarrow p}$ and $\psi \in \mathsf{For}_{\downarrow \neg p}$

(2) $\varphi, \psi \in \mathsf{For}_{\downarrow \chi}$, for some $\chi \in \mathsf{For}$.

For all $\varphi, \psi \in \mathsf{For}$, R_2 is defined as R_1 except we change (1) in the following way:

(1′) $\varphi \in \mathsf{For}_{\downarrow \neg\neg p}$ and $\psi \in \mathsf{For}_{\downarrow \neg p}$.

Let us consider two r-models $\mathfrak{M}_1 = \langle v, R_1\rangle$ and $\mathfrak{M}_2 = \langle v, R_2\rangle$. We have the following fact:

Fact 4.16. *1. For $\langle v, R_1\rangle$ the following conditions are not satisfied: (R_{AC1}), (R_{C}). (More specifically, $R_1(p, p)$ and $R_1(p, \neg p)$).*

2. For $\langle v, R_2\rangle$ the following conditions are not satisfied: (R_{AC2}), (R_{C}). (More specifically, $R_2(\neg\neg p, \neg\neg p)$ and $R_2(\neg\neg p, \neg p)$).

3. For $\langle v, R_1\rangle, \langle v, R_2\rangle$ the following conditions are satisfied: (R_{CR0}), (R_{CR1}), (R_{CR2}), (R_{DD0}), ($\mathsf{R}_{AC1^{\vee}}$), ($\mathsf{R}_{AC2^{\vee}}$) and ($\mathsf{R}_{C^{\vee}}$).

Proof. 1 and 2 follow straightforwardly by the definitions of R_1 and R_2.

For 3. In what follows we focus on the relation R_1. For R_2 we can reason in a similar way.

For (R_{CR0}). Suppose $R_1(\varphi \to \varphi, \psi)$. By the definition of R_1 one of the following holds: 1) $\varphi \to \varphi \in \mathsf{For}_{\downarrow p}$ and $\psi \in \mathsf{For}_{\downarrow \neg p}$, 2) $\varphi \to \varphi, \psi \in \mathsf{For}_{\downarrow \chi}$ for some $\chi \in \mathsf{For}$. In case 1), by Fact 2.1.2, $\varphi \in \mathsf{For}_{\downarrow p}$. In case 2), by Fact 2.1.2, $\varphi \in \mathsf{For}_{\downarrow \chi}$. Thus, by the definition of R_1, $R_1(\varphi, \psi)$ By reasoning in a similar way, also by Fact 2.1.2, we obtain that if $R_1(\varphi, \psi)$, then $R_1(\varphi \to \varphi, \psi)$.

For (R_{CR1}). By Fact 2.1.1, $\varphi \in \mathsf{For}_{\downarrow \varphi}$. Thus, by the definition of R_1, $R_1(\varphi, \varphi)$.

For (R_{CR2}) similarly as for (R_{CR0}).

For (R_{DD0}). Suppose $R(\varphi, \psi)$ and $R(\psi, \chi)$. By the definition of R_1 we have that one of the following holds: 1) $\varphi \in \mathsf{For}_{\downarrow p}$ and $\psi \in \mathsf{For}_{\downarrow \neg p}$, and $\psi, \chi \in$

$\mathsf{For}_{\downarrow\chi'}$, for some $\chi' \in \mathsf{For}$, 2) $\varphi \in \mathsf{For}_{\downarrow p}$ and $\psi \in \mathsf{For}_{\downarrow\neg p}$, and $\psi \in \mathsf{For}_{\downarrow p}$ and $\chi \in \mathsf{For}_{\downarrow\neg p}$, 3) $\varphi, \psi \in \mathsf{For}_{\downarrow\chi'}$, for some $\chi' \in \mathsf{For}$ and $\psi \in \mathsf{For}_{\downarrow p}$ and $\chi \in \mathsf{For}_{\downarrow\neg p}$, 4) $\varphi, \psi \in \mathsf{For}_{\downarrow\chi'}$, for some $\chi' \in \mathsf{For}$ and $\psi, \chi \in \mathsf{For}_{\downarrow\chi''}$, for some $\chi'' \in \mathsf{For}$. In case 1), since $\mathsf{For}_{\downarrow\neg p} \cap \mathsf{For}_{\downarrow\chi'} \neq \emptyset$, by Corollary 2.2.1, $\chi \in \mathsf{For}_{\downarrow\neg p}$. Thus, by the definition of R_1, $R_1(\varphi, \chi)$. Case 2) is excluded by Fact 2.1.3, since $\downarrow p \neq \downarrow\neg p$. In cases 3) and 4) we reason similarly as in case 1).

For ($\mathsf{R}_{\mathtt{AC1}^{\vee}}$). Suppose $R_1(\varphi, \neg\varphi)$. By the definition of R_1 one of the following holds: 1) $\varphi \in \mathsf{For}_{\downarrow p}$ and $\neg\varphi \in \mathsf{For}_{\downarrow\neg p}$, 2) $\varphi, \neg\varphi \in \mathsf{For}_{\downarrow\chi}$ for some $\chi \in \mathsf{For}$. In case 1), by the definition of $\mathsf{For}_{\downarrow\neg p}$, $\neg\varphi = \neg p$, so $\varphi = p$. By the definition of $\mathfrak{M}_1$, $\mathfrak{M}_1 \models p$. Case 2 is excluded by Corollary 2.2.2.

For ($\mathsf{R}_{\mathtt{AC2}^{\vee}}$). Suppose $R_1(\neg\varphi, \varphi)$. By the definition of R_1 one of the following holds: 1) $\neg\varphi \in \mathsf{For}_{\downarrow p}$ and $\varphi \in \mathsf{For}_{\downarrow\neg p}$, 2) $\neg\varphi, \varphi \in \mathsf{For}_{\downarrow\chi}$ for some $\chi \in \mathsf{For}$. Case 1) is excluded by Corollary 2.2.3, since $\downarrow p \neq \neg\varphi$. Case 2) is excluded by Corollary 2.2.2.

For ($\mathsf{R}_{\mathsf{C}^{\vee}}$). Suppose $R_1(\varphi, \psi)$. By the definition of R_1 one of the following holds: 1) $\varphi \in \mathsf{For}_{\downarrow p}$ and $\psi \in \mathsf{For}_{\downarrow\neg p}$, 2) $\varphi, \psi \in \mathsf{For}_{\downarrow\chi}$, for some $\chi \in \mathsf{For}$. Assume also that $R_1(\varphi, \neg\psi)$. By the definition of R_1 one of the following holds: 1$'$) $\varphi \in \mathsf{For}_{\downarrow p}$ and $\neg\psi \in \mathsf{For}_{\downarrow\neg p}$, 2$'$) $\varphi, \neg\psi \in \mathsf{For}_{\downarrow\chi'}$, for some $\chi' \in \mathsf{For}$. By Corollary 2.2.2 either 1) holds or 1$'$) holds. By Corollary 2.2 cases 1 and 2, either 2) holds or 2$'$) holds. But if either 1) holds or 1$'$) holds, by the definition of $\mathfrak{M}_1$, $\mathfrak{M}_1 \models \varphi$. ■

By Facts 4.6, 4.10 and 4.16 we have the following corollary:

Corollary 4.17. *1.* $\mathfrak{M}_1 \models$*(A1$^{\vee}$) and* $\mathfrak{M}_2 \models$*(A1$^{\vee}$).*

2. $\mathfrak{M}_1 \models$*(A2$^{\vee}$) and* $\mathfrak{M}_2 \models$*(A2$^{\vee}$).*

3. $\mathfrak{M}_1 \models$*(wB$^{\vee}$) and* $\mathfrak{M}_2 \models$*(wB$^{\vee}$).*

4. $\mathfrak{M}_1 \not\models \neg(p \to (\neg p \to \neg p))$ *and* $\mathfrak{M}_1 \not\models (p \to (p \to p)) \supset \neg(p \to (\neg p \to \neg p))$.

