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From Semantic Games to Provability: The Case of Gödel Logic

Abstract. We present a semantic game for Gödel logic and its extensions, where the players' interaction stepwise reduces arbitrary claims about the relative order of truth degrees of complex formulas to atomic ones. The paper builds on a previously developed game for Gödel logic with projection operator in Fermüller *et al.* (in: M.-J. Lesot, S. Vieira, M.Z. Reformat, J.P. Carvalho, A. Wilbik, B. Bouchon-Meunier, and R.R. Yager, (eds.), *Information processing and management of uncertainty in knowledge-based systems*, Springer, Cham, 2020, pp. 257–270). This game is extended to cover Gödel logic with involutive negations and constants, and then lifted to a provability game using the concept of disjunctive strategies. Winning strategies in the provability game, with and without constants and involutive negations, turn out to correspond to analytic proofs in a version of SeqGZL (A. Ciabattoni, and T. Vetterlein, *Fuzzy Sets and Systems* 161(14):1941–1958, 2010) and in a sequent-of-relations calculus (M. Baaz, and Ch.G. Fermüller, in: N.V. Murray, (ed.), *Automated reasoning with analytic tableaux and related methods*, Springer, Berlin, 1999, pp. 36–51) respectively.

Keywords: Gödel logic, Fuzzy logic, Semantic games, Provability game, Analytic calculus.

1. Introduction

The use of games in logic has a long and varied tradition, wherein the types of games considered have been as manifold as the logics to which they have been applied. This paper focuses on two major games types, *semantic games* and *provability games*, and the relation between them. Semantic games are defined with respect to concrete interpretations of the language in question and seek to determine whether a formula is true under this interpretation. Provability games on the other hand refer to a higher level of abstraction in that they aim for a characterization of validity, i.e. truth of a formula under all possible interpretations. In both cases, game-semantic truth or validity is witnessed by the existence of a winning strategy in a carefully designed game.

The main idea of the paper is to start with a semantic game and subsequently lift it to a provability game using *disjunctive strategies*. So what is a disjunctive strategy? When we talk about an ordinary strategy for a

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player **P**, it comprises all possible moves by her rival **O** but only one choice of **P** at each choice point for **P**. When playing with disjunctive strategies, we keep all possible choices of **P** open. Intuitively, **P** then plays not on a single state but on a set of 'disjunctive states', each of which represents a possibility of how the game might have developed so far. This makes it possible for **P** to win games without knowing then interpretation of the atomic formulas beforehand. The crucial feature of disjunctive strategies is that they have a finite representation, even though they handle infinitely many interpretations at once. As it turns out, disjunctive *winning* strategies can be seen as proofs in an analytic calculus.

We apply the above method to *Gödel logic*, one of the most prominent fuzzy logics. The name 'fuzzy logic' encompasses a large family of logics developed for the task of reasoning under vague information. Fuzzy logics usually presuppose reasoning about the 'degrees of truth' rather than binary distinction between 'true' and 'false', and correspondingly their truth functions are often interpreted as functions over the unit interval $[0, 1]$ (where 0 stands for absolute falsity and 1 for absolute truth). As an example, in *Łukasiewicz logic* conjunction is modelled by the function $x *_{\mathrm{L}} y = \max(0, x + y - 1)$, whereas in Gödel logic it is $x *_G y = \min(x, y)$. It is a distinguishing feature of Gödel logic that every formula evaluates to either 0, 1, or to the value of one of the propositional variables occurring in it. Moreover, it is the only t-norm based fuzzy logic where the truth of a formula (i.e., whether it evaluates to 1) does not depend on the particular values in $[0, 1]$ that interpret the propositional variables, but only on their relative ordering. For these reasons, Gödel logic can rightfully be called a logic of order.

The present paper builds on the previously published [13], and also has a precursor in [15]. A similar result for Łukasiewicz logic has already been shown in [14]. Apart from many improvements in the definitions and proofs, this paper extends [13,15] by the treatment of constants and involutive negations.

We structure our results as follows. There are two major sections consisting of further subsections.

In Section 2, we introduce Gödel logic $\mathsf{G}^{\triangle}$ (G extended with the $\triangle$-operator) together with a truth degree comparison game for it, where a player **P** seeks to uphold, against attacks by opponent **O**, a claim of the form $F < G$ or $F \leq G$, expressing that the truth value of F is smaller (or equal) to that of G under a given interpretation. The interaction of **P** and **O** stepwise reduces the initial truth comparison claim to an atomic claim that can be immediately checked. Then we lift the game from truth degree comparison claims for concrete interpretations to the level of validity, i.e., to comparison claims that hold under every interpretation. Following the general clue given above, the key ingredient is the notion of disjunctive states, triggering disjunctive strategies. It turns out that disjunctive winning strategies for **P** correspond to proofs in an analytic proof system, called sequents-of-relations calculus, introduced in [6].

In Section 3, we extend $\mathsf{G}^{\triangle}$ with an involutive negation and constants. We discuss the general properties of involutions pertinent to our game approach and add corresponding rules to the game. Lifting the game rules to disjunctive ones, we obtain a provability game for $\mathsf{G}_*^{\triangle}(S)$. In Section 3.3, a purely syntactic criterion for winning states is introduced and relations between Gödel logics with different involutive negations and sets of constants are discussed. These results are later used in Section 3.4 to prove polynomial checkability of final disjunctive states of a game. Thus, the provability game with disjunctive rules can be interpreted as an analytic calculus, which turns out to be quite close to an existing calculus for Gödel logic with involutive negation and constants proposed in [10].

We conclude with a brief summary of our results, followed by suggestions for future research in this area.

2. A Truth Degree Comparison Game for Gödel Logic

2.1. Gödel Logic with Δ as a Logic of Order

Propositional Gödel logic can be presented in two ways:

1. As the extension of propositional intuitionistic logic by the *linearity axiom* $(A \to B) \vee (B \to A)$, and correspondingly as the logic of linearly ordered Kripke frames;
2. As one of the basic t-norm fuzzy logics with truth values in the interval $[0, 1]$.

