Andrea Iacona

Connexivity in the Logic of Reasons

Abstract. This paper discusses some key connexive principles construed as principles about reasons, that is, as principles that express logical properties of sentences of the form 'p is a reason for q'. Its main goal is to show how the theory of reasons outlined by Crupi and Iacona, which is based on their evidential account of conditionals, yields a formal treatment of such sentences that validates a restricted version of the principles discussed, overcoming some limitations that affect most extant accounts of conditionals.

Keywords: Reasons, Conditionals, Aristotle's thesis, Boethius thesis, Abelard's first principle, Aristotle's second thesis, Supraclassicality.

1. Preliminary Clarifications

Let us start with the following list of connexive principles:

P1 $\neg(\neg p \to p)$

P2 $\neg(p \to \neg p)$

P3 $\neg((p \to q) \wedge (p \to \neg q))$

P4 $(p \to q) \to \neg(p \to \neg q)$

P5 $(p \to \neg q) \to \neg(p \to q)$

P6 $\neg((p \to q) \wedge (\neg p \to q))$

P7 $(p \to q) \to \neg(\neg p \to q)$

P8 $(\neg p \to q) \to \neg(p \to q)$

P1 and P2 are known as *Aristotle's Thesis*.[1] P3 is known as *Abelard's First Principle*, and is sometimes phrased in conditional form as *Weak Boethius*

[1] The reference is to Aristotle, *Prior Analytics* 57b14, where a statement of the form 'If not-p, p' is rejected as impossible.

Special Issue: Frontiers of Connexive Logic
Edited by: **Hitoshi Omori and Heinrich Wansing**.

Studia Logica (2024) 112: 325–342
https://doi.org/10.1007/s11225-023-10041-w
© The Author(s) 2023

Thesis: $(p \to q) \supset \neg(p \to \neg q)$.[2] P4 and P5 are alternative formulations of *Boethius Thesis*, which is stronger than Weak Boethius Thesis on the assumption that $\to$ is stronger than $\supset$.[3] P6 is sometimes called *Aristotle's Second Thesis*.[4] Finally, P7 and P8 may be called *Boethius Left Thesis*.[5] These two principles bear to P6 the same relation that P4 and P5 bear to P3: if one replaces the main connective in P7 and P8 with $\supset$, one obtains a weaker claim which is equivalent to P6.

The principles listed above—especially Aristotle's Thesis and Boethius Thesis—have been discussed mostly in relation to the logic of conditionals, so their plausibility has been measured against the standard of the ordinary use of 'if'. But the symbol $\to$ can be construed in different ways, and this paper focuses precisely on one alternative interpretation of it. The expression $p \to q$ will be used here to represent a sentence of the form 'p is a reason for q'. To mark the difference between conditionals and sentences about reasons, 'If p, then q' will be represented as $p > q$.

More specifically, the interpretation of $\to$ that will be considered concerns epistemic reasons, that is, reasons for belief. So it is not intended to model practical reasons, that is, reasons for action. For any two propositions p and q, to say that p is a reason for q in the sense that matters here is to say that assuming p provides a justification for believing q. For example, assuming that Fido is a dog provides a justification for believing that Fido can bark. In other words, $p \to q$ holds when p supports q.

A reason in this sense is a sufficient reason. To assert $p \to q$ is to say that the justification provided by p suffices for believing q: Fido's being a dog suffices for believing that he can bark. It is important to note, however, that sufficiency so understood is consistent with defeasibility. The relation expressed by $\to$ is non-monotonic, in that it can happen that $p \to q$ holds but $(p \wedge r) \to q$ does not hold for some r. In this case r acts as a *defeater* for $p \to q$. For example, on the assumption that Fido is a dog and is mute, it is not reasonable to think that Fido can bark. So 'sufficient' is to be read as 'defeasibly sufficient'.

How is $\to$ to be defined in order to provide an adequate account of the logical properties of sentences about reasons? This question, which is crucial

[2] See Wansing [33]. This principle is not mentioned in McCall's characterization of connexivity, see McCall [19], p. 435. In Angell [2], P3 is called 'principle of subjunctive contrariety'.

[3] The reference is to Boethius, *De Syllogismo Hypothetico* 843D, see Wansing [33].

[4] This label is used in McCall [20].

[5] In Francez [8], P7 and P8 are called 'Boethius $\neg l$ Thesis', where l stands for 'left'.

to any formal study of reasons, has been addressed by different authors and from different angles.[6] However, the role of connexivity in the logic of reasons has not yet received the attention it deserves, or so it appears. It is quite natural to wonder whether P1–P8 can be part of a theory of reasons, because they look very plausible when one considers ordinary statements about reasons.

Consider P1 and P2. Clearly, Fido's not being a dog is not a reason for thinking that he is a dog, and Fido's being a dog is not a reason for thinking that he is not a dog. Consider P3. Clearly, it is not the case that Fido's being a dog is both a reason for thinking that he can bark and a reason for thinking that he cannot bark. P4 is equally compelling: assuming that Fido's being a dog is a reason for thinking that he can bark, it is reasonable to deny that his being a dog is also a reason for thinking that he cannot bark. P5 is similar to P4. Consider P6. Clearly, it is not the case that both Fido's being a dog and his not being a dog are reasons for thinking that he can bark. P7 is equally compelling: assuming that Fido's being a dog is a reason for thinking that he can bark, it is reasonable to deny that Fido's not being a dog is also a reason for thinking that he can bark. P8 is simlar to P7.

