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Connexive Logic, Probabilistic Default Reasoning, and Compound Conditionals

Abstract. We present two approaches to investigate the validity of connexive principles and related formulas and properties within coherence-based probability logic. Connexive logic emerged from the intuition that conditionals of the form *if not-A, then A*, should not hold, since the conditional's antecedent *not-A* contradicts its consequent A. Our approaches cover this intuition by observing that the only coherent probability assessment on the conditional event $A|\overline{A}$ is $p(A|\overline{A}) = 0$. In the first approach we investigate connexive principles within coherence-based probabilistic default reasoning, by interpreting defaults and negated defaults in terms of suitable probabilistic constraints on conditional events. In the second approach we study connexivity within the coherence framework of compound conditionals, by interpreting connexive principles in terms of suitable conditional random quantities. After developing notions of validity in each approach, we analyze the following connexive principles: *Aristotle's theses*, *Aristotle's Second Thesis*, *Abelard's First Principle*, and *Boethius' theses*. We also deepen and generalize some principles and investigate further properties related to connexive logic (like *non-symmetry*). Both approaches satisfy *minimal* requirements for a connexive logic. Finally, we compare both approaches conceptually.

Keywords: Coherence, Compounds of conditionals, Conditional events, Conditional random quantities, Connexive principles, Default reasoning, Iterated conditionals, Probability logic.

1. Introduction

Connexive logics emerged from the intuition that conditionals of the form *if not-A, then A*, denoted by $\sim A \to A$, should not hold, since the conditional's antecedent *not-A* contradicts its consequent A. Indeed, experimental psychological data show that people believe that sentences of the form *if not-A*,

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then A are false (e.g., [53,55]), which supports the psychological plausibility of this intuition. Connexive principles were developed to rule out such self-contradictory conditionals (for overviews, see e.g., [49,51,72]). Many of these principles can be traced back to antiquity or the middle ages, which is reflected by the names of these principles, for example, Aristotle's Thesis or Abelard's First Principle (see Table 1).

In classical logic, however, Aristotle's Thesis, i.e. $\sim(\sim A \to A)$, is not a theorem since the corresponding material conditional is contingent because $\sim(\sim\sim A \vee A)$ is logically equivalent to $\sim A$ (which is not necessarily true). Moreover, connexive logics aim to capture the intuition that conditionals should express some "connection" between the antecedent and the consequent or, in terms of inferences, validity should require some connection between the premise set and the conclusion. There is quite some agreement in the literature that connexive logic should at least validate Aristotles' theses (AT) and (AT′) and Boethius' theses (BT) and (BT′) but Symmetry should be a non-theorem. Symmetry (or "Non-Symmetry of Implication") is $(B \to A) \to (A \to B)$ which, according to connexive logicians, should fail to be a theorem in any connexive logic ([18,72]). In our contribution we study prominent candidate principles for connexive logic (listed in Table 1 and like Symmetry also other debated candidates for connexive non-theorems (i.e., Contraposition, Denying a Consequent, Improper Transposition).

The connexive intuition that conditionals of the form *if not-A, then A* should not hold is covered in subjective probability theory. Specifically, we cover this intuition by the observation that for any event A, with $\overline{A} \neq \bot$ (where $\bot$ denotes the impossible event), the only coherent assessment on the conditional event $A|\overline{A}$ is $p(A|\overline{A}) = 0$.

The aim of our contribution is to investigate selected connexive principles within the framework of *coherence-based probability logic*.[1] The coherence approach to (subjective) probability was originated by Bruno de Finetti (see, e.g., [14,16]) and has been generalised to the conditional probability and to previsions of conditional random quantities (see, e.g., [2,3,8,12,30,33,40,46,67,71]). In the present framework, we present two approaches to connexivity within coherence-based probability logic. In the first approach we analyze connections between antecedents and consequents in terms of probabilistic constraints on conditional events (in the sense of defaults or negated defaults [25,59–61,63]). In the second approach, based the recently developed more general framework of compounds of conditionals and iterated conditionals

[1] This paper is a revised and substantially expanded version of the work presented at ECSQARU 2021 ([62]).

Table 1. Selected connexive principles (see also [72])

Name	Abbreviation	Connexive principle
Aristotle's Thesis	(AT)	$\sim(\sim A \to A)$
Aristotle's Thesis′	(AT′)	$\sim(A \to \sim A)$
Abelard's First Principle	(AB)	$\sim((A \to B) \wedge (A \to \sim B))$
Aristotle's Second Thesis	(AS)	$\sim((A \to B) \wedge (\sim A \to B))$
Boethius' Thesis	(BT)	$(A \to B) \to \sim(A \to \sim B)$
Boethius' Thesis′	(BT′)	$(A \to \sim B) \to \sim(A \to B)$
Reversed Boethius' Thesis	(RBT)	$\sim(A \to \sim B) \to (A \to B)$
Reversed Boethius' Thesis′	(RBT′)	$\sim(A \to B) \to (A \to \sim B)$
Boethius Variation 3	(B3)	$(A \to B) \to \sim(\sim A \to B)$
Boethius Variation 4	(B4)	$(\sim A \to B) \to \sim(A \to B)$

According to [18], connexive logics should at least validate (AT), (AT′), (BT), and (BT′) while not validating Symmetry

([27,28,31,33,37]), we define these connections in terms of constraints on suitable conditional random quantities. After developing different notions of negations and notions of validity, we analyze the connexive principles given in Table 1 within both approaches.

The coherence principle plays a key role in probabilistic reasoning and allows for probabilistic inferences of a further conditional event (the conclusion) from any coherent probabilistic assessment on an arbitrary family of conditional events (the premises). We recall that in standard approaches to probability, the conditional probability $p(C|A)$ is defined by the ratio $\frac{p(A \wedge C)}{p(A)}$, which requires positive probability of the conditioning event, $p(A) > 0$. However, in the framework of coherence, conditional probability $p(C|A)$, as a degree of belief, is a primitive notion and it is properly defined even if the conditioning event has probability zero, i.e., $p(A) = 0$. Analogously, within coherence, previsions of conditional random quantities, are primitive and properly defined even if the conditioning event has probability zero. Therefore, coherence is a more general approach to conditional probabilities compared to approaches which require positive probability for the conditioning events. The only requirement is that the conditioning event must be logically possible. Thus, although $p(C|A)$ is well defined even if $p(A) = 0$, it is undefined if $A \equiv \bot$. This is in line with the reading that Boethius and Aristotle thought that principles like (BT) and (AT), respectively, hold only when the conditional's antecedent is possible (see [47] who argues that the "ancient logicians most likely meant their theses as applicable only to 'normal' conditionals with antecedents which are not self-contradictory"; p. 16). It is also in line with the Ramsey test, which is expressed in his famous footnote: "If two people are arguing 'If A will C?' and are both in doubt as to A, they are adding A hypothetically to their stock of knowledge and arguing on that basis about C; so that in a sense 'If A, C' and 'If A, $\overline{C}$' are contradictories. We can say they are fixing their degrees of belief in C given A. If A turns out false, these degrees of belief are rendered *void*" [66, p. 155, we adjusted the notation]. The quantitative interpretation of the Ramsey test became a cornerstone of the conditional probability interpretation of conditionals. Adding a contradiction to your stock of knowledge does not make sense (as, traditionally, knowledge implies truth). Moreover, Ramsey's thought that conditionals with contradicting consequents C and $\overline{C}$ contradict each other coincides with the underlying intuition of (AB).

The structure of the paper is as follows. In Section 2, we recall and adapt some probabilistic preliminary notions and results on coherence-based probability theory and the theory of compound conditional events. Then, in Section 3, we present our first approach to connexivity (Approach 1) in terms of probabilistic default reasoning. Specifically, we define and investigate the validity of connexive principles and further properties of connexivity in terms of probabilistic constraints on conditional events (in the sense of defaults, or negated defaults). Section 4 presents the second approach to connexivity (Approach 2) in terms of compound conditionals in the framework of conditional random quantities. In Approach 2 validity is defined by means of constant compound conditionals. Then, we investigate the validity of connexive principles and some further properties of connexivity. We also study the validity of connexive principles under alternative notions of conjunction of conditionals. Finally, in Section 5 we summarize the main results and compare both approaches from a conceptual point of view. We conclude the paper with a brief evaluation of the psychological plausibility of both approaches and with directions of future research.

2. Preliminary Probabilistic Notions and Results

In this section we recall some preliminary notions and results of the coherence approach to conditional probability assessments (Section 2.1) and of the theory of logical operations among conditional events as conditional random quantities (Section 2.2). Section 2.1 will be relevant for both approaches to connexivity and Section 2.2 will be relevant for Approach 2.

2.1. Coherent Conditional Probability Assessments

In the following paragraphs we recall basic notion of coherence for conditional probability assessments. Specifically, we recall the trivalent notion of the conditional event, the coherence semantics of probability assessments in terms of degrees of belief in suitable bets and in geometrical terms. We also give an example by illustrating how to check coherence of a conditional probability assessments, which will be also used later in the paper. Finally, we recall the probabilistic interpretation of defaults and negated defaults which will be used in Section 3.

2.1.1. Conditional Events and Coherence in the Betting Framework We recall that an event A is a two-valued logical entity which can be *true*, or *false*. An event E is an uncertain fact described by a (non ambiguous) logical proposition; in formal terms E is a two-valued logical entity which can be *true*, or *false*. The sure event and impossible event are denoted by $\top$ and $\bot$, respectively. Given two events A and B, we denote by $A \wedge B$, or simply by AB, (resp., $A \vee B$) the logical conjunction (resp., the logical disjunction).

The negation of A is denoted $\overline{A}$. We simply write $A \subseteq B$ to denote that A logically implies B, that is $A\overline{B} = \bot$.

DEFINITION 1. Given two events A and H, with $H \neq \bot$, the *conditional event* $A|H$ (read: A given H) is defined as a three-valued logical entity which is *true* if AH is true, *false* if $\overline{A}H$ is true, and *void* if H is false.

We observe that $A|H$ assumes the logical value (*true* or *false*) of A, when H is true, and it is *void*, otherwise. There is a long history of how to deal with negations (see, e.g., [41]). In our context, we use the following inner negation of a conditional event:

DEFINITION 2. The *negation of the conditional event* "A given H", denoted by $\overline{A|H}$, is the conditional event $\overline{A}|H$, that is "the negation of A" given H.

We use the inner negation to preserve for conditional events the usual property of negating unconditional events: $p(\overline{A}) = 1 - p(A)$. In the subjective approach to probability based on the betting scheme, a conditional probability assessment $p(A|H) = x$ corresponds to a degree of belief, meaning that, for every real number s, you are willing to pay an amount $s \cdot x$ and to receive s, or 0, or $s \cdot x$ (money back), according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true (bet called off), respectively. The random gain, which is the difference between the (random) amount that you receive and the amount that you pay, is $G = (sAH + 0\overline{A}H + sx\overline{H}) - sx = sAH + sx(1 - H) - sx = sH(A - x)$.

Given a probability function p defined on an arbitrary family $\mathcal{K}$ of conditional events, consider a finite subfamily $\mathcal{F} = \{A_1|H_1, \ldots, A_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{P} = (x_1, \ldots, x_n)$, where $x_i = p(A_i|H_i)$, $i = 1, \ldots, n$. Based on the betting scheme, with the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i (A_i - x_i)$. We denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$, i.e., the set of values of G when $\mathcal{H}_n$ is true. Then, we recall below the notion of coherence in the context of the *betting scheme*.

DEFINITION 3. The function p defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geq 1$, $\forall\, s_1, \ldots, s_n$, $\forall\, \mathcal{F} = \{A_1|H_1, \ldots, A_n|H_n\} \subseteq \mathcal{K}$, it holds that: $min\ \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq max\ \mathcal{G}_{\mathcal{H}_n}$.

