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A Simple Way to Overcome Hyperconnexivity

Abstract. The term 'hyperconnexive logic' (or 'hyperconnexivity' in general) in relation to a certain logical system was coined by Sylvan to indicate that not only do Boethius' theses hold in such a system, but also their converses. The plausibility of the latter was questioned by some connexive logicians. Without going into the discussion regarding the plausibility of hyperconnexivity and the converses of Boethius' theses, this paper proposes a quite simple way to escape the hyperconnexivity within the semantic framework of Wansing-style constructive connexive logics. In particular, we present a working method for escaping hyperconnexivity of constructive connexive logic **C**, discuss the problem that creates an obstacle to using the same method in the case of logic **C3** and provide a possible solution to this problem that allows us to construct a logical theory which is similar to **C3** and free from hyperconnexivity. All new logics introduced in this paper are equipped with sound and complete Hilbert-style calculi, and their relationships with other well-known connexive logics are discussed.

Keywords: Connexive logic, Hyperconnexivity, Mesoconnexivity, Constructivity, Boethius' theses, Completeness and soundness.

1. Introductory Section

It is hard to say that there exists a uniform and undisputed criterion of connexivity. Moreover, attempts to work out such a criterion have led to the extreme diversity of views among connexive logicians. For instance, while it is widely agreed that connexive logics should contain the complete set of Aristotle and Boethius' theses as valid,

$$\neg(A \rightarrow \neg A), \tag{$\mathcal{A}_1$}$$

$$\neg(\neg A \rightarrow A), \tag{$\mathcal{A}_2$}$$

$$(A \rightarrow \neg B) \rightarrow \neg(A \rightarrow B), \tag{$\mathcal{B}_1$}$$

$$(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B), \tag{$\mathcal{B}_2$}$$

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it is controversial whether they should endow the following converses of Boethius' Theses with the same status.

$$\neg(A \to B) \to (A \to \neg B), \qquad (\mathcal{HB}_1)$$

$$\neg(A \to \neg B) \to (A \to B). \qquad (\mathcal{HB}_2)$$

In the works of some authors, we can find enough arguments for and against $\mathcal{HB}_1$ and $\mathcal{HB}_2$ (see, for example, [16, 23]). We are not going to enter into this discussion here, but it is fair to say that the validity of these formulas becomes redundant if we assume what Estrada-Gonzáles and Ramirez-Cámara call 'minimal connexivity' [11], i.e. the invalidity of $(A \to B) \to (B \to A)$ and the validity of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, and $\mathcal{B}_2$. Sylvan[1] observed something similar and introduced the term 'hyperconnexivity' to indicate that a corresponding logical system validates more than enough, namely $\mathcal{HB}_1$ and $\mathcal{HB}_2$ [20]. Henceforth, we will call these formulas *Hyper-Boethius' theses*.

For the sake of convenience, let us introduce another useful term, '*mesoconnexivity*', to indicate the situation when the requirements of minimal connexivity are fulfilled and the hyperconnexivity is blocked, i.e. neither $\mathcal{HB}_1$, nor $\mathcal{HB}_2$ hold.

Probably, one of the most significant achievements of modern connexive logic is Wansing's approach to modelling the falsity condition for implication [21]. It states that a conditional sentence is false iff the truth of its antecedent implies the falsity of the consequent. This semantic condition differs from the traditional (classical or material) approach in one fundamental respect – the latter does not involve any kind of conditionality between the truth of the antecedent and the falsity of the consequent.

Incorporating Wansing's idea within a certain semantic framework gives rise to different minimally connexive logics that are well-known in the literature: Wansing's **C** and **MC** [21], Omori and Wansing's **C3** [19], Cantwell's **CN** [3], Cooper's **OD** [5], etc. But what is important is that the vast majority of logics obtained by Wansing's method (or, as Estrada-Gonzáles remarked [9], by the Bochum-plan) is hyperconnexive in the above sense.

In this paper, we are concerned with this problem. Leaving aside the discussion of the intuitive plausibility of Hyper-Boethius' theses, we are going to reveal some hidden features of Wansing's account that might help one to overcome the hyperconnexivity if one has some reasons to do this. The main question driving our paper is the following.

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[1] Richard Sylvan changed his name in 1983 from 'Routley' to 'Sylvan'; the majority of his published work is under the latter name.

(Q) Is Wansing's account of the falsity condition for implication flexible enough to overcome hyperconnexivity?

We continue and significantly extend the previous work on the four-valued 'minimal material connexive logic' **MMC** that was done in [2]. In particular, we propose a working method for escaping the hyperconnexivity of constructive connexive logic **C**, discuss the problem that creates an obstacle to using the same method in the case of logic **C3** and provide a possible solution to this problem that allows us to construct a logical theory which is similar to **C3** and free from hyperconnexivity.

The paper is structured as follows. In Section 2, we recapitulate the semantics of **C** and **C3**. In Section 3, the semantics for a mesoconnexive variant of **C** is introduced, the problem of extending our method to the case of **C3** is discussed and a mesoconnexive logic similar to **C3** is introduced. In Section 4, Hilbert-style calculi for both logics are introduced. In Section 5, completeness and soundness results for the corresponding calculi are proven. Section 6 contains the proofs of constructivity and decidability of a mesoconnexive variant of **C**. In Section 7, we discuss relationships between the mesoconnexive variant of **C** and other related connexive logics. In Section 8, we compare our approach of escaping hyperconnexivity with the approach developed by Omori in [18]. In Section 9, we finish our paper by making some final conclusions and discussing problems for further investigations.

2. Semantics of Hyperconnexive Logics

The connexive logic **C** is a logic of fundamental importance because it seems to be the first connexive logic that enjoys intuitively plausible semantics. Let us briefly review it.

We shall consider logical systems built on a propositional language $\mathscr{L}$ which contains conjunction '$\wedge$', disjunction '$\vee$', implication '$\rightarrow$', and negation '$\neg$'. The notion of a formula is defined in the standard inductive way. Let $\mathscr{P}$ be the set of all propositional variables of $\mathscr{L}$ and $\mathscr{F}$ be the set of all formulae of $\mathscr{L}$.

A *pre-ordered frame* for $\mathscr{L}$ is a pair $\langle W, \leq \rangle$, where

- W is a non-empty set of 'informational states',
- $\leq$ is a reflexive and transitive binary relation on W.

A **C**-*model* for $\mathscr{L}$ is a triple $\langle W, \leq, V \rangle$, where $\langle W, \leq \rangle$ is a pre-ordered frame and V is a **C**-*valuation*, defined as

$$V : \mathscr{P} \times W \to \{\{1,0\},\{1\},\{0\},\varnothing\}.$$

We require V to satisfy the following hereditary condition: for all $p \in \mathscr{P}$, for all $w, w' \in W$, for all $i \in \{1,0\}$,

$$\text{if } i \in V(p,w) \text{ and } w \leq w' \text{ then } i \in V(p,w').$$

The following semantic conditions are employed to extend V to the interpretation function I:

$$I(p,w) = V(p,w),$$
$$1 \in I(\neg A, w) \Leftrightarrow 0 \in I(A,w), \qquad (\neg_1)$$
$$0 \in I(\neg A, w) \Leftrightarrow 1 \in I(A,w), \qquad (\neg_0)$$
$$1 \in I(A \wedge B, w) \Leftrightarrow\ 1 \in I(A,w) \text{ and } 1 \in I(B,w), \qquad (\wedge_1)$$
$$0 \in I(A \wedge B, w) \Leftrightarrow 0 \in I(A,w) \text{ or } 0 \in I(B,w), \qquad (\wedge_0)$$
$$0 \in I(A \vee B, w) \Leftrightarrow\ 0 \in I(A,w) \text{ and } 0 \in I(B,w), \qquad (\vee_0)$$
$$1 \in I(A \vee B, w) \Leftrightarrow 1 \in I(A,w) \text{ or } 1 \in I(B,w), \qquad (\vee_1)$$
$$1 \in I(A \to B, w) \Leftrightarrow\ \text{for all } w' \text{ such that } w \leq w',\ \text{it holds that}$$
$$\text{if } 1 \in I(A,w') \text{ then } 1 \in I(B,w'), \qquad (\to_1)$$
$$0 \in I(A \to B, w) \Leftrightarrow\ \text{for all } w' \text{ such that } w \leq w',\ \text{it holds that}$$
$$\text{if } 1 \in I(A,w') \text{ then } 0 \in I(B,w'). \qquad (\mathcal{W})$$

Now, we define a formula A to be a *logical consequence* of a set of formulae Γ in **C** (in symbols $\Gamma \vDash_{\mathbf{C}} A$) if and only if, for all **C**-models $\langle W, \leq, V\rangle$, and for every $w \in W$, if $1 \in I(B,w)$ (for every $B \in \Gamma$), then $1 \in I(A,w)$. Finally, a formula A is called *valid* in **C** only if $1 \in I(A,w)$ in every **C**-model $\langle W, \leq, V\rangle$ and every $w \in W$.