As it emerges from the examples just provided, P1–P8 are *prima facie* plausible when $\rightarrow$ is read as 'is a reason for'. Or at least, they are no less intuitive than they are when $\rightarrow$ is understood as 'if'. This should not be surprising. Sentences about reasons are typically expressed by means of hypothetical constructions, so it is sensible to expect that the logic of reasons is somehow related to the logic of conditionals. The material presented in the following sections is intended to shed some light on the relation between reasons and conditionals.

2. Classicality

In order to assess the prospects of an account of conditionals for the purpose of providing an analysis of 'p is a reason for q', at least three key issues must be addressed. The first concerns the link between $\rightarrow$ and logical consequence as understood in classical logic. Here the symbol $\models$ will be used to indicate the latter relation. Most extant accounts of conditionals validate the principle known as *Supraclassicality*, according to which $p \models q$ entails $p > q$. So it is quite natural to ask whether the same principle should hold for reasons,

[6]The works on the logic of reasons include Pollock [22,23], Reiter [25], Horty [10,11], Spohn [29].

that is, whether $p \models q$ should entail $p \rightarrow q$. Yet there is no obvious answer to this question, for it is an open issue whether every case in which $p \models q$ is plausibly a case in which p is a reason for q.

In particular, two notoriously controversial cases are to be considered, the case in which p is logically false and the case in which q is logically true. Consider the following sentences:

(1) If it is raining and it is not raining, God exists

(2) If God exists, either it is raining or it is not raining

As long as one is inclined to grant that in (1) and (2) the antecedent entails the consequent in the sense of ‘entails’ that matters to logic, one will also be apt to think that the antecedent supports the consequent in virtue of that relation. By contrast, if one is inclined to question (1) and (2) as genuine cases of entailment, arguing that they lack the right sort of relevance link, one will be willing to claim that there are classically valid arguments in which the premises do not support the conclusion.

As long as the cases of the two kinds considered are left aside, it is definitely less controversial that $p \models q$ entails $p \rightarrow q$. Consider the following principle, which may be called *Restricted Classicality*: if p is not logically false, q is not logically true, and $p \models q$, then p is a reason for q. Restricted Classicality is weaker than Supraclassicality in that it rules out cases in which q logically follows from p merely in virtue of some property—logical truth or logical falsity—that one of them possesses independently of the other. So it guarantees that the combination of the truth of p with the falsity of q is ruled out by some logical relation between p and q, which plausibly justifies the claim that p is a reason for q.

Arguably, Restricted Classicality follows from two basic assumptions that any theory of reasons should grant. The first is that there is an essential conceptual link between reasons and inferences: to say that p is a reason for q is to say that the inference from p to q is justified. The second is that the inference from p to q is justified when p is not logically false, q is not logically true, and $p \models q$. Logical consequence—once potentially controversial cases such as those considered above are left aside—may be regarded as the strongest form of support that a premise p can provide for a conclusion q. Given these two assumptions, it seems correct to conclude that Restricted Classicality should hold.

Of course, the further question remains of whether Supraclassicality should hold as well. Supraclassicality, unlike Restricted Classicality, conflicts with P1–P8. Consider P1. If p is a classical tautology, we have that $\neg p \models p$. By

Supraclassicality it follows that $\neg p \to p$ is true, hence $\neg(\neg p \to p)$ is false. Similar counterexamples can be found for P2–P8. Note that, in a modal semantics, it is not even necessary to use classical tautologies, or contradictions, to generate such counterexamples, for it suffices to consider formulas that are true, or false, in all worlds in some model. For example, $\neg(\neg p \to p)$ is false in a world in a model if p is true in all worlds in that model. In any case, one cannot have both Supraclassicality and P1–P8.[7]

This dilemma opens two divergent routes for a theory of reasons. One option is to maintain Supraclassicality and opt for some suitably restricted version of P1–P8. The other is to maintain P1–P8 and replace Supraclassicality with Restricted Classicality. The first route is for those who have classical inclinations about (1) and (2), while the second is for those who regard (1) and (2) as seriously problematic. Although both routes deserve careful consideration, here I will explore only the first.

The idea that guides this choice is that the initial plausiblity of P1–P8 can be explained without assuming that they hold unrestrictedly. More precisely, they can be explained by the following restricted versions of P1–P8, which are consistent with Supraclassicality:

P9 $\Diamond \neg p \supset \neg(\neg p \to p)$

P10 $\Diamond p \supset \neg(p \to \neg p)$

P11 $\Diamond p \supset \neg((p \to q) \wedge (p \to \neg q))$

P12 $\Diamond p \supset ((p \to q) \to \neg(p \to \neg q))$

P13 $\Diamond p \supset ((p \to \neg q) \to \neg(p \to q))$

P14 $\Diamond \neg q \supset \neg((p \to q) \wedge (\neg p \to q))$

P15 $\Diamond \neg q \supset ((p \to q) \to \neg(\neg p \to q))$

P16 $\Diamond \neg q \supset ((\neg p \to q) \to \neg(p \to q))$

Note that the examples used in Section 1 to show the initial plausibility of P1–P8 are cases in which the antecedents of P9–P16 are satisfied. Arguably, this holds in general for any example that might convincingly be invoked in support of P1–P8, for there seem to be no clear intuitions about the cases in which the antecedents of P9–P16 are *not* satisfied.[8]

---

[7] A key assumption of the reasoning just stated, of course, is that the falsity of $\neg(\neg p \to p)$ entails its non-truth. That entailment, however, might not hold, for example as in Wansing's C, in which case there would be no counterexample.