In betting terms, the coherence of conditional probability assessments means that in any finite combination of n bets, after discarding the case where all the bets are called off, the values of the random gain are neither all positive nor all negative (i.e., *no Dutch Book*). In particular, coherence of $x = p(A|H)$ is defined by the condition $min\ \mathcal{G}_H \leq 0 \leq max\ \mathcal{G}_H$, $\forall\, s$, where $\mathcal{G}_H$ is the set of values of G restricted to H (that is when the bet is not called off). Depending on the logical relations between A and H (with $H \neq \bot$), the set Π of all coherent conditional probability assessments $x = p(A|H)$ is:

$$\Pi = \begin{cases} [0,1], \text{ if } & \bot \neq AH \neq H, \\ \{0\}, \text{ if } & AH = \bot, \\ \{1\}, \text{ if } & AH = H. \end{cases} \quad (1)$$

2.1.2. Geometrical Characterization of Coherence We recall that coherence can be characterized geometrically. This characterization will be used, for instance, in Section 3.3 to show the non-validity of selected properties. Let $\mathcal{F} = (E_1|H_1, \ldots, E_n|H_n)$. As $E_jH_j \vee \overline{E}_jH_j \vee \overline{H}_j$ coincides with sure event $\top$, $j = 1, \ldots, n$, it holds that $\top = \bigwedge_{j=1}^{n}(E_jH_j \vee \overline{E}_jH_j \vee \overline{H}_j)$. By applying the distributive property it follows that $\top$ can also be written as the disjunction of 3^n logical conjunctions, some of which may be impossible. The remaining ones are the constituents, generated by $\mathcal{F}$ and, of course, form a partition of $\top$. We denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n$ and (if $\mathcal{H}_n \neq \top$) by C_0 the remaining constituent $\overline{\mathcal{H}}_n = \overline{H}_1 \cdots \overline{H}_n$, so that

$$\mathcal{H}_n = C_1 \vee \cdots \vee C_m, \quad \top = \overline{\mathcal{H}}_n \vee \mathcal{H}_n = C_0 \vee C_1 \vee \cdots \vee C_m, \quad m+1 \leq 3^n.$$

Let $\mathcal{P} = (x_1, \ldots, x_n)$, where $x_i = p(E_i|H_i)$, $i = 1, \ldots, n$. For each constituent C_h, $h = 1, \ldots, m$, we associate a point $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = 1$, or 0, or x_i, according to whether $C_h \subseteq E_iH_i$, or $C_h \subseteq \overline{E}_iH_i$, or $C_h \subseteq \overline{H}_i$. The point $Q_0 = \mathcal{P}$ is associated with C_0. We say that the points $Q_0, Q_1, \ldots, Q_m$ are associated with the pair $(\mathcal{F}, \mathcal{P})$.

Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, by a suitable alternative theorem (Theorem 2.9 in [20]), the condition $\mathcal{P} \in \mathcal{I}$ is equivalent to the condition $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$ given in Definition 3 (see, e.g., [22,30]). Moreover, the condition $\mathcal{P} \in \mathcal{I}$ amounts to the solvability of the following system (Σ) in the unknowns $\lambda_1, \ldots, \lambda_m$

$$(\Sigma): \qquad \textstyle\sum_{h=1}^{m} q_{hi}\lambda_h = x_i\,, \;\; i \in J_n\,; \;\; \sum_{h=1}^{m} \lambda_h = 1\,; \;\; \lambda_h \geq 0\,, \; h \in J_m\,, \quad (2)$$

where, $J_n = \{1, 2, \ldots, n\}$, for every integer n. We say that system (Σ) is associated with the pair $(\mathcal{F}, \mathcal{P})$ and of course its solvability is a necessary condition for the coherence of $\mathcal{P}$ on $\mathcal{F}$. Given a probability assessment $\mathcal{P} = (x_1, \ldots, x_n)$ on $\mathcal{F} = (E_1|H_1, \ldots, E_n|H_n)$, let $\mathcal{S}$ be the set of solutions of the form $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of the system (Σ). Then, assuming $\mathcal{S} \neq \emptyset$, we define

$$\begin{array}{l} \Phi_j(\Lambda) = \Phi_j(\lambda_1, \ldots, \lambda_m) = \sum_{r: C_r \subseteq H_j} \lambda_r\,, \quad j \in J_n\,;\; \Lambda \in \mathcal{S}\,; \\ M_j = \max_{\Lambda \in \mathcal{S}} \; \Phi_j(\Lambda)\,, \quad j \in J_n, \end{array} \quad (3)$$

and

$$I_0 = \{j \in J_n \,:\, M_j = 0\}. \tag{4}$$

Assuming $\mathcal{P}$ coherent, each solution $\Lambda = (\lambda_1, \dots, \lambda_m)$ of system (Σ) is a coherent extension of the assessment $\mathcal{P}$ on $\mathcal{F}$ to the sequence $(C_1|\mathcal{H}_n, C_2|\mathcal{H}_n, \dots, C_m|\mathcal{H}_n)$. Then, for each solution Λ of system (Σ) the quantity $\Phi_j(\Lambda)$ is a coherent extension of the conditional probability $p(H_j|\mathcal{H}_n)$. Moreover, the quantity M_j is the upper probability $p''(H_j|\mathcal{H}_n)$ over all the solutions Λ of system (Σ). Of course, $j \in I_0$ if and only if $p''(H_j|\mathcal{H}_n) = 0$. Notice that I_0 is a strict subset of J_n. If I_0 is nonempty, we set $\mathcal{F}_0 = (E_i|H_i \in \mathcal{F}, i \in I_0)$ and $\mathcal{P}_0 = (p(E_i|H_i), i \in I_0)$. We say that the pair $(\mathcal{F}_0, \mathcal{P}_0)$ is associated with I_0. Then, we have (Theorem 3.3 in [21]):

Theorem 1. *(Characterization of coherence) The assessment $\mathcal{P}$ on $\mathcal{F}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) associated with the pair $(\mathcal{F}, \mathcal{P})$ is solvable; (ii) if $I_0 \neq \emptyset$, then $\mathcal{P}_0$ is coherent.*

Let $\mathcal{S}'$ be a nonempty subset of the set of solutions $\mathcal{S}$ of system (Σ). We denote by I_0' the set I_0 defined as in (4), where $\mathcal{S}$ is replaced by $\mathcal{S}'$, that is

$$I_0' = \{j \in J_n \,:\, M_j' = 0\}, \text{ where } M_j' = \max_{\Lambda \in \mathcal{S}'} \Phi_j(\Lambda)\,, \quad j \in J_n. \tag{5}$$

Moreover, we denote by $(\mathcal{F}_0', \mathcal{P}_0')$ the pair associated with I_0'. Then, we obtain (see, e.g., Theorem 7 in [5])

Theorem 2. *The assessment $\mathcal{P}$ on $\mathcal{F}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) associated with the pair $(\mathcal{F}, \mathcal{P})$ is solvable; (ii) if $I_0' \neq \emptyset$, then $\mathcal{P}_0'$ is coherent.*

2.1.3. Illustration of How to Check Coherence For an illustration of Theorem 2 we consider an example with two *logically independent* events A, B, which will be also useful in Section 3.3 to prove the non-validity of Symmetry (i.e., $(B \to A) \to (A \to B)$ fails to be a theorem in connexive logic [18,72]).

We recall that, given n events $E_1, \dots, E_n$ they are said to be *logically independent* if there are no logical relations among them. This means that any conjunction constructed from the n events, such that their respective conjuncts are either affirmed (E_i) or negated $(\overline{E}_i)$, is not impossible, that is $E_1^* \wedge \dots \wedge E_n^* \neq \bot$, where $E_i^* \in \{E_i, \overline{E}_i\}$, $i = 1, \dots, n$.

Example 1. Let $\mathcal{F} = (E_1|H_1, E_2|H_2) = (A|B, B|A)$, where A, B are logically independent events, and $\mathcal{P} = (x, y)$ be a probability assessment on $\mathcal{F}$. We show that $\mathcal{P}$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0, 1]^2$. It holds that:

$$\top = (AB \vee \overline{A}B \vee \overline{B}) \wedge (AB \vee A\overline{B} \vee \overline{A}) = C_0 \vee C_1 \vee C_2 \vee C_3,$$

where the constituents are $C_1 = AB$, $C_2 = A\overline{B}$, $C_3 = \overline{A}B$, and $C_0 = \overline{A}\,\overline{B}$. We observe that $\mathcal{H}_2 = C_1 \vee \cdots \vee C_3 = A \vee B$. Moreover, the points $Q_1 = (1,1)$, $Q_2 = (x,0)$, $Q_3 = (0,y)$, $Q_0 = \mathcal{P} = (x,y)$ are associated with $(\mathcal{F},\mathcal{P})$. We distinguish two cases: (i) $(x,y) \neq (0,0)$; (ii) $(x,y) = (0,0)$.

In case (i), it follows that the system (Σ) associated with $(\mathcal{F},\mathcal{P})$ is solvable with a solution given by $\Lambda = (\lambda_1,\lambda_2,\lambda_3)$, where $\lambda_1 = \frac{xy}{x+y-xy}$, $\lambda_2 = \frac{x-xy}{x+y-xy}$, $\lambda_3 = \frac{y-xy}{x+y-xy}$. Indeed,

$$\mathcal{P} = (x,y) = \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3,$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $\lambda_i \geq 0$, $i = 1,2,3$. We consider the following subcases: (i_1) $x > 0$ and $y > 0$; (i_2) $x = 0$ and $y > 0$; (i_3) $x > 0$ and $y = 0$.

In case (i_1) it holds that $\Phi_1(\Lambda) = \lambda_1 + \lambda_3 = \frac{y}{x+y-xy} > 0$ and $\Phi_2(\Lambda) = \lambda_1 + \lambda_2 = \frac{x}{x+y-xy} > 0$. Then, by setting $\mathcal{S}' = \{\Lambda\}$ it holds that $M_1' > 0$, $M_2' > 0$ and hence $I_0' = \emptyset$. Thus, by Theorem 2 the assessment (x,y), with $x > 0$, $y > 0$, is coherent.

In case (i_2) it holds that $\Phi_1(\Lambda) = \lambda_1 + \lambda_3 = 1$ and $\Phi_2(\Lambda) = \lambda_1 + \lambda_2 = 0$. Then, by setting $\mathcal{S}' = \{\Lambda\}$ it holds that $M_1' > 0$, $M_2' = 0$ and hence $I_0' = \{2\}$. We observe that the subassesment $\mathcal{P}_0 = y > 0$ on $\mathcal{F}_0 = B|A$ is coherent. Thus, by Theorem 2 the assessment $(0,y)$, with $y > 0$, is coherent.

In case (i_3) it holds that $\Phi_1(\Lambda) = \lambda_1 + \lambda_3 = 0$ and $\Phi_2(\Lambda) = \lambda_1 + \lambda_2 = \frac{x}{x+y-xy}$. Then, by setting $\mathcal{S}' = \{\Lambda\}$ it holds that $M_1' = 0$, $M_2' > 0$ and hence $I_0' = \{1\}$. We observe that the subassesment $\mathcal{P}_0 = x > 0$ on $\mathcal{F}_0 = A|B$ is coherent. Thus, by Theorem 2 the assessment $(x,0)$, with $x > 0$, is coherent.

In case (ii), we observe that $\mathcal{P} = (0,0) = Q_2 = Q_3$. Then, $\Lambda = (0,\frac{1}{2},\frac{1}{2})$ is a solution of (Σ), with $\Phi_1(\Lambda) = \lambda_1 + \lambda_3 = \frac{1}{2} > 0$ and $\Phi_2(\Lambda) = \lambda_1 + \lambda_2 = \frac{1}{2} > 0$. Then, by setting $\mathcal{S}' = \{\Lambda\}$ it holds that $M_1' > 0$, $M_2' > 0$ and hence $I_0' = \emptyset$. Thus, by Theorem 2 the assessment $(0,0)$ is coherent.

Therefore the assessment (x,y) on $(A|B, B|A)$ is coherent for every $(x,y) \in [0,1]^2$.

The following remark on Example 1 will be used in Section 3.3 to show the non-validity of Contraposition.

Remark 1. As $p(\overline{A}|B) = 1 - p(A|B)$ and $p(\overline{B}|A) = 1 - p(B|A)$, it follows from Example 1 that the probability assessment $(p(\overline{A}|B), p(\overline{B}|A))$ is coherent for every value in the unit square $[0,1]^2$. Moreover, by replacing B with $\overline{B}$, it follows that the probability assessment (u,v) on $(\overline{A}|\overline{B}, B|A)$ is coherent for every $(u,v) \in [0,1]^2$.