In [19], H. Omori and H. Wansing investigated an extension of **C** that, from a semantic point of view, can be seen as a result of excluding 'gappy' valuations from **C**-models. The corresponding logical system is known as **C3**. More rigorously, a **C3**-model is obtained from a **C**-model by using a restricted valuation function $V : \mathscr{P} \times W \mapsto \{\{1,0\},\{1\},\{0\}\}$ and leaving all semantic conditions of **C** unchanged. The notions of logical consequence and validity in **C3** are defined analogously to **C**.

Both **C** and **C3** are hyperconnexive logics because they validate Hyper-Boethius' Theses. It can be shown using the same argument in both cases. Let us consider the case of **C** as an example.

For any **C**-model, for any $w \in W$ the following holds

$$
\begin{aligned}
&1 \in I(\neg(A \to B), w)\\
&\quad \Leftrightarrow 0 \in I(A \to B, w)\\
&\quad \Leftrightarrow \text{for all } w' \text{ s. t. } w \leq w' : \text{if } 1 \in I(A, w') \text{ then } 0 \in I(B, w')\\
&\quad \Leftrightarrow \text{for all } w' \text{ s. t. } w \leq w' : \text{if } 1 \in I(A, w') \text{ then } 1 \in I(\neg B, w')\\
&\quad \Leftrightarrow 1 \in I(A \to \neg B, w).\\
&1 \in I(\neg(A \to \neg B), w)\\
&\quad \Leftrightarrow 0 \in I(A \to \neg B, w)\\
&\quad \Leftrightarrow \text{for all } w' \text{ s. t. } w \leq w' : \text{if } 1 \in I(A, w') \text{ then } 0 \in I(\neg B, w')\\
&\quad \Leftrightarrow \text{for all } w' \text{ s. t. } w \leq w' : \text{if } 1 \in I(A, w') \text{ then } 1 \in I(B, w')\\
&\quad \Leftrightarrow 1 \in I(A \to B, w).
\end{aligned}
$$

3. Semantics for Mesoconnexive Logics

3.1. Mesoconnexive Variant of C

In this section, we present a mesoconnexive variant of Wansing's logic **C**. Let us briefly recall that a logic is called *mesoconnexive* if it is subject to two conditions:

- it is minimally connexive, i.e. it validates all Aristotle and Boethius' theses and invalidates the symmetry of implication;
- it invalidates Hyper-Boethius' theses.

In order to escape hyperconnexivity of **C**, one should find a way of breaking the foregoing equivalences, showing the validity of Hyper-Boethius' theses. In [2], the authors proposed a solution to the analogous problem with respect to the four-valued logic **MC** introduced by Wansing [22]. The key idea is to modify Wansing-style falsity condition for implication in such a way that it takes into account not only the information about the falsity of the consequent but also the information about its untruth. Let us extend this idea to the case of **C**.

A **MeC**-*model* for $\mathscr{L}$ is a triple $\langle W, \leq, V\rangle$, where $\langle W, \leq\rangle$ is a pre-ordered frame and V is a **MeC**-*valuation*, defined as

$$V \ : \ \mathscr{P} \times W \to \{\{1,0\},\{1\},\{0\},\varnothing\}.$$

We require V to satisfy the following hereditary condition: for all $p \in \mathscr{P}$, for all $w, w' \in W$, for all $i \in \{1, 0\}$,

$$\text{if } i \in V(p, w) \text{ and } w \leq w' \text{ then } i \in V(p, w').$$

The valuation function V can be extended to the interpretation function I by the same semantic conditions as with **C**, excepting that the falsity condition $(\mathcal{W})$ should be replaced with the following.

$$\begin{aligned} 0 \in I(A \to B, w) \Leftrightarrow \ &\text{for all } w' \text{ such that } w \leq w', \text{ it holds that} \\ &\text{if } 1 \in I(A, w') \text{ then } (0 \in I(B, w') \text{ or } 1 \notin I(B, w')). \end{aligned} \tag{$\to_0$}$$

Remark 1. It is worthwhile to observe that the condition $(\to_0)$ can be equivalently rewritten as

$$\begin{aligned} 0 \in I(A \to B, w) \Leftrightarrow \ &\text{for all } w' \text{ such that } w \leq w', \text{ it holds that} \\ &\text{if } 1 \in I(A, w') \text{ then } (\text{if } 1 \in I(B, w') \text{ then } 0 \in I(B, w')). \end{aligned}$$

We define a formula A to be a *logical consequence* of a set of formulae Γ in **MeC** (in symbols $\Gamma \vDash_{\mathbf{MeC}} A$) if and only if, for all **MeC**-models $\langle W, \leq, V \rangle$, and for every $w \in W$, if $1 \in I(B, w)$ (for every $B \in \Gamma$), then $1 \in I(A, w)$.

Finally, a formula A is called *valid* in **MeC** if and only if $1 \in I(A, w)$ in every **MeC**-model $\langle W, \leq, V \rangle$ and every $w \in W$.

Using simple induction on the complexity of formula A, the following lemma can be proven in a standard manner.

Lemma 1. *For any* **MeC**-*model* $\langle W, \leq, V \rangle$, *for all* $w, w' \in W$, *for any* $A \in \mathscr{F}$, *for all* $i \in \{1, 0\}$, *if* $i \in V(A, w)$ *and* $w \leq w'$ *then* $i \in V(A, w')$.

It is not difficult to find a counter-model that falsifies Hyper-Boethius' theses in **MeC**. For example, in order to falsify the formula $\neg(p \to q) \to (p \to \neg q)$, consider the following **MeC**-model: $W = \{w\}$, $\leq$ is defined as $\{\langle w, w \rangle\}$, and $\{1\} = V(p, w)$, $\varnothing = V(q, w)$.

Proposition 1. *Boethius' theses are valid in* **MeC**.

Proof. We consider the case of $\mathcal{B}_1$. Assume, for some **MeC**-model $\langle W, \leq, V \rangle$, for some $w \in W$, that $1 \notin I((p \to \neg q) \to \neg(p \to q), w)$. Then, by $(\to_1)$ and $(\neg_1)$, there exists $w \in W$ such that $w \leq w'$, $1 \in I(p \to \neg q, w')$, and $0 \notin I(p \to q, w')$. From the latter it follows, according to $(\to_0)$, that there exists such $w'' \in W$ that $w' \leq w''$, $1 \in I(p, w'')$, $0 \notin I(q, w'')$, and $1 \in I(q, w'')$. But in the light of $(\to_1)$ and $(\neg_1)$ this contradicts with what follows from $1 \in I(p \to \neg q, w')$; namely that for all $w'' \in W$ such that $w' \leq w''$ it holds that if $1 \in I(p, w'')$ then $0 \in I(q, w'')$. The case of $\mathcal{B}_2$ is similar. ■

PROPOSITION 2. *Aristotle's theses are valid in* **MeC**.

PROOF. Again, we consider only the case of $\mathcal{A}_1$, the case of $\mathcal{A}_2$ is similar. For some **MeC**-model $\langle W, \leq, V \rangle$ and for some $w \in W$ we assume that $1 \notin I(\neg(p \to \neg p), w)$. According to $(\neg_1)$, it is equivalent to $0 \notin I(p \to \neg p, w)$. But in the light of $(\to_0)$, $(\neg_1)$, and $(\neg_0)$, it follows that there exists such $w' \in W$ that $w \leq w'$, $1 \in I(p, w')$, $1 \notin I(p, w')$, and $0 \in I(p, w')$, which is a contradiction. ∎

3.2. The Problem with a C3-Like Extension

Then, a natural question arises: is it possible to provide a similar modification of **C3** that would result in a mesoconnexive counterpart of **C3**? Unfortunately, such a modification doesn't work in the case of **C3**. In the absence of truth-value gaps, the non-truth of a sentence implies its falsity, hence a **C3**-model in which $(\mathcal{W})$ is replaced with $(\to_0)$ collapses back to the **C3**-model. However, this doesn't mean that there is no other simple way to obtain a mesoconnexive logic relative to **C3**.