[8] This is essentially the view advocated in Iacona [12]. Restricted forms of connexivity are discussed in Unterhuber [32], Kapsner [14], and Lenzen [15].

3. Aristotle's Second Thesis

The second issue to be addressed concerns Aristotle's Second Thesis. Most extant accounts of conditionals do not preserve this principle, no matter whether it is restricted or not. Here it will suffice to consider three well known accounts that rely on the Ramsey Test, the idea that in order to assess $p > q$ one must check whether q holds on the supposition that p holds. The first is the probabilistic view developed by Adams, which defines the acceptability of a conditional as a function of the conditional probability of its consequent given its antecedent. On this view, $p > q$ is acceptable to the extent that $P(q|p)$ is high.[9] The second is the possible-world view advocated by Stalnaker and Lewis, according to which $p > q$ is true just in case q is true in the closest world, or worlds, in which p is true.[10] The third is the belief revision view elaborated by Gärdenfors and others. On this view, $p > q$ is acceptable relative to a belief state K if and only if $q \in f(K, p)$, where f is a function that takes belief states and sentences as arguments and yields revised belief states as values.[11]

The three accounts just mentioned invalidate P14 because they imply that both $p > q$ and $\neg p > q$ can hold, typically when q is very likely independently of p. The first contemplates cases in which $P(q|p)$ and $P(q|\neg p)$ are both high, the second contemplates cases in which q is true both in the closest worlds in which p is true and in the closest worlds in which p is false, and the third contemplates cases in which $q \in f(K, p)$ and $q \in f(K, \neg p)$. Consider for example the following conditionals:

(3) If the coin lands heads, Fido can bark

(4) If the coin lands tails, Fido can bark

On each of the three accounts considered, (3) and (4) turn out to be both acceptable, given that very likely Fido can bark regardless of the outcome of the coin toss.[12]

Independently of whether this result is desirable within an account of conditionals, it is certainly not desirable as part of an analysis of 'p is a reason for q', for the cases of the kind described are intuitively cases in which p does *not* support q. It would be implausible to say that the coin

---

[9] Adams [1].

[10] Stalnaker [31], Lewis [17].

[11] Gärdenfors [9], Levi [16], Arlo-Costa [3].

[12] Another account of conditionals that invalidates P14 is the trivalent theory developed in Egré et al. [7].

landing heads, or tails, provides a reason for thinking that Fido can bark. More generally, as long as we rule out the limiting situation in which q is necessary, it seems that $p \rightarrow q$ and $\neg p \rightarrow q$ cannot both hold. This suggests that none of the three accounts considered can provide a fully satisfactory interpretation of $\rightarrow$.

A fourth account, which fares better in this respect, is the difference-making view of conditionals suggested by Rott and embedded in Spohn's theory of reasons. This view can be phrased in terms of possible worlds as follows: $p > q$ is acceptable if and only if (i) q holds in the closest worlds in which p holds, and (ii) it is not the case that q holds in the closest worlds in which $\neg p$ holds. (i) is the Ramsey Test as understood by Stalnaker and Lewis, while (ii) is an additional condition designed to capture the intuition that p must be relevant for q. As is easy to see, (ii) rules out cases of irrelevance such as those considered. For example, (3) and (4) turn out to be unacceptable, because plausibly Fido can bark both in the closest worlds in which the coin lands tails and in those in which it lands heads. More generally, Rott's account makes $\neg p \rightarrow q$ incompatible with $p \rightarrow q$, so it validates P14.[13]

As the foregoing remarks suggest, Aristotle's Second Thesis provides an interesting measure of adequacy for an interpretation of in terms of support. This emerges with clarity if one compares P14 with P11. While all the four accounts of conditionals considered, as well as others, validate P11, only the fourth validates both P11 and P14. But as far as the logic of reasons is concerned, P14 is at least as important as P11, or so is reasonable to believe. Similar considerations hold for P15 and P16, which are stronger than P14 in the same way in which P12 and P14 are stronger than P11. The next section deals precisely with these stronger principles.

4. Boethius Thesis and Boethius Left Thesis

The third issue to be addressed concerns Boethius Thesis and Boethius Left Thesis, which involve embedded occurrences of $\rightarrow$. Quite often, theorists of conditionals do not deal with such constructions, either because they have qualms about their truth or assertibility conditions, or simply because they want to avoid technical complications. A well known example is Adams' probabilistic semantics, which is designed for conditional formulas whose

[13]Rott [26,27], Spohn [30]. This formulation of the difference-making view is in terms of possible worlds, although Rott and Spohn rely on the AGM formalism and ranking functions respectively.

constituents are strictly Boolean formulas. Similar limitations affect other accounts of conditionals, such as Rott's difference-making view. So, not all theories of conditionals have the syntactic resources to handle Boethius Thesis or Boethius Left Thesis.[14]

Yet it would definitely make sense to include these principles in a theory of reasons, because they display interesting connections between different orders of reasons, so to say. Consider P12. As we saw in Section 1, the assumption that Fido's being a dog is a reason for thinking that he can bark is itself a reason for thinking that it is not the case that Fido's being a dog is a reason for thinking that he cannot bark. If we call p a *first-order reason* for q when p is not itself a proposition about reasons, we call p a *second-order reason* for q when p is a proposition about first-order reasons, and so on, then P12 expresses a relation between different orders of reasons: assuming that p is first-order, $p \to q$ is a second-order reason for $\neg(p \to \neg q)$, or equivalently, $p \to q$ is a second-order reason against $p \to \neg q$. Similar considerations hold for P13, P15, and P16.