Because of this dual perspective, Gödel logic is also called *Intuitionistic Fuzzy Logic* in the literature (e.g., [8]). Intuitionistic Fuzzy Logic, however, was developed by Gaisi Takeuti together with Satoko Titani in [20] completely independently of the literature on Gödel logic. Gödel logic appeared for the first time implicitly in an article by Kurt Gödel [16] where he proved that intuitionistic logic is not a finite-valued logic. It was axiomatized and further investigated by Michael Dummett [11]. As a t-norm based fuzzy logic, it is characterised by taking the minimum

$$\|A \wedge B\|_{\mathcal{J}} = \min(\|A\|_{\mathcal{J}}, \|B\|_{\mathcal{J}})$$

as t-norm modelling conjunction $\wedge$. The truth functions for disjunction $\vee$ and $\rightarrow$ (which is the residuum of min) are:

$$\|A \vee B\|_{\mathcal{J}} = \max(\|A\|_{\mathcal{J}}, \|B\|_{\mathcal{J}})$$

$$\|A \rightarrow B\|_{\mathcal{J}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{J}} \leq \|B\|_{\mathcal{J}} \\ \|B\|_{\mathcal{J}} & \text{otherwise.} \end{cases}$$

These truth functions extend any *interpretation*, i.e., any assignment $\mathcal{J}$ of *truth values* in $[0,1]$ to propositional variables, to compound formulas. We include the propositional constants $\bot$ and $\top$ in G, interpreted by $\|\bot\|_{\mathcal{J}} = 0$ and $\|\top\|_{\mathcal{J}} = 1$. Negation in G is a defined connective, given by $\neg A = A \rightarrow \bot$. We moreover extend G to $\mathsf{G}^{\triangle}$ by including the following *projection operator* [2]:

$$\|\triangle A\|_{\mathcal{J}} = \begin{cases} 1 & \text{if } \|A\|_{\mathcal{J}} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The *atomic formulas* of $\mathsf{G}^{\triangle}$ are the propositional variables and the propositional constants. The set of all $[0,1]$-valued interpretations is denoted $\mathbf{Int}^{[0,1]}$. An interpretation $\mathcal{J} \in \mathbf{Int}^{[0,1]}$ *satisfies* a formula F (written $\mathcal{J} \models F$) if $\|F\|_{\mathcal{J}} = 1$. F is *valid* if all interpretations satisfy F.

To a formula A of $\mathsf{G}^{\triangle}$ in n variables we can associate its truth function $f_A : [0,1]^n \rightarrow [0,1]$, which takes as input a tuple $(a_1, \ldots, a_n)$ and outputs $\|A\|_{\mathcal{J}}$ where $\mathcal{J}$ is any interpretation mapping the i-th variable in A to a_i. The following two features are characteristic of the truth functions of Gödel logic:

1. $f_A(a_1, \ldots, a_n) \in \{0, 1, a_1, \ldots, a_n\}$;
2. Whether $f_A(a_1, \ldots, a_n) = b$ holds for some $b \in \{0, 1, a_1, \ldots, a_n\}$ depends only on the ordering of $0, 1, a_1, \ldots, a_n$, but not on their exact values.

Call *order-projective* any function $f : [0,1]^n \rightarrow [0,1]$ satisfying (1) and (2) above. Then one can prove the following:

THEOREM 1. (functional completeness of $\mathsf{G}^{\triangle}$) *A function $f : [0,1]^n \rightarrow [0,1]$ is order-projective iff it is the truth function of some $\mathsf{G}^{\triangle}$-formula.*

PROOF. See [7]. ■

We want to develop a game-theoretic characterization telling us when a formula F is satisfied by an interpretation $\mathcal{J}$, i.e. when it takes value 1. For this it will be useful to look at the more general problem of comparing the truth value of two formulas. Below we focus on *truth degree comparison* *claims*, or just *claims*, of the form $F \leq G$ or $F < G$, where F and G are $\mathsf{G}^{\triangle}$-formulas. Such a claim is called *atomic* if both F and G are atomic $\mathsf{G}^{\triangle}$-formulas. An interpretation $\mathcal{J}$ *satisfies* a claim if $\|F\|_{\mathcal{J}} \leq \|G\|_{\mathcal{J}}$ or $\|F\|_{\mathcal{J}} < \|G\|_{\mathcal{J}}$, respectively. In particular, $\mathcal{J}$ satisfies a formula F iff $\mathcal{J}$ satisfies the claim $\top \leq F$.

We introduce a semantic game for the stepwise reduction of arbitrary truth degree comparison claims to atomic ones. Game states consist of truth degree comparison claims $F \lhd G$, where $\lhd$ is either $\leq$ or $<$. Furthermore each non-atomic claim carries a *marking* which points to a non-atomic formula in the claim (either F or G). Given an interpretation $\mathcal{J}$, at any state $F \lhd G$ player **P** seeks to defend and **O** to refute the claim that $\mathcal{J}$ satisfies $F \lhd G$.

At each non-atomic state of the game, **P** and **O** make moves according to the rules below resulting in a successor claim where the marked formula has been decomposed. If the successor claim is not atomic, then in an implicit step, a *regulation function* ρ marks one of the non-atomic formulas in the successor claim. The resulting claim is the *successor state* of the game.

The precise definition of a regulation function will turn out to be inessential for the theory we are interested in, but for the sake of definedness, we can think of it as a function ρ from the set of all claims to a 2-element set $\{l, r\}$, where the value $\rho(F \lhd G)$ indicates whether the *l*eft or *r*ight formula of the claim is to be marked.

The game now goes on until an atomic state $F \lhd G$ is reached, where **P** wins (and **O** loses) if $\|F\|_{\mathcal{J}} \lhd \|G\|_{\mathcal{J}}$.

To motivate the choice of the game rules, consider the example claim $A \to B \leq C$. For any interpretation $\mathcal{J}$, an easy calculation shows us the following equivalence:

$$\mathcal{J} \models A \to B \leq C \,\text{iff}\, \mathcal{J} \models \top \leq C \quad \textit{or} \quad (\mathcal{J} \models A \leq B \,\text{and}\, \mathcal{J} \models C < \top)$$

This can be read as a reduction of a claim to simpler (the outermost $\to$ disappears) subclaims, which are connected with ‘and’ and ‘or’. Interpreting the ‘or’ as a choice of **P** and the ‘and’ as a choice of **O**, we arrive at the following game rule:

P chooses between:

(1): the game continues with $\top \leq C$;

(2): **O** chooses whether the game continues with $B < A$ or with $B \leq C$.

Figure 1 presents similar game rules for all other game states. Their soundness will be proved in Proposition 3. For each connective there are four rules, depending on whether the connective appears in a marked formula on

$A \wedge B \lhd C$**:** **P** chooses whether the game continues with $A \lhd C$ or with $B \lhd C$.

$C \lhd A \wedge B$**:** **O** chooses whether the game continues with $C \lhd A$ or with $C \lhd B$.

$A \vee B \lhd C$**:** **O** chooses whether the game continues with $A \lhd C$ or with $B \lhd C$.

$C \lhd A \vee B$**:** **P** chooses whether the game continues with $C \lhd A$ or with $C \lhd B$.

$A \to B \leq C$**:** **P** chooses between:
(1): the game continues with $\top \leq C$;
(2): **O** chooses whether the game continues with $B < A$ or with $B \leq C$.

$C \leq A \to B$**:** **P** chooses whether the game continues with $A \leq B$ or with $C \leq B$.

$A \to B < C$**:** **O** chooses whether the game continues with $B < A$ or with $B < C$.

$C < A \to B$**:** **P** chooses between
(1): the game continues with $C < B$;
(2): **O** chooses whether the game continues with $A \leq B$ or with $C < \top$.

$\triangle A \leq C$**:** **P** chooses whether to continue with $A < \top$ or with $\top \leq C$.

$C \leq \triangle A$**:** **P** chooses whether to continue with $\top \leq A$ or with $C \leq \bot$.

$\triangle A < C$**:** **O** chooses whether to continue with $A < \top$ or with $\bot < C$.