A **qMeC3**-*model* for $\mathscr{L}$ is a triple $\langle W, \leq, V \rangle$, where $\langle W, \leq \rangle$ is a preordered frame and V is a **qMeC3**-*valuation*, defined as

$$V \ : \ \mathscr{P} \times W \to \{\{1,0\}, \{1\}, \{0\}\}.$$

We require V to satisfy the following hereditary condition: for all $p \in \mathscr{P}$, for all $w, w' \in W$, for all $i \in \{1, 0\}$,

$$\text{if } i \in V(p, w) \text{ and } w \leq w' \text{ then } i \in V(p, w').$$

qMeC3-valuation is extended to the interpretation I by employing all semantic conditions used in **C3**, excepting $(\mathcal{W})$. The latter should be replaced with the following:

$$0 \in I(A \to B, w) \Leftrightarrow \text{ for all } w' \text{ such that } w \leq w', \text{ it holds that if } 0 \notin I(A, w') \text{ then } 0 \in I(B, w'). \quad (\mathcal{F})$$

REMARK 2. Notice that the condition $(\mathcal{F})$ can be equivalently rewritten as

$$0 \in I(A \to B, w) \Leftrightarrow \text{ for all } w' \text{ such that } w \leq w', \text{ it holds that } 0 \in I(A, w') \text{ or } 0 \in I(B, w').$$

As an anonymous referee correctly pointed out, this condition looks similar to the falsity condition of $\to_O$ from [10]. Let σ be a function from $\mathscr{P}$ into $\{\{1\}, \{1,0\}, \{0\}\}$, then the falsity and truth conditions for $\to_O$ from

[10] are defined as

$$0 \in \sigma(A \to_O B) \Leftrightarrow 0 \in \sigma(A) \text{ or } 0 \in \sigma(B),$$
$$1 \in \sigma(A \to_O B) \Leftrightarrow 0 \in \sigma(A) \text{ or } 1 \in \sigma(B).$$

These conditions give rise to the following three-valued table for $\to_O$.

$\to_O$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

It should be noticed, however, that if we rewrite our 'dynamic' truth condition ($\to_1$) in such a three-valued framework, we obtain the following.

$$1 \in \sigma(A \to_O B) \Leftrightarrow 1 \notin \sigma(A) \text{ or } 1 \in \sigma(B).$$

And this, coupled with the foregoing falsity condition, results in a different three-valued table

$\to_F$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

which was discovered by Farrell in [12]. This is what motivates the name ($\mathcal{F}$) for the falsity condition of implication in **qMeC3**.

A **qMeC3**-*consequence relation* is defined in a standard way: $\Gamma \vDash_{\mathbf{qMeC3}} A \Leftrightarrow$ for all **qMeC3**-models $\langle W, \leq, V\rangle$, and for every $w \in W$, if $1 \in I(B, w)$ (for every $B \in \Gamma$), then $1 \in I(A, w)$. Finally, a formula A is called **qMeC3**-*valid* only if $1 \in I(A, w)$ in every **qMeC3**-model and every $w \in W$.

The following lemma can be easily proven.

Lemma 2. *For any* **qMeC3**-*model* $\langle W, \leq, V\rangle$, *for all* $w, w' \in W$, *for any* $A \in \mathscr{F}$, *for all* $i \in \{1, 0\}$, *if* $i \in V(A, w)$ *and* $w \leq w'$ *then* $i \in V(A, w')$.

Lemma 3. *For any* **qMeC3**-*model* $\langle W, \leq, V\rangle$, *for any* $w \in W$, *for any* $A \in \mathscr{F}$, *either* $1 \in I(A, w)$ *or* $0 \in I(A, w)$.

Proof. By induction on the complexity of A. We consider only the case of $A = B \to C$. We have to show that $1 \in I(B \to C, w)$ or $0 \in I(B \to C, w)$. Let us suppose the contrary, i.e. $1 \notin I(B \to C, w)$ and $0 \notin I(B \to C, w)$. Notice that in the light of the inductive hypothesis either $1 \in I(B, w)$ or $0 \in I(B, w)$; and either $1 \in I(C, w)$ or $0 \in I(C, w)$. We now apply ($\to_1$), ($\mathcal{F}$) and get the following two statements:

- There exists w' such that $w \leq w'$, $1 \in I(B, w')$, and $1 \notin I(C, w')$;
- There exists w'' such that $w \leq w''$, $0 \notin I(B, w'')$, and $0 \notin I(C, w'')$.

Eliminating existential quantifiers and applying Lemma 2, we obtain $1 \notin I(C, w)$ and $0 \notin I(C, w)$ which leads us to a contradiction. ■

Again, it is not difficult to find a counter-model for the Hyper-Boethius' theses in **qMeC3**. Consider the following **qMeC3**-model that falsifies $\neg(p \to q) \to (p \to \neg q)$: $W = \{w\}$, $\leq$ is defined as $\{\langle w, w\rangle\}$, and $V(p, w) = \{1, 0\}$, $V(q, w) = \{1\}$.

Proposition 3. *Boethius's theses are valid in* **qMeC3**.

Proof. We consider the case of $\mathcal{B}_2$ and left $\mathcal{B}_1$ to the interested reader. Suppose that, for some **qMeC3**-model $\langle W, \leq, V\rangle$, for some $w \in W$, that $1 \notin I((p \to q) \to \neg(p \to \neg q), w)$. According to $(\to_1)$ and $(\neg_1)$, there exists $w \in W$ such that $w \leq w'$, $1 \in I(p \to q, w')$, and $0 \notin I(p \to \neg q, w')$. From the latter, it follows, according to $(\mathcal{F})$ and $(\neg_0)$, that there exists such $w'' \in W$ that $w' \leq w''$, $0 \notin I(p, w'')$, $1 \notin I(q, w'')$. Note that $0 \notin I(p, w'')$ implies $1 \in I(p, w'')$. But in the light of $(\to_1)$ this contradicts with what follows from $1 \in I(p \to q, w')$; namely that for all $w'' \in W$ such that $w' \leq w''$ it holds that if $1 \in I(p, w'')$ then $1 \in I(q, w'')$. ■

Proposition 4. *Aristotle's theses are valid in* **qMeC3**.

Proof. As an example, we consider only $\mathcal{A}_2$. Suppose, for some **qMeC3**-model $\langle W, \leq, V\rangle$ and for some $w \in W$ we assume that $1 \notin I(\neg(\neg p \to p), w)$. According to $(\neg_1)$, it is equivalent to $0 \notin I(\neg p \to p, w)$. But in the light of $(\mathcal{F})$ and $(\neg_0)$, it follows that there exists such $w' \in W$ that $w \leq w'$, $1 \notin I(p, w')$ and $0 \notin I(p, w')$, which is a contradiction since I is a total function. ■

Though **qMeC3** matches the requirements of mesoconnexivity, it seems to us hasty to treat it as a mesoconnexive counterpart of **C3**. This is what exactly motivates the usage of 'q' in its name, abbreviating 'quasi'. The reason is that the genuine counterpart of **C3** would have to be in the same relation with **MeC**, in which **C3** stands for **C**. It is known that **C3** is an extension of **C**. In turn, **qMeC3** is not an extension of **MeC**. Consider the following formula

$$q \to (\neg(p \to q) \to (p \to \neg q)),$$

which is **MeC**-valid but not **qMeC3**-valid.

For a counter-model in **qMeC3**, one can use exactly the same model that we used above to falsify Hyper-Boethius' theses. To show the validity of this formula in **MeC**, let us suppose, for some **MeC**-model $\langle W, \leq, V\rangle$ and for some $w \in W$, that $1 \notin I(q \to (\neg(p \to q) \to (p \to \neg q)), w)$. According to

$(\to_1)$, it means that there exists such $w' \in W$ that $w \leq w'$, $1 \in I(q, w')$ and $1 \notin I(\neg(p \to q) \to (p \to \neg q), w')$. Then, according to $(\to_1)$ and $(\neg_1)$, there exists such $w'' \in W$ that $w' \leq w''$, $0 \in I(p \to q, w'')$, and $1 \notin I(p \to \neg q, w'')$. Now, from $0 \in I(p \to q, w'')$, according to $(\to_0)$, it follows that for all w''', such that $w'' \leq w'''$ it holds that if $1 \in I(p, w''')$ then either $0 \in I(q, w''')$ or $1 \notin I(q, w''')$. In turn, according to $(\to_1)$ and $(\neg_1)$, $1 \notin I(p \to \neg q, w'')$ implies the existence of such w''' that $q \in I(p, w''')$ and $0 \notin I(q, w''')$. The latter fact together with the fact that $1 \in I(q, w')$ implies $1 \in I(q, w''')$, by the heredity condition, leads us to a contradiction.