One remarkable fact about the relations displayed by these principles is that they show some characteristic ways in which first-order reasons may act as defeaters. The discussions on the non-monotonicity of conditionals mostly focus on *rebutting* defeaters: a defeater r for $p \to q$ is rebutting if it provides a reason against q, thus preventing $p \wedge r \to q$ from being acceptable. For example, Fido's being mute is a reason against the conclusion that Fido can bark, and this is why the conjunction of Fido being a dog and his being mute does not support that conclusion. However, there is another kind of defeaters which is no less important for the purposes of a theory of reasons, namely, *undercutting* defeaters: a defeater r for $p \to q$ is undercutting if it questions the connection between p and q, thus providing a reason against $p \to q$.[15] Boethius Thesis and Boethius Left Thesis may be regarded as higher-order principles about undercutting defeaters, where r is itself a proposition of the form $p \to q$.

[14] An interesting exception is the trivalent theory mentioned in footnote 11. This theory allows embeddings, however it invalidates Boethius Left Thesis.

[15] The distinction between rebutting and undercutting defeaters goes back to Pollock [21], pp. 73–74.

5. The Evidential Interpretation

As it emerges from Sections 2–4, there seems to be no clear answer to the question of how $\rightarrow$ is to be interpreted in order to validate P9–P16. Although some extant account of conditionals validate P9–P11, the same does not hold for P12–P16. Now it will be shown that there is at least one coherent analysis of 'p is a reason for q' that yields the logical properties desired: it is the analysis articulated by Crupi and Iacona on the basis of their evidential account of conditionals. According to this analysis—that I will call *evidential interpretation*—p is a reason for q when a suitably defined relation of incompatibility holds between p and $\neg q$.[16]

The idea that a conditional is true when its antecedent is incompatible with the negation of its consequent goes back to Chrysippus. Sextus Empiricus reports Chrysippus' view as follows:

> Those who introduce connectedness say that a conditional is sound when the opposite of its consequent conflicts with its antecedent.[17]

This notion of incompatibility is inextricably tied to the idea of connexivity. As a matter of fact, when McCall introduced the label "connexive implication", he did it precisely by making reference to Sextus Empiricus's report of Chrysippus view.[18]

The incompatibility condition suggested by Crupi and Iacona is spelled out in terms of possible worlds as follows:

Definition 1. p and $\neg q$ are incompatible iff for every world where p is true and q is false,

(a) p and q have the same value in some of the closest worlds;

(b) in the closest worlds in which p is true, q is also true;

(c) in the closest worlds in which $\neg q$ is true, $\neg p$ is also true.

To explain this definition, let us consider (a)–(c) one by one. (a) requires that p and $\neg q$ have different values at least in some of the closest worlds,

---

[16] Crupi and Iacona [6] presents the theory of reasons. The account of conditionals is developed in Crupi and Iacona [4] and in Raidl et al. [24].

[17] Sextus Empiricus, *Outlines of Scepticism*, II, 111, edited and translated by J. Annas and J. Barnes, in Sextus Empiricus [28], p. 96. The attribution of this view Chrysippus is based on further sources, such as Cicero *De Fato*, 12, and Diogenes Laertius, *Lives of Eminent Philosophers*, vii, 73.

[18] McCall [18], p. 151.

which may be regarded as a minimal condition for their incompatibility. (b) is the Ramsey Test as understood by Stalnaker and Lewis. This condition implies that if p is true, $\neg q$ cannot easily be true. Note that, given (b), the only interesting case ruled out by (a) is that in which p is false and q is true in all the closest worlds. So, (a) prevents the incompatibility condition from obtaining when (b) and (c) are satisfied only because p is very unlikely and q is very likely for independent reasons. Finally, (c) reverses the Ramsey Test, as it requires that the closest worlds in which $\neg q$ is true make p false: if $\neg q$ is true, p cannot easily be true.[19]

To say that (a)–(c) are jointly satisfied is to say that the combination of p and $\neg q$ is a remote possibility. So the incompatibility condition stated in Definition 1 may be phrased as follows: if there are worlds in which p and $\neg q$ are both true, such worlds are comparatively remote. Note that when there are no worlds in which p is true and q is false, the conditional is vacuously true. In this case p and $\neg q$ are absolutely incompatible, as it were. Instead, when there are worlds in p is true and q is false, the satisfaction of (a)–(c) ensures that p and $\neg q$ are relatively incompatible, as it were. Absolute incompatibility between p and $\neg q$ amounts to p being a conclusive reason for q, while relative incompatibility between p and $\neg q$ amounts to p being a non-conclusive reason for q.