$C < \triangle A$**:** **O** chooses whether to continue with $\top \leq A$ or with $C < \top$.

Figure 1. Game rules for the truth degree comparison game

the left or on the right, and whether the truth degree comparison is strict or non-strict, i.e., of the form $F < G$ or $F \leq G$. Some of the rules can be represented in a uniform manner using $\lhd$ to stand for either $<$ or $\leq$ (consistently within the rule). In the following, the exhibited compound formula is the marked formula of the state.[1]

We denote by $\tau_\rho^{\mathcal{J}}[F \lhd G]$ the game which is based on the interpretation $\mathcal{J}$ and regulation function ρ, and which starts with the claim $F \lhd G$.

Inspecting the game rules in Figure 1, we observe that the successor claims always decrease in logical complexity (i.e., number of logical symbols). From this we immediately see that:

Lemma 2. (Termination) *The game $\tau_\rho^{\mathcal{J}}[F \lhd G]$ always terminates at an atomic state.*

Two alternative presentations of a game rule will be utilized below.

[1] This convention will be followed often throughout the article.

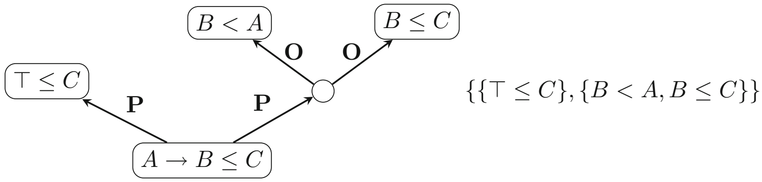


Figure 2. Decision tree and power set of a game rule

First, we can picture a game rule as a decision tree. E.g, the rule for the game state $A \to B \le C$ corresponds to the tree in Figure 2. The leaves of this tree, i.e., $\top \le C, B < A$ and $B \le C$, are the possible successor claims of $A \to B \le C$.

Second, every game rule can be represented as a set of **P**-*powers*.[2] A set X of claims is called a **P**-power of the non-atomic game state $F \lhd G$ if it is a subset-minimal set of claims such that in the game starting with $F \lhd G$, **P** can enforce that the successor claim is among the claims in X. We denote by $\mathtt{Pow}(F \lhd G)$ the collection of all such powers. For example, the state $A \to B \le C$ has the **P**-powers $\{\top \le C\}$ and $\{B < A, B \le C\}$. Similarly we have $\mathtt{Pow}(A \wedge B \lhd C) = \{\{A \lhd C\}, \{B \lhd C\}\}$ and $\mathtt{Pow}(C \lhd A \wedge B) = \{\{C \lhd A, C \lhd B\}\}$. For an atomic state $F \lhd G$, we formally set $\mathtt{Pow}(F \lhd G) = \{\{F \lhd G\}\}$.

The next Proposition is formulated in terms of **P**-powers. It provides the justification for the previously presented game rules of Figure 1. Put another way, the game rules are designed so that the following Proposition becomes true:

PROPOSITION 3. (Soundness of game rules) *For any game state $F \lhd G$ and any $\mathcal{J} \in \mathbf{Int}^{[0,1]}$, $\mathcal{J} \models F \lhd G$ iff for some $X \in \mathtt{Pow}(F \lhd G)$, $\mathcal{J}$ satisfies all claims in X.*

PROOF. For an atomic state $F \lhd G$, this holds by definition. For non-atomic $F \lhd G$, this is proved for all 12 types of game states separately. Consider for example the state $A \to B \le C$ and its **P**-power

$$\{\{\top \le C\}, \{B < A, B \le C\}\}.$$

Let $\mathcal{J} \in \mathbf{Int}^{[0,1]}$. Then either $\|A\|_{\mathcal{J}} \le \|B\|_{\mathcal{J}}$, in which case $\|A \to B\|_{\mathcal{J}} = 1$ and so $\mathcal{J}$ satisfies the claim $A \to B \le C$ iff $\mathcal{J}$ satisfies $\top \le C$. Or $\|A\|_{\mathcal{J}} > \|B\|_{\mathcal{J}}$: Then $\|A \to B\|_{\mathcal{J}} = \|B\|_{\mathcal{J}}$ and so $\mathcal{J}$ satisfies the claim $A \to B \le C$ iff $\mathcal{J}$ satisfies $B \le C$.

[2]This notion is similar to the general definition of a power in game theory, cf. [22].

We prove the equivalence for the other two examples given above. For the game state $A \wedge B \lhd C$ with $\texttt{Pow}(A \wedge B \lhd C) = \{\{A \lhd C\}, \{B \lhd C\}\}$ we observe that an interpretation $\mathcal{J}$ satisfies $A \wedge B \lhd C$ if and only if either $\|A\|_{\mathcal{J}} < \|B\|_{\mathcal{J}}$ and $\mathcal{J}$ satisfies $A \lhd C$, or $\|A\|_{\mathcal{J}} \geq \|B\|_{\mathcal{J}}$ and $\mathcal{J}$ satisfies $B \lhd C$.

For the game state $C \lhd A \wedge B$ with $\texttt{Pow}(C \lhd A \wedge B) = \{\{C \lhd A, C \lhd B\}\}$ we observe that an interpretation $\mathcal{J}$ satisfies $C \lhd A \wedge B$ if and only if $\mathcal{J}$ satisfies both $C \lhd A$ and $C \lhd B$.

The remaining 9 cases can be shown similarly. ■

We now need the notion of a winning strategy (henceforth often abbreviated 'ws') for **P**. This is exactly what one expects: A strategy telling **P** which moves to make (given the history of all previous moves) and always leading to a winning state. To have a formal notion, consider again the decision tree for a game state $F \lhd G$. Given the regulation ρ, we can further expand every non-atomic successor claim into a decision tree according to the game rules. Repeating this process, we eventually arrive (by Lemma 2) at a fully expanded tree which is called the *game tree* of $\tau_\rho^{\mathcal{J}}[F \lhd G]$; Its branches are the possible runs of the game. So the notation indicates that the initial claim is $F \lhd G$, and the regulation function and interpretation underlying the game are ρ and $\mathcal{J}$ respectively. Note that the choice of $\mathcal{J}$ does not change the game rules, but only the winning condition in the atomic game states. A winning strategy for **P** may then be represented as a subtree of the game tree, where every node has at most one outgoing edge labelled **P**, and all the leaves are winning states.

PROPOSITION 4. *For any interpretation $\mathcal{J}$ and regulation ρ, if* **P** *has a winning strategy in $\tau_\rho^{\mathcal{J}}[F \lhd G]$ then $\mathcal{J}$ satisfies $F \lhd G$.*

PROOF. Assume that **P** has a winning strategy in $\tau_\rho^{\mathcal{J}}[F \lhd G]$. If $F \lhd G$ is atomic, there are no further moves in the game. Consequently it must be the case that $F \lhd G$ is a winning state in $\tau_\rho^{\mathcal{J}}[F \lhd G]$, and by definition this means that $\mathcal{J}$ satisfies $F \lhd G$.