2.1.4. Defaults and Negated Defaults In order to validate the connexive principles in Approach 1 we recall the probabilistic interpretation of defauls and negated defaults. A default $A \mid\!\sim C$ can be read as C *is a plausible consequence of* A and is interpreted by the probability constraint $p(C|A) = 1$ ([25]).[2] A negated default $\sim(A \mid\!\sim C)$ (*it is not the case that:* C *is a plausible consequence of* A; also denoted by $A \not\mid\!\sim C$ in [19,45]), which is interpreted by the probabilistic constraint $p(C|A) \neq 1$ and corresponds to the wide scope negation of negating conditionals ([25]).

2.2. Conditional Random Quantities: Logical Operations among Conditional Events

In the following paragraphs we recall key notions for logical operations among conditional events in the framework of conditional random quantities, which will be used mainly for constructing Approach 2 in Section 4. Specifically, we recall the indicator of a conditional event, conditional random quantities, the conjunction and the iteration of conditional events, and illustrate these concepts with some properties.

2.2.1. The Indicator of a Conditional Event In numerical terms, once $x = p(A|H)$ is assessed by the betting scheme, the indicator of $A|H$, denoted by the same symbol, is defined as 1, or 0, or x, according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true. Then, by setting $p(A|H) = x$,

$$A|H = AH + x\overline{H} = \begin{cases} 1, \text{ if } & AH \quad \text{is true}, \\ 0, \text{ if } & \overline{A}H \quad \text{is true}, \\ x, \text{ if } & \overline{H} \quad \text{is true}. \end{cases} \tag{6}$$

Note that since the three-valued numerical entity $A|H$ is defined by the betting scheme once the value $x = p(A|H)$ is assessed, the definition of (the indicator of) $A|H$ is not circular. The third value of the random quantity $A|H$ (subjectively) depends on the assessed probability $p(A|H) = x$. Moreover, the value x coincides with the corresponding conditional prevision, denoted by $\mathbb{P}(A|H)$, because $\mathbb{P}(A|H) = \mathbb{P}(AH + x\overline{H}) = p(AH) + xp(\overline{H}) = p(A|H)p(H) + xp(\overline{H}) = xp(H) + xp(\overline{H}) = x$.

In the special case where $AH = H$, it follows by (1) that $x = 1$ is the only coherent assessment for $p(A|H)$; then, for the indicator $A|H$ it holds

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[2] According to ε-semantics (see, e.g., [1,52]) a default $A \mid\!\sim C$ is interpreted by $p(C|A) \geq 1 - \varepsilon$, with $\varepsilon > 0$ and $p(A) > 0$. Gilio introduced a coherence-based probability semantics for defaults by also allowing ε and $p(A)$ to be zero ([23]). In this context, defaults in terms of probability 1 can be used to give a alternative definition of p-entailment which preserve the usual non-monotonic inference rules like those of System P ([4,23,29,30], see also [12,13]). For the psychological plausibility of the coherence-based semantics of non-monotonic reasoning, see, e.g., [54,56–58,65].

that

$$A|H = AH + x\overline{H} = H + \overline{H} = 1, \quad \text{if } AH = H. \tag{7}$$

In particular (7) holds when $A = \top$ (i.e., the sure event), since $\top \wedge H = H$ and hence $\top|H = H|H = 1$. Likewise, if $AH = \bot$, it follows by (1) that $x = 0$ is the only coherent assessment for $p(A|H)$; then,

$$A|H = 0 + 0\overline{H} = 0, \quad \text{if } AH = \bot. \tag{8}$$

In particular (8) holds when $A = \bot$, since $\bot \wedge H = \bot$ and hence $\bot|H = 0$. We observe that conditionally on H be true, for the (indicator of the) negation it holds that $\overline{A|H} = \overline{A} = 1 - A = 1 - A|H$. Conditionally on H be false, by coherence, it holds that $\overline{A|H} = p(\overline{A}|H) = 1 - p(A|H) = 1 - A|H$. Thus, in all cases it holds that

$$\overline{A|H} = \overline{A}|H = (1 - A)|H = 1 - A|H. \tag{9}$$

2.2.2. Conditional Random Quantities We denote by X a *random quantity*, with a finite set of possible values. Given any event $H \neq \bot$, agreeing to the betting metaphor, if you assess the prevision $\mathbb{P}(X|H) = \mu$ means that for any given real number s you are willing to pay an amount $s\mu$ and to receive sX, or $s\mu$, according to whether H is true, or false (bet called off), respectively. In particular, when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = p(A|H)$. The notion of coherence can be generalized to the case of prevision assessments on a family of conditional random quantities (see, e.g., [34,69]). Given a random quantity X and an event $H \neq \bot$, with prevision $\mathbb{P}(X|H) = \mu$, likewise formula (6) for the indicator of a conditional event, an extended notion of a conditional random quantity, denoted by the same symbol $X|H$, is defined as follows $X|H = XH + \mu\overline{H}$. We recall now the notion of conjunction of two (or more) conditional events within the framework of conditional random quantities in the setting of coherence ([28,31,33,35], for alternative approaches see also, e.g., [43,50]).

2.2.3. Conjunction of Conditional Events Given a coherent probability assessment (x, y) on $\{A|H, B|K\}$, we consider the random quantity $AHBK + x\overline{H}BK + y\overline{K}AH$ and we set $\mathbb{P}[(AHBK + x\overline{H}BK + y\overline{K}AH)|(H \vee K)] = z$. Then we define the conjunction $(A|H) \wedge (B|K)$ as follows:

Definition 4. Given a coherent prevision assessment $p(A|H) = x$, $p(B|K) = y$, and $\mathbb{P}[(AHBK + x\overline{H}BK + y\overline{K}AH)|(H \vee K)] = z$, the conjunction $(A|H) \wedge (B|K)$ is the conditional random quantity defined as

$$(A|H) \wedge (B|K) = (AHBK + x\overline{H}BK + y\overline{K}AH)|(H \vee K) = \\ = \begin{cases} 1, \text{ if } & AHBK \quad \text{is true,} \\ 0, \text{ if } & \overline{A}H \vee \overline{B}K \quad \text{is true,} \\ x, \text{ if } & \overline{H}BK \quad \text{is true,} \\ y, \text{ if } & AH\overline{K} \quad \text{is true,} \\ z, \text{ if } & \overline{H}\,\overline{K} \quad \text{is true.} \end{cases} \tag{10}$$

Notice that, by definition, $z = \mathbb{P}[(AHBK + x\overline{H}BK + y\overline{K}AH)|(H \vee K)] = \mathbb{P}[(A|H) \wedge (B|K)]$. Here, differently from conditional events which are three-valued objects, the conjunction $(A|H) \wedge (B|K)$ is not any longer a three-valued object, but a five-valued object with values in $[0, 1]$. We observe that $(A|H) \wedge (A|H) = A|H$ and $(A|H) \wedge (B|K) = (B|K) \wedge (A|H)$. Moreover, if $H = K$, then $(A|H) \wedge (B|H) = AB|H$.

Coherence requires that the Fréchet-Hoeffding bounds for prevision of the conjunction are preserved ([31]), i.e,

$$\max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}, \tag{11}$$

like in the case of unconditional events, where $P(A) = x, P(B) = y$, and $P(AB) = z$. Differently from the other notions of conjunctions of conditional events which yield conditional events, the conjunction $\wedge$ yields a conditional random quantity and preserves the classical logical and probabilistic properties which hold for unconditional events (see, e.g., [37]).

We notice that if conjunctions of conditional events are defined as suitable conditional events (see, e.g., [1,6,10,11,39]), classical probabilistic properties, like the Fréchet-Hoeffding bounds, are not preserved ([68]).

2.2.4. Iterated Conditionals In analogy to formula (6), where the indicator of a conditional event "A given H" is defined as $A|H = A \wedge H + p(A|H)\overline{H}$, the iterated conditional "$B|K$ given $A|H$" is defined as follows (see, e.g., [27,28,31]):

Definition 5. (Iterated conditioning) Given any pair of conditional events $A|H$ and $B|K$, with $AH \neq \bot$, the iterated conditional $(B|K)|(A|H)$ is defined as the conditional random quantity $(B|K)|(A|H) = (A|H) \wedge (B|K) + \mu\overline{A}|H$, where $\mu = \mathbb{P}[(B|K)|(A|H)]$.

As the iterated conditional $(B|K)|(A|H)$ has values in $[0, 1]$, it follows also that $\mu \in [0, 1]$. Furthermore, we recall that the compound prevision theorem is preserved, that is

$$\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(B|K)|(A|H)]p(A|H). \tag{12}$$

Notice that, in the context of betting scheme, $\mu = \mathbb{P}[(B|K)|(A|H)]$ represents the amount you agree to pay, with the proviso that you will receive the quantity

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu\,\overline{A}|H = \begin{cases} 1, & \text{if} \quad AHBK \quad \text{is true,} \\ 0, & \text{if} \quad AH\overline{B}K \quad \text{is true,} \\ y, & \text{if} \quad AH\overline{K} \quad \text{is true,} \\ x + \mu(1-x), & \text{if} \quad \overline{H}BK \quad \text{is true,} \\ \mu(1-x), & \text{if} \quad \overline{H}\,\overline{B}K \quad \text{is true,} \\ z + \mu(1-x), & \text{if} \quad \overline{H}\,\overline{K} \quad \text{is true,} \\ \mu, & \text{if} \quad \overline{A}H \quad \text{is true,} \end{cases} \tag{13}$$

where $x = p(A|H)$, $y = p(B|K)$, $z = \mathbb{P}[(A|H) \wedge (B|K)]$, and by (12), $z + \mu(1-x) = \mu$.

We now show that the iterated conditional $(B|K)|(B|K)$ is constant and equal to 1, which will be exploited in Section 4, for example where we show the validity of Boethius theses.

Remark 2. In the particular case where $A|H = B|K$, with $BK \neq \bot$, as $(B|K) \wedge (B|K) = B|K$, by setting $p(B|K) = y$, Equation (13) reduces to

$$(B|K)|(B|K) = B|K + \mu(\overline{B}|K) = \begin{cases} 1, & \text{if} \quad BK \quad \text{is true,} \\ \mu, & \text{if} \quad \overline{B}K \quad \text{is true,} \\ y + \mu(1-y), & \text{if} \quad \overline{K} \quad \text{is true.} \end{cases}$$

By the linearity of prevision we obtain $\mu = \mathbb{P}[(B|K)|(B|K)] = y + \mu(1-y)$ and hence

$$(B|K)|(B|K) = B|K + \mu(\overline{B}|K) = \begin{cases} 1, & \text{if} \quad BK \quad \text{is true,} \\ \mu, & \text{if} \quad \overline{B}K \vee \overline{K} \quad \text{is true.} \end{cases} \tag{14}$$

From (12) it holds that $y = \mu y$, and hence $\mu = 1$ when $y > 0$. However, we show in betting terms that coherence requires $\mu = 1$ in general, even if $y = 0$. We observe from (14) that a bet on $(B|K)|(B|K)$, with $\mu = \mathbb{P}[(B|K)|(B|K)]$, is called off when $\overline{B}K \vee \overline{K}$ is true, because in this case one receives back the quantity μ initially paid, for any initially paid value μ. Then, in order to check coherence, we only consider the set $\mathcal{G}_{BK}$ of values of the random gain $G = (B|K)|(B|K) - \mu$ restricted to the case when the bet is not called off, that is when BK is true. As $\mathcal{G}_{BK} = \{1 - \mu\}$, coherence requires that $\min \mathcal{G}_{BK} \cdot \max \mathcal{G}_{BK} = (1-\mu)^2 \leq 0$. Then $\mu = 1$ and hence from (14) we obtain

$$(B|K)|(B|K) = 1, \tag{15}$$

that is $(B|K)|(B|K)$ is constant and equal to 1 (for more general cases see [36, Theorem 6]).