To see how the evidential interpretation differs from Rott's difference-making account of conditionals, let us compare (c) with condition (ii) of that account. Like (ii), (c) yields the desired result in cases of irrelevance such as (3) and (4). (3) fails because it is not the case that the coin does not land heads in the closest possible worlds in which Fido does not bark, and (4) fails for a similar reason. Nonetheless, (c) does not entail difference-making. As far as Definition 1 is concerned, p can be incompatible with $\neg q$ even though it does not make a difference for q in Rott's sense, and arguably this is a virtue of the evidential interpretation. Consider the following conditional about a series of coin tosses:

(5) If there are at least 3 heads in the first 10 tosses, there are at least 4 heads in the first 20 tosses

Since the closest worlds in which there are less than 3 heads in the first 10 tosses are still worlds in which there are at least 4 heads in the first 20 tosses, (ii) is not satisfied, hence Rott's account predicts that (5) is unacceptable,

[19] The formulation just provided differs from the one given in [6] as it includes (a) in addition to (b) and (c), see [5] for further details. Nothing important will depend on this difference, though, given that (a) is not necessary for the proofs that follow.

in spite of the fact that there is an obvious relation of support between its antecedent and its consequent. The incompatibility condition stated in Definition 1, by contrast, is satisfied because (c) holds in addition to (a) and (b): plausibly, the closest worlds in which there are less than 4 heads in the first 20 tosses are worlds in which there are less than 3 heads in the first 10 tosses.

All things considered, the evidential interpretation provides a fairly close approximation to the notion of relevance typically involved in the ordinary use of sentences about reasons. This is not quite the same thing as to say that it matches perfectly well our intuitions about relevance. For example, it might sill be contended that in (1) and (2), as well as in similar examples of conditionals with impossible antecedent or necessary consequent, the antecedent is intuitively not relevant for the consequent, in spite of the fact that the incompatibility condition stated in Definition 1 is vacuously satisfied. Note, however, that the same objection would apply to any account of reasons that grants the classical understanding of conclusive inference as necessary truth preservation, in line with Supraclassicality, so it would not specifically concern the evidential interpretation. Moreover, and more importantly, it is reasonable to expect that no formal analysis of sentences about reasons can match our intuitions about relevance perfectly well, so it is an open question whether such an analysis can coherently disqualify sentences such as (1) and (2) without yielding consequences that are at least as counterintuitive.

6. Definitions

Now it will be shown how the evidential interpretation can be defined in a proper formal framework. Let L be a language whose alphabet is constituted by a set P of sentence letters $p, q, r, ...$, the connectives $\neg, \supset, \Box, \rightarrow$, and the brackets $(,)$. The formulas of L are defined as follows: the sentence letters are atomic formulas; if α is a formula, then $\neg\alpha$ and $\Box\alpha$ are formulas; if α and β are formulas, then $\alpha \supset \beta$ and $\alpha \rightarrow \beta$ are formulas. The connectives $\wedge, \vee, \Diamond$ are definable in terms of $\neg, \supset, \Box$, as usual.

L differs from the language adopted by Crupi and Iacona in one important respect, namely, that it allows embedding for $\rightarrow$. Although Crupi and Iacona do not consider complex formulas containing multiple occurrences of $\rightarrow$, nothing prevents their language from being extended so as to include such constructions.

The semantics of L is a preferential semantics, in that it is based on models defined in the following way:

Definition 2. A model M is a quadruple $\langle W, A, \prec, V\rangle$, where

- W is a nonempty set
- A assigns to each $x \in W$ a subset W_x of W;
- $\prec$ assigns to each $x \in W$ an irreflexive and transitive relation $\prec_x$ on W_x
- V assigns to each $x \in W$ and $\alpha \in P$ one element of $\{0, 1\}$.

W is a set of worlds. A is a function that determines a sphere of accessibility W_x for each $x \in W$. $\prec$ is a function that assigns to each $x \in W$ an order of preference. To say that $y \prec_x z$ is to say that y is preferred to z relative to x, or equivalently that y is strictly closer than z relative to x. This order implies that, for any $S \subseteq W$, some worlds are *x-minimal* with respect to S:

Definition 3. $Min_x(S)$ is the set of all $y \in S \cap W_x$ such that there is no $z \in S \cap W_x$ such that $z \prec_x y$.

For the sake of simplicity, I will write $Min_x(\alpha)$ for $Min_x(S)$ when S is $||\alpha||$, the set of worlds in which α is true. When S is W_x itself, I will simply write Min_x.

Definitions 2 and 3 are very general, in that they apply to a wide variety of models. Here, however, I will restrict considerations to models that satisfy the following conditions, which hold in the semantics adopted by Crupi and Iacona:

(Uni) $W_x = W$

(LA) If $||\alpha|| \cap W_x \neq \emptyset$, then $Min_x(\alpha) \neq \emptyset$.

(Uni) is *Universality*: every world is accessible from any world. (LA) is the *Limit Assumption*, which ensures that we always reach x-minimality for every α, ruling out infinitely descending chains.

The truth of a formula in a world x in a model is defined as follows:

Definition 4.

1 $[\alpha]_x = 1$ iff $V(x, \alpha) = 1$ for every $\alpha \in P$;

2 $[\neg\alpha]_x = 1$ iff $[\alpha]_x = 0$;

3 $[\alpha \supset \beta]_x = 1$ iff $[\alpha]_x = 0$ or $[\beta]_x = 1$;

4 $[\Box\alpha]_x = 1$ iff $[\alpha]_y = 1$ for all $y \in W_x$;

5 $[\alpha \to \beta]_x = 1$ iff for every $y \in W_x$, if $[\alpha]_y = 1$ and $[\beta]_y = 0$, then

(a) some $z \in Min_x$ is such that $[\alpha]_z = [\beta]_z$;
(b) for every $z \in Min_x(\alpha)$, $[\beta]_z = 1$;
(c) for every $z \in Min_x(\neg\beta)$, $[\neg\alpha]_z = 1$.