5. $\mathfrak{M}_2 \not\models \neg(\neg\neg p \to (\neg p \to \neg p))$ *and* $\mathfrak{M}_2 \not\models (\neg\neg p \to (\neg p \to \neg p)) \supset \neg(\neg\neg p \to (\neg\neg p \to \neg\neg p))$.

4.1.6. Soundness and Completeness of the Considered wBCCRLs and BCCRLs Let us now consider the main logics of our interest, i.e., some wBCCRLS and BCCRLs. We already introduced all relational conditions which enable to determine all wBCCRLs and BCCRLs analysed in the article. Based on the analysis presented above, we obtain the following soundness theorem:

THEOREM 4.18. *1.* **wBCCR** *(***BCCR***) is sound with respect to the class of all such r-models that (*R_{CR0}*), (*R_{CR1}*), (*R_{CR2}*), (*R_C*) (and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied, and either (*R_{S0}*) or (*R_{DD0}*) is satisfied.*

2. **wBCS** *(***BCS***) is sound with respect to the class of all such r-models that (*R_{CR1}*), (*R_{CR2}*), (*R_C*), (*R_{S0}*),* ($R_{S2\supset}$), ($R_{S3\supset}$), ($R_{S4\supset}$) *(and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied.*

3. **wBCDD** *(***BCDD***) is sound with respect to the class of all such r-models that (*R_{CR1}*), (*R_{CR2}*), (*R_C*), (*R_{DD0}*),* ($R_{DD2\subset}$), ($R_{DD3\subset}$), ($R_{DD4\subset}$), *(and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied.*

4. **wBCDD**$^+$ *(***BCDD**$^+$*) is sound with respect to the class of all such r-models that (*R_{CR1}*), (*R_{CR2}*), (*R_C*), (*R_{DD0}*),* ($R_{DD2\subset}$), ($R_{DD3\subset}$), ($R_{DD4\subset}$), ($R_{S2'\supset}$), ($R_{S3'\supset}$), ($R_{S4'\supset}$) *(and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied.*

5. **wBCS**$^+$ *(***BCS**$^+$*) is sound with respect to the class of all such r-models that (*R_{CR1}*), (*R_{CR2}*), (*R_C*), (*R_{S0}*),* ($R_{S2\supset}$), ($R_{S3\supset}$), ($R_{S4\supset}$), *(*R_{DD0}*) (and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied.*

PROOF. We can reason as in the case of proof for Theorem 4.12. ■

We prove completeness as in the previous cases, to this end we use Corollary 4.11 and Lemma 4.13:

FACT 4.19. *1. If* $\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus}(C)$*, then R satisfies (*$R_{AC1}$*) and (*$R_{AC2}$*).*

2. If $\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus}(C)$*, then R satisfies (*$R_C$*).*

3. If $\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus}(BC1)$*, then R satisfies (*$R_{BC1}$*).*

4. If $\Sigma \in \mathsf{Max}_{\mathbf{CR}\oplus}(BC2)$*, then R satisfies (*$R_{BC2}$*).*

As in the previous cases, we can now prove the completeness theorem for the considered logics:

THEOREM 4.20. *1.* **wBCCR** *(***BCCR***) is complete with respect to the class of all such r-models that (*R_{CR0}*), (*R_{CR1}*), (*R_{CR2}*), (*R_C*) (and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied, and either (*R_{S0}*) or (*R_{DD0}*) is satisfied.*

2. **wBCS** *(***BCS***) is complete with respect to the class of all such r-models that (*R_{CR1}*), (*R_{CR2}*), (*R_C*), (*R_{S0}*),* ($R_{S2\supset}$), ($R_{S3\supset}$), ($R_{S4\supset}$) *(and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied.*

3. **wBCDD** *(***BCDD***) is complete with respect to the class of all such r-models that (*R_{CR1}*), (*R_{CR2}*), (*R_C*), (*R_{DD0}*),* ($R_{DD2\subset}$), ($R_{DD3\subset}$), ($R_{DD4\subset}$), *(and additionally (*R_{BC1}*), (*R_{BC2}*)) are satisfied.*

4. **wBCDD**$^+$ (**BCDD**$^+$) *is complete with respect to the class of all such r-models that* (R_{CR1}), (R_{CR2}), (R_C), (R_{DD0}), ($\mathsf{R}_{DD2\subset}$), ($\mathsf{R}_{DD3\subset}$), ($\mathsf{R}_{DD4\subset}$), ($\mathsf{R}_{S2'\supset}$), ($\mathsf{R}_{S3'\supset}$), ($\mathsf{R}_{S4'\supset}$) *(and additionally* (R_{BC1}), (R_{BC2})*) are satisfied.*
5. **wBCS**$^+$ (**BCS**$^+$) *is complete with respect to the class of all such r-models that* (R_{CR1}), (R_{CR2}), (R_C), (R_{S0}), ($\mathsf{R}_{S2\supset}$), ($\mathsf{R}_{S3\supset}$), ($\mathsf{R}_{S4\supset}$), (R_{DD0}) *(and additionally* (R_{BC1}), (R_{BC2})*) are satisfied.*

Proof. We reason as in proofs for Theorems 4.5, 4.15. We use Facts 4.14 and 4.19 to show that the considered BCCRL-canonical-model belongs to the suitable class of models. ■

4.1.7. Content Relationship Let us now specify in what sense the considered logics are content relationship logics. To achieve this goal we introduce the notion of a *set-assignment*, that is any function $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$, where S is a suitable non-empty set of contents.[5] Thus, following the idea of Epstein (see [3]), the outputs of set-assignments can be used to represent sentential contents. In our analysis, we assume that representation of content satisfies at least two properties: non-emptiness and insensitiveness to the repetition of sentences within conditional sentences. We say that a set-assignment s is a *content set-assignment* (a *c-assignment*) iff for all $\varphi \in \mathsf{For}$ the following conditions are satisfied:

$$s(\varphi) \neq \emptyset \tag{c1}$$

$$s(\varphi \to \varphi) = s(\varphi). \tag{c2}$$

In the analysis of content relationship, Epstein used a special kind of c-assignment, the so-called union set-assignment by means of which he wanted to capture the principle of compositionality for content.[6] A *union set-assignment* (a *u-assignment*) is a c-assignment $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ such that for all $\varphi, \psi \in \mathsf{For}$ the following conditions are satisfied:

$$s(\neg\varphi) = s(\varphi) \tag{u1}$$

$$s(\varphi \wedge \psi) = s(\varphi) \cup s(\psi) \tag{u2}$$

[5]What contents are is left to the reader to decide. Epstein mentions as plausible candidates for contents Lewisian subject matters or even state-descriptions. Krajewski [17] also speaks of "topics". We only require here that they can be grouped into sets that can be subject to the usual operations and satisfy some conditions necessary for a content representation.