3. Approach 1: Connexive Principles and Default Reasoning

In Approach 1 we exploit the probabilistic interpretation of defaults and negated defaults. In order to validate the connexive principles we interpret a conditional $A \to C$ where A and C are two events, with $A \neq \bot$, by the default $A \mid\!\sim C$, which is interpreted by the constraint $p(C|A) = 1$. Then, the conditional $A \to {\sim}C$ is interpreted by the default $A \mid\!\sim \overline{C}$. Likewise, ${\sim}A \to C$ is interpreted by $\overline{A} \mid\!\sim C$. Moreover, a negated conditional ${\sim}(A \to C)$ is interpreted by the negated default ${\sim}(A \mid\!\sim C)$, that is by the constraint $p(C|A) \neq 1$. The conjunction of two conditionals, denoted by $(A \to B) \wedge (C \to D)$, is interpreted by the sequence of their associated defaults $(A \mid\!\sim B, C \mid\!\sim D)$, which represents in probabilistic terms the constraint $(p(B|A) = 1, p(D|C) = 1)$, that is $(p(B|A), p(D|C)) = (1, 1)$. Then, the negation of the conjunction of two conditionals, denoted by ${\sim}((A \to B) \wedge (C \to D))$ (i.e., in terms of defaults ${\sim}(A \mid\!\sim B) \wedge (C \mid\!\sim D))$ is interpreted by the negation of the probabilistic constraint $(p(B|A), p(D|C)) = (1, 1)$, that is $(p(B|A), p(D|C)) \neq (1, 1)$. Table 2 summarizes the interpretations.

We now introduce the definition of validity for non-iterated connexive principles (e.g., (AT), (AT′), (AB)) in Approach 1.

Definition 6. We say that a non-iterated connexive principle is *valid in Approach 1* (*valid*$_1$) if and only if the probabilistic constraint associated with the connexive principle is satisfied by every coherent assessment on the involved conditional events.

3.1. Non-iterated Connexive Principles in Approach 1

In this section we check the validity in terms of Definition 6 of the non-iterated connexive principles in Table 1.

3.1.1. Aristotle's Thesis (AT): ${\sim}({\sim}A \to A)$. We interpret the principle ${\sim}({\sim}A \to A)$ by the negated default ${\sim}(\overline{A} \mid\!\sim A)$ with the following associated probabilistic constraint: $p(A|\overline{A}) \neq 1$. We observe that $p(\overline{A}|A) = 0$ is the unique precise coherent assessment on $\overline{A}|A$. Then, (AT) is valid$_1$ because every coherent precise assessment $p(A|\overline{A})$ is such that $p(A|\overline{A}) \neq 1$.

3.1.2. Aristotle's Thesis′ (*AT′*): ${\sim}(A \to {\sim}A)$.
Like (AT), it can be shown that (AT)′ is valid$_1$.

3.1.3. Abelard's First Principle (AB): ${\sim}((A \to B) \wedge (A \to {\sim}B))$. The structure of this principle is formalized by ${\sim}((A \mid\!\sim B) \wedge (A \mid\!\sim \overline{B}))$ which expresses the constraint $(p(B|A), p(\overline{B}|A)) \neq (1, 1)$. We recall that coherence requires $p(B|A) + p(\overline{B}|A) = 1$. Then, (AT) is valid$_1$ because each coherent assessment on $(B|A, \overline{B}|A)$ is necessarily of the form $(x, 1-x)$, with $x \in [0,1]$, which of course satisfies $(p(B|A), p(\overline{B}|A)) \neq (1,1)$.

3.1.4. Non-validity of Aristotle's Second Thesis (AS): $\sim((A \to B) \wedge (\sim A \to B))$ is not valid$_1$. The structure of this principle is formalized by $\sim((A \mathrel{|\!\sim} B) \wedge (\overline{A} \mathrel{|\!\sim} B))$ which expresses the constraint $(p(B|A), p(B|\overline{A})) \neq (1,1)$. We recall that, given two logically independent events A and B every assessment $(x,y) \in [0,1]^2$ on $(B|A, B|\overline{A})$ is coherent. In particular, $(p(B|A), p(B|\overline{A})) = (1,1)$ is a coherent assessment which does not satisfy the probabilistic constraint $(p(B|A), p(B|\overline{A})) \neq (1,1)$. Thus, (AS) is not valid$_1$.

Remark 3. (A restricted version of (AS)) We observe that (AS) can be validated in Approach 1 under some suitable further probabilistic assumptions. We recall that $p(B) = p(B|A)p(A) + p(B|\overline{A})p(\overline{A})$. Thus, $P(B) = 1$ when $p(B|A) = p(B|\overline{A}) = 1$. Then, with contraposition, under the further probabilistic constraint that $p(B) \neq 1$, it follows that $p(B|A)$ and $p(B|\overline{A})$ cannot both be equal to 1 and hence the assessment $(1,1)$ on $(B|A, B|\overline{A})$ is no longer coherent. Therefore, (AS) is valid$_1$ under the further probabilistic constraint $p(B) \neq 1$.[3] This restricted version of (AS) can be written in terms of defaults and negated defaults as follows: $\sim((A \mathrel{|\!\sim} B) \wedge (\sim A \mathrel{|\!\sim} B)) \wedge (\top \mathrel{|\!\not\sim} B)$.

3.2. Iterated Connexive Principles in Approach 1

In this section we check the validity of the iterated connexive principles in Table 1 within Approach 1: (BT), (BT′), (RBT), (RBT′), (B3), (B4). We interpret the main connective ($\to$) of iterated connexive principles as the implication ($\Rightarrow$) from the probabilistic constraint on the premise to the probabilistic constraint on the conclusion. Then, for instance, the iterated conditional $(A \to B) \to (C \to D)$ is interpreted by the implication $A \mathrel{|\!\sim} B \Rightarrow C \mathrel{|\!\sim} D$, that is $p(B|A) = 1 \;\Rightarrow\; p(D|C) = 1$. We now define validity for iterated connexive principles in Approach 1.

Definition 7. An iterated connexive principle $\bigcirc \Rightarrow \Box$ is *valid in Approach 1* (*valid*$_1$) if and only if the probabilistic constraint of the conclusion $\Box$ is satisfied by every coherent extension to the conclusion from any coherent probability assessment satisfying the constraint of the premise $\bigcirc$.

We check the validity in terms of Definition 7 of the iterated connexive principles in Table 1.

[3]Thanks to Nic Wilson for a comment inspiring this observation during ECSQARU 2021.

Table 2. Probabilistic interpretations of logical operation on conditionals in terms of defaults or negated defaults

Conditional object	Default interpretation	Probabilistic interpretation
$A \rightarrow C$	$A \mid\!\sim C$	$p(C\|A) = 1$
$\sim(A \rightarrow C)$	$\sim(A \mid\!\sim C)$	$p(C\|A) \neq 1$
$(A \rightarrow B) \wedge (C \rightarrow D)$	$(A \mid\!\sim B, C \mid\!\sim D)$	$(p(B\|A), p(D\|C)) = (1, 1)$
$\sim((A \rightarrow B) \wedge (C \rightarrow D))$	$\sim(A \mid\!\sim B, C \mid\!\sim D)$	$(p(B\|A), p(D\|C)) \neq (1, 1)$
$(A \rightarrow B) \rightarrow (C \rightarrow D)$	$A \mid\!\sim B \Rightarrow C \mid\!\sim D$	$p(B\|A) = 1 \Rightarrow p(D\|C) = 1$

3.2.1. Boethius' Thesis (BT): $(A \to B) \to \sim(A \to \sim B)$. This is interpreted by the implication $A \mid\!\sim B \Rightarrow \sim(A \mid\!\sim \overline{B})$, that is $p(B|A) = 1 \Rightarrow p(\overline{B}|A) \neq 1$. We observe that, by setting $p(B|A) = 1$, $p(\overline{B}|A) = 1 - p(B|A) = 0$ is the unique coherent extension to $\overline{B}|A$. Then, as $p(B|A) = 1 \Rightarrow p(\overline{B}|A) = 0 \neq 1$, the iterated connexive principle (BT) is valid$_1$.

3.2.2. Boethius' Thesis′ (*BT′*): $(A \to \sim B) \to \sim(A \to B)$. Like (BT), it can be shown that (BT′) is valid$_1$.

3.2.3. Non-validity of Reversed Boethius' Thesis (RBT) $\sim(A \to \sim B) \to (A \to B)$is not valid$_1$. This is interpreted by $\sim(A \mid\!\sim \overline{B}) \Rightarrow A \mid\!\sim B$, that is $p(\overline{B}|A) \neq 1 \Rightarrow p(B|A) = 1$. We observe that, by setting $p(\overline{B}|A) = x$ it holds that $p(B|A) = 1 - x$ is the unique coherent extension to $B|A$. In particular by choosing $x \in]0, 1[$, it holds that $p(\overline{B}|A) \neq 1$ and $p(B|A) \neq 1$. Thus, $p(\overline{B}|A) \neq 1 \not\Rightarrow p(B|A) = 1$ and hence (RBT) is not valid$_1$.

3.2.4. Non-validity of Reversed Boethius' Thesis' (*RBT*') $\sim(A \to B) \to (A \to \sim B)$. Like (RBT), it can be shown that (RBT′) is not valid$_1$.

3.2.5. Non-validity of Boethius Variation (B3) $(A \to B) \to \sim(\sim A \to B)$ is not valid$_1$. This is interpreted by $A \mid\!\sim B \Rightarrow \sim(\overline{A} \mid\!\sim B)$, that is $p(B|A) = 1 \Rightarrow p(B|\overline{A}) \neq 1$. We observe that, by setting $p(B|A) = 1$, any value $p(B|\overline{A}) \in [0, 1]$ is a coherent extension to $B|\overline{A}$, because the assessment $(1, y)$ on $(B|A, B|\overline{A})$ is coherent for every $y \in [0, 1]$. In particular the assessment $(1, 1)$ on $(B|A, B|\overline{A})$ is coherent. Therefore, as $p(B|A) = 1 \not\Rightarrow p(B|\overline{A}) \neq 1$, (B3) is not valid$_1$.

3.2.6. Non-validity of Boethius Variation (B4) $(\sim A \to B) \to \sim(A \to B)$is not valid$_1$. Like (B3), it can be shown that (B4) is not valid$_1$.

We summarize the previous results on connexive principles within Approach 1 in Table 3.

3.3. Further Properties of Approach 1

In this section we study the validity of selected properties which are discussed in the literature on connexive logic (see, e.g., [18,49,72]). In particular, we investigate Symmetry, Minimality, Kapsner-strong, Simplification, Conjunction-Idempotence, paradoxes of material conditional, Improper Transposition, Contraposition, and Denying Conjunct within Approach 1. Moreover, we study generalized versions of Aristotle's theses.

Table 3. Connexive principles of Table 1 in the framework of defaults and probabilistic constraints (Approach 1)

Principle	Default interpretation	Probabilistic constraint	Valid$_1$
(AT)	$\sim(\overline{A} \mid\!\sim A)$	$p(A\vert\overline{A}) \neq 1$	Yes
(AT')	$\sim(A \mid\!\sim \overline{A})$	$p(\overline{A}\vert A) \neq 1$	Yes
(AB)	$\sim(A \mid\!\sim B, A \mid\!\sim \overline{B})$	$(p(B\vert A), p(\overline{B}\vert A)) \neq (1,1)$	Yes
(AS)	$\sim(A \mid\!\sim B, \overline{A} \mid\!\sim B)$	$(p(B\vert A), p(B\vert\overline{A})) \neq (1,1)$	No
(BT)	$A \mid\!\sim B \Rightarrow \sim(A \mid\!\sim \overline{B})$	$p(B\vert A) = 1 \Rightarrow p(\overline{B}\vert A) \neq 1$	Yes
(BT')	$A \mid\!\sim \overline{B} \Rightarrow \sim(A \mid\!\sim B)$	$p(\overline{B}\vert A) = 1 \Rightarrow p(B\vert A) \neq 1$	Yes
(RBT)	$\sim(A \mid\!\sim \overline{B}) \Rightarrow A \mid\!\sim B$	$p(\overline{B}\vert A) \neq 1 \Rightarrow p(B\vert A) = 1$	No
(RBT')	$\sim(A \mid\!\sim B) \Rightarrow A \mid\!\sim \overline{B}$	$p(B\vert A) \neq 1 \Rightarrow p(\overline{B}\vert A) = 1$	No
(B3)	$A \mid\!\sim B \Rightarrow \sim(\overline{A} \mid\!\sim B)$	$p(B\vert A) = 1 \Rightarrow p(B\vert\overline{A}) \neq 1$	No
(B4)	$\overline{A} \mid\!\sim B \Rightarrow \sim(A \mid\!\sim B)$	$p(B\vert\overline{A}) = 1 \Rightarrow p(B\vert A) \neq 1$	No

3.3.1. Non-validity of Symmetry As mentioned above, there is a common agreement that connexive logics should be non-symmetric, i.e.,

$$(B \to A) \to (A \to B) \tag{16}$$

fails to be a theorem ([18,72]). In Approach 1, to show that (16) is not valid$_1$ amounts to show that $(B \mathrel{|\!\sim} A) \Rightarrow (A \mathrel{|\!\sim} B)$ is not valid$_1$, that is $p(A|B) = 1 \not\Rightarrow p(B|A) = 1$. In Example 1 we showed that the assessment (x, y) on $(A|B, B|A)$ is coherent for every $(x, y) \in [0, 1]^2$. Hence, as the assessment $p(A|B) = 1$ and $p(B|A) = 0$ is coherent, Symmetry is not valid$_1$.