Clauses 1–4 are standard. Clause 5 is the crucial one, as it specifies the meaning of $\to$ in accordance with Definition 1: $p \to q$ is true just in case p and $\neg q$ are incompatible in the sense defined.

Logical consequence, indicated by the symbol $\models$, is defined in the usual way as preservation of truth in every world in every model:

Definition 5. $\Gamma \models \alpha$ iff for any model, there is no x such that $[\beta]_x = 1$ for every $\beta \in \Gamma$ and $[\alpha]_x = 0$.

7. Proof of P9–P16

Now it will be shown that the semantics just outlined validates P9–P16.

Fact 1. $\models \Diamond\alpha \supset \neg(\alpha \to \neg\alpha)$

Proof. Assume that $[\Diamond\alpha]_x = 1$, that is, $[\alpha]_y = 1$ for some $y \in W_x$. In this case, $[\alpha \to \neg\alpha]_x = 0$ because $[\alpha]_y = 1$ and $[\neg\alpha]_y = 0$, but conditions (a)–(c) of clause 5 of Definition 4 cannot be satisfied. Therefore, $[\neg(\alpha \to \neg\alpha)]_x = 1$. ■

Fact 2. $\models \Diamond\alpha \supset \neg(\neg\alpha \to \alpha)$

Proof. Like the proof of Fact 1, replacing α with $\neg\alpha$. ■

Fact 3. $\models \Diamond\alpha \supset \neg((\alpha \to \beta) \wedge (\alpha \to \neg\beta))$

Proof. Assume that $[\Diamond\alpha]_x = 1$, that is, $[\alpha]_y = 1$ for some $y \in W_x$. Then either $[\alpha]_y = 1$ and $[\beta]_y = 0$, or $[\alpha]_y = 1$ and $[\neg\beta]_y = 0$, which entails that, for at least one of the formulas $\alpha \to \beta$ and $\alpha \to \neg\beta$, clause 5 of Definition 4 is not vacuously satisfied. So, three cases are to be considered.
Case 1. Clause 5 of definition 3 is vacuously satisfied for neither of the two formulas. In this case we get that $[\alpha \to \beta]_x = 0$ or $[\alpha \to \neg\beta]_x = 0$ because (b) cannot hold for both formulas: for any $z \in Min_x(\alpha)$, it cannot be the case that $[\beta]_z = 1$ and $[\neg\beta]_z = 1$.
Case 2. Clause 5 of definition 3 is vacuously satisfied only for $\alpha \to \neg\beta$. In this case $[\alpha \to \beta]_x = 0$ because there is no $z \in W_x$ such that $[\alpha]_z = [\beta]_z = 1$, so (b) does not hold.

Case 3. Clause 5 of definition 3 is vacuously satisfied only for $\alpha \to \beta$. In this case $[\alpha \to \neg\beta]_x = 0$ because there is no $z \in W_x$ such that $[\alpha]_z = [\neg\beta]_z = 1$, so (b) does not hold.
From cases 1–3 we get that $[(\alpha \to \beta) \wedge (\alpha \to \neg\beta)]_x = 0$. Therefore, $[\neg((\alpha \to \beta) \wedge (\alpha \to \neg\beta))]_x = 1$. ■

Fact 4. $\models \Diamond\alpha \supset ((\alpha \to \beta) \to \neg(\alpha \to \neg\beta))$

Proof. Assume that $[\Diamond\alpha]_x = 1$, that is, $[\alpha]_y = 1$ for some $y \in W_x$. As the proof of Fact 3 shows, for any $z \in W_x$, $[(\alpha \to \beta) \wedge (\alpha \to \neg\beta)]_z = 0$. This means that there is no $z \in W_x$ such that $[\alpha \to \beta]_z = 1$ and $[\neg(\alpha \to \neg\beta)]_z = 0$. But then $[(\alpha \to \beta) \to \neg(\alpha \to \neg\beta)]_x = 1$ because clause 5 of Definition 4 is vacuously satisfied. ■

Fact 5. $\models \Diamond\alpha \supset ((\alpha \to \neg\beta) \to \neg(\alpha \to \beta))$

Proof. Like the proof of Fact 4, replacing β with $\neg\beta$. ■

Fact 6. $\models \Diamond\neg\beta \supset \neg((\alpha \to \beta) \wedge (\neg\alpha \to \beta))$