[6]Although the union set-assignments are used in Epstein's main analysis of content relationships, other set-assignments are needed to interpret some well-known non-classical logics in terms of set-assignments (see [3], cf. also [16]).

$$s(\varphi \vee \psi) = s(\varphi) \cup s(\psi) \tag{u3}$$

$$s(\varphi \rightarrow \psi) = s(\varphi) \cup s(\psi). \tag{u4}$$

Taking any c-assignment s we define the following relations that specify the understanding of content relationship:

$$R_s^{\cap}(\varphi, \psi) \text{ iff } s(\varphi) \cap s(\psi) \neq \emptyset \tag{Def $R_s^{\cap}$}$$

$$R_s^{\subseteq}(\varphi, \psi) \text{ iff } s(\varphi) \subseteq s(\psi) \tag{Def $R_s^{\subseteq}$}$$

$$R_s^{\supseteq}(\varphi, \psi) \text{ iff } s(\varphi) \supseteq s(\psi) \tag{Def $R_s^{\supseteq}$}$$

$$R_s^{=}(\varphi, \psi) \text{ iff } s(\varphi) = s(\psi), \tag{Def $R_s^{=}$}$$

These relations represent four concepts of content relationship: having a shared content, containment of content in two variants and equality of content.

We can easily check that the relations defined with respect to c-assignments in one of the presented ways meet some of the considered conditions that allow defining some of the distinguished CRLs:

FACT 4.21. *1. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ is a c-assignment, then $R_s^{\cap}$ satisfies (R_{CR1}), (R_{CR2}), (R_{S0}).*

2. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ is a c-assignment, then $R_s^{\subseteq}$ satisfies (R_{CR0}), (R_{CR1}), (R_{CR2}), (R_{DD0}).

3. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ is a c-assignment, then $R_s^{\supseteq}$ satisfies (R_{CR0}), (R_{CR1}), (R_{CR2}), (R_{DD0}).

4. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ is a c-assignment, then $R_s^{=}$ satisfies (R_{CR1}), (R_{CR2}), (R_{DD0}), (R_{S0}).

5. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ is a u-assignment, then $R_s^{\cap}$ satisfies (R_{CR1}), (R_{S0})–(R_{S4}).

6. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ is a u-assignment, then $R_s^{\subseteq}$ satisfies (R_{CR1}), (R_{DD0}), ($R_{S1'}$), ($R_{DD1'}$), (R_{DD2})–(R_{DD4}).

PROOF. For 1. For (R_{CR1}). By (c1) we have $s(\varphi) \cap s(\varphi) \neq \emptyset$. Thus, by (Def $R_s^{\cap}$), $R_s^{\cap}(\varphi, \varphi)$.

For (R_{CR2}). Suppose $R_s^{\cap}(\varphi, \psi \rightarrow \psi)$. By (Def $R_s^{\cap}$), $s(\varphi) \cap s(\psi \rightarrow \psi) \neq \emptyset$. Therefore, by (c2), $s(\varphi) \cap s(\psi) \neq \emptyset$. Thus, by (Def $R_s^{\cap}$), $R_s^{\cap}(\varphi, \psi)$. Suppose $R_s^{\cap}(\varphi, \psi)$. By (Def $R_s^{\cap}$), $s(\varphi) \cap s(\psi) \neq \emptyset$. Therefore, by (c2), $s(\varphi) \cap s(\psi \rightarrow \psi) \neq \emptyset$. Thus, by (Def $R_s^{\cap}$), $R_s^{\cap}(\varphi, \psi \rightarrow \psi)$.

For (R_{S0}). We have, $R_s^{\cap}(\varphi, \psi)$, by (Def $R_s^{\cap}$), iff $s(\varphi) \cap s(\psi) \neq \emptyset$ iff $s(\psi) \cap s(\varphi) \neq \emptyset$, by (Def $R_s^{\cap}$), iff $R_s^{\cap}(\psi, \varphi)$.

For 2. For ($\mathsf{R_{CR1}}$). We have $s(\varphi) \subseteq s(\varphi)$. Thus, by (Def $R_s^{\subseteq}$), $R_s^{\subseteq}(\varphi, \varphi)$.

For ($\mathsf{R_{CR0}}$). Suppose $R_s^{\subseteq}(\varphi \to \varphi, \psi)$. By (Def $R_s^{\subseteq}$), $s(\varphi \to \varphi) \subseteq s(\psi)$. Therefore, by (c1), $s(\varphi) \subseteq s(\psi)$. Thus, by (Def $R_s^{\subseteq}$), $R_s^{\subseteq}(\varphi, \psi)$. Suppose $R_s^{\subseteq}(\varphi, \psi)$. By (Def $R_s^{\subseteq}$), $s(\varphi) \subseteq s(\psi)$. Therefore, by (c2), $s(\varphi) \subseteq s(\psi \to \psi)$. Thus, by (Def $R_s^{\subseteq}$), $R_s^{\subseteq}(\varphi, \psi \to \psi)$.

For ($\mathsf{R_{CR2}}$) we reason similarly as for ($\mathsf{R_{CR0}}$).

For ($\mathsf{R_{DD0}}$). Suppose $R_s^{\subseteq}(\varphi, \psi)$ and $R_s^{\subseteq}(\psi, \chi)$. By (Def $R_s^{\subseteq}$), $s(\varphi) \subseteq s(\psi)$ and $s(\psi) \subseteq s(\chi)$. Then, $s(\varphi) \subseteq s(\chi)$. Thus, by (Def $R_s^{\subseteq}$), $R_s^{\subseteq}(\varphi, \chi)$.

For 3 and 4 we reason similarly as for 1 and 2.

For 5 and 6 see [3, pp. 68–70, 122], [16, pp. 596–598]. ■

We call a relation $R \subseteq \mathsf{For} \times \mathsf{For}$ a *content relation* (a *c-relation*) iff there is a c-assignment $s\colon \mathsf{For} \longrightarrow \mathcal{P}(S)$, there is $* \in \{\cap, \subseteq, \supseteq, =\}$ such that for all $\varphi, \psi \in \mathsf{For}$ $R(\varphi, \psi)$ iff $R_s^{*}(\varphi, \psi)$. We can now prove that all of CRLs we considered are determined by some c-relations. The following fact holds:

Fact 4.22. *1. Let* $R \subseteq \mathsf{For} \times \mathsf{For}$ *satisfies* ($\mathsf{R_{CR1}}$), ($\mathsf{R_{CR2}}$), ($\mathsf{R_{S0}}$). *We define* $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ *in the following way:* $s(\varphi) = \{\{\psi, \chi\} \subseteq \mathsf{For} :$ *either* $\psi \in \mathsf{For}_{\downarrow\varphi}$ *or* $\chi \in \mathsf{For}_{\downarrow\varphi}$ *and* $R(\psi, \chi)\}$. *Then, (a)* s *is a c-assignment and (b)* $R = R_s^{\cap}$.

2. Let $R \subseteq \mathsf{For} \times \mathsf{For}$ *satisfies* ($\mathsf{R_{CR0}}$), ($\mathsf{R_{CR1}}$), ($\mathsf{R_{CR2}}$), ($\mathsf{R_{DD0}}$). *We define* $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ *in the following way:* $s(\varphi) = \{\psi \in \mathsf{For} : R(\psi, \varphi)\}$. *Then, (a)* s *is a c-assignment and (b)* $\mathsf{R} = \mathsf{R}_s^{\subseteq}$.

3. Let $R \subseteq \mathsf{For} \times \mathsf{For}$ *satisfies* ($\mathsf{R_{CR0}}$), ($\mathsf{R_{CR1}}$), ($\mathsf{R_{CR2}}$), ($\mathsf{R_{DD0}}$). *We define* $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ *in the following way:* $s(\varphi) = \{\psi \in \mathsf{For} : R(\varphi, \psi)\}$. *Then, (a)* s *is a c-assignment and (b)* $\mathsf{R} = \mathsf{R}_s^{\supseteq}$.