3.3.2. Minimality Since we have shown that Approach 1 validates (AT), (AT′), (BT), and (BT′) but not Symmetry, Approach 1 satisfies all of Estrada-González and Ramírez-Cámara's minimal criteria for any connexive logic ([18, p. 346]) which means that Approach 1 provides a semantics for connexive logic. However, Approach 1 does not support *subminimal connexive logic*, which "is a logic which satisfies at least some but not all" of these criteria [18, p. 347].

3.3.3. Kapsner-Strong We recall that a connexive logic is called *Kapsner-strong* ([18]) when the two conditions (i) $A \to \sim A$ *is not satisfiable* and (ii) $A \to B$ *and* $A \to \sim B$ *are not simultaneously satisfiable* are met ([42]). Naturally, in Approach 1 satisfiability of the conditional $A \to B$ amounts to the satisfiability of the probabilistic constraint associated with the corresponding default $A \mathrel{|\!\sim} B$, that is $p(B|A) = 1$ is a coherent assessment. Approach 1 is Kapsner-strong because

- $A \to \sim A$ is unsatisfiable. Indeed, as $p(\overline{A}|A) = 0$ for every coherent probability p, the probabilistic constraint $p(\overline{A}|A) = 1$, associated with the corresponding default $A \mathrel{|\!\sim} \sim A$, cannot be coherent.
- $A \to B$ and $A \to \sim B$ are not simultaneously satisfiable. Indeed, as $p(B|A) + p(\overline{B}|A) = 1$ for every coherent probability p, the probability assessment $p(B|A) = 1$ and $p(\overline{B}|A) = 1$, associated with $(A \mathrel{|\!\sim} B)$ and $(A \mathrel{|\!\sim} \sim B)$, cannot be coherent.

3.3.4. Simplification Approach 1 also satisfies the two versions of Simplification $(A \wedge B) \to A$ and $(A \wedge B) \to B$. Indeed, Simplification, interpreted as $(A \wedge B) \mathrel{|\!\sim} A$ and $(A \wedge B) \mathrel{|\!\sim} B$, is valid$_1$ because $p(B|AB) = p(A|AB) = p(AB|AB) = 1$.

3.3.5. Conjunction-Idempotence Of course, $(A \wedge A) \to A$ and $A \to (A \wedge A)$ are trivially valid$_1$ because $p(A|AA) = p(AA|A) = p(A|A) = 1$.

3.3.6. Non-validity of Paradoxes of the Material Conditional We show the non validity of the following paradoxes of the material conditional in Approach 1: $B \to (A \to B)$; $\sim A \to (A \to B)$. We observe that $B \Rightarrow (A \mathrel{|\!\sim} B)$ is not valid$_1$ because $p(B) = 1 \not\Rightarrow p(B|A) = 1$. Indeed, even if $p(B|A) = 1$, when $p(A) > 0$ and $p(B) = 1$, however in general as $p(A)$ could be zero, every $p(B|A) \in [0,1]$ is a coherent extension of $p(B) = 1$. Moreover, $\sim A \Rightarrow (A \mathrel{|\!\sim} B)$ is not valid$_1$ because $p(\overline{A}) = 1 \not\Rightarrow p(B|A) = 1$. Indeed, every $p(B|A) \in [0,1]$ is a coherent extension of $p(\overline{A}) = 1$ (for more details see [54]).

3.3.7. Non-validity of Improper Transposition Improper Transposition is

$$(A \to B) \to (\sim A \to \sim B). \tag{17}$$

Within Approach 1 formula (17) is interpreted by $(A \mathrel{|\!\sim} B) \Rightarrow (\overline{A} \mathrel{|\!\sim} \overline{B})$, which is not valid$_1$ because $p(B|A) = 1$ does not imply that $p(\overline{B}|\overline{A}) = 1$. Indeed, as shown below, the assessment $\mathcal{P} = (x, y)$ on $(B|A, \overline{B}|\overline{A})$ is coherent for every $(x, y) \in [0,1]^2$. The constituents C_h and the associated points Q_h are: $C_1 = AB$, $C_2 = A\overline{B}$, $C_3 = \overline{A}B$, $C_4 = \overline{A}\,\overline{B}$, and $Q_1 = (1, y)$, $Q_2 = (0, y)$, $Q_3 = (x, 0)$, $Q_4 = (x, 1)$. We observe that the disjunction of the conditioning events is $A \vee \overline{A} = \top$ and that system (Σ) is solvable because

$$\mathcal{P} = (x, y) = xQ_1 + (1 - x)Q_2.$$

Then, as $I_0 = \emptyset$, from Theorem 1, the assessment $\mathcal{P} = (x, y)$ on $(\overline{AB}|\top, B|\overline{A})$ is coherent for every $(x, y) \in [0,1]^2$. Since the assessment $(1, 0)$ on $(B|A, \overline{B}|\overline{A})$ is coherent, formula (17) is not valid$_1$ in Approach 1.

3.3.8. Non-validity of Contraposition Contraposition is

$$(A \to B) \to (\sim B \to \sim A). \tag{18}$$

Within Approach 1 formula (18) is interpreted by $(A \mathrel{|\!\sim} B) \Rightarrow (\overline{B} \mathrel{|\!\sim} \overline{A})$. As shown in Remark 1, the probability assessment $\mathcal{P} = (x, y)$ on $(B|A, \overline{A}|\overline{B})$ is coherent for every $(x, y) \in [0,1]^2$. In particular $(p(B|A), p(\overline{A}|\overline{B})) = (1, 0)$ is coherent. Therefore, Contraposition is not valid$_1$, because $p(B|A) = 1$ does not imply that $p(\overline{A}|\overline{B})$ is necessarily equal to 1.

3.3.9. Non-validity of Denying a Conjunct Denying a Conjunct is

$$\sim(A \wedge B) \to (\sim A \to B). \tag{19}$$

In Approach 1 formula (19) is interpreted by the relation $(\top \mid\!\sim \overline{AB}) \Rightarrow (\overline{A} \mid\!\sim B)$. We observe that $p(\overline{AB}|\top) = p((\overline{A} \vee \overline{B})|\top) = 1 \nRightarrow p(B|\overline{A}) = 1$ and hence (19) is not valid$_1$. Indeed, we show below by exploiting the geometrical characterization of coherence, that the assessment $\mathcal{P} = (x, y)$ on $(\overline{AB}|\top, B|\overline{A})$ is coherent for every $(x, y) \in [0, 1]^2$. The constituents C_h and the associated points Q_h are: $C_1 = AB$, $C_2 = A\overline{B}$, $C_3 = \overline{A}B$, $C_4 = \overline{A}\,\overline{B}$, and $Q_1 = (0, y)$, $Q_2 = (1, y)$, $Q_3 = (1, 1)$, $Q_4 = (1, 0)$. We observe that the disjunction of the conditioning events is $\top \vee \overline{A} = \top$ and that system (Σ) is solvable because $\mathcal{P} = (x, y) = (1 - x)Q_1 + xQ_2$. Then, as $I_0 = \emptyset$, based on Theorem 1, the assessment $\mathcal{P} = (x, y)$ on $(\overline{AB}|\top, B|\overline{A})$ is coherent for every $(x, y) \in [0, 1]^2$. Since the assessment $(1, 0)$ on $(\overline{AB}|\top, B|\overline{A})$ is coherent, formula (19) is not valid$_1$.

3.3.10. Generalized Aristotle's Theses We recall that a basic connexive intuition is that conditionals of the form *if not-A then A* should not hold. We now generalize this intuition by studying iterated versions of Aristotle's thesis. Specifically, we replace A by the conditional $A \rightarrow B$ in (AT) and (AT)′ and obtain the respective iterated versions:

(IAT) $\sim\!(\sim\!(A \rightarrow B) \rightarrow (A \rightarrow B))$;

(IAT)′ $\sim\!((A \rightarrow B) \rightarrow \sim\!(A \rightarrow B))$.

We interpret the negation of an inference, that is $\sim\!(\bigcirc \Rightarrow \square)$, by $\bigcirc \Rightarrow \sim\!\square$. Then, (IAT) is interpreted by the inference $\sim\!(A \rightarrow B) \Rightarrow \sim\!(A \rightarrow B)$, that is

$$\sim\!(A \mid\!\sim B) \Rightarrow \sim\!(A \mid\!\sim B),$$

which of course holds since $p(B|A) < 1 \Rightarrow p(B|A) < 1$. Likewise, (IAT)′ is interpreted by the inference $A \mid\!\sim B \Rightarrow A \mid\!\sim B$, which holds because $p(B|A) = 1 \Rightarrow p(B|A) = 1$. Therefore (IAT) and (IAT)′ are both valid$_1$.

In this section we analyzed connexivity (Sections 3.1 and 3.2) and related further properties (Sect. 3.3) in terms of the probabilistic interpretation of defaults and negated defaults. In the next section we consider another approach to connexivity within our framework of coherence.

4. Approach 2: Connexive Principles and Compounds of Conditionals

In this section we analyze connexive principles within the theory of logical operations among conditional events (Approach 2). Specifically, we analyze connections between antecedents and consequents in terms of constraints on compounds of conditionals and iterated conditionals. In this approach, a basic conditional $A \to C$ is interpreted by (the indicator of) the conditional event $C|A$ (instead of a probabilistic constraint on conditional events as in Approach 1) which is a three-valued object (see Section 2.2): $C|A \in \{1, 0, x\}$, where $x = p(C|A)$. The negation $\sim(A \to C)$ is interpreted by $\overline{C}|A$ (which is the narrow scope negation of negating conditionals). Then, $\sim(A \to \sim C)$ amounts to $\overline{\overline{C}|A}$ which coincides with $C|A$. We recall that logical operations among conditional events do not yield a conditional event, rather they yield conditional random quantities with more than three possible values (see, e.g., [31]). Then, we interpret the results of the logical operations in the connexive principles by suitable conditional random quantities. In particular, the conjunction $(A \to B) \wedge (C \to D)$ (resp., $\sim((A \to B) \wedge (C \to D))$) is interpreted by $(B|A) \wedge (D|C)$ (resp., by $\overline{(B|A) \wedge (D|C)}$), and the iterated conditional $(A \to B) \to (C \to D)$ is interpreted by $(D|C)|(B|A)$. Moreover, we define validity of connexive principles within Approach 2.

Definition 8. A connexive principle is *valid in Approach 2* (*valid*$_2$) if and only if the associated conditional random quantity is constant and equal to 1.

We now check the validity of the connexive principles in Table 1 according to Definition 8.