4. Axiomatic Formulation

In this section, we present axiomatic calculi for **MeC** and **qMeC3**.

MeC is axiomatized by the calculus, containing the following set of axiomatic schemata and rules of inference ($A \leftrightarrow B$ abbreviates $(A \to B) \wedge (B \to A)$).

$$A \to (B \to A), \tag{A1}$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)), \tag{A2}$$

$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)), \tag{A3}$$

$$A \to (A \vee B), \tag{A4}$$

$$B \to (A \vee B), \tag{A5}$$

$$(A \to B) \to ((A \to C) \to (A \to (B \wedge C))), \tag{A6}$$

$$(A \wedge B) \to A, \tag{A7}$$

$$(A \wedge B) \to B, \tag{A8}$$

$$\neg\neg A \leftrightarrow A, \tag{A9}$$

$$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B), \tag{A10}$$

$$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B), \tag{A11}$$

$$\neg(A \to B) \leftrightarrow, (A \to (B \to \neg B)), \tag{A12}$$

$$\frac{A \to B, \quad A}{B}. \tag{MP}$$

In order to obtain a calculus, formalizing **qMeC3**, it is sufficient to replace (A12) with

$$\neg(A \to B) \leftrightarrow (\neg A \vee \neg B), \tag{A13}$$

$$A \vee \neg A. \tag{A14}$$

We use the standard notions of proof in both systems. Let **L** denotes **MeC** or **qMeC3**.

A *proof* of a formula A in the calculus of **L** is a sequence of formulas $A_1, \ldots, A_n, A$, where $0 \leq n$, such that every formula in the sequence $A_1, \ldots, A_n, A$ either (1) is an axiom of the calculus of **L**, or (2) is obtained with the help of (MP) from the preceding formulas.

We write $\vdash_{\mathbf{L}} A$ to indicate that A has a proof in the calculus of **L** (such A is called a *theorem*). $\Gamma \vdash_{\mathbf{L}} A$ means that A has a *proof from hypotheses* Γ in the calculus of **L**, i. e. there is a sequence of formulas $A_1, \ldots, A_n, A$, where $0 \leq n$, such that every formula in the sequence $A_1, \ldots, A_n, A$ either (1) belongs to Γ, or (2) is an axiom of the calculus of **L**, or (3) is obtained with the help of (MP) from the preceding formulas.

It is worth noticing that in the light of the presence of (A1), (A2) and (MP) in both systems the deduction theorem holds in both logics, so we omit its proof.

Theorem 1. *Let* $\mathbf{L} \in \{\mathbf{MeC}, \mathbf{qMeC3}\}$. *If* $\Gamma, A \vdash_{\mathbf{L}} B$ *then* $\Gamma \vdash_{\mathbf{L}} A \to B$.

Some remarks on **MeC**, **qMeC3** and related systems are in order.

- A Hilbert-style calculus for Nelson's logic **N4** (also referred to in the literature as $\mathbf{N}^-$) is obtained from **MeC** by replacing (A12) with

$$\neg(A \to B) \leftrightarrow (A \wedge \neg B). \tag{A15}$$

  N4 was originally introduced in [1]. For a more detailed discussion of its proof-theoretical formulations and properties, the reader may wish to consult [14].

- A Hilbert-style calculus for Wansing's **C** is obtained from **MeC** by replacing (A12) with

$$\neg(A \to B) \leftrightarrow (A \to \neg B). \tag{A16}$$

- A Hilbert-style calculus for Omori-Wansing **C3** is obtained by adding (A14) to the calculus of **C**.

- (A12) is a characteristic axiom schema of **MeC**. It is worth noticing that Boethius' theses (and hence Aristotle's theses too) are provable in the presence of (A12); the general schema of their proofs can be found in Section 7. But (A12) is insufficient to prove Hyper-Boethius' theses and this guarantees the mesoconnexivity. The right-to-left direction of (A12) is contra-classical because it is invalid in classical logic. This 'thesis' is completely novel to the field of connexive logics. Despite all of the differences between it and Boethius' theses, we can provide a plausible connexive reading of this formula. In contrast to (A16), stating that the implication from the truth of the antecedent to the falsity of the consequent is necessary and sufficient to the falsity of the whole conditional, (A12) expresses a stronger requirement. Some connexivists tend to interpret a statement of the form $A \to \neg A$ as kind of a statement saying that A is incompatible with itself. This interpretation goes back to McCall's analysis of the well-known definition of a sound conditional by Chryssipus [16]. From this point of view, (A12) encodes that the falsity of a conditional statement is necessary and sufficiently established by the fact that the truth of its antecedent implies the consequent which is 'incompatible with itself'. Notice, however, that here we use 'incompatible' in a broad sense of the term, and refer the reader to a careful historical discussion by Lenzen [15] where he illuminates some subtleties in how the notion of incompatibility was used by ancient logicians and some inaccuracies of McCall's interpretation of the history of connexive implication.
- (A13) is a characteristic axiom schemata of **qMeC3**, and it reflects quite an unusual feature of the implication of **qMeC3**. Clearly, (A13) together with (A10) imply the equivalence between the falsity of implication and the falsity of conjunction. For a more detailed discussion of implication connectives that shares similarities with conjunctions, the reader can consult [7,8].
- It is well-known that **N4** and **C** are constructive logics. This means that both of them enjoy the following disjunction and the constructible falsity properties (let $\mathbf{L} \in \{\mathbf{N4}, \mathbf{C}\}$):
  - If $\vdash_{\mathbf{L}} A \vee B$, then $\vdash_{\mathbf{L}} A$ or $\vdash_{\mathbf{L}} B$ (disjunction property);
  - If $\vdash_{\mathbf{L}} \neg(A \wedge B)$, then $\vdash_{\mathbf{L}} \neg A$ or $\vdash_{\mathbf{L}} \neg B$ (constructible falsity property).

  Comparing **MeC** with **N4** and **C**, it can be questioned whether **MeC** is constructive. We provide a positive answer to this question in Section 6.

As to **qMeC3**, it can be easily shown that (A14) destroys the constructivity. Consider $p \vee \neg p$ for arbitrary $p \in \mathscr{P}$. Clearly, neither p nor $\neg p$ is provable in **qMeC3**, and hence the disjunction property does not hold. A similar argument can be used to show that the constructible falsity does not hold in **qMeC3** as well.

5. Completeness and Soundness

Before turning to the completeness result we need to lay down some auxiliary notions. A set of formulas $\mathcal{T}$ is called an **L**-*theory* if it is closed under $\vdash_{\mathbf{L}}$, i.e. if $\mathcal{T} \vdash_{\mathbf{L}} A$ then $A \in \mathcal{T}$. An **L**-theory $\mathcal{T}$ is called *prime* iff, for every formulas A and B, it holds that $A \vee B \in \mathcal{T}$ implies $A \in \mathcal{T}$ or $B \in \mathcal{T}$. An **L**-theory $\mathcal{T}$ is called *non-trivial* iff there exists a formula A such that $A \notin \mathcal{T}$. An **L**-theory $\mathcal{T}$ is called *negation complete* iff, for every formula A, it holds that $A \in \mathcal{T}$ or $\neg A \in \mathcal{T}$.

The following variants of the Lindenbaum Lemma are well-known, hence their proofs are omitted.

Lemma 4. *If* $\Gamma \nvdash_{\mathbf{MeC}} A$ *then there exists a non-trivial prime* **MeC**-*theory* Γ', *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \nvdash_{\mathbf{MeC}} A$.

Lemma 5. *If* $\Gamma \nvdash_{\mathbf{qMeC3}} A$ *then there exists a non-trivial prime negation complete* **qMeC3**-*theory* Γ', *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \nvdash_{\mathbf{qMeC3}} A$.

We next prove a series of lemmas.

Lemma 6. *For any* **L** *from* $\{\mathbf{MeC}, \mathbf{qMeC3}\}$, *if* Π *is a non-trivial prime* **L**-*theory and* $A \to B \notin \Pi$, *then there exists a non-trivial prime* **L**-*theory* Π' *such that* $\Pi \subseteq \Pi'$, $A \in \Pi'$ *and* $B \notin \Pi'$.