Assume now that $F \lhd G$ is not atomic, and let X be the set of successor claims of $F \lhd G$ in $\tau_\rho^{\mathcal{J}}[F \lhd G]$ which can possibly occur if **P** plays according to her winning strategy. Note that $X \in \texttt{Pow}(F \lhd G)$. In each $S \in X$ player **P** has a winning strategy, namely the restriction of her original winning strategy to the subgame starting with S. These 'substrategies', written in tree form, have lower depth than the original winning strategy. So we may assume by induction that $\mathcal{J}$ satisfies each $S \in X$. We conclude by Proposition 3 that $\mathcal{J}$ satisfies $F \lhd G$. ■

PROPOSITION 5. *If an interpretation* $\mathcal{J}$ *satisfies* $F \lhd G$ *then* **P** *has a winning strategy in* $\tau_\rho^{\mathcal{J}}[F \lhd G]$ *for any regulation* ρ.

PROOF. If $\mathcal{J}$ satisfies $F \lhd G$, then by Proposition 3 there is a power $X \in \mathtt{Pow}(F \lhd G)$ (where the marking in $F \lhd G$ is set according to ρ) such that $\mathcal{J}$ satisfies all claims in X. So **P** can enforce that the successor state of $F \lhd G$ in the game $\tau_\rho^{\mathcal{J}}[F \lhd G]$ is contained in X.

Repeating the same kind of reasoning, we see that **P** can always move ensuring that the resulting game state is satisfied by $\mathcal{J}$. By Lemma 2, eventually an atomic state ν will be reached using this strategy. Then ν must be a winning state in $\tau_\rho^{\mathcal{J}}[F \lhd G]$ for **P**. ∎

Combining Propositions 5 and 4, we obtain:

THEOREM 6. *An interpretation* $\mathcal{J}$ *satisfies* $F \lhd G$ *iff for some (or equivalently, any) regulation* ρ, **P** *has a winning strategy in* $\tau_\rho^{\mathcal{J}}[F \lhd G]$.

From this formulation it also becomes apparent that the regulation functions play only a minor role: Having a winning strategy for any regulation implies having one for all regulations.

REMARK 7. The validity of a formula F corresponds to the non-strict truth degree comparison claim $1 \leq F$, and the strict comparison $<$ comes into play only when such validity claims are decomposed. In fact, the only non-strict claim in the base language of Gödel logic which has strict successor claims in our game rules is $A \to B \leq C$. This begs the question if the appearance of $<$ is necessary, or if it could be avoided altogether by choosing a different game rule for $A \to B \leq C$.

As it turns out, the use of $<$ is necessary in the following sense: There is no game rule for $A \to B \leq C$ which has only non-strict successor claims such that the corresponding version of Proposition 3 holds. Indeed, consider the game state $x \to y \leq z$ where x, y, z are pairwise different variables, and let $\mathcal{P}$ be an arbitrary set of sets of *non-strict* atomic claims including x, y, z and the constants 0 and 1. We think of $\mathcal{P}$ as a candidate for the **P**-power of an alternative game rule for $x \to y \leq z$. Note that $\mathcal{P}$ is necessarily finite. If the game rule encoded by the set $\mathcal{P}$ was as desired we would have the following for any interpretation $\mathcal{J}$: $\mathcal{J} \models (x \to y \leq z)$ iff for some $X \in \mathcal{P}$, $\mathcal{J}$ satisfies all claims in X. However, the following argument shows that this cannot be the case.

Picture interpretations $\mathcal{J}$ as points in a three-dimensional cube $[0, 1]^3$, where the coordinates are given as $(\mathcal{J}(x), \mathcal{J}(y), \mathcal{J}(z))$. The set Π of interpretations which make the claim $x \to y \leq z$ true is not topologically closed: We have that $(0.5, 0.5, 0.5) \notin \Pi$, but for every $\epsilon > 0$, $(0.5 + \epsilon, 0.5, 0.5) \in \Pi$.

On the other hand, every non-strict claim between $x, y, z, 1$ and 0 is satisfied on a closed set in $[0,1]^3$. Therefore, the set Σ of interpretations which make all claims in X true for some $X \in \mathcal{P}$ is a finite union of intersections of closed sets, and therefore closed itself. In particular $\Sigma \neq \Pi$.

2.2. Disjunctive Winning Strategies

In the previous section we established a game-theoretic characterization of truth of a claim under an interpretation $\mathcal{J}$. We now want to obtain a similar characterization for the *validity* of a claim, that is, truth of a claim under all interpretations. From Theorem 6, we know that a claim $F \lhd G$ is valid if for every interpretation $\mathcal{J}$ and every regulation τ, **P** has a winning strategy in $\tau_\rho^{\mathcal{J}}[F \lhd G]$. This description of validity can hardly be called constructive, as there are infinitely many different interpretations $\mathcal{J}$. But the crucial observation is that notwithstanding the infinitude of interpretations, the space of winning strategies is a finite one.

In this section, we will define *disjunctive winning strategies*, which can be seen as combinations of finitely many winning strategies in the truth degree comparison game. We then show that a single disjunctive winning strategy witnesses the validity of a claim.

Disjunctive strategies can be approached informally in a game-theoretic way as follows. We imagine that the game $\tau_\rho^{\mathcal{J}}[F \lhd G]$ is played, but the interpretation $\mathcal{J}$ is unknown to player **P** and will be revealed to her only in the very last move. In return, **P** is granted the option to keep all her options open whenever she is to make a decision in the game. This leads to **P** playing a number of games *in parallel*, defending the claim that she will win at least one of the games. A disjunctive winning strategy is then nothing but a winning strategy in this generalized game. Led by this intuition, we will introduce disjunctive strategies syntactically in the following.

First, define a *disjunctive state* D to be a finite nonempty multiset of claims written $D = S_1 \bigvee \ldots \bigvee S_n$. A disjunctive state is called *atomic* if all of its disjuncts are atomic claims. We say that an interpretation $\mathcal{J}$ satisfies a disjunctive state D, and write $\mathcal{J} \models D$, if $\mathcal{J}$ satisfies at least one of the disjuncts of D. A disjunctive state D is called *winning* if it is an atomic state satisfied by every interpretation.