4.1. Connexive Principles in Approach 2

4.1.1. Aristotle's hesis (AT): $\sim(\sim A \to A)$. We interpret the principle $\sim(\sim A \to A)$ by the negation of the conditional event $A|\overline{A}$, that is by $\overline{A|\overline{A}}$. Then, based on equations (9) and (7), it follows that $\overline{A|\overline{A}} = 1 - A|\overline{A} = \overline{A}|\overline{A} = 1$. Therefore, (AT) is valid$_2$ because the conditional random quantity $\overline{A|\overline{A}}$, which also coincides with the conditional event $\overline{A}|\overline{A}$, is constant and equal to 1.

4.1.2. Aristotle's Thesis′ (*AT′*): $\sim(A \to \sim A)$. We interpret the principle $\sim(A \to \sim A)$ by the negation of the conditional event $\overline{A}|A$, that is by $\overline{\overline{A}|A}$. Like in (AT), it holds that $\overline{\overline{A}|A} = 1 - \overline{A}|A = A|A = 1$, which validates (AT′) in Approach 2. Notice that, the validity of (AT′) also follows from (AT) when A is replaced by $\overline{A}$ (of course $\overline{\overline{A}} = A$).

4.1.3. Abelard's First Principle (AB): $\sim((A \to B) \wedge (A \to \sim B))$. The structure of this principle is formalized by the conditional random quantity $\overline{(B|A) \wedge (\overline{B}|A)}$, where $A \neq \bot$. We observe that $(B|A) \wedge (\overline{B}|A) = (B \wedge \overline{B})|A = \bot|A$. Then, $\overline{(B|A) \wedge (\overline{B}|A)} = \overline{\bot|A} = \overline{\bot}|A = \top|A = 1$. Therefore, (AB) is valid$_2$.

4.1.4. Non-validity of Aristotle's Second Thesis (AS) $\sim((A \to B) \wedge (\sim A \to B))$is not valid$_2$.

The structure of this principle is formalized by the random quantity $\overline{(B|A) \wedge (B|\overline{A})}$ (where $A \neq \bot$ and $\overline{A} \neq \bot$, that is A is contingent). By setting $p(B|A) = x$ and $p(B|\overline{A}) = y$, it follows that [28,32]

$$(B|A) \wedge (B|\overline{A}) = (B|A) \cdot (B|\overline{A}) = \begin{cases} 0, & \text{if } A\overline{B} \vee \overline{A}\,\overline{B} \text{ is true,} \\ y, & \text{if } AB \text{ is true,} \\ x, & \text{if } \overline{A}B \text{ is true.} \end{cases}$$

Then, $\overline{(B|A) \wedge (B|\overline{A})} = 1 - (B|A) \wedge (B|\overline{A}) = 1 - (yAB + x\overline{A}B)$, which is not constant and can therefore not necessarily be equal to 1. In particular, by choosing the coherent assessment $x = y = 1$, it follows that $\overline{(B|A) \wedge (B|\overline{A})} = 1 - AB - \overline{A}B = 1 - B = \overline{B}$, which is not necessarily equal to 1 as it could be either 1 or 0, according to whether $\overline{B}$ is true or false, respectively. Therefore, (AS) is not valid$_2$. Moreover, by setting $\mathbb{P}[(B|A) \wedge (B|\overline{A})] = \mu$, it holds that $\mu = y\,p(AB) + x\,p(\overline{A}B) = y\,p(B|A)p(A) + x\,p(B|\overline{A})p(\overline{A}) = xy\,p(A) + xy\,p(\overline{A}) = xy$. Then, $\mathbb{P}[\overline{(B|A) \wedge (B|\overline{A})}] = 1 - xy$. We also observe that, in the special case where $x = y = 0$, it follows that $\overline{(B|A) \wedge (B|\overline{A})} = 1$.

4.1.5. Boethius' Thesis (BT): $(A \to B) \to \sim(A \to \sim B)$.

This principle is formalized by the iterated conditional $(\overline{\overline{B}|A})|(B|A)$ (where $AB \neq \bot$). We recall that $(\overline{\overline{B}|A}) = B|A$. Then $(\overline{\overline{B}|A})|(B|A) = (B|A)|(B|A)$.

Then, from Remark 2 it follows that

$$(B|A)|(B|A) = 1. \tag{20}$$

Therefore $(\overline{\overline{B}|A})|(B|A)$ is constant and equal to 1 and hence (BT) is valid$_2$.

4.1.6. Boethius' Thesis′ (*BT′*): $(A \to \sim B) \to \sim(A \to B)$. This principle is formalized by the iterated conditional $(\overline{B|A})|(\overline{B}|A)$, where $A\overline{B} \neq \bot$. From (20), by replacing B with $\overline{B}$, it holds that

$$(\overline{B}|A)|(\overline{B}|A) = 1. \tag{21}$$

Then, by observing that $\overline{B|A} = \overline{B}|A$, it follows that $(\overline{B|A})|(\overline{B}|A) = (\overline{B}|A)|(\overline{B}|A)$ is constant and equal to 1. Therefore, (BT′) is valid$_2$.

4.1.7. Reversed Boethius' Thesis (RBT): $\sim(A \to \sim B) \to (A \to B)$. This principle is formalized by the iterated conditional $(B|A)|(\overline{\overline{B}|A})$, where $AB \neq \bot$. As $(\overline{\overline{B}|A}) = B|A$ it follows that $(B|A)|(\overline{\overline{B}|A}) = (B|A)|(B|A)$, which is constant and equal to 1 because of (20). Therefore, (RBT) is valid$_2$. Notice that the interpretation of (BT) and (RBT) amount to the same iterated conditional (i.e., $(B|A)|(B|A)$).

4.1.8. Reversed Boethius' Thesis′ (*RBT′*): $\sim(A \to B) \to (A \to \sim B)$. This principle is formalized by the iterated conditional $(\overline{B}|A)|(\overline{B|A})$, where $A\overline{B} \neq \bot$. As $(\overline{B|A}) = \overline{B}|A$ it follows that $(\overline{B}|A)|(\overline{B|A}) = (\overline{B}|A)|(\overline{B}|A)$, which is constant and equal to 1 because of (21). Therefore (RBT′) is valid$_2$. Like for (BT) and (RBT), we observe that the interpretation of (BT′) and (RBT′) amount to the same iterated conditional (i.e., $(\overline{B}|A)|(\overline{B}|A)$).

4.1.9. Non-validity of Boethius Variation (B3) $(A \to B) \to \sim(\sim A \to B)$ is not valid$_2$. This principle is formalized by the iterated conditional $(\overline{B|\overline{A}})|(B|A)$, where $AB \neq \bot$. As $\overline{B|\overline{A}} = 1 - B|\overline{A} = \overline{B}|\overline{A}$, it follows that

$$(\overline{B|\overline{A}})|(B|A) = (\overline{B}|\overline{A})|(B|A). \tag{22}$$

By setting $p(B|A) = x$, $p(\overline{B}|\overline{A}) = y$, and $\mathbb{P}[(\overline{B}|\overline{A})|(B|A)] = \mu$, it holds that

$$(\overline{B}|\overline{A})|(B|A) = (\overline{B}|\overline{A}) \wedge (B|A) + \mu(1 - B|A) = \begin{cases} y, & \text{if } AB \text{ is true,} \\ \mu, & \text{if } A\overline{B} \text{ is true,} \\ \mu(1-x), & \text{if } \overline{A}B \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} \tag{23}$$

which is not constant and can therefore not necessarily be equal to 1. For example, if we choose the coherent assessment $x = y = 1$, it follows that

$$(\overline{B}|\overline{A})|(B|A) = (\overline{B}|\overline{A}) \wedge (B|A) + \mu(1 - B|A) = \begin{cases} 1, & \text{if } AB \text{ is true,} \\ \mu, & \text{if } A\overline{B} \text{ is true,} \\ 0, & \text{if } \overline{A}B \text{ is true,} \\ 1, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases}$$

is not constant and hence not necessarily equal to 1. Therefore, (B3) is not valid$_2$.

4.1.10. Non-validity of Boethius Variation (B4): $(\sim A \to B) \to \sim(A \to B)$is not valid$_2$. This principle is formalized by the iterated conditional $(\overline{B|A})|(B|\overline{A})$, where $\overline{A}B \neq \bot$. We observe that $(\overline{B|A})|(B|\overline{A})$ is not constant and not necessarily equal to 1 because it is equivalent to (B3) when A is replaced by $\overline{A}$. Therefore, (B4) is not valid$_2$.

Connexive principles and their interpretation in terms of compounds of conditionals or iterated conditionals are illustrated in Table 4.

Table 4. Connexive principles in the framework of compounds of conditionals and iterated conditionals (Approach 2). Value denotes whether the conditional random quantity is constant and equal to 1 (denoted by $= 1$) or not constant (which implies that it is not necessarily equal to 1)

	Connexive principle	Interpretation	Value	Valid_2
(AT)	$\sim(\sim A \to A)$	$\overline{A\|\overline{A}}$	$= 1$	Yes
(AT′)	$\sim(A \to \sim A)$	$\overline{\overline{A}\|A}$	$= 1$	Yes
(AB)	$\sim((A \to B) \wedge (A \to \sim B))$	$\overline{(B\|A) \wedge (\overline{B}\|A)}$	$= 1$	Yes
(AS)	$\sim((A \to B) \wedge (\sim A \to B))$	$\overline{(B\|A) \wedge (B\|\overline{A})}$	not constant	No
(BT)	$(A \to B) \to \sim(A \to \sim B)$	$\overline{(\overline{B}\|A)}\|(B\|A)$	$= 1$	Yes
(BT′)	$(A \to \sim B) \to \sim(A \to B)$	$\overline{(B\|A)}\|(\overline{B}\|A)$	$= 1$	Yes
(RBT)	$\sim(A \to \sim B) \to (A \to B)$	$(B\|A)\|\overline{(\overline{B}\|A)}$	$= 1$	Yes
(RBT′)	$\sim(A \to B) \to (A \to \sim B)$	$(\overline{B}\|A)\|\overline{(B\|A)}$	$= 1$	Yes
(B3)	$(A \to B) \to \sim(\sim A \to B)$	$\overline{(B\|\overline{A})}\|(B\|A)$	not constant	No
(B4)	$(\sim A \to B) \to \sim(A \to B)$	$\overline{(B\|A)}\|(B\|\overline{A})$	not constant	No

4.2. Further Properties of Approach 2

We study in this section the validity of selected further properties relevant to connexivity within Approach 2. In analogy to Section 3.3, we investigate Symmetry, Minimality, Kapsner-strong, Simplification, Conjunction-Idempotence, paradoxes of material conditional, Improper Transposition, Contraposition, and Denying Conjunct. Moreover, we study generalized versions of Aristotle's theses in terms of iterated conditionals.

Non-validity of Symmetry: $(B \to A) \to (A \to B)$ is not valid$_2$. We recall that connexive logics fail to be symmetric. In Approach 2, Symmetry is interpreted by the iterated conditional $(B|A)|(A|B)$. We observe that

$$(B|A)|(A|B) = (A|B) \wedge (B|A) + \mu(\overline{A}|B), \tag{24}$$

where $\mu = \mathbb{P}[(B|A)|(A|B)]$. The conjunction $(A|B) \wedge (B|A)$ is also known as the *biconditional event* and it holds that $(A|B) \wedge (B|A) = AB|(A \vee B)$ ([70]). Then, by setting $z = p(AB|(A \vee B))$ and $x = p(A|B)$, we obtain that

$$(B|A)|(A|B) = (AB)|(A \vee B) + \mu(\overline{A}|B) = \begin{cases} 1, & \text{if } AB \text{ is true,} \\ \mu(1-x), & \text{if } A\overline{B} \text{ is true,} \\ \mu, & \text{if } \overline{A}B \text{ is true,} \\ z + \mu(1-x), & \text{if } \overline{A}\,\overline{B} \text{ is true.} \end{cases}$$

By the linearity of prevision, $\mu = z + \mu(1-x)$ and hence

$$(B|A)|(A|B) = \begin{cases} 1, & \text{if } AB \text{ is true,} \\ \mu(1-x), & \text{if } A\overline{B} \text{ is true,} \\ \mu, & \text{if } \overline{A} \text{ is true.} \end{cases} \tag{25}$$

We observe that by setting, for instance $x = 1$, the iterated conditional $(B|A)|(A|B)$ assumes the value 0, when $A\overline{B}$ is true. Thus, the iterated conditional $(B|A)|(A|B)$ is not constant and hence not necessarily equal to 1. Therefore, Symmetry does not hold in Approach 2.