4. Let $R \subseteq \mathsf{For} \times \mathsf{For}$ *satisfies* ($\mathsf{R_{CR1}}$), ($\mathsf{R_{CR2}}$), ($\mathsf{R_{DD0}}$), ($\mathsf{R_{S0}}$). *We define* $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ *in the following way:* $s(\varphi) = \{\psi \in \mathsf{For} : R(\psi, \varphi)\}$. *Then, (a)* s *is a c-assignment and (b)* $\mathsf{R} = \mathsf{R}_s^{=}$.

5. Let $R \subseteq \mathsf{For} \times \mathsf{For}$ *satisfies* ($\mathsf{R_{CR1}}$), ($\mathsf{R_{S0}}$)–($\mathsf{R_{S4}}$). *We define* $s\colon \mathsf{For} \longrightarrow \{\{\varphi, \psi\} : \varphi, \psi \in \mathsf{For}\}$ *in the following way: for any* $\varphi \in \mathsf{Var}$, $s(\varphi) = \{\{\varphi, \psi\} \in \mathsf{For} : R(\varphi, \psi)\}$, *we extend* s *on* For *it the following way:* $s(\varphi) = \bigcup\{s(\psi)\colon \psi$ *is a variable of* $\varphi\}$. *Then, (a)* s *is a u-assignment and (b)* $R = R_s^{\cap}$.

6. Let $R \subseteq \mathsf{For} \times \mathsf{For}$ *satisfies* ($\mathsf{R_{CR1}}$), ($\mathsf{R_{DD0}}$), ($\mathsf{R_{S1'}}$), ($\mathsf{R_{DD1'}}$), ($\mathsf{R_{DD2}}$)–($\mathsf{R_{DD4}}$). *We define* $t\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ *in the following way:* $t(\varphi) = \{\psi \in \mathsf{For} : R(\psi, \varphi)\}$, *and then we define* $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ *in the following way:* $s(\varphi) = \mathsf{For} \backslash t(\varphi)$. *Then, (a)* s *is a u-assignment and (b)* $\mathsf{R} = \mathsf{R}_s^{\subseteq}$.

Proof. For 1(a). For (c1). By ($\mathsf{R_{CR1}}$) $R(\varphi, \varphi)$. Thus, by definition of s and Fact 2.1.1, $\{\varphi, \varphi\} \in s(\varphi)$, and so $s(\varphi) \neq \emptyset$. For (c2). We have $\{\psi, \chi\} \in s(\varphi)$, by the definition of s and Fact 2.1.1, iff $\{\psi, \chi\} \in s(\varphi \to \varphi)$.

For 1(b). Suppose $R(\varphi, \psi)$. By ($\mathsf{R_{S0}}$) we also have $R(\psi, \varphi)$. By Fact 2.1.1, $\varphi \in \mathsf{For}_{\downarrow\varphi}$ and $\psi \in \mathsf{For}_{\downarrow\psi}$. Thus, by the definition of s, $\{\varphi, \psi\} \in s(\varphi)$ and $\{\varphi, \psi\} \in s(\psi)$. By the definition of s, $s(\varphi) \cap s(\psi) \neq \emptyset$. Suppose, $s(\varphi) \cap s(\psi) \neq \emptyset$. Thus, $\{\chi, \chi'\} \in s(\varphi)$, and $\{\chi, \chi'\} \in s(\psi)$. By the definition of s, $R(\chi, \chi')$, where either $\chi \in \mathsf{For}_{\downarrow\varphi}$ or $\chi' \in \mathsf{For}_{\downarrow\varphi}$ and either $\chi \in \mathsf{For}_{\downarrow\psi}$ or $\chi' \in \mathsf{For}_{\downarrow\psi}$. If $\chi \in \mathsf{For}_{\downarrow\varphi}$ and $\chi \in \mathsf{For}_{\downarrow\psi}$, or $\chi' \in \mathsf{For}_{\downarrow\varphi}$ and $\chi' \in \mathsf{For}_{\downarrow\psi}$, then by Corollary 2.2.1, $\mathsf{For}_{\downarrow\varphi} = \mathsf{For}_{\downarrow\psi}$. By ($\mathsf{R_{CR1}}$) $R(\varphi, \varphi)$. Thus, by ($\mathsf{R_{CR0}}$) or ($\mathsf{R_{CR2}}$) depending on $\downarrow$-complexity of φ and ψ, $R(\varphi, \psi)$. Suppose $\chi \in \mathsf{For}_{\downarrow\varphi}$ and $\chi' \in \mathsf{For}_{\downarrow\psi}$. Thus, by ($\mathsf{R_{CR2}}$) and ($\mathsf{R_{CR0}}$), $R(\varphi, \psi)$. Suppose $\chi' \in \mathsf{For}_{\downarrow\varphi}$ and $\chi \in \mathsf{For}_{\downarrow\psi}$. Thus, by ($\mathsf{R_{CR2}}$), ($\mathsf{R_{CR0}}$) and ($\mathsf{R_{S0}}$), $R(\varphi, \psi)$.

For 2(a). For (c1). By ($\mathsf{R_{CR1}}$) $R(\varphi, \varphi)$. Thus, by definition of s, $\varphi \in s(\varphi)$, and so $s(\varphi) \neq \emptyset$. For (c2). We have, $\psi \in s(\varphi)$, by definition of s, iff $R(\psi, \varphi)$, by ($\mathsf{R_{CR2}}$), iff $R(\psi, \varphi \to \varphi)$, by definition of s, $\psi \in s(\varphi \to \varphi)$.

For 2(b). Suppose $R(\varphi, \psi)$ and $\chi \in s(\varphi)$. By the definition of s, $R(\chi, \varphi)$. By ($\mathsf{R_{DD0}}$), $R(\chi, \psi)$. Thus, by the definition of s, $\chi \in s(\psi)$. Suppose $s(\varphi) \subseteq s(\psi)$. By the definition of s and ($\mathsf{R_{CR1}}$), $\varphi \in s(\psi)$. Thus, by the definition of s, $R(\varphi, \psi)$.

For 3 and 4 we reason similarly as for 2. For 5 and 6, see [3, pp. 68–70, 122] and [16, pp. 589–590, 592–594, 600–601]. ■

4.2. Set-Assignment Semantics

A *set-assignment structure* (*sa-structure*) is an ordered pair $\langle v, s\rangle$ such that $v \colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a classical valuation and $s \colon \mathsf{For} \longrightarrow \mathcal{P}(S)$ is a set-assignment. For any logic Λ, in order to define sa-models for Λ we use sa-structures, specifying set-assignments on which the given structures are supposed to be based. Then, by means of some set-theoretic condition F with two variables we define a relation $R^s_\Lambda \subseteq \mathsf{For} \times \mathsf{For}$ in the following way:

$$R^s_\Lambda(\varphi, \psi) \text{ iff } F(s(\varphi), s(\psi)).$$

For instance, R^s_Λ might be defined as in (Def $R^{\cap}_s$), (Def $R^{\supseteq}_s$), (Def $R^{\subseteq}_s$) or (Def $R^{=}_s$).