For a set $\mathcal{P} = \{X_1, \ldots, X_n\}$ where each X_i is a set of claims, we define $\bigvee \mathcal{P}$ as the set of all disjunctive states

$$S_1 \vee \ldots \vee S_n$$

where for each $i \leq n$, $S_i \in X_i$.

$$\frac{D \bigvee C \lhd A \qquad D \bigvee C \lhd B}{D \bigvee C \lhd A \wedge B}\ (\lhd\wedge) \qquad \frac{D \bigvee A \lhd C \bigvee B \lhd C}{D \bigvee A \wedge B \lhd C}\ (\wedge\lhd)$$

$$\frac{D \bigvee A \lhd C \qquad D \bigvee B \lhd C}{D \bigvee A \vee B \lhd C}\ (\vee\lhd) \qquad \frac{D \bigvee C \lhd A \bigvee C \lhd B}{D \bigvee C \lhd A \vee B}\ (\lhd\vee)$$

$$\frac{D \bigvee \top \leq C \bigvee B < A \qquad D \bigvee \top \leq C \bigvee B \leq C}{D \bigvee C \leq A \to B}\ (\to\leq) \qquad \frac{D \bigvee B < A \qquad D \bigvee B < C}{D \bigvee A \to B < C}\ (\to<)$$

$$\frac{D \bigvee C < B \bigvee A \leq B \qquad D \bigvee C < B \bigvee C < \top}{D \bigvee C < A \to B}\ (<\to) \qquad \frac{D \bigvee A \leq B \bigvee C \leq B}{D \bigvee C \leq A \to B}\ (\leq\to)$$

$$\frac{D \bigvee A < \top \qquad D \bigvee \bot < C}{D \bigvee \triangle A < C}\ (\triangle <) \qquad \frac{D \bigvee \top \leq A \bigvee C \leq \bot}{D \bigvee C \leq \triangle A}\ (\triangle \leq)$$

$$\frac{D \bigvee \top \leq A \bigvee C \ \leq \bot}{D \bigvee C \leq \triangle A}\ (\leq \triangle) \qquad \frac{D \bigvee \top \leq A \qquad D \bigvee C < \top}{D \bigvee C < \triangle A}\ (< \triangle)$$

Figure 3. Disjunctive rules

Definition 8. (Disjunctive rule) Let S be a non-atomic claim and D a disjunctive state. A *disjunctive rule* is a rule of the form

$$\frac{D \bigvee D_1 \quad \dots \quad D \bigvee D_k}{D \bigvee S}$$

where for a game state S' obtained from marking a formula in S, the sequence $D_1, \dots, D_k$ is an enumeration of $\bigvee \texttt{Pow}(S')$.

As an example, let S be the claim $A \to B \leq C$ and S' the corresponding game state where $A \to B$ is marked. Recall that

$$\texttt{Pow}(A \to B \leq C) = \{\{\top \leq C\}, \{B < A, B \leq C\}\}$$

and so

$$\bigvee \texttt{Pow}(A \to B \leq C) = \{(\top \leq C \vee B < A), (\top \leq C \vee B \leq C)\}.$$

The corresponding disjunctive rule is thus:

$$\frac{D \bigvee (\top \leq C) \bigvee (B < A) \quad D \bigvee (\top \leq C) \bigvee (B \leq C)}{D \bigvee (A \to B \leq C)}$$

Figure 3 contains the disjunctive rules corresponding to all 12 types of game states.

Definition 9. (Disjunctive strategy) Let D be a disjunctive state. A *disjunctive strategy for* $\mathbf{P}$ *in* D is a tree of disjunctive states built using disjunctive rules, and with root D. A disjunctive strategy is called *winning strategy* if all its leaves are disjunctive winning states.

We immediately remark the following analogue to Lemma 2:

LEMMA 10. (Finiteness) *Every disjunctive strategy is finite.*

PROOF. Consider a disjunctive rule instance as given in Definition 8. Let d be the logical complexity (i.e. number of logical connectives) of S, and let $D \bigvee D_i$ be one of the premises. We can observe the following: Moving from $D \bigvee S$ to $D \bigvee D_i$,

- the number of claims with complexity d strictly decreases; and
- the number of claims with complexity $> d$ is constant.

From this it follows that any sequence of disjunctive rule applications eventually leads to an atomic state. ∎

PROPOSITION 11. (Soundness of disjunctive rules) *Let $\mathcal{J} \in \mathbf{Int}^{[0,1]}$. Then $\mathcal{J}$ satisfies the conclusion of a disjunctive rule iff $\mathcal{J}$ satisfies all of its premises.*

PROOF. Let the disjunctive rule be presented as in Definition 8. Assume first that $\mathcal{J} \vDash D \bigvee S$. If $\mathcal{J} \vDash D$, then clearly $\mathcal{J}$ satisfies all premises of the disjunctive rule as well. On the other hand, if $\mathcal{J} \vDash S$, then by Proposition 3 there exists a power $X \in \mathtt{Pow}(S)$ such that all claims in X are satisfied by $\mathcal{J}$. It follows that $\mathcal{J} \vDash D_i$ for every $i \leq n$ because D_i contains a disjunct from X.

For the other direction, assume that $\mathcal{J} \nvDash D \bigvee S$. Then $\mathcal{J} \nvDash D$ and $\mathcal{J} \nvDash S$. The latter implies, again by Proposition 3, that every power $X \in \mathtt{Pow}(S)$ contains a state not satisfied by $\mathcal{J}$. The disjunctive combination of all these failing states is one of the D_i's, and so $\mathcal{J}$ does not satisfy the premise $D \bigvee D_i$. ∎

THEOREM 12. *$F \lhd G$ is valid in $\mathsf{G}^{\triangle}$ iff there is a disjunctive ws for $\mathbf{P}$ in $F \lhd G$.*

PROOF. Given the claim $F \lhd G$ (seen as a disjunctive state with one component), start by exhaustively applying disjunctive rules to it in any order. By Lemma 10 we eventually obtain a disjunctive strategy with atomic leaves. By Proposition 11 (and a simple induction on the height of the tree), all leaves of this tree will be disjunctive winning states because $F \lhd G$ is valid by assumption. Conversely, if there is a disjunctive ws for $\mathbf{P}$ in $F \lhd G$, then by definition all of its leaves are winning states. Again by Proposition 11 and a simple induction on the tree height, it follows that all disjunctive states in the ws are valid. Hence in particular, the claim $F \lhd G$ is valid. ∎

2.3. Proof-Theoretical Content of Disjunctive Strategies

Theorem 12 suggests that the disjunctive winning strategies can serve as a proof system for the logic $\mathsf{G}^{\triangle}$: Its tree-shaped proofs are constructed from the disjunctive winning states (taken as axioms) by applying disjunctive rules. However this view of disjunctive winning strategies as syntactic proofs has a flaw, as the criterion for deciding whether an atomic disjunctive state is winning (i.e., an axiom) is a semantic instead of a syntactic one. We will fix this issue in Section 3.3, where we show that it is indeed possible to efficiently decide axioms in a purely syntactic way, thereby extending a result in [3].

Meanwhile, let us give the name $\mathsf{DWS}(\mathsf{G}^{\triangle})$ to the proof system of disjunctive winning strategies. Then $\mathsf{DWS}(\mathsf{G}^{\triangle})$ is quite close to the *sequents-of-relations* calculus $\mathbf{R}\mathsf{G}_{\infty}$, and its extension $\mathbf{R}\mathsf{G}_{\infty}^{\triangle}$ capturing the $\triangle$-projection operator, as developed in [3,5,6]. The approach there is algebraic rather then game-theoretic.

On a purely notational level, the sequents-of-relations calculus differs from the disjunctive ws by the use of the symbol $\mid$ (borrowed from the *hypersequent calculus* [4]) instead of $\bigvee$.