Proof. The proof of this lemma is the same for both logics. Let Π be a non-trivial prime **L**-theory and $A \to B \notin \Pi$. According to Theorem 1, we have $\Pi \cup \{A\} \nvdash_{\mathbf{L}} B$. By Lemma 4 (and Lemma 5), there exists a non-trivial prime (in case of $\mathbf{L} = \mathbf{qMeC3}$ it is negation complete as well) **L**-theory Π', such that $\Pi \cup \{A\} \subseteq \Pi'$ and $\Pi' \nvdash_{\mathbf{L}} B$. Clearly, $A \in \Pi'$. Due to the reflexivity of $\vdash_{\mathbf{L}}$ we also have $B \notin \Pi'$, as desired. ■

Lemma 7. *If* Π *is a non-trivial prime* **MeC**-*theory and* $\neg(A \to B) \notin \Pi$, *then there exists a non-trivial prime* **MeC**-*theory* Π', *such that* $\Pi \subseteq \Pi'$, $A \in \Pi'$, $\neg B \notin \Pi'$ *and* $B \in \Pi'$.

Proof. Let Π be a non-trivial prime **MeC**-theory and $\neg(A \to B) \notin \Pi$. Then, in the view of (A12), we have $A \to (B \to \neg B) \notin \Pi$. After a double application of Lemma 6, we obtain that there exists a non-trivial prime **MeC**-theory Π', such that $A \in \Pi'$, $B \in \Pi'$ and $\neg B \notin \Pi'$. ■

Lemma 8. *If* Π *is a non-trivial prime negation complete* **qMeC3**-*theory and* $\neg(A \to B) \notin \Pi$*, then there exists a non-trivial prime negation complete* **qMeC3**-*theory* Π' *such that* $\Pi \subseteq \Pi'$*,* $\neg A \notin \Pi'$ *and* $\neg B \notin \Pi'$*.*

Proof. Let Π be a non-trivial prime negation complete **qMeC3**-theory and $\neg(A \to B) \notin \Pi$. Then, due to the closure of Π under $\vdash_{\mathbf{qMeC3}}$, we obtain $\Pi \nvdash_{\mathbf{qMeC3}} \neg(A \to B)$. From this, using Lemma 5, we have that there exists a non-trivial prime negation complete **qMeC3**-theory Π' such that $\Pi \subseteq \Pi'$ and $\Pi' \nvdash_{\mathbf{qMeC3}} \neg(A \to B)$. Then, using the reflexivity of $\vdash_{\mathbf{qMeC3}}$ and subsequently applying (A13), (A4) and (A5), we obtain $\neg A \notin \Pi'$ and $\neg B \notin \Pi'$ ■

A *canonical model for* **MeC** is the structure $\langle W_c, \leq_c, I_c \rangle$ defined as follows:

- W_c is the set of all non-trivial prime **MeC**-theories;
- for any $\mathcal{T}, \mathcal{T}' \in W_c$, $\mathcal{T} \leq_c \mathcal{T}'$ iff $\mathcal{T} \subseteq \mathcal{T}'$;
- I_c is the canonical valuation defined for any $p \in \mathscr{P}$ and $\mathcal{T} \in W_c$ so that

$$1 \in I_c(p, \mathcal{T}) \ \Leftrightarrow \ p \in \mathcal{T}, \qquad 0 \in I_c(p, \mathcal{T}) \ \Leftrightarrow \ \neg p \in \mathcal{T}.$$

In the next lemma, it will be shown that the canonical valuation has a suitable generalization for all formulae of $\mathscr{L}$.

Lemma 9. *Let* $\langle W_c, \leq_c, I_c \rangle$ *be a canonical model for* **MeC***. Then, for all* $A \in \mathscr{F}$ *and all* $\mathcal{T} \in W_c$*:*

$$1 \in I_c(A, \mathcal{T}) \ \Leftrightarrow \ A \in \mathcal{T}, \qquad 0 \in I_c(A, \mathcal{T}) \ \Leftrightarrow \ \neg A \in \mathcal{T}.$$

Proof. By induction on the complexity of A. We abbreviate ‘inductive hypothesis’ by ‘IH’. The basis case is by the foregoing definition of canonical valuation. Since the cases for negative, conjunctive, and disjunctive formulas are provable in a similar way to **C** and **C3**, we consider only the cases for implication.

Let $A = B \to C$. We start with the case of

$$0 \in I_c(B \to C, \mathcal{T}) \ \Leftrightarrow \ \neg(B \to C) \in \mathcal{T}.$$

From left to right, assume that $\neg(B \to C) \notin \mathcal{T}$. Then, by Lemma 7, there exists a non-trivial prime **MeC**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $B \in \mathcal{T}'$, $\neg C \notin \mathcal{T}'$ and $C \in \mathcal{T}'$. Applying (IH), we obtain that there exists a non-trivial prime **MeC**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $1 \in I_c(B, \mathcal{T}')$, $0 \notin I_c(C, \mathcal{T}')$ and $1 \in I_c(C, \mathcal{T}')$. From this, we obtain $0 \notin I_c(B \to C, \mathcal{T})$.

From right to left, let's suppose that $\neg(B \to C) \in \mathcal{T}$, $B \in \mathcal{T}'$ and $C \in \mathcal{T}'$ for any $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$. From the latter, it follows that $\neg(B \to C) \in \mathcal{T}'$. Then, applying (A12), we have $B \to (C \to \neg C) \in \mathcal{T}'$, and then, after double application of (MP), we obtain $\neg C \in \mathcal{T}'$. Applying (IH), we have the sufficient condition to obtain $0 \in I_c(B \to C, \mathcal{T})$.

Finally, we consider the case of

$$1 \in I_c(B \to C, \mathcal{T}) \;\Leftrightarrow\; B \to C \in \mathcal{T}.$$

From left to right, assume that $1 \in I_c(B \to C, \mathcal{T})$ and $B \to C \notin \mathcal{T}$. From the latter, using Lemma 6, we obtain that there exists a non-trivial prime **MeC**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $B \in \mathcal{T}'$ and $C \notin \mathcal{T}'$. Applying (IH), we obtain that there exists a non-trivial prime **MeC**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $1 \in I_c(B, \mathcal{T}')$ and $1 \notin I_c(C, \mathcal{T}')$. This implies that $1 \notin I_c(B \to C, \mathcal{T})$, thereby producing a contradiction with the initial assumption.

From right to left, assume $B \to C \in \mathcal{T}$ and $1 \notin I_c(B \to C, \mathcal{T})$. From the latter, by the truth condition of implication, we obtain that there exists a non-trivial prime **MeC**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $1 \in I_c(B, \mathcal{T}')$ and $1 \notin I_c(C, \mathcal{T}')$. Applying (IH), we obtain that there exists a non-trivial prime **MeC**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $B \in \mathcal{T}'$ and $C \notin \mathcal{T}'$. Notice that $\mathcal{T} \subseteq \mathcal{T}'$ implies $B \to C \in \mathcal{T}'$. From this and $B \in \mathcal{T}'$, applying (MP), follows $C \in \mathcal{T}'$. A contradiction. ■

Now, we move toward **qMeC3**. A *canonical model for* **qMeC3** is a structure $\langle W_c, \leq_c, I_c \rangle$ defined as follows:

- W_c is the set of all non-trivial prime negation complete **qMeC3**-theories;
- for any $\mathcal{T}, \mathcal{T}' \in W_c$, $\mathcal{T} \leq_c \mathcal{T}'$ iff $\mathcal{T} \subseteq \mathcal{T}'$;
- I_c is the canonical valuation defined for any $p \in \mathscr{P}$ and $\mathcal{T} \in W_c$ so that

$$1 \in I_c(p, \mathcal{T}) \;\Leftrightarrow\; p \in \mathcal{T}, \qquad 0 \in I_c(p, \mathcal{T}) \;\Leftrightarrow\; \neg p \in \mathcal{T}.$$

Again, we have to show that the canonical valuation of **qMeC3** has a suitable generalization for all formulae of $\mathscr{L}$.

Lemma 10. *Let $\langle W_c, \leq_c, I_c \rangle$ be a canonical model for* **qMeC3**. *Then, for all $A \in \mathscr{F}$ and all $\mathcal{T} \in W_c$:*

$$1 \in I_c(A, \mathcal{T}) \;\Leftrightarrow\; A \in \mathcal{T}, \qquad 0 \in I_c(A, \mathcal{T}) \;\Leftrightarrow\; \neg A \in \mathcal{T}.$$

Proof. The method of proving this lemma is identical to the method we used in Lemma 9. We use the same abbreviation for 'inductive hypothesis' and consider only the case of implication.