4.2.1. Minimality Like in Approach 1, also in Approach 2 (AT), (AT′), (BT), and (BT′) are valid$_2$, while Symmetry is not valid$_2$. Therefore, Approach 2 satisfies all minimal any connexive logic should satisfy ([18, p. 346]).

4.2.2. Kapsner-Strong Approach 2 is also *Kapsner-strong* because the respective conditions (i) $A \to {\sim}A$ *is not satisfiable* and (ii) $A \to B$ *and* $A \to {\sim}B$ *are not simultaneously satisfiable* are met. We first define satisfiability for Approach 2. We say that a conditional $A \to B$ is *satisfiable* iff the conditional event $B|A$ is not constant and necessarily equal to zero. Then,

- (i) $A \to {\sim}A$ is not satisfiable because $\overline{A}|A$ is constant and equal to 0.
- (ii) $A \to B$ and $A \to {\sim}B$ are not simultaneously satisfiable because, as $B|A + \overline{B}|A = 1$, the conditional events $(B|A)$ and $(\overline{B}|A)$ cannot both be equal to 1.

Simplification Approach 2 also satisfies the two versions of Simplification: $(A \wedge B) \to A$ and $(A \wedge B) \to B$. Indeed, as $A|AB = B|AB = AB|AB = 1$, Simplification is valid$_2$.

4.2.3. Conjunction-Idempotence Of course, $(A \wedge A) \to A$ and $A \to (A \wedge A)$ are trivially valid$_2$ because $A|AA = AA|A = A|A = 1$.

4.2.4. Non-validity of Paradoxes of the Material Conditional We show the non-validity of the following paradoxes of the material conditional in Approach 2: $B \to (A \to B)$ (positive paradox); ${\sim}A \to (A \to B)$ (negative paradox). We observe that

$$(B|A)|B = (B|A) \wedge B + \mu\overline{B} = \begin{cases} 1, & \text{if } AB \text{ is true,} \\ x, & \text{if } \overline{A}B \text{ is true,} \\ \mu, & \text{if } \overline{B} \text{ is true,} \end{cases}$$

where $x = P(B|A)$ and $\mu = \mathbb{P}((B|A)|B)$. This random quantity is not necessarily constant and equal to 1 (since x could be less than 1). Therefore, the positive paradox is not valid$_2$.[4] Moreover, it holds that

$$(B|A)|\overline{A} = (B|A) \wedge \overline{A} + \eta A = \begin{cases} x, & \text{if } \overline{A} \\ \eta, & \text{if } A \text{ is true,} \end{cases}$$

where $\eta = \mathbb{P}((B|A)|\overline{A})$. As coherence requires that $\eta = x$ (see, e.g., [70, Section 8]) and $x = P(B|A) \in [0, 1]$, it holds that the negative paradox is not valid$_2$. Therefore, positive and negative paradoxes are both not valid$_2$.

4.2.5. Non-validity of Improper Transposition $(A \to B) \to ({\sim}A \to {\sim}B)$ is not valid$_2$. Improper Transposition is interpreted by $(\overline{B}|\overline{A})|(B|A)$. This iterated conditional, which coincides with the interpretation of Boethius Variation (B3), is not constant and hence not necessarily equal to 1 (see Equation (23)). Therefore, Improper Transposition is not valid$_2$.

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[4] We recall that the Import–Export principle does not hold in our approach ([31,69]), that is $(B|A)|H \neq B|AH$. Note that if the Import–Export principle would hold, then we would inherit the positive paradox of the material conditional, since in this case we would have that $(B|A)|B$ coincides with $B|AB$, which is constant and equal to 1 since $B|AB = B|B$.

4.2.6. Non-validity of Contraposition $(A \to B) \to (\sim B \to \sim A)$ is not valid$_2$. Contraposition is interpreted by the iterated conditional $(\overline{A}|\overline{B})|(B|A)$. By setting $p(B|A) = x$, $p(\overline{A}|\overline{B}) = y$, and $\mathbb{P}[(\overline{A}|\overline{B}) \wedge (B|A)] = z$, we obtain

$$(\overline{A}|\overline{B})|(B|A) = (\overline{A}|\overline{B}) \wedge (B|A) + \mu(\overline{B}|A) = \begin{cases} y, & \text{if } AB \text{ is true,} \\ \mu, & \text{if } A\overline{B} \text{ is true,} \\ z + \mu(1-x), & \text{if } \overline{A}B \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases}$$

where $\mu = \mathbb{P}[(\overline{A}|\overline{B})|(B|A)]$. Notice that, by the linearity of prevision, $\mu = z + \mu(1-x)$. Then,

$$(\overline{A}|\overline{B})|(B|A) = \begin{cases} y, & \text{if } AB \text{ is true,} \\ \mu, & \text{if } A\overline{B} \vee \overline{A}B \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} \tag{26}$$

which is not necessarily equal to 1. For instance, by the coherent assessment $x = 1$ and $y = 0$, formula (26) becomes

$$(\overline{A}|\overline{B})|(B|A) = \begin{cases} 0, & \text{if } AB \text{ is true,} \\ \mu, & \text{if } A\overline{B} \vee \overline{A}B \text{ is true,} \\ 1, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} \tag{27}$$

which coincides with 1, when $\overline{A}\,\overline{B}$ is true and with 0, when AB and hence $(\overline{A}|\overline{B})|(B|A)$ is not constant and hence not necessarily equal to 1. Therefore, Contraposition is not valid$_2$.

4.2.7. Non-validity of Denying a Conjunct $\sim(A \wedge B) \to (\sim A \to B)$ is not valid$_2$. Denying a Conjunct is interpreted by the iterated conditional $(B|\overline{A})|(\overline{AB})$. By setting $y = p(B|\overline{A})$ we obtain that

$$(B|\overline{A})|(\overline{AB}) = (\overline{AB}) \wedge (B|\overline{A}) + \mu \cdot (1 - \overline{AB}) = \begin{cases} \mu, & \text{if } AB \text{ is true,} \\ y, & \text{if } A\overline{B} \text{ is true,} \\ 1, & \text{if } \overline{A}B \text{ is true,} \\ 0, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} \tag{28}$$

where $\mu = \mathbb{P}[(B|\overline{A})|\overline{AB}]$. While the Import–Export principle does not hold in general ([31,69]), it holds in this case because $\overline{A}$ logically implies $\overline{A} \vee \overline{B} = \overline{AB}$. Then, we have that $(B|\overline{A})|\overline{AB} = B|\overline{A}$ ([27, Proposition 1]). In other words, coherence requires that $\mu = y$ in formula (28). Then,

$$(B|\overline{A})|\overline{AB} = B|\overline{A} = \begin{cases} y, & \text{if } A \text{ is true,} \\ 1, & \text{if } \overline{A}B \text{ is true,} \\ 0, & \text{if } \overline{A}\,\overline{B} \text{ is true,} \end{cases} \tag{29}$$

which is not constant and hence not necessarily equal to 1. Therefore, Denying a Conjunct is not valid$_2$.

4.2.8. Generalized Aristotle's Theses In this paragraph we study the iterated versions of Aristotle's theses:

(IAT) $\sim(\sim(A \to B) \to (A \to B))$;

(IAT)$'$ $\sim((A \to B) \to \sim(A \to B))$.

Principle (IAT) is formalized by the negated iterated conditional $\overline{(B|A)|\overline{(B|A)}}$ (with $A\overline{B} \neq \bot$). For showing the validity of (IAT) in Approach 2 we prove that this object is constant and equal to 1. We first introduce the negation of an iterated conditional in analogy to (9).

Definition 9. The negation of the iterated conditional $(B|K)|(A|H)$, denoted by $\overline{(B|K)|(A|H)}$, is defined as follows:

$$\overline{(B|K)|(A|H)} = \overline{(B|K)}|(A|H) = (\overline{B}|K)|(A|H). \tag{30}$$

Notice that the negation of the iterated conditional coincides with the narrow scope interpretation of negating conditionals. Then, the principle (IAT) is formalized in Approach 2 by the negated iterated conditional $\overline{(B|A)|\overline{(B|A)}}$, where $A\overline{B} \neq \bot$. From Definition 9 and equations (9) and (21), we obtain

$$\overline{(B|A)|\overline{(B|A)}} = \overline{(B|A)}|\overline{(B|A)} = (\overline{B}|A)|(\overline{B}|A) = 1,$$

that is $\overline{(B|A)|\overline{(B|A)}}$ is constant and equal to 1. Therefore, (IAT) is valid$_2$. We observe that our interpretation of (IAT) (i.e., $(\overline{B}|A)|(\overline{B}|A)$) coincides within our interpretation of (BT)$'$ and (RBT)$'$ in Approach 2.

We interpret (IAT)$'$ by the negated iterated conditional $\overline{\overline{(B|A)}|(B|A)}$ (with $AB \neq \bot$). Analogously to (IAT), From Definition 9 and equations (9) and (20), we obtain

$$\overline{\overline{(B|A)}|(B|A)} = (B|A)|(B|A) = 1.$$

Therefore, (IAT)$'$ is valid$_2$. We observe that our interpretation of (IAT)$'$ coincides with our interpretation of (BT) and (RBT) within Approach 2.

For validating (IAT) and (IAT)$'$ in Approach 2 we used Definition 9. In the next remark we show that, for each given iterated conditional $(B|K)|(A|H)$, it holds that $(B|K)|(A|H) + \overline{(B|K)|(A|H)} = 1$. This equation can be seen as a proper generalization from non-iterated conditionals of equation (9) to iterated conditionals. For validating (IAT) and (IAT)$'$ in Approach 2 we used Definition 9. In the next remark we show that, for each given iterated conditional $(B|K)|(A|H)$, it holds that

$$(B|K)|(A|H) + \overline{(B|K)|(A|H)} = 1.$$

This equation can be seen as a proper generalization of equation (9) from non-iterated conditionals to iteretad conditionals.

Remark 4. We observe that $\overline{(B|K)|(A|H)} = 1 - (B|K)|(A|H)$ by proving that $(\overline{B}|K)|(A|H) = 1 - (B|K)|(A|H)$. Indeed, by setting, $x = p(A|H)$, $\mu = \mathbb{P}[(B|K)|(A|H)]$, and $\mu' = \mathbb{P}[(\overline{B}|K)|(A|H)]$, it holds that $\mu + \mu' = \mathbb{P}[(B|K)|(A|H) + (\overline{B}|K)|(A|H)]$ and

$$\begin{aligned}
&(B|K)|(A|H) + (\overline{B}|K)|(A|H) \\
&= (A|H) \wedge (B|K) + \mu(\overline{A}|H) + (A|H) \wedge (\overline{B}|K) + \mu(\overline{A}|H) \\
&= (A|H) + (\mu + \mu')(\overline{A}|H) = \begin{cases} 1, \text{ if } AH \text{ is true,} \\ \mu + \mu', \text{ if } \overline{A}H \text{ is true,} \\ x + (\mu + \mu')(1 - x) \text{ if } \overline{H} \text{ is true.} \end{cases}
\end{aligned}$$

By the linearity of prevision, $\mu + \mu' = x + (\mu + \mu')(1 - x)$, and hence

$$(B|K)|(A|H) + (\overline{B}|K)|(A|H) = \begin{cases} 1, \text{ if } AH \text{ is true,} \\ \mu + \mu', \text{ if } \overline{A}H \vee \overline{H} \text{ is true.} \end{cases}$$

By coherence, it can be proved that $\mu + \mu' = 1$ and hence it follows that $(\overline{B}|K)|(A|H) = 1 - (B|K)|(A|H)$. Then, from Definition 9 we obtain

$$\overline{(B|K)|(A|H)} = (\overline{B}|K)|(A|H) = 1 - (B|K)|(A|H). \tag{31}$$

Moreover, from (31) and by the linearity of prevision, it holds that

$$\mathbb{P}[\overline{(B|K)|(A|H)}] = \mathbb{P}[(\overline{B}|K)|(A|H)] = 1 - \mathbb{P}[(B|K)|(A|H)].$$

4.3. Connexivity under Alternative Notions of Conjunction of Conditional Events

In Section. 4.1 we showed that (AB) is valid$_2$, while (AS) is not valid$_2$. In this section we study whether both connexive principles can be validated when we use alternative definitions of the conjunction among conditional events. Specifically, we study the validity of (AB) and (AS) when the conjunction $\wedge$ of conditional events is replaced by the Kleene-Lukasiewicz-Heyting conjunction (denoted by $\wedge_K$) and the Sobocinsky conjunction (denoted by $\wedge_S$). In contrast to $\wedge$, the conjunctions $\wedge_K$ and $\wedge_S$ yield a conditional event. Of course, (AT) and (AT)$'$ are still valid$_2$ under these different interpretations of conjunction, because these principles do not involve conjunctions of conditionals.