For any logic Λ and any sa-structure $\langle v, s\rangle$, we assume the following truth conditions:

$$\langle v, s\rangle \models \varphi \text{ iff } v(\varphi) = 1, \text{ if } \varphi \in \mathsf{Var}$$
$$\langle v, s\rangle \models \neg\varphi \text{ iff not } \langle v, s\rangle \models \varphi \text{ (i.e. } \langle v, s\rangle \not\models \varphi)$$

$$\langle v, s\rangle \models \varphi \wedge \psi \text{ iff } \langle v, s\rangle \models \varphi \text{ and } \langle v, s\rangle \models \psi$$
$$\langle v, s\rangle \models \varphi \vee \psi \text{ iff } \langle v, s\rangle \models \varphi \text{ or } \langle v, s\rangle \models \psi$$
$$\langle v, s\rangle \models \varphi \rightarrow \psi \text{ iff either } \langle v, s\rangle \not\models \varphi \text{ or } \langle v, s\rangle \models \psi, \text{ and } R^s_{\Lambda}(\varphi, \psi).$$

And so, in order to define a Λ-sa-model, for any logic Λ, we use sa-structures specifying the set-assignments we want to focus on and the connection relation defined with respect to the images of set-assignments from the considered sa-structures.[7] For example, we have:

- $\langle v, s\rangle$ is a **S**-sa-model iff s is a u-assignment and for any u-assignment t, $R^t_{\mathbf{S}} = R^{\cap}_t$
- $\langle v, s\rangle$ is a **DD**-sa-model iff s is a u-assignment and for any u-assignment t, $R^t_{\mathbf{S}} = R^{\subseteq}_t$.

Having defined the notion of a Λ-sa-model for a given logic Λ, we can define the relation of semantic consequence, validity, soundness and completeness as before for the class of r-models.[8]

4.2.1. Soundness and Completeness of Logics wBCDD$^+$, BCDD$^+$, wBCS$^+$ and BCS$^+$ Let us now show how we can define set-assignment semantics for the following CRLs: **wBCDD**$^+$, **BCDD**$^+$, **wBCS**$^+$ and **BCS**$^+$. We obtain the proper semantics by modifying the notion of u-assignment.

We first define a version of partial u-assignment. A *partial union set-assignment* (*pu-assignment*) is a c-assignment $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ such that the following conditions are satisfied:

$$\varphi \in s(\varphi) \tag{pu1}$$

$$s(\varphi) \subseteq s(\varphi \rightarrow \varphi) \tag{pu2}$$

$$s(\neg\varphi) \subseteq \mathsf{For}\backslash s(\varphi) \tag{pu3}$$

$$s(\varphi \wedge \psi) \subseteq s(\varphi) \cup s(\psi) \tag{pu4}$$

$$s(\varphi \vee \psi) \subseteq s(\varphi) \cup s(\psi) \tag{pu5}$$

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[7] This method for specifying the set-assignment semantics for a given logic was proposed in [8]. In the given approach, we do not consider the possibility of determining the set-assignment semantics for a given logic by intersecting or summing the classes of different set-assignment models.

[8] All of Epstein's logics determinable by means of r-models might be also determined by means of the proper sa-structures (see [3, pp. 74–75, 130–122], [16, pp. 569–598]). By Theorems 4.12, 4.15 and Facts 4.22, 4.21 we get that **S** is determined by the class of all **S**-sa-models and **DD** is determined by the class of all **DD**-sa-models (see [3, pp. 122-123, 135–139], [16, pp. 598–605]).

$$s(\varphi \to \psi) \subseteq s(\varphi) \cup s(\psi). \tag{pu6}$$

A *content inclusion pu-assignment* (*ipu-assignment*) is a pu-assignment $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ such that the following condition is satisfied:

$$\varphi \in s(\psi) \Rightarrow s(\varphi) \subseteq s(\psi). \tag{ipu}$$

A *common content pu-assignment* (*cpu-assignment*) is a pu-assignment $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ such that the following condition is satisfied:

$$s(\varphi) \cap s(\psi) \neq \emptyset \Rightarrow (\psi \in s(\varphi) \text{ and } \varphi \in s(\psi)). \tag{cpu}$$

We need two more conditions for an sa-assignment. A c-assignment $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ a *b-assignment* iff the following conditions are satisfied:

$$s(\varphi \to \psi) \subseteq s(\neg(\varphi \to \neg\psi)) \tag{b1}$$
$$s(\varphi \to \neg\psi) \subseteq s(\neg(\varphi \to \psi)). \tag{b2}$$

Let us note an interesting fact about cpu-assignments.

Fact 4.23. *If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ is a cpu-assignment, then $R^{\cap}_s = R^{=}_s$.*

Proof. Suppose $s(\varphi) \cap s(\psi) \neq \emptyset$. By (cpu), $\varphi \in s(\psi)$. Let $\chi \in s(\varphi)$. Then, by (pu1), $s(\varphi) \cap s(\chi) \neq \emptyset$. Thus, by (cpu), $\varphi \in s(\chi)$. Hence, $s(\psi) \cap s(\chi) \neq \emptyset$. By (cpu), $\chi \in s(\psi)$. We can similarly prove the inclusion $s(\psi) \subseteq s(\varphi)$. Suppose $s(\varphi) = s(\psi)$. By (pu1), $s(\varphi) \cap s(\psi) \neq \emptyset$. ■

According to Fact 4.23, the relation defined by (Def $R^{\cap}_s$) with respect to cpu-assignments enables us to represent content relationship understood as content equality, i.e., stronger than content sharing.

We can prove thats some set-assignments defined with respect to proper relations enable us to obtain some pu-assignments. We call a relation $R \subseteq \mathsf{For} \times \mathsf{For}$:

- a *wbcdd+-relation* (a *bcdd+-relation*) iff R satisfies conditions ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{CR2}}$), ($\mathsf{R}_{\mathtt{C}}$), ($\mathsf{R}_{\mathtt{DD0}}$), ($\mathsf{R}_{\mathtt{DD2}\subset}$)– ($\mathsf{R}_{\mathtt{DD4}\subset}$), ($\mathsf{R}_{\mathtt{S2'}\supset}$)– ($\mathsf{R}_{\mathtt{S4'}\supset}$) (and additionally ($\mathsf{R}_{\mathtt{BC1}}$), ($\mathsf{R}_{\mathtt{BC2}}$)).
- a *wbcs+-relation* (a *bcs+-relation*) iff R satisfies conditions ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{CR2}}$), ($\mathsf{R}_{\mathtt{C}}$), ($\mathsf{R}_{\mathtt{S0}}$), ($\mathsf{R}_{\mathtt{S2}\supset}$)– ($\mathsf{R}_{\mathtt{S4}\supset}$), ($\mathsf{R}_{\mathtt{DD0}}$) (and additionally ($\mathsf{R}_{\mathtt{BC1}}$), ($\mathsf{R}_{\mathtt{BC2}}$)).[9]

The class of all r-models with wbcdd+-relations (bcdd+-relations) is denoted by rWBCDD^{+} (rBCDD^{+}). The class of all r-models with wbcs+-relations (bcs+-relations) is denoted by rWBCS^{+} (rBCS^{+}).

[9] And thus, by Theorems 4.12 and 4.15, $\mathbf{wBCDD}^{+}$ ($\mathbf{BCDD}^{+}$) is determined by the class of all r-models with wbcdd+-relations (bcdd+-relation) and $\mathbf{wBCS}^{+}$ ($\mathbf{BCS}^{+}$) is determined by the class of all r-models with wbcs+-relations (bcs+-relations).

Let $R \subseteq \mathsf{For} \times \mathsf{For}$. We define $s_R\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ in the following way:

$$s_R(\varphi) := \{\psi \in \mathsf{For} : R(\psi, \varphi)\}. \qquad \text{(Def } s_R\text{)}$$

We have the following fact:

FACT 4.24. *1. If R is a wbcdd+-relation, then (a)* s_R *is an ipu-assignment and (b)* $R = R^{\subseteq}_{s_R}$.