The other differences are: $\mathbf{R}\mathsf{G}_{\infty}^{\triangle}$ includes the structural rules

$$\frac{D}{D \mid S}\,ew \qquad \text{and} \qquad \frac{D \mid S \mid S}{D \mid S}\,ec$$

of *external weakening* and *external contraction*, and it features the logical rules

$$\frac{D \mid (\top \leq C) \mid (B < A) \quad D \mid (B \leq C)}{D \mid (A \to B \leq C)} \to\leq^* \qquad \frac{D \mid (C < B) \mid (A \leq B) \quad D \mid (C < \top)}{D \mid (C < A \to B)} <\to^*$$

instead of our rules $\to\leq$ and $<\to$ (cf. Figure 3). All other rules are the same.

To show the equivalence of both calculi, we can proceed as follows. First, for the rule variants $\to\leq^*$ and $<\to^*$ the analogue of Proposition 11 can be shown:

LEMMA 13. *Let $\mathcal{J} \in \mathbf{Int}^{[0,1]}$. Then $\mathcal{J}$ satisfies the conclusion of the rule $\to\leq^*$ (resp. $<\to^*$) iff $\mathcal{J}$ satisfies all of the premises of $\to\leq^*$ (resp. $<\to^*$).*

PROOF. Assume $\mathcal{J} \nvDash D$ (otherwise the statement is obvious).

If $\|A\|_{\mathcal{J}} \leq \|B\|_{\mathcal{J}}$, then $\mathcal{J}$ satisfies the conclusion of $\to\leq^*$ iff $\|C\|_{\mathcal{J}} = 1$, and this is equivalent to the statement that $\mathcal{J}$ satisfies the premises of $\to\leq^*$, since $\mathcal{J} \nvDash D$ and $\mathcal{J} \nvDash (B < A)$. If on the other hand $\|A\|_{\mathcal{J}} > \|B\|_{\mathcal{J}}$, then $\mathcal{J}$ satisfies the conclusion of $\to\leq^*$ iff $\|B\|_{\mathcal{J}} \leq \|C\|_{\mathcal{J}}$. This in turn is equivalent to saying that $\mathcal{J}$ satisfies the premises of $\rightarrow\leq^*$ since it satisfies the left premise by assumption, and the right premise reduces to $B \leq C$ since $\mathcal{J} \not\models \mathcal{D}$.

The argument for the rule $<\rightarrow^*$ is similar. ■

It follows that the proof of Theorem 12 goes through if we use $\rightarrow\leq^*$ and $\rightarrow\leq^*$ as disjunctive rules instead of their non-starred versions.

The additional structural rules *ew* and *ec* are in fact redundant, since already the system without them is complete for $\mathsf{G}^{\triangle}$. Note however that the inclusion of redundant rules might lead to shorter proofs. More such rules for the sequents-of-relations calculus are discussed in [6]. It is worthwile to consider the game-theoretic interpretation of the structural rules *ew* and *ec*. Recall that in the reading of sequents-of-relations as disjunctive states, $S_1 \mid S_2$ corresponds to a parallel play of truth comparison games S_1 and S_2, at least one of which must be won by **P**. The rule

$$\frac{S_1}{S_1 \mid S_2}\ ew$$

then denotes a situation where **P** *discards* the subgame S_2. This is a resonable move if **P** already knows that she can win S_1, as in the case

$$\frac{A \leq A \vee B}{A \leq A \vee B \mid S_2}\ ew\ .$$

However, in our definition of disjunctive strategies, all games must be played to the very end.

The rule

$$\frac{S \mid S}{S}\ ec$$

on the other hand corresponds to **P** creating a new copy of the game. This move is only necessary if **P** plans to make different moves in both copies of the games. And indeed, our disjunctive rules incorporate the creation of copies only where it matters, namely in rule applications where **P** has to make a choice. Compare, e.g.,

$$\frac{A < B \bigvee A < C}{A < B \vee C} < \vee \quad \text{with} \quad \cfrac{\cfrac{\cfrac{A < B \mid A < C}{A < B \mid A < B \vee C} < \vee}{A < B \vee C \mid A < B \vee C} < \vee}{A < B \vee C}\ ec\ .$$

3. Extensions of Gödel Logic

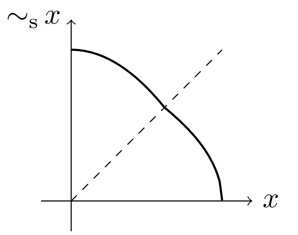
3.1. Involutive Negations and Constants

An *involutive negation* [12] is a function $(.)^* : [0,1] \to [0,1]$ that maps 0 to 1, is decreasing ($x \leq y$ implies $y^* \leq x^*$) and involutive ($x^{**} = x$). Every such function is bijective, continuous, maps 1 to 0 and has a unique fixed-point [9]. We will denote the fixed-point of $(.)^*$ by τ_*.

EXAMPLE 14. The archetype of an involutive negation is $\sim_{Ł} x = 1 - x$, called Łukasiewicz-negation, with fixed point $\tau_{\sim_Ł} = 1/2$. To have another, non-linear example, set

$$\sim_{\mathrm{s}} x = \begin{cases} 1 - x^2 & \text{for } x \leq \frac{\sqrt{5}-1}{2}, \\ \sqrt{1-x} & \text{for } x > \frac{\sqrt{5}-1}{2}. \end{cases}$$

One can readily verify the required properties and check that $\tau_{\sim_{\mathrm{s}}} = (\sqrt{5} - 1)/2$. The corresponding graph is depicted below.



EXAMPLE 15. We call an *automorphism* a continuous and strictly increasing function $f : [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$ [19]. Given an automorphism $f : [0,1] \to [0,1]$, the function $x \mapsto f^{-1}(1 - f(x))$ is an involutive negation with fixed point $f^{-1}(\frac{1}{2})$. Trillas [21] proved that all involutive negations arise in this way. Moreover, for every two involutions n_1, n_2 of the interval $[0,1]$, there exists an automorphism f, s.t. $n_1(x) = f^{-1}(n_2(f(x)))$.

For the rest of the paper, we fix an involutive negation $*$ and extend our logic $\mathsf{G}^{\triangle}$ to $\mathsf{G}^{\triangle}_*$ by setting

$$\|\sim A\|_{\mathcal{J}} = (\|A\|_{\mathcal{J}})^*.$$

Another extension we wish to consider is by adding truth constants to our logic: Let $S \subseteq [0,1]$. We denote by $\mathsf{G}^{\triangle}_*(S)$ the logic augmented by truth constants $\bar{s}$ for all $s \in S$, where

$$\|\bar{s}\|_{\mathcal{J}} = s.$$

Note that the symbols $\bot$ and $\top$ correspond to the truth constants 0 and 1 respectively and have already been included in our logic. Hence, we will always require $S \supseteq \{0, 1\}$. We will tacitly identify S with its set $\{\bar{s} \mid s \in S\}$ of associated constants.