Let $A = B \to C$. Again, we start with the case of

$$0 \in I_c(B \to C, \mathcal{T}) \;\Leftrightarrow\; \neg(B \to C) \in \mathcal{T}.$$

From left to right, assume that $\neg(B \to C) \notin \mathcal{T}$. Then, by Lemma 8, there exists a non-trivial prime negation complete **qMeC3**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $\neg B \notin \mathcal{T}'$ and $\neg C \notin \mathcal{T}'$. Applying (IH), we obtain that there exists a non-trivial prime negation complete **qMeC3**-theory $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, $0 \notin I_c(B, \mathcal{T}')$ and $0 \notin I_c(C, \mathcal{T}')$. From this, we obtain $0 \notin I_c(B \to C, \mathcal{T})$.

From right to left, assume $\neg(B \to C) \in \mathcal{T}$ and $\neg B \notin \mathcal{T}'$ for any $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$. From $\mathcal{T} \subseteq \mathcal{T}'$ and $\neg(B \to C) \in \mathcal{T}$, we obtain $\neg(B \to C) \in \mathcal{T}'$. Then, using (A13) and the primeness of $\mathcal{T}'$, we obtain $\neg C \in \mathcal{T}'$. Applying (IH), we obtain that for any $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$: if $0 \notin I_c(B, \mathcal{T}')$ then $0 \in I_c(C, \mathcal{T}')$. From this, it follows that $0 \in I_c(B \to C, \mathcal{T})$.

The proof of the case of

$$1 \in I_c(B \to C, \mathcal{T}) \;\Leftrightarrow\; (B \to C) \in \mathcal{T}$$

is similar to the one from Lemma 9. Notice that Lemma 6 plays a crucial role therein. ■

We can now state the following completeness theorem.

THEOREM 2. *For any* **L** *from* $\{\mathbf{MeC}, \mathbf{qMeC3}\}$, *if* $\Gamma \vDash_{\mathbf{L}} A$ *then* $\Gamma \vdash_{\mathbf{L}} A$.

PROOF. Using Lemma 4, 5, Lemma 9, and Lemma 10. ■

Finally, we can state the following soundness theorem, whose proof is standard and left for an interested reader.

THEOREM 3. *For any* **L** *from* $\{\mathbf{MeC}, \mathbf{qMeC3}\}$, *if* $\Gamma \vdash_{\mathbf{L}} A$ *then* $\Gamma \vDash_{\mathbf{L}} A$.

6. Constructivity and Decidability of MeC

In this section, we prove constructivity and decidability of **MeC**. In doing so, we rely on the embedding technique used by many authors; see, for example, [13,17,21].

We define a formula A to be in a *negative normal form* if it contains negations only before propositional variables; that is, if $\neg B$ is a subformula of A then B is a propositional variable.

DEFINITION 1. A *transformation* $\overline{(.)}$ of formulas in $\mathscr{L}$ is defined as follows

1. $\overline{p} := p, \quad \overline{\neg p} := \neg p$, for any $p \in \mathscr{P}$;
2. $\overline{\neg\neg A} := \overline{A}$, for any $A \in \mathscr{F}$;

3. $\overline{A \diamond B} := \overline{A} \diamond \overline{B}$, for any $A, B \in \mathscr{F}$ and $\diamond \in \{\wedge, \vee, \rightarrow\}$;
4. $\overline{\neg(A \wedge B)} := \overline{\neg A} \vee \overline{\neg B}$, for any $A, B \in \mathscr{F}$;
5. $\overline{\neg(A \vee B)} := \overline{\neg A} \wedge \overline{\neg B}$, for any $A, B \in \mathscr{F}$;
6. $\overline{\neg(A \rightarrow B)} := \overline{A} \rightarrow (\overline{B} \rightarrow \overline{\neg B})$, for any $A, B \in \mathscr{F}$;

LEMMA 11. *For any $A \in \mathscr{F}$, it holds that $\vdash_{\mathbf{MeC}} A \leftrightarrow \overline{A}$.*

PROOF. By induction on the complexity of A. We consider the most distinctive case. Let $A = \neg(B \rightarrow C)$. Then, by Definition 1, $\overline{\neg(B \rightarrow C)} := \overline{B} \rightarrow (\overline{C} \rightarrow \overline{\neg C})$. Applying the inductive hypothesis, we obtain that $\overline{B} \rightarrow (\overline{C} \rightarrow \overline{\neg C})$ is equivalent to $B \rightarrow (C \rightarrow \neg C)$. Finally, applying (A12), we obtain $\neg(B \rightarrow C) \leftrightarrow (B \rightarrow (C \rightarrow \neg C))$. Therefore, $\neg(B \rightarrow C) \leftrightarrow \overline{\neg(B \rightarrow C)}$. ■

Given the set $\mathscr{P}$ of propositional variables of $\mathscr{L}$, we define the set $\mathscr{P}' = \{p' \mid p \in \mathscr{P}\}$. We now define the language $\mathscr{L}^+$ of positive intuitionistic logic by deleting $\neg$ from $\mathscr{L}$ and adding $\mathscr{P}'$. For a language $\mathscr{L}^+$, we write $\mathscr{F}^+$ for the set of all formulas of $\mathscr{L}^+$.

DEFINITION 2. The transformation $(.)^*$ of formulas from $\mathscr{F}$ into $\mathscr{F}^+$ is defined as follows.

1. For any $p \in \mathscr{P}$, $(p)^* := p$, $\quad (\neg p)^* := p'$, where $p' \in \mathscr{P}'$;
2. $(A \diamond B)^* := (A)^* \diamond (B)^*$, for all $A, B \in \mathscr{F}$ in negative normal form and $\diamond \in \{\wedge, \vee, \rightarrow\}$;
3. $(A)^* := (\overline{A})^*$, for any $A \in \mathscr{F}$ not in negative normal form.

Let us denote positive intuitionistic propositional logic as $\mathbf{Int}^+$ whose axiomatization can be obtained by dropping (A9)-(A12) from the axiomatization of **MeC**. An expression $(\Gamma)^*$ denotes the result of replacing every occurrence of a formula B in Γ with an occurrence of $(B)^*$.

THEOREM 4. $\Gamma \vdash_{\mathbf{MeC}} A$ *iff* $(\Gamma)^* \vdash_{\mathbf{Int}^+} (A)^*$.

PROOF. By induction of the length of proofs. Again, we consider only the most distinctive case. Let A be axiom (A12) of the calculus of **MeC**; that is, $A = (B \rightarrow (C \rightarrow \neg C)) \leftrightarrow \neg(B \rightarrow C)$. Then, by several applications of Definition 2, we obtain

$$((B)^* \rightarrow ((C)^* \rightarrow (\neg C)^*)) \leftrightarrow (\overline{\neg(B \rightarrow C)})^*.$$

Applying Definition 1, we obtain

$$((B)^* \rightarrow ((C)^* \rightarrow (\neg C)^*)) \leftrightarrow (\overline{B} \rightarrow (\overline{C} \rightarrow \overline{\neg C}))^*.$$

Since $(.)^*$ preserves all positive connectives and due to Lemma 11, for any $D \in \mathscr{F}$, $\vdash_{\mathbf{MeC}} D \leftrightarrow \overline{D}$, we have

$$((B)^* \to ((C)^* \to (\neg C)^*)) \leftrightarrow (B \to (C \to \neg C))^*.$$

Again, applying Definition 2, we obtain

$$((B)^* \to ((C)^* \to (\neg C)^*)) \leftrightarrow ((B)^* \to ((C)^* \to (\neg C)^*))$$

which is a theorem of $\mathbf{Int}^+$. The other cases are provable in a similar manner. ■

In light of the well-known results regarding $\mathbf{Int}^+$ (see, for example, [4, Section 2]), Theorem 4 implies several fundamental results concerning **MeC**.