2. If R is a bcdd+-relation, then (a) s_R *is a bipu-assignment and (b)* $R = R^{\subseteq}_{s_R}$.

3. If R is a wbcs+-relation, then (a) s_R *is a cpu-assignment and (b)* $R = R^{\cap}_{s_R}$.

4. If R is a bcs+-relation, then (a) s_R *is a bcpu-assignment and (b)* $R = R^{\cap}_{s_R}$.

PROOF. For 1 and 2.

For (a). For (pu1). By ($\mathsf{R_{CR1}}$), $R(\varphi, \varphi)$. Thus, by (Def s_R), $\varphi \in s_R(\varphi)$.

For (pu2). We have, $\psi \in s_R(\varphi)$, by (Def s_R), iff $R(\psi, \varphi)$, by ($\mathsf{R_{CR2}}$), iff $R(\psi, \varphi \to \varphi)$, by (Def s_R), $\psi \in s_R(\varphi \to \varphi)$.

For (pu3). Suppose $\chi \in s_R(\neg\varphi)$. By (Def s_R), $R(\chi, \neg\varphi)$. Thus, by ($\mathsf{R_C}$), $\sim R(\chi, \varphi)$. Hence, by (Def s_R), $\chi \notin s_R(\varphi)$.

For (pu1), (pu2) and (pu3). Let $* \in \{\wedge, \vee, \to\}$. Suppose $\chi \in s_R(\varphi * \psi)$. By (Def s_R), $R(\chi, \varphi * \psi)$. By ($\mathtt{S2'}^{\supset}$), ($\mathtt{S3'}^{\supset}$) and ($\mathtt{S4'}^{\supset}$), $R(\chi, \varphi)$ or $R(\chi, \psi)$. Thus, by (Def s_R), $\chi \in s_R(\varphi)$ or $\chi \in s_R(\psi)$.

For (ipu). Suppose $\varphi \in s_R(\psi)$ and $\chi \in s_R(\varphi)$. By (Def s_R), $R(\varphi, \psi)$ and $R(\chi, \varphi)$. Thus, by ($\mathsf{R_{DDO}}$), $R(\chi, \psi)$. By (Def s_R), $\chi \in s_R(\psi)$.

For (b1) and (b2). By ($\mathsf{R_{BC1}}$), ($\mathsf{R_{BC2}}$), $R(\varphi \to \psi, \neg(\varphi \to \neg\psi))$ and $R(\varphi \to \neg\psi, \neg(\varphi \to \psi))$. Suppose $\chi \in s_R(\varphi \to \psi)$ and $\chi' \in s_R(\varphi \to \psi)$. Thus, by (Def s_R), $R(\chi, \varphi \to \psi)$ and $R(\chi', \varphi \to \neg\psi)$. By ($\mathsf{R_{DDO}}$) $R(\chi, \neg(\varphi \to \neg\psi))$ and $R(\chi', \neg(\varphi \to \psi))$ Thus, by (Def s_R), $\chi \in s_R(\neg(\varphi \to \neg\psi))$ and $\chi' \in s_R(\neg(\varphi \to \neg\psi))$.

For (b) we reason similarly as for Fact 4.22.2.

For 3 and 4.

For (a). For (pu1)–(pu6), (b1) and (b2) as above. For (cpu). Suppose $s_R(\varphi) \cap s_R(\psi) \neq \emptyset$. Then, by (Def s_R), $R(\chi, \varphi)$ and $R(\chi, \psi)$. Thus, by ($\mathsf{R_{SO}}$), $R(\varphi, \chi)$ and $R(\psi, \chi)$. Hence, by ($\mathsf{R_{DDO}}$), $R(\psi, \varphi)$ and $R(\varphi, \psi)$. Then, by (Def s_R), $\psi \in s_R(\varphi)$ and $\varphi \in s_R(\psi)$.

For (b). Suppose $R(\varphi, \psi)$. Thus, by (Def s_R), $\varphi \in s_R(\psi)$. By ($\mathsf{R_{CR1}}$), $R(\varphi, \varphi)$. By (Def s_R), $\varphi \in s_R(\varphi)$. Hence, $s(\varphi) \cap s(\psi) \neq \emptyset$. Suppose, $s_R(\varphi) \cap$

$s_R(\psi) \neq \emptyset$. Thus, $\chi \in s_R(\varphi)$ and $\chi \in s_R(\psi)$. By (Def s_R), $R(\chi, \varphi)$ and $R(\chi, \psi)$. Hence, by ($\mathsf{R}_{\mathtt{S0}}$) and ($\mathsf{R}_{\mathtt{DD0}}$), $R(\varphi, \psi)$. ■

We can also prove that relations defined according to (Def $R^{\cap}_s$) and (Def $R^{\subseteq}_s$) with respect to pu-assignments under consideration satisfy some of the relational conditions studied in this subsection, and are thus elements of some of the considered classes of relations.

FACT 4.25. *1. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ is an ipu-assignment, then $R^{\subseteq}_s$ is a wbcdd+-relation.*

2. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ is a bipu-assignment, then $R^{\subseteq}_s$ is a bcdd+-relation.

3. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ is a cpu-assignment, then $R^{\cap}_s$ is a wbcs+-relation.

4. If $s\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ is a bcpu-assignment, then $R^{\cap}_s$ is a bcs+-relation.

PROOF. For 1 and 2.

For ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{CR2}}$), ($\mathsf{R}_{\mathtt{CR0}}$) and ($\mathsf{R}_{\mathtt{DD0}}$) by Fact 4.21.2.

For ($\mathsf{R}_{\mathtt{AC1}}$) and ($\mathsf{R}_{\mathtt{AC2}}$). By (pu1), $\varphi \in s(\varphi)$ and $\neg\varphi \in s(\neg\varphi)$. Thus, by (pu3), $\varphi \notin s(\neg\varphi)$ and $\neg\varphi \notin s(\varphi)$. Hence, $s(\varphi) \nsubseteq s(\neg\varphi)$ and $s(\neg\varphi) \nsubseteq s(\varphi)$. By (Def $R^{\subseteq}_s$), $\sim R^{\subseteq}_s(\varphi, \neg\varphi)$ and $\sim R^{\subseteq}_s(\neg\varphi, \varphi)$.

For ($\mathsf{R}_{\mathtt{C}}$). Suppose $R^{\subseteq}_s(\varphi, \psi)$. By (Def $R^{\subseteq}_s$), $s(\varphi) \subseteq s(\psi)$. Thus, by (pu1), $\varphi \in s(\psi)$. Therefore, by (pu3), $\varphi \notin s(\neg\psi)$. Thus, $s(\varphi) \nsubseteq s(\neg\psi)$. By (Def $R^{\subseteq}_s$), $\sim R^{\subseteq}_s(\varphi, \neg\psi)$.

For ($\mathtt{DD2}^{\subset}$), ($\mathtt{DD3}^{\subset}$) and ($\mathtt{DD4}^{\subset}$). Let $* \in \{\wedge, \vee, \rightarrow\}$. Suppose $R^{\subseteq}_s(\varphi, \chi)$ and $R^{\subseteq}_s(\psi, \chi)$. Then, by (Def $R^{\subseteq}_s$), $s(\varphi) \subseteq s(\chi)$ and $s(\psi) \subseteq s(\chi)$. By (pu4), (pu5) and (pu6), $s(\varphi * \psi) \subseteq s(\chi)$. By (Def $R^{\subseteq}_s$), $R^{\subseteq}_s(\varphi * \psi, \chi)$.