Proof. Assume that $[\Diamond\neg\beta]_x = 1$, that is, $[\beta]_y = 0$ for some $y \in W_x$. Then either $[\alpha]_y = 1$ and $[\beta]_y = 0$, or $[\neg\alpha]_y = 1$ and $[\beta]_y = 0$, which entails that, for at least one of the formulas $\alpha \to \beta$ and $\neg\alpha \to \beta$, clause 5 of Definition 4 is not vacuously satisfied. So, three cases are to be considered.
Case 1. Clause 5 of Definition 4 is vacuously satisfied for neither of the two formulas. In this case we get that $[\alpha \to \beta]_x = 0$ or $[\neg\alpha \to \beta]_x = 0$ because (c) cannot hold for both formulas: for any $z \in Min_x(\neg\beta)$, it cannot be the case that $[\neg\alpha]_z = 1$ and $[\neg\neg\alpha]_z = 1$.
Case 2. Clause 5 of definition 3 is vacuously satisfied only for $\neg\alpha \to \beta$. In this case $[\alpha \to \beta]_x = 0$ because there is no $z \in W_x$ such that $[\neg\alpha]_z = [\neg\beta]_z = 1$, so (c) does not hold.
Case 3. Clause 5 of definition 3 is vacuously satisfied only for $\alpha \to \beta$. In this case $[\neg\alpha \to \beta]_x = 0$ because there is no $z \in W_x$ such that $[\neg\neg\alpha]_z = [\neg\beta]_z = 1$, so (c) does not hold.
From cases 1–3 we get that $[(\alpha \to \beta) \wedge (\neg\alpha \to \beta)]_x = 0$. Therefore, $[\neg((\alpha \to \beta) \wedge (\neg\alpha \to \beta))]_x = 1$. ■

Fact 7. $\models \Diamond\neg\beta \supset ((\alpha \to \beta) \to \neg(\neg\alpha \to \beta))$

Proof. Assume that $[\Diamond\neg\beta]_x = 1$, that is, $[\beta]_y = 0$ for some $y \in W_x$. As the proof of Fact 6 shows, for any $z \in W_x$, $[(\alpha \to \beta) \wedge (\neg\alpha \to \beta)]_z = 0$. This means that there is no $z \in W_x$ such that $[\alpha \to \beta]_z = 1$ and $[\neg(\neg\alpha \to \beta)]_z = 0$. But then $[(\alpha \to \beta) \to \neg(\neg\alpha \to \beta)]_x = 1$ because clause 5 of Definition 4 is vacuously satisfied. ■

Fact 8. $\models \Diamond\neg\beta \supset ((\neg\alpha \to \beta) \to \neg(\alpha \to \beta))$

Proof. Like the proof of Fact 7, replacing α with $\neg\alpha$. ∎

8. Contraposition

Sections 1–4 suggest that there is an interesting symmetry between P11–P13 and P14–P16, as it is plausible to expect that both P11–P13 and P14–P16 hold. As we have seen, the evidential interpretation preserves this symmetry, as it entails Facts 3–8. So it significantly differs from the four accounts of conditionals discussed above.

One straightforward way to explain this result is to point out a distinctive property of the account of reasons outlined, namely, that it validates *Contraposition*:

Fact 9. $\alpha \to \beta \models \neg\beta \to \neg\alpha$

Proof. Assume that $[\alpha \to \beta]_x = 1$. Then either clause 5 is vacuously satisfied or it isn't. If it is, there is no $y \in W_x$ such that $[\neg\beta]_y = 1$ and $[\neg\alpha]_y = 0$. Therefore, $[\neg\beta \to \neg\alpha]_x = 1$ for the same reason. If it isn't, then (a) some $z \in Min_x$ is such that $[\neg\beta]_z = [\neg\alpha]_z$, (b) for every $z \in Min_x(\neg\beta)$, $[\neg\alpha]_z = 1$, and (c) for every $z \in Min_x(\neg\neg\alpha)$, $[\neg\neg\beta]_z = 1$. Therefore, again, $[\neg\beta \to \neg\alpha]_x = 1$. ∎

Since any formula α is logically equivalent to $\neg\neg\alpha$ by Definition 4, from Fact 9 we get that $\neg\beta \to \neg\alpha \models \alpha \to \beta$, hence that $\alpha \to \beta$ is logically equivalent to $\neg\beta \to \neg\alpha$. Accordingly, if one assumes P11, one can obtain P14 by simple substitution of logical equivalents, and the other way round. The same goes for P12 and P15, and for P13 and P16. In other words, Contraposition makes each negation-right principle equivalent to a corresponding negation-left principle. So it may be regarded as a more fundamental fact which constitutes the source of the symmetry.

9. A Final Remark

The theory of reasons presented in Sections 5–8 rests on the idea that p is a reason for q if and only if the conditional that has p as antecedent and q as consequent is true. On the assumption that the inference from p to q is justified just in case p is a reason for q—the first of the two assumptions in the argument for Restricted Classicality outlined in Section 2—this idea boils down to the equivalence that I have called *Stoic Thesis*: the argument from p to q is valid if and only if the conditional that has p as antecedent and q as consequent is true. As I have argued in a previous work, insofar as validity is construed in a fairly broad sense, which is not limited to deductive reasoning and includes defeasible inference, the Stoic Thesis is appreciably more credible than is usually believed.[20]

The Stoic Thesis provides a straightforward answer to our initial question concerning the relation between conditionals and sentences about reasons: sentences about reasons are nothing but conditionals from the logical point of view. So, the apparent logical similarity between the two kinds of sentences is explained simply by saying that the logic of $\rightarrow$ and the logic of $>$ are one and the same logic.

However, it is important to understand that the main points made in Sections 5–8 do not essentially depend on the Stoic Thesis, as the evidential interpretation could equally be appreciated without postulating an equivalence between $\rightarrow$ and $>$. One might be unwilling to endorse the account of conditionals advocated by Crupi and Iacona, and still adopt the definition of $\rightarrow$ suggested here as an adequate analysis of 'p is a reason for q'. In that case one could consistently maintain that some of the facts proved above about $\rightarrow$ do not hold $>$.