Corollary 1. *The logic* **MeC** *is decidable.*

Corollary 2. *The logic* **MeC** *satisfies the following disjunction and the constructible falsity properties:*

- *If* $\vdash_{\mathbf{MeC}} A \vee B$*, then* $\vdash_{\mathbf{MeC}} A$ *or* $\vdash_{\mathbf{MeC}} B$*;*
- *If* $\vdash_{\mathbf{MeC}} \neg(A \wedge B)$*, then* $\vdash_{\mathbf{MeC}} \neg A$ *or* $\vdash_{\mathbf{MeC}} \neg B$*.*

7. Mesoconnexive Logics and Their Relatives

Just like Nelson's logic **N4**, Wansing's **C** has some interesting extensions. As we've remarked in the course of the present paper, one of such extensions is Omori and Wansing's **C3**. It can be obtained by adding the law of excluded middle (A16) to the calculus of **C**. Another simple modification of **C** results in what is known as 'material connexive logic' **MC**, also developed by Wansing [22]. It can be obtained by adding Peirce's law ($\mathcal{P}$) to **C**.

$$((A \to B) \to A) \to A. \tag{$\mathcal{P}$}$$

Lastly, we can get Cantwell's logic **CN** [3] by adding (A14) to **MC**. These facts are thoroughly discussed in the relevant literature (see, e.g. [19]).

It can be questioned whether there are similar extensions of **MeC**. After some scrutinizing, we will show that some of them are not just similar but identical.

Let us start with an extension of **MeC** obtained by adding ($\mathcal{P}$). We will denote this system as $\mathbf{MeC}^p$. Since adding ($\mathcal{P}$) to **C** results in a four-valued logic **MC**, by parity of reasoning, we might expect that $\mathbf{MeC}^p$ coincides with a four-valued logic **MMC** from [2]. Despite the fact that this is actually true, the original axiomatization of **MMC** is essentially different from $\mathbf{MeC}^p$, as it doesn't contain (A12) among the set of axiomatic schemata but instead, it contains the following ones.

$$(A \to \neg B) \to \neg(A \to B), \tag{$\mathcal{B}_1$}$$

$$B \to (\neg(A \to B) \to (A \to \neg B)), \tag{1}$$

$$B \vee \neg(A \to B). \tag{2}$$

Nevertheless, it can be proven that the two systems are equivalent. The following proof schemata show that (A12) is provable in the original formulation of **MMC** from [2]. We use 'D. T.' to denote an application of deduction theorem.

1. $\neg(A \to B)$ (assumption)
2. A (assumption)
3. B (assumption)
4. $B \to (\neg(A \to B) \to (A \to \neg B))$ (Ax. of **MMC**)
5. $\neg(A \to B) \to (A \to \neg B)$ (4,3, (MP))
6. $A \to \neg B$ (5,1, (MP))
7. $\neg B$ (6,2, (MP))
8. $B \to \neg B$ (D. T.)
9. $A \to (B \to \neg B)$ (D. T.)
10. $\neg(A \to B) \to (A \to (B \to \neg B))$ (D. T.)

1. $A \to (B \to \neg B)$ (assumption)
2. B (assumption)
3. $(A \to \neg B) \to \neg(A \to B)$ (Ax. of **MMC**)
4. $B \vee \neg(A \to B)$ (Ax. of **MMC**)
5. $(A \to (B \to \neg B)) \to (B \to (A \to \neg B))$ (provable in **MMC**)
6. $B \to (A \to \neg B)$ (5,1, (MP))
7. $A \to \neg B$ (6,2, (MP))
8. $\neg(A \to B)$ (3,7, (MP))
9. $B \to \neg(A \to B)$ (D. T.)
10. $\neg(A \to B) \to \neg(A \to B)$ (provable in **MMC**)
11. $(B \vee \neg(A \to B)) \to \neg(A \to B)$ (using (A3), 9, 10, (MP))
12. $\neg(A \to B)$ (11,4, (MP))
13. $(A \to (B \to \neg B)) \to \neg(A \to B)$ (D. T.)

We now show that all theorems of **MMC** are provable in $\mathbf{MeC}^p$; it suffices to prove $\mathcal{B}_1$, (1), and (2) in $\mathbf{MeC}^p$. The proofs are in order.

1. $A \to \neg B$ (assumption)
2. $(A \to (B \to \neg B)) \to \neg(A \to B)$ (A12)
3. $(B \to (A \to \neg B)) \leftrightarrow (A \to (B \to \neg B))$ (provable in $\mathbf{MeC}^p$)
4. $(A \to \neg B) \to (B \to (A \to \neg B))$ (A1)
5. $B \to (A \to \neg B)$ (4,1, (MP))
6. $A \to (B \to \neg B)$ (3,5, (MP))
7. $\neg(A \to B)$ (2,6, (MP))
8. $(A \to \neg B) \to \neg(A \to B)$ (D. T.)

1. B (assumption)
2. $\neg(A \to B)$ (assumption)
3. $\neg(A \to B) \to (A \to (B \to \neg B))$ (A12)
4. $A \to (B \to \neg B)$ (3,2, (MP))

5. $(A \to (B \to \neg B)) \to (B \to (A \to \neg B))$ (provable in $\mathbf{MeC}^p$)
6. $B \to (A \to \neg B)$ (5,4, (MP))
7. $A \to \neg B$ (6,1, (MP))
8. $\neg(A \to B) \to (A \to \neg B)$ (D. T.)
9. $B \to (\neg(A \to B) \to (A \to \neg B))$ (D. T.)

1. $(B \vee \neg(A \to B)) \to (A \to \neg B)$ (assumption)
2. B (assumption)
3. $B \to (B \vee \neg(A \to B))$ (A4)
4. $B \vee \neg(A \to B)$ (3,2, (MP))
5. $A \to \neg B$ (1,4, (MP))
6. $B \to (A \to \neg B)$ (D. T.)
7. $(B \to (A \to \neg B)) \to (A \to (B \to \neg B))$ (provable in $\mathbf{MeC}^p$)
8. $A \to (B \to \neg B)$ (7,6, (MP))
9. $(A \to (B \to \neg B)) \to \neg(A \to B)$ (A12)
10. $\neg(A \to B)$ (9,8, (MP))
11. $\neg(A \to B) \to (B \vee \neg(A \to B))$ (A5)
12. $B \vee \neg(A \to B)$ (11,10, (MP))
13. $((B \vee \neg(A \to B)) \to (A \to \neg B)) \to (B \vee \neg(A \to B))$ (D. T.)
14. $(((B \vee \neg(A \to B)) \to (A \to \neg B)) \to (B \vee \neg(A \to B))) \to (B \vee \neg(A \to B))$ $(\mathcal{P})$
15. $B \vee \neg(A \to B)$ (14,13, (MP))

It follows that **MMC** from [2] is deductively equivalent to $\mathbf{MeC}^p$.

PROPOSITION 5. $\Gamma \vdash_{\mathbf{MMC}} A$ *if and only if* $\Gamma \vdash_{\mathbf{MeC}^p} A$.

But what is more interesting is what would be the effect of adding the law of excluded middle to both $\mathbf{MeC}^p$ and **MeC**. One would expect that the resulting systems would be new but, in fact, they are not. Partially we have already touched on this issue in Section 3.2 when we discussed the problem of extending $(\to_0)$ to the case of **C3**.

Let's take the calculus of $\mathbf{MeC}^p$, supply it with (A14) and denote the resulting system by $\mathbf{MeC}^p_{em}$. It is easy to verify that all axioms of $\mathbf{MeC}^p_{em}$ preserve the validity in the following **CN**-matrices (the designated values are 1 and 1/2).

x	$f_\neg$
1	0
1/2	1/2
0	1

$f_\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$f_\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$f_\to$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

But this means, given the semantic completeness of **CN**, that there should be a proof of $\mathcal{HB}_1$ in $\mathbf{MeC}^p_{em}$ which heavily relies on (A14). Now, consider the following proof-schemata in $\mathbf{MeC}^p_{em}$.

1. $B \vee \neg B$ (A14)
2. $(\neg B \rightarrow (A \rightarrow \neg B)) \rightarrow [\neg(A \rightarrow B) \rightarrow (\neg B \rightarrow (A \rightarrow \neg B))]$ (A1)
3. $\neg B \rightarrow (A \rightarrow \neg B)$ (A1)
4. $\neg(A \rightarrow B) \rightarrow (\neg B \rightarrow (A \rightarrow \neg B))$ (2, 3, MP)
5. $[\neg(A \rightarrow B) \rightarrow (\neg B \rightarrow (A \rightarrow \neg B))] \rightarrow [\neg B \rightarrow (\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg B))]$ (provable in $\mathbf{MeC}^p_{em}$)
6. $\neg B \rightarrow (\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg B))$ (5,4, (MP))
7. $B \rightarrow (\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg B))$ (provable in $\mathbf{MeC}^p_{em}$)
8. $(B \vee \neg B) \rightarrow (\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg B))$ (6,7, (A3), (MP))
9. $\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg B)$ (8,1, (MP))

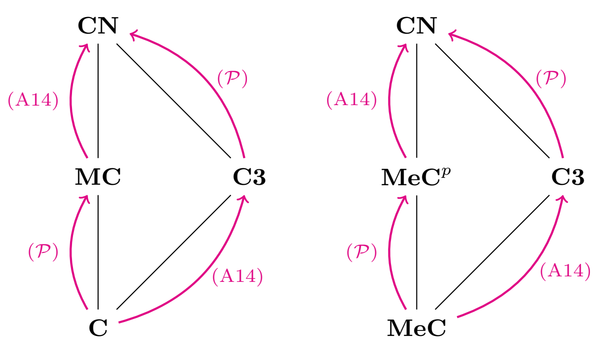
It follows that **CN** is deductively equivalent to $\mathbf{MeC}^p_{em}$.