For ($\mathtt{S2'}^{\supset}$), ($\mathtt{S3'}^{\supset}$) and ($\mathtt{S4'}^{\supset}$). Let $* \in \{\wedge, \vee, \rightarrow\}$. Suppose $R^{\subseteq}_s(\varphi, \psi * \chi)$. Then, by (Def $R^{\subseteq}_s$), $s(\varphi) \subseteq s(\psi * \chi)$. By (pu4), (pu5) and (pu6), $s(\varphi) \subseteq s(\psi) \cup s(\chi)$. Thus, by (ipu), $s(\varphi) \subseteq s(\psi)$ or $s(\varphi) \subseteq s(\chi)$. By (Def $R^{\subseteq}_s$), $R^{\subseteq}_s(\varphi, \psi)$ or $R^{\subseteq}_s(\varphi, \chi)$.

For ($\mathsf{R}_{\mathtt{BC1}}$) and ($\mathsf{R}_{\mathtt{BC2}}$). By (b1) and (b2), $s(\varphi \rightarrow \psi) \subseteq s(\neg(\varphi \rightarrow \neg\psi))$ and $s(\varphi \rightarrow \neg\psi) \subseteq s(\neg(\varphi \rightarrow \psi))$. By (Def $R^{\subseteq}_s$), $R^{\subseteq}_s(\varphi \rightarrow \psi, \neg(\varphi \rightarrow \neg\psi))$ and $R^{\subseteq}_s(\varphi \rightarrow \neg\psi, \neg(\varphi \rightarrow \psi))$.

For 3 and 4.

For ($\mathsf{R}_{\mathtt{CR0}}$), ($\mathsf{R}_{\mathtt{CR1}}$), ($\mathsf{R}_{\mathtt{CR2}}$) and ($\mathsf{R}_{\mathtt{S0}}$) by Fact 4.21.1. For ($\mathsf{R}_{\mathtt{DD0}}$) by Facts 4.21.4 and 4.23.

For ($\mathsf{R}_{\mathtt{AC1}}$) and ($\mathsf{R}_{\mathtt{AC2}}$). By (pu1), $s(\neg\varphi) \neq \emptyset \neq s(\varphi)$. Thus, by (pu3), $s(\varphi) \cap s(\neg\varphi) = \emptyset$ and $s(\neg\varphi) \cap s(\varphi) = \emptyset$. By (Def $R^{\cap}_s$), $\sim R^{\cap}_s(\varphi, \neg\varphi)$ and $\sim R^{\cap}_s(\neg\varphi, \varphi)$.

For ($\mathsf{R}_{\mathtt{C}}$). Suppose $R^{\cap}_s(\varphi, \psi)$. By (Def $R^{\cap}_s$), $s(\varphi) \cap s(\psi) \neq \emptyset$. Thus, by (cpu), $\varphi \in s(\psi)$ and $\psi \in s(\varphi)$. Therefore, by (pu3), $\varphi \notin s(\neg\psi)$. Thus, by (cpu), $s(\varphi) \cap s(\neg\psi) = \emptyset$. By (Def $R^{\subseteq}_s$), $\sim R^{\subseteq}_s(\varphi, \neg\psi)$.

For (S2$^{\supset}$), (S3$^{\supset}$) and (S4$^{\supset}$). Let $* \in \{\wedge, \vee, \rightarrow\}$. Suppose $R^{\cap}_{s}(\varphi * \psi, \chi)$. Then, by (Def $R^{\cap}_{s}$), $s(\varphi * \psi) \cap s(\chi) \neq \emptyset$. By (pu4), (pu5) and (pu6), $s(\varphi) \cap s(\chi) \neq \emptyset$ or $s(\psi) \cap s(\chi) \neq \emptyset$. By (Def $R^{\cap}_{s}$), $R^{\cap}_{s}(\varphi, \chi)$ or $R^{\cap}_{s}(\psi, \chi)$.

For ($\mathsf{R}_{\mathtt{BC1}}$) and ($\mathsf{R}_{\mathtt{BC2}}$). By (pu1), (b1) and (b2), $\emptyset \neq s(\varphi \rightarrow \psi) \subseteq s(\neg(\varphi \rightarrow \neg\psi))$ and $\emptyset \neq s(\varphi \rightarrow \neg\psi) \subseteq s(\neg(\varphi \rightarrow \psi))$. Thus, $s(\varphi \rightarrow \psi) \cap s(\neg(\varphi \rightarrow \neg\psi)) \neq \emptyset$ and $s(\varphi \rightarrow \neg\psi) \cap s(\neg(\varphi \rightarrow \psi)) \neq \emptyset$. By (Def $R^{\cap}_{s}$), $R^{\cap}_{s}(\varphi \rightarrow \psi, \neg(\varphi \rightarrow \neg\psi))$ and $R^{\cap}_{s}(\varphi \rightarrow \neg\psi, \neg(\varphi \rightarrow \psi))$. ∎

Let us now specify models with set-assignments for the logics considered above. Let $\langle v, s\rangle$ be an as-structure. Then:

- $\langle v, s\rangle$ is an *sa-*$\mathbf{wBCDD}^{+}$*-model* iff s is an ipu-assignment and for any ipu-assignment t, $R^{t}_{\mathbf{wBCDD}^{+}} = R^{\subseteq}_{t}$
- $\langle v, s\rangle$ is an *sa-*$\mathbf{BCDD}^{+}$*-model* iff s is a bipu-assignment and for any bipu-assignment $R^{s}_{\mathbf{wBCDD}^{+}} = R^{\subseteq}_{t}$
- $\langle v, s\rangle$ is an *sa-*$\mathbf{wBCS}^{+}$*-model* iff s is a cpu-assignment and for any cpu-assignment t, $R^{s}_{\mathbf{wBCS}^{+}} = R^{\cap}_{s}$
- $\langle v, s\rangle$ is an *sa-*$\mathbf{BCS}^{+}$*-model* iff s is a bcpu-assignment and for any bcpu-assignment t, $R^{s}_{\mathbf{wBCS}^{+}} = R^{\cap}_{s}$.

The class of all sa-$\mathbf{wBCDD}^{+}$-models (sa-$\mathbf{BCDD}^{+}$-models) is denoted by $\mathsf{saWBCDD}^{+}$ (saBCDD^{+}). The class of all sa-$\mathbf{wBCS}^{+}$-models (sa-$\mathbf{BCS}^{+}$-models) is denoted by saWBCS^{+} (saBCS^{+}).

We now prove the fact expressing a correspondence, with respect to truths, between r-models and sa-models of the considered type:

LEMMA 4.26. *1. For all* $\langle v, R\rangle \in \mathsf{rWBCDD}^{+}$ *(*$\langle v, R\rangle \in \mathsf{rBCDD}^{+}$*):*

a. $\langle v, s_R\rangle \in \mathsf{saWBCDD}^{+}$ *(*$\langle v, s_R\rangle \in \mathsf{saBCDD}^{+}$*)*

b. for all $\varphi \in \mathsf{For}$, $\langle v, R\rangle \models \varphi$ *iff* $\langle v, s_R\rangle \models \varphi$.

2. For all $\langle v, s\rangle \in \mathsf{saWBCDD}^{+}$ *(*$\langle v, s\rangle \in \mathsf{saBCDD}^{+}$*):*

a. $\langle v, R^{\subseteq}_{s}\rangle \in \mathsf{rWBCDD}^{+}$ *(*$\langle v, R^{\subseteq}_{s}\rangle \in \mathsf{rBCDD}^{+}$*)*

b. for all $\varphi \in \mathsf{For}$, $\langle v, s\rangle \models \varphi$ *iff* $\langle v, R^{\subseteq}_{s}\rangle \models \varphi$.