4.3.1. Abelard's First Principle and Aristotle's Second Thesis Under $\wedge_K$
We recall that the Kleene-Lukasiewicz-Heyting conjunction $\wedge_K$ among conditional events (see, e.g. [11], which also coincides with the conjunction in the trivalent logic of de Finetti [15]; see also [17]) is the conditional event which is

- *true* if all conjuncts are true;
- *false* if at least one conjunct is false;
- *void* otherwise.

In particular, for the two conditional events $A|H$ and $B|K$, the conjunction $(A|H) \wedge_K (B|K)$ is given by

$$(A|H) \wedge_K (B|K) = AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K). \tag{32}$$

Of course, $(A|H) \wedge_K (B|H) = AB|H$. Then, as $(B|A) \wedge_K (\overline{B}|A) = \bot|H = 0$, $\overline{(B|A) \wedge_K (\overline{B}|A)}$ is constant and equal to 1. Therefore (AB) is valid$_2$ under $\wedge_K$.

We now show the validity of (AS) under $\wedge_K$. The structure of (AS) is then formalized by the conditional event $\overline{(B|A) \wedge_K (B|\overline{A})}$, where $A \neq \bot$ and $\overline{A} \neq \bot$. From (32) we obtain

$$(B|A) \wedge_K (B|\overline{A}) = \bot|(A\overline{B} \vee \overline{A}\,\overline{B}) = \bot|\overline{B} = 0.$$

Thus $\overline{(B|A) \wedge_K (B|\overline{A})} = 1 - (B|A) \wedge_K (B|\overline{A}) = 1$, which validates (AS) under $\wedge_K$. From the viewpoint of *some* connexive logics the validity of (AS) and (AB) is desirable, which is obtained by using $\wedge_K$. However, the use of $\wedge_K$ in probability logic is problematic, since it leads to mismatches with basic probabilistic properties, as we show now.

Recall that, in case of logical independence of the events A, B, H, K, the assessment $z_K = p[(A|H) \wedge_K (B|K)]$ is a coherent extension of $p(A|H)$ and $p(B|K)$ if and only if ([68, Theorem 3])

$$z_K \in [0, \min\{p(A|H), p(B|K)\}].$$

Then, the Fréchet-Hoeffding bounds (11), which hold in the unconditional case, are not preserved when $\wedge_K$ is used. For instance, the assessment $(1, 1, 0)$ on $(A|H, B|K, (A|H) \wedge_K (B|K))$ is coherent, while it is not coherent on (A, B, AB), because $p(AB) = 1$ is the only coherent extension of $(1, 1)$ on (A, B). Notice that, in particular, as $(B|A) \wedge_K (B|\overline{A}) = 0$, we have that $z_K = 0$ is the only coherent extension of $p(B|A) = p(B|\overline{A}) = 1$ on $(B|A) \wedge_K (B|\overline{A})$. This mismatch between the validity of (AS) and the violation of the Fréchet-Hoeffding bounds is not desirable.

4.3.2. Abelard's First Principle and Aristotle's Second Thesis Under $\wedge_S$
We recall that the Sobocinsky conjunction $\wedge_S$ among conditional events (see, e.g., [11], which coincides with Adams' Quasi conjunction [1]), is the conditional event which is

- *true*, if at least one conjunct is true and all other conjuncts are not false;
- *false*, if at least one conjunct is false;
- *void*, otherwise (i.e., all conjuncts are void).

In particular, for the two conditional events $A|H$ and $B|K$,

$$(A|H) \wedge_S (B|K) = ((\overline{H} \vee A) \wedge (\overline{K} \vee B))|(H \vee K). \tag{33}$$

Concerning (AB) we observe that $(A|H) \wedge_S (B|H) = AB|H$. Then, as $(B|A) \wedge_S (\overline{B}|A) = \bot|H = 0$, $\overline{(B|A) \wedge_S (\overline{B}|A)}$ is constant and equal to 1. Therefore (AB) is valid$_2$ under $\wedge_S$.

We formalize the structure of (AS) by the conditional event $\overline{(B|A) \wedge_S (B|\overline{A})}$, where $A \neq \bot$ and $\overline{A} \neq \bot$. We observe that, from (33), it holds that $(B|A) \wedge_S (B|\overline{A}) = B$. Then,

$$\overline{(B|A) \wedge_S (B|\overline{A})} = 1 - (B|A) \wedge_S (B|\overline{A}) = \overline{B},$$

which does not coincide with 1 and hence (AS) is not valid$_2$ under $\wedge_S$.

Like $\wedge$, the conjunction $\wedge_S$, satisfies (AB) but does not satisfy (AS). However, like $\wedge_K$, the conjunction $\wedge_S$ is problematic for probability logic, as it violates the probabilistic property (11). Indeed, under logical independence of the events A, B, H, K, $z_S = P[(A|H) \wedge_S (B|K)]$ is a coherent extension of (x, y) on $(A|H, B|K)$ if and only if $z_S \in [z'_S, z''_S]$, where ([24,30])

$$z'_S = \max\{x + y - 1, 0\} \;\text{ and }\; z''_S = \begin{cases} \frac{x+y-2xy}{1-xy}, & (x, y) \neq (1, 1), \\ 1, & (x, y) = (1, 1). \end{cases}$$

Then, the Fréchet-Hoeffding (11) bounds are not preserved when $\wedge_S$ is used. We also recall that z''_S is an Hamacher t-conorm (with the parameter $\lambda = 0$). Then, as $\max(x, y)$ is the smallest t-conorm ([44, p. 13]), it holds that

$$z''_S \geq \max\{x, y\} \geq \min\{x, y\} \quad \forall (x, y) \in [0, 1]^2. \tag{34}$$

We observe that the assessment $(1, 0, 1)$ on $(A|H, B|K, (A|H) \wedge_S (B|K))$ is coherent, while it is not coherent on (A, B, AB), because $p(AB) = 0$ is the only coherent extension of $(1, 0)$ on (A, B). This is not desirable from a probabilistic point of view concerning conjunction.

In this section we analyzed connexivity (Section 4.1) and related further properties (Section 4.2) in terms of compounds of conditionals. Moreover, we studied connexivity under alternative notions of conjoined conditionals within this framework (Section 4.3).

5. Concluding Remarks

We presented two approaches to investigate connexive principles. In Approach 1, we investigated connexive principles within coherence-based probabilistic default reasoning, by interpreting defaults and negated defaults in terms of suitable probabilistic constraints on conditional events. Within this approach we showed that the connexive principles (AT), (AT′), (AB), (BT), and (BT′) are valid$_1$, whereas (AS), (RBT), (RBT′), (B3), and (B4) are not valid$_1$ (see Table 3). In Approach 2 we study connexivity within the coherence framework of compound and iterated conditionals, by interpreting connexive principles in terms of suitable conditional random quantities. Here, we demonstrated that, like in Approach 1, (AT), (AT′), (AB), (BT), and (BT′) are valid$_2$, whereas (AS), (B3), and (B4) are not valid$_2$. Contrary to Approach 1, (RBT) and (RBT′) are valid$_2$ (see Table 4).

Moreover, both approaches satisfy Simplification and Conjunction-Idempotence. However, Symmetry, paradoxes of the material conditional, Improper Transposition, Contraposition, and Denying a Conjunct are neither valid$_1$ nor valid$_2$.

Approach 1 and Approach 2 satisfy the minimality conditions for connexive logic, since both Aristotles' theses (AT, AT′) and Boethius' theses (BT, BT′) are valid$_1$ and valid$_2$, while Symmetry is neither valid$_1$ nor valid$_2$. We also showed that both Approach 1 and Approach 2 are *Kapsner-strong*.

We also presented generalised versions of Aristotle's theses in terms of iterated conditionals.

Since we observed that Aristotle's Second Thesis is neither valid$_1$ nor valid$_2$, we also studied selected ways under which it can be validated within both approaches, by making suitable additional assumptions. Specifically, in Approach 1, (AS) can be validated under the assumption that the probability of the consequent of the involved conditionals is not equal to 1 (i.e, $p(B) \neq 1$). For Approach 2 we studied the validity of (AS) (and also (AB)) under different interpretation of the conjunction of conditionals in trivalent logic. However, changing the notion of conjunction yields to the loss of basic probabilistic properties, which is undesirable from a probability-logical point of view.

From a conceptual point of view, Approach 1 is characterized by employing concepts from coherence-based probability theory and probabilistic interpretations of defaults and negated defaults. Conditionals, interpreted as defaults, are negated by the wide scope negation. We gave two notions of validity, namely for non-iterated and iterated connexive principles, respectively. Approach 2, however, is characterized by interpreting conditionals as conditional events in the framework of conditional random quantities. This allows for dealing with logical operations on conditional events and avoids (see, e.g., [69]) the well known Lewis' triviality results (see, e.g., [48]). It therefore offers a more unified approach to connexive principles, which is reflected by a unique definition of validity for both, iterated and non-iterated connexive principles. Moreover, Approach 2 negates conditionals by the narrow scope negation. Thus, validity depends on how conditionals and negation are defined.

One might wonder why neither of the two approaches validates all connexive principles. Apart from the insight obtained from our proofs, this is not surprising since also not all connexive principles are valid in all systems of connexive logic. Moreover, some rules which are valid in classical logic (e.g., transitivity, contraposition, and premise strengthening) are not valid in probability logic, while, for example, the rules of the basic nonmonotonic System P are valid within coherence-based probability logic ([12,23,25,26,33]). We have also shown that Aristotle's Second Thesis can be validated, when further assumptions are made or when the definition of logical operations among conditional events is altered. In particular, (AS) can be validated under the further constraint that the probability of the consequent B of the involved conditionals is not equal to 1 in Approach 1. (AS) can also be validated in Approach 2 when the conjunction of conditional events ($\wedge$) is replaced by the Kleene-Lukasiewicz-Heyting-de Finetti conjunction ($\wedge_K$). Of course, also Aristotle's theses and Abelard's First Principle are valid$_2$ when $\wedge_K$ is used. This validity result in Approach 2, however, comes with the cost that basic logical and probabilistic properties of conjunction ([38]) are then lost. Similar losses arise when alternatives to Definition 5 are used for the iterated conditional (like in [7,15]; for a discussion see [9]).

Finally we note that strong empirical support for the psychological plausibility of Approach 2 can be observed: an experimental-psychological study on all connexive principles in Table 1 shows that Table 4 correctly predicts for each connexive principle the majority of participants' responses [64]. In particular, human inference is consistent with the validity of (RBT) and (RBT$'$), which speaks empirically for Approach 2 and against Approach 1. Among other formulas, the experimental data are also consistent with the validity of Simplification, and Generalized Aristotle's theses as well as with the non-validity of Improper Transposition, and Denying a Conjunct [64].

That people reject Contraposition is reported in [58]. All this experimental evidence speaks for Approach 2. The only prediction, which was not yet convincingly confirmed experimentally, is Symmetry: only 40% of the participants responded as predicted, with "cannot tell" (i.e., non-validity of Symmetry) and 16% responded with "does not hold". However, still 44% responded that Symmetry "holds" [64]. This calls for further experimental research, especially on Symmetry.

We have shown that coherence-based probability logic offers a rich language to investigate the validity of various connexive principles and thereby provides a new semantics of connexive conditionals.

Future work will be devoted to investigations on other intuitively plausible logical principles contained in alternative and non-classical logics.

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