REMARK 16. One can readily observe that every $\mathsf{G}_*^{\triangle}(S)$-formula evaluates to either a constant in S, the value of one of its variables, or the involutive negation of the value of one of its variables. In contrast to $\mathsf{G}^{\triangle}$, it is not true that truth functions in $\mathsf{G}_*^{\triangle}(S)$ depend solely on the ordering of the input values, due to the fact that an ordering of $\{a_1, ..., a_n, 0, 1\}$ does not completely determine the ordering of $\{a_1, ..., a_n, a_1^*, ..., a_n^*, 0, 1\} \cup S$.

For example, consider the formula $\triangle(y \to \sim x)$ in the logic $\mathsf{G}_{\sim_Ł}^{\triangle}$. The corresponding truth function takes the following different values, even though the ordering of $\mathcal{J}(x)$ and $\mathcal{J}(y)$ does not change:

$\mathcal{J}(x)$	$\mathcal{J}(y)$	$\mathcal{J}(\sim x)$	$\mathcal{J}(\triangle(y \to \sim x))$
0.2	0.7	0.8	1
0.4	0.7	0.6	0

Hence truth functions of $\mathsf{G}_*^{\triangle}(S)$ do not correspond to order-projective functions anymore, but rather to functions depending on the ordering of the whole set $\{a_1, ..., a_n, a_1^*, ..., a_n^*, 0, 1\} \cup S$, for inputs $a_1, ..., a_n$.

The implication-free fragment of $\mathsf{G}_{\sim_Ł}^{\triangle}([0, 1])$ has been considered in [10] in order to investigate the fuzzy logical content of CADIAG-2 — an expert system with the purpose of assisting physicians in medical decision making. In the course of this investigation, a sequent-of-relations calculus SeqGZL was given. We will see in Section 3.4 that this calculus is closely related to our game-theoretic approach.

3.2. Adding New Game Rules

Like in the *truth comparison game* for logic $\mathsf{G}^{\triangle}$, we will focus on truth degree- comparison claims $F \leq G$ or $F < G$. However, now F and G are $\mathsf{G}_*^{\triangle}(S)$- formulas. We add three additional rules to deal with involutive negation.

$A \lhd \sim B$: the game continues with $B \lhd \sim A$.

$\sim A \lhd B$: the game continues with $\sim B \lhd A$.

$\sim A \lhd \sim B$: the game continues with $B \lhd A$.

Two minor differences to the game $\mathsf{G}^{\triangle}$ arise.

First, to account for the rule for $\sim A \lhd \sim B$ the notion of a marking should be adapted so that it can point at both formulas in a claim if they are both negated, indicating the subsequent choice of the third negation rule above.

Second, the above rules do not suffice to reduce arbitrary claims to atomic ones. But we can get quite close, namely to states $F \lhd G$ where either F is atomic and G is the negation of an atomic formula, or G is atomic and F is the negation of an atomic formula. We will call such states *quasi-atomic*. Note that quasi-atomic states may contain constants from S.

Now if $F \lhd G$ is quasi-atomic, the game is in a *quasi-atomic state*, where **P** wins (and **O** loses) if $\|F\|_{\mathcal{J}} \lhd \|G\|_{\mathcal{J}}$.[3]

That being said, we can prove the analogue of Proposition 3.

PROPOSITION 17. (Soundness of game rules with involutive negation) *Let F and G be formulas of $\mathsf{G}_*^{\triangle}(S)$. For any game state $F \lhd G$ and any $\mathcal{J} \in$ $\mathbf{Int}^{[0,1]}$, $\mathcal{J} \models F \lhd G$ iff for some $X \in \mathtt{Pow}(F \lhd G)$, $\mathcal{J}$ satisfies all formulas in X.*

PROOF. We have proved this (cf. Proposition 3) for the rules without involutive negation in Section 2. Thus, we need to prove it only for the 3 new rules for involutive negation. For example, given the game state $\sim A \lhd B$ and its **P**-power $\{\{\sim B \lhd A\}\}$ we calculate

$$\mathcal{J} \models \sim A \lhd B \iff \mathcal{J} \models \sim B \lhd \sim\sim A \iff \mathcal{J} \models \sim B \lhd A$$

using the decreasing (first step) and involutive (second step) property of involutive negations. The other two rules are proved sound in a similar way. ∎

Unlike in the game for $\mathsf{G}^{\triangle}$, careless application of the negation rules can lead to non-terminating runs of the game. We may avoid this by considering only regulations which are defined on states which are not quasi-atomic, and which never mark single negated formulas in a claim unless the other formula of the claim is atomic. This prevents loops such as

$$F \lhd \sim G, \quad \sim F \lhd G, \quad F \lhd \sim G, \quad \dots$$

Calling these regulations *termination-friendly*, one can show the following:

LEMMA 18. *Let ρ be a termination-friendly regulation. Then the game $\tau_\rho^{\mathcal{J}}[F \lhd G]$ always terminates at a quasi-atomic state.*

---

[3]This definition of quasi-atomic state differs from that of the *basic sequent-of-relations* in [10] where there is no restriction on the number of involutively negated atomic formulas.

THEOREM 19. *An interpretation* $\mathcal{J}$ *satisfies* $F \lhd G$ *iff for some (or equivalently, any) termination-friendly regulation* ρ, $\mathbf{P}$ *has a winning strategy in* $\tau_\rho^{\mathcal{J}}[F \lhd G]$.

PROOF. Similar to the proof of Theorem 6. ∎

Following definitions in Section 2, we extend the notions of disjunctive winning strategy and *provability game* to include the following three disjunctive rules for negation:

$$\frac{D \bigvee (B \lhd \sim A)}{D \bigvee (A \lhd \sim B)} \quad \frac{D \bigvee (\sim B \lhd A)}{D \bigvee (\sim A \lhd B)} \quad \frac{D \bigvee (B \lhd A)}{D \bigvee (\sim A \lhd \sim B)}$$

One should slightly modify the notion of a *winning disjunctive state*. A disjunctive state D is called winning if it is a *quasi-atomic* state satisfied by every interpretation. A disjunctive state is called *quasi-atomic* if all of its disjuncts are quasi-atomic claims.

THEOREM 20. $F \lhd G$ *is valid in* $\mathsf{G}_*^{\triangle}(S)$ *iff there is a disjunctive ws for* $\mathbf{P}$ *in* $F \lhd G$.

PROOF. Similar to Theorem 12. ∎

3.3. A Purely Syntactic Criterion for Winning States

So far, our definition of a winning state is a semantic: It is a quasi-atomic disjunctive state satisfied by every interpretation. In the present section, we develop a syntactic criterion which can be employed to efficiently check whether a state is a winning state.

Given a quasi-atomic disjunctive state $D = A_1 \lhd_1 B_1 \bigvee ... \bigvee A_k \lhd_k B_k$, we consider the set $\Gamma = \{A_1 \not\lhd_1 B_1, ..., A_k \not\lhd_k B_k\}$, where $\not\leq$ is $>$ and $\not<$ is $\geq$. We call any set Γ arising in this way a *quasi-atomic set of inequalities.* Note that D is a winning state in $\mathsf{G}_*^{\triangle}(S)$ (i.e. valid) iff Γ is not satisfiable in $\mathsf{G}_*^{\triangle}(S)$.