3. For all $\langle v, R\rangle \in \mathsf{rWBCS}^{+}$ *(*$\langle v, R\rangle \in \mathsf{rBCS}^{+}$*):*

a. $\langle v, s_R\rangle \in \mathsf{saWBCS}^{+}$ *(*$\langle v, s_R\rangle \in \mathsf{saBCS}^{+}$*)*

b. for all $\varphi \in \mathsf{For}$, $\langle v, R\rangle \models \varphi$ *iff* $\langle v, s_R\rangle \models \varphi$.

4. For all $\langle v, s\rangle \in \mathsf{saWBCS}^{+}$ *(*$\langle v, s\rangle \in \mathsf{saBCS}^{+}$*):*

a. $\langle v, R^{\cap}_{s}\rangle \in \mathsf{rWBCS}^{+}$ *(*$\langle v, R^{\cap}_{s}\rangle \in \mathsf{rBCS}^{+}$*)*

b. for all $\varphi \in \mathsf{For}$, $\langle v, s\rangle \models \varphi$ *iff* $\langle v, R^{\cap}_{s}\rangle \models \varphi$.

Proof. For 1.a. By Fact 4.24 cases 1 and 2. For 1.b. Proof by induction. We consider only the inductive step and the case $\varphi = \psi \to \chi$. We have, $\langle v, R\rangle \models \psi \to \chi$, by truth conditions, iff either $\langle v, R\rangle \not\models \psi$ or $\langle v, R\rangle \models \chi$ and $R(\psi, \chi)$, by inductive hypothesis, iff either $\langle v, s_R\rangle \not\models \psi$ or $\langle v, s_R\rangle \models \chi$, and $R(\psi, \chi)$, by Fact 4.24 cases 1 and 2, respectively, either $\langle v, s_R\rangle \not\models \psi$ or $\langle v, s_R\rangle \models \chi$, and $s_R(\psi) \subseteq s_R(\chi)$, by truth conditions, iff $\langle v, s_R\rangle \models \psi \to \chi$. For the other cases, the proof is standard, we use truth conditions and the inductive hypothesis straightforwardly.

For 2.a. By Facts 4.25.1 and 4.25.2. For 2.b. Proof by induction. We consider only the inductive step and the case $\varphi = \psi \to \chi$. We have, $\langle v, s\rangle \models \psi \to \chi$, by truth conditions, iff either $\langle v, s\rangle \not\models \psi$ or $\langle v, s\rangle \models \chi$ and $s(\psi) \subseteq s(\chi)$, by inductive hypothesis, iff either $\langle v, R_s^{\subseteq}\rangle \not\models \psi$ or $\langle v, R_s^{\subseteq}\rangle \models \chi$, and $s(\psi) \subseteq s(\chi)$, by Fact 4.25 cases 1 and 2, respectively, either $\langle v, R_s^{\subseteq}\rangle \not\models \psi$ or $\langle v, R_s^{\subseteq}\rangle \models \chi$, and $R_s^{\subseteq}(\psi, \chi)$, by truth conditions, iff $\langle v, R_s^{\subseteq}\rangle \models \psi \to \chi$. The proof for the other cases is standard, we use truth conditions and the inductive hypothesis straightforwardly.

For 3 and 4 we reason as for 1 and 2. ■

By Lemma 4.26 we obtain the following result:

Theorem 4.27. *1.* $\Sigma \models_{\mathsf{saWBCDD}^+} \varphi$ *iff* $\Sigma \models_{\mathsf{rWBCDD}^+} \varphi$.

2. $\Sigma \models_{\mathsf{saBCDD}^+} \varphi$ *iff* $\Sigma \models_{\mathsf{rBCDD}^+} \varphi$.

3. $\Sigma \models_{\mathsf{saWBCS}^+} \varphi$ *iff* $\Sigma \models_{\mathsf{rWBCS}^+} \varphi$.

4. $\Sigma \models_{\mathsf{saBCS}^+} \varphi$ *iff* $\Sigma \models_{\mathsf{rBCS}^+} \varphi$.

Proof. For 1. Suppose $\Sigma \models_{\mathsf{saWBCDD}^+} \varphi$. Let $\langle v, R\rangle \in \mathsf{rWBCDD}^+$ and assume that for all $\psi \in \Sigma$, $\langle v, R\rangle \models \psi$. By Lemma 4.26.1 $\langle v, s_R\rangle \in \mathsf{saWBCDD}^+$ and for all $\psi \in \Sigma$, $\langle v, s_R\rangle \models \psi$. Then, by the definition of the relation of semantic consequence, $\langle v, s_R\rangle \models \varphi$. By Lemma 4.26.1 $\langle v, R\rangle \models \varphi$. Suppose $\Sigma \models_{\mathsf{rWBCDD}^+} \varphi$. Let $\langle v, s\rangle \in \mathsf{saWBCDD}^+$ and assume that for all $\psi \in \Sigma$, $\langle v, s\rangle \models \psi$. By Lemma 4.26.2 $\langle v, R_s^{\subseteq}\rangle \in \mathsf{rWBCDD}^+$ and for all $\psi \in \Sigma$, $\langle v, R_s^{\subseteq}\rangle \models \psi$. Then, by the definition of relation of semantic consequence, $\langle v, R_s^{\subseteq}\rangle \models \varphi$. By Lemma 4.26.2 $\langle v, s\rangle \models \varphi$.

For 2, 3 and 4 we reason as for 1. ■

Straightforwardly, by the presented analysis we obtain the following corollary:

Theorem 4.28.

1. $\mathbf{wBCDD}^+$ *(*$\mathbf{BCDD}^+$*) is determined by* $\mathsf{saWBCDD}^+$ *(*saBCDD^+*).*

2. $\mathbf{wBCS}^+$ *(*$\mathbf{BCS}^+$*) is determined by* saWBCS^+ *(*saBCS^+*).*

Proof. By Theorems 4.27, 4.18 and 4.20. ■

5. Summary and Future Work

In the article, we presented a selection of logics belonging to the Boolean connexive logic family defined by Jarmużek and Malinowski. The logics that modified the logics of the content relationship proposed by Epstein were of particular interest to us. All the logics analyzed here can be determined by relating models. We have also introduced set-assignment semantics for four of them, namely $\mathbf{wBCDD}^+$, $\mathbf{BCDD}^+$, $\mathbf{wBCS}^+$ and $\mathbf{BCS}^+$. Structures of this type were first proposed by Epstein as a tool for analyzing the content relationship.

In our analysis, we took into account two of Epstein's logics, the logics **S** and **DD**, that are determined by structures that enable the representation of a content relationship as a non-empty intersection of content and the containment of content, respectively. As we have shown, it is also possible to use structures expressing the containment of content to determine $\mathbf{wBCDD}^+$ and $\mathbf{BCDD}^+$. In the case of $\mathbf{wBCS}^+$ and $\mathbf{BCS}^+$, we can use structures expressing non-empty intersection of contents. In this case, the analyzed structures also allow us to capture the relationship understood as the identity of contents. On this basis, we can conclude that in the case of $\mathbf{wBCDD}^+$ and $\mathbf{BCDD}^+$, connexivity is understood as content inclusion, and in the case of $\mathbf{wBCS}^+$ and $\mathbf{BCS}^+$ as content identity.

An open problem is whether we can modify the notion of union set-assignment in a similar way as presented in the article to define set-assignment semantics also for the proper connexive counterparts of **S** and **DD**, namely **wBCS**, **BCS** and **wBCDD**, **BCDD**, respectively. Another one is to examine the connexive counterparts of Epstein's logics **D** and **Eq**. The presented semantic approach, defined by means of relating models, certainly allows for such analysis. Once again, however, the open problem is to define the appropriate set-assignment models. Finally, a topic for future research is the systematic comparison of the relationship of content presented here with other relationships proposed to be at the base of connexivity, like compatibility or relevance.

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