PROPOSITION 6. $\Gamma \vdash_{\mathbf{CN}} A$ *if and only if* $\Gamma \vdash_{\mathbf{MeC}^p_{em}} A$.

Now, we can confidently state that adding the law of excluded middle to $\mathbf{MeC}^p$ results in **CN**. But actually, we can state something else. The effect of adding excluded middle to the calculus of **MeC** is similar. If we denote such a system as $\mathbf{MeC}_{em}$ then the proof of $\mathcal{HB}_1$ in $\mathbf{MeC}_{em}$ will be exactly the same. Therefore, $\mathbf{MeC}_{em}$ and **C3** are deductively equivalent as well.

PROPOSITION 7. $\Gamma \vdash_{\mathbf{C3}} A$ *if and only if* $\Gamma \vdash_{\mathbf{MeC}_{em}} A$.

All of these observations are summarized in the following diagrams (purple arrows indicate what should be added to a system at the bottom of the arrow, in order to obtain a system placed at the top of the arrow).



8. Comparison with Omori's Logic WBK

In [18], Omori studied another way to escape hyperconnexivity. His approach stems from experimental studies of the negation of indicative conditionals carried out by Politzer and Egré in [6]. Roughly, the idea is to consider the following 'weak' formula $A \to \Diamond\neg B$ as equivalent to $\neg(A \to B)$ and provide a suitable semantic formalization of this equivalence. We briefly review the basics of the corresponding logic **WBK** presented by Omori and compare his approach with the one developed in the present paper and in [2].

Omori's logic **WBK** is based on a modal propositional language $\mathscr{L}_m$, extending $\mathscr{L}$ with the falsity constant $\bot$, necessity $\Box$ and possibility $\Diamond$ operators. We will write $\mathscr{F}_m$ and $\mathscr{P}_m$ for the sets of all formulas and propositional variables of $\mathscr{L}_m$, respectively.

A **WBK**-model for $\mathscr{L}_m$ is a triple $\langle W, R, V\rangle$ where W is a non-empty set of states, R is a binary relation on W, and a valuation function V is defined as follows.

$$V \ : \ \mathscr{P}_m \times W \to \{\{1,0\},\{1\},\{0\},\varnothing\}.$$

Valuations are then extended to interpretations I in a standard manner, here we only mention the semantic clause for the falsity of $\to$ because it captures the core idea.

$$0 \in I(A \to B, w) \Leftrightarrow \text{ if } 1 \in I(A,w) \text{ then for some } w' \in W \text{ it holds that } (wRw' \text{ and } 0 \in I(B,w')). \qquad (\mathcal{O})$$

The semantic conditions for other connectives, definitions of consequence and validity can be found in [18].

One of the interesting features of **WBK** is that it is not a connexive logic, but it allows one to obtain connexive theses under certain assumptions.

PROPOSITION 8. (Proposition 1, Proposition 3, [18]) *The following holds for* **WBK***:*

1. $\vdash_{\mathbf{WBK}} \neg(A \to \neg A) \leftrightarrow (A \to \Diamond A)$*;*
2. $\vdash_{\mathbf{WBK}} \neg(A \to B) \to (A \to \neg B)$ *if and only if* $\vdash_{\mathbf{WBK}} \Diamond B \to B$*;*
3. $\vdash_{\mathbf{WBK}} (A \to \neg B) \to \neg(A \to B)$ *if and only if* $\vdash_{\mathbf{WBK}} B \to \Diamond B$*.*

Proposition 8, as Omori observed, implies that **WBK** collapses into Wansing's **MC** if and only if the possibility operator is trivialized in **WBK**. This fact is also interesting in the context of escaping hyperconnexivity. It turns out that the right amount of connexive theses can be obtained by admitting quite a natural requirement of $\vdash_{\mathbf{WBK}} B \to \Diamond B$ – that the truth of B implies its possibility. In turn, hyperconnexivity depends on the provability of a rather controversial principle $\vdash_{\mathbf{WBK}} \Diamond B \rightarrow B$. We conjecture that a proper mesoconnexive variant of **WBK** can be obtained simply by requiring the reflexivity of the accessibility relation R but, to the best of our knowledge, such a logic has not yet been studied in the literature.

On the other hand, we think it is fair to observe that though **MeC** and **MMC** are mesoconnexive in a proper sense, the hyperconnexivity can be regained under certain assumptions, as the following formulas are provable in both systems.

$$\begin{aligned}
&B \rightarrow (\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg B)),\\
&\neg B \rightarrow (\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg B)),\\
&B \rightarrow (\neg(A \rightarrow \neg B) \rightarrow (A \rightarrow B)),\\
&\neg B \rightarrow (\neg(A \rightarrow \neg B) \rightarrow (A \rightarrow B)).
\end{aligned}$$

Thus, in **MeC** and **MMC** the truth or the falsity of the consequent of a negated conditional is sufficient to the provability of Hyper-Boethius theses. As we have seen in the previous section, this fact has interesting consequences when we add the law of excluded middle to both **MeC** and **MMC** (or $\mathbf{MeC}^p$).

In a sense, one might say that Omori's technique of modalizing the consequent of negated conditionals and the approach that was developed in the present paper and [2] lead to quite similar results. However, **WBK** is not connexive in a proper sense, after all. From this point of view, a technique that underlies **MeC** and **MMC** seems to us more preferable and simple. In order to define **WBK** it is crucial to use the modal language and this means that Omori's idea cannot be expressed in purely propositional languages. This, again, speaks in favour of the simplicity and general applicability of our approach.

It should be noted however that the need to use modality doesn't appear in the void. As we remarked earlier, it is motivated by the experimental data supporting the usage of modality in the negation of conditional sentences among ordinary reasoners. Thus, for a more fair and detailed comparison of the two approaches it is interesting to consider their combination. More particularly, one can define semantic clause for $\rightarrow$ on **WBK**-models in the following way.

$$\begin{aligned}
&0 \in I(A \rightarrow B, w) \Leftrightarrow \text{ if } 1 \in I(A, w) \text{ then for some } w' \in W \text{ it holds}\\
&\quad \text{that } (wRw' \text{ and } (0 \in I(B, w') \text{ or } 1 \notin I(B, w'))).
\end{aligned}$$

The study of the effect of replacing $(\mathcal{O})$ with this condition is beyond the scope of the present paper, but it constitutes an interesting problem for further investigation.

9. Concluding Remarks

As promised, we have given a simple way of obtaining logical systems that escape hyperconnexivity, i.e. block the validity of the converses of Boethius' Theses and remain minimally connexive in the sense of [11]. We showed that the falsity condition for implication, borrowed from [2], can be successfully adapted to the case of constructive connexive logic **C**. This enabled us to introduce a new logical system **MeC**. However, we have seen that the falsity condition from [2] is not suitable in the case of **C3**. This forced us to revisit it to obtain a **C3**-like mesoconnexive logic **qMeC3**. Both **MeC** and **qMeC3** were formalized by means of sound and complete axiomatic proof-systems. The logic **MeC** was shown to be constructive and decidable. This allows us to treat **MeC** as a proper mesoconnexive modification of **C**.

We have seen that the presence of the law of excluded middle in **qMeC3** destroys the constructivity and this is what connects **qMeC3** with **C3**. However, we cannot say that **qMeC3** is a mesoconnexive modification of **C3** because it does not bear analogous relation to **MeC** as **C3** does to **C**. We remarked that the semantic conditions for implication in **qMeC3** are closely related to the three-valued implication introduced by Farrell [12]. This observation indicates that **qMeC3** and its yet unknown relatives form quite a distinctive family of mesoconnexive logical systems that deserves independent study and create a variety of problems for further investigations.

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