Consider first the case that $* = \sim_{\mathrm{L}}$. Then Γ is a system of *linear* inequalities, whose satisfiability can be checked in polynomial time using methods from linear programming (assuming that the constants are presented in a computable way). To deal with other involutive negations, we now develop a purely syntactic criterion which will equip us with an effective tool to tell whether a quasi-atomic disjunctive state is winning (c.f. Section 3.4). It also allows for easy equivalence proofs for different involutive negations.

Given a literal A, let $A^{\sim}$ denote the syntactic negation of the literal A, i.e. $A^{\sim} = \sim A$ if A is unnegated and $A^{\sim} = B$ if $A = \sim B$.

DEFINITION 21. (Constraint Graph) Let Γ be a quasi-atomic set of inequalities. A Γ-literal is a literal built from a variable or constant appearing in Γ. Define the *constraint graph* $G_*(\Gamma)$ as follows. The nodes of $G_*(\Gamma)$ are the Γ-literals. The edges of $G_*(\Gamma)$ are defined by the clauses below. Every edge can either be marked or unmarked.

1. (A, B) is an edge if $A \lhd B$ or $B^{\sim} \lhd A^{\sim}$ appears in Γ, and it is marked if $\lhd =<$.
2. (s, t) is and edge if s and t are constant literals (i.e., constants or negated constants) and $s \lhd t$ is true when $\sim$ is replaced by $*$, and it is marked if $\lhd =<$.
3. $(0, 1)$ is a marked edge.
4. $(0, A)$ and $(A, 1)$ are edges for every node A.

Furthermore, write $\preceq$ for the reachability relation on $G_*(\Gamma)$ and define $A \prec B$ if there is a path in $G_*(\Gamma)$ from A to B which traverses a marked edge. Finally, call Γ $*$-*consistent* if the relation $\prec$ on $G_*(\Gamma)$ is irreflexive. Call Γ *complete* if for every pair A, B of literals using only variables from Γ and constants from S, Γ contains $A < B$ or $B \leq A$.

THEOREM 22. *Let Γ be a quasi-atomic set of inequalities with constants from S. Then Γ is satisfiable in $\mathsf{G}_*^{\triangle}(S)$ iff Γ is $*$-consistent.*

PROOF. This proof is an extension of [3]. The new part is the treatment of constants and involutive negation.

First, if $\mathcal{J}$ is a satisfying assignment for Γ, then it is straightforward to check that $\mathcal{J}(A) < \mathcal{J}(B)$ whenever $A \prec B$ in $G_*(\Gamma)$. It follows that there cannot be a node A such that $A \prec A$, and hence Γ must be $*$-consistent. ■

For the other direction, let A and B be two Γ-literals. We first show the following:

Claim: If Γ is $*$-consistent, then at least one of $\Gamma \cup \{A < B\}$ and $\Gamma \cup \{B \leq A\}$ is $*$-consistent as well.

PROOF. We write $\preceq_\Gamma$ and $\prec_\Gamma$ for the corresponding relations on $G_{*,S}(\Gamma)$ to distinguish from relations on extensions of this graph. If in $G_{*,S}(\Gamma \cup \{A < B\})$ we have $C \prec C$ for some literal C appearing in Γ, then we already have had $C \preceq_\Gamma A$ and $B \preceq_\Gamma C$ before the extension. Hence, $B \preceq_\Gamma A$, which shows that $\Gamma \cup \{B \leq A\}$ is $*$-consistent. On the other hand, if in $G_*(\Gamma \cup \{B \leq A\})$ we have $C \prec C$, then we already have had $C \prec_\Gamma A$ and $B \preceq_\Gamma C$ or $C \preceq_\Gamma A$ and $B \prec_\Gamma C$. In both cases, $A \prec_\Gamma B$, which shows that $\Gamma \cup \{A < B\}$ is $*$-consistent. ■

From the claim we can conclude that every $*$-consistent set Γ can be extended to a complete $*$-consistent set $\bar{\Gamma}$. Hence it suffices to show that such $\bar{\Gamma}$ is satisfiable.

Consider the relation $\preceq_{\bar{\Gamma}}$ and $\prec_{\bar{\Gamma}}$. Since $\bar{\Gamma}$ is $*$-consistent and complete, $\preceq_{\bar{\Gamma}}$ is a complete preorder on the set of Γ-literals, and $\prec_{\bar{\Gamma}}$ is its strict part. Let $A \approx_{\bar{\Gamma}} B$ if $A \preceq_{\bar{\Gamma}} B$ and $B \preceq_{\bar{\Gamma}} A$. Call a Γ-literal A 'left-of-τ' if $A \prec_{\bar{\Gamma}} A^{\sim}$, and call it 'equal-to-$\tau$' if $A \approx_{\bar{\Gamma}} A^{\sim}$. Note that it can never be the case that both a literal A and its negation $A^{\sim}$ are 'left-of-τ'.

Now let τ denote the fixed-point of $*$. In a first step, we embed the 'left-of-τ' literals (with their ordering $\preceq_{\bar{\Gamma}}$) in the interval $[0, \tau)$ such that every constant literal is mapped to its value in $[0, 1]$. There are two things to check. First, that $\preceq_{\bar{\Gamma}}$ agrees with the correct ordering of the evaluated literal constants. This is true because of clause (2) in the definition of $G_*(\Gamma)$. Second, the value of a 'left-of-τ' constant literal must indeed be $< \tau$. To see this, let $s \in [0, 1]$ and assume towards a contradiction that $\tau \leq s < s^*$. Then by monotonicity $\tau^* \geq s^* > s^{**}$, i.e. $\tau \geq s^* > s$ by idempotency and the fact that $\tau^* = \tau$. But now we have $s < s$, a contradiction.

Next, we assign to all 'equal-to-τ' literals the value τ. Finally, for any literal A not assigned yet, $A^{\sim}$ is a 'left-of-τ' literal and has been given some value x. We therefore assign to A the value x^*. This finishes the construction of the satisfying assignment. ■

COROLLARY 23. *The logics* $\mathsf{G}_*^{\triangle}$ *and* $\mathsf{G}_\diamond^{\triangle}$ *coincide for every pair* $*, \diamond$ *of involutive negations.*

PROOF. The validity of a formula in $\mathsf{G}_*^{\triangle}$ is equivalent to the validity of a set of quasi-atomic disjunctive states, and the validity of a quasi-atomic disjunctive state is equivalent to the unsatisfiability of its associated set of quasi-atomic inequalities. The same holds for $\mathsf{G}_\diamond^{\triangle}$.

Therefore is suffices to show that a quasi-atomic set G of inequalities is satisfiable over $\mathsf{G}_*^{\triangle}$ iff it is satisfiable over $\mathsf{G}_\diamond^{\triangle}$. Indeed, since there are no constants but 0 and 1, the constraint graphs $G_*(\Gamma)$ and $G_\diamond(\Gamma)$ are identical, and so by Theorem 22, Γ is satisfiable over $\mathsf{G}_*^{\triangle}$ if and only if it is satisfiable over