In order to properly question the adequacy or usefulness of the formal semantics offered here, one would have to provide independent arguments that specifically concern the logical properties of sentences about reasons. One could either contend that these sentences do not behave in the way predicted as to the connexive principles discussed, or offer some alternative interpretation of $\rightarrow$ that yields similar results. In any case, the three issues raised in Sections 2–4 would still stand. In particular, Aristotle's Second thesis, Boethius Thesis, and Boethius Left Thesis pose an interesting challenge to any theory of reasons that aims at a high level of generality.

Funding Open access funding provided by Università degli Studi di Torino within the CRUI-CARE Agreement.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use

[20] Iacona [13].

is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Adams, E. W., The Logic of Conditionals, *Inquiry* 8:166–197, 1965.
[2] Angell, R. B., A propositional logic with subjunctive conditionals, *Journal of Symbolic Logic* 27:327–343, 1962.
[3] Arlo-Costa, H., Epistemic Conditionals, Snakes and Stars, in G. Crocco, L. Farinas del Cerro, and A. Herzig, (eds.), *Conditionals, from Philosophy to Computer Science*, Oxford University Press, 1995, pp. 193–239.
[4] Crupi, V., and A. Iacona, The Evidential Conditional, *Erkenntnis*, 2020.
[5] Crupi, V., and A. Iacona, Conditionals: Inferentialism Explicated, *manuscript*, 2022a.
[6] Crupi, V., and A. Iacona, Outline of a Theory of Reasons, *Philosophical Quarterly* forthcoming, 2022b.
[7] Egré, P., L. Rossi, and J. Sprenger, De Finettian Logics of Indicative Conditionals Part I, *Journal of Philosophical Logic* 50:187–213, 2021.
[8] Francez, N., Natural deduction for two connexive logics, *IFCoLog Journal of Logics and their Applications* 3:479–504, 2016.
[9] Gärdenfors, P., *Knowledge in Flux*, MIT Press, 1988.
[10] Horty, J., Reasons as defaults, *Philosopher's Imprint* 7:1–28, 2007.
[11] Horty, J., *Reasons as defaults*, Oxford University Press, 2012.
[12] Iacona, A., Strictness and Connexivity, *Inquiry* 64:1024–1037, 2012.
[13] Iacona, A., Valid Arguments as True Conditionals, *Mind*, 2023. https://academic.oup.com/mind/advance-article-abstract/doi/10.1093/mind/fzac026/6991807.
[14] Kapsner, A., Humble Connexivity, *Logic and Logical Philosophy* 28:513–536, 2019.
[15] Lenzen, W., Rewriting the History of Connexive Logic, *Journal of Philosophical Logic*, online, 2022.
[16] Levi, I., Iteration of Conditionals and the Ramsey Test, *Synthese* 76:49–81, 1988.
[17] Lewis, D., Counterfactuals. Blackwell, 1973.
[18] McCall, S., A New Variety of Implication, *Journal of Symbolic Logic* 29:151–152, 1964.
[19] McCall, S., Connexive Implication, in A.R. Anderson, and N.D. Belnap, (eds.), *Entailment. The Logic of Relevance and Necessity*, Princeton University Press, 1975, pp. 434–446.
[20] McCall, S., A history of connexivity, in D.M. Gabbay, F. Pelletier, and J. Woods, (eds.), *Handbook of the History of Logic, vol. 11, Logic: A History of its Central Concepts*, Elsevier, 2012, pp. 415–449.
[21] Pollock, J. L., The Structure of Epistemic Justification, *Studies in the Theory of Knowledge 4, American Philosophical Quarterly Monograph Series*, 2012, pp. 62–78.

[22] Pollock, J. L., Defeasible Reasoning, *Cognitive Science* 11:481–518, 1987.

[23] Pollock, J. L., *Cognitive Carpentry: A Blueprint for How to Build a Person*, MIT Press, 1995.

[24] Raidl, E., A. Iacona, and V. Crupi, The Logic of the Evidential Conditional, *Review of Symbolic Logic*, 2021.

[25] Reiter, R., A logic for default reasoning, *Artificial Intelligence* 13:81–132, 1980.

[26] Rott, H., Ifs, though, and because, *Erkenntnis* 25:345–370, 1986.

[27] Rott, H., Difference-making conditionals and the relevant Ramsey test, *Review of Symbolic Logic* 15:133–164, 2022.

[28] Sextus, E., Outlines of Scepticism, in J. Annas, and J. Barnes, (eds.), *Cambridge Texts in the History of Philosophy*, 2nd ed., Cambridge University Press, 2000.

[29] Spohn, W., *The Laws of Belief*, Oxford University Press, 2012.

[30] Spohn, W., Conditionals: a unifying ranking-theoretic perspective, *Philosophers' Imprint* 15:1–30, 2015.

[31] Stalnaker, R., A Theory of Conditionals, in N. Rescher, (ed.), *Studies in Logical Theory*, Blackwell, 1968, pp. 98–112.

[32] Unterhuber, M., Beyond System P. Hilbert-style convergence results for conditional logics with a connexive twist, *IFCOLog Journal of Logics and their Applications* 3:377–412, 2016.

[33] Wansing, H., Connexive Logic, in E. Zalta (eds.), *Stanford Encyclopedia of Philosophy*, Connexive Logic, 2020. http://plato.stanford.edu/archives/spr2019/entries/logicconnexive/.

A. Iacona
University of Turin
Turin
Italy
andrea.iacona@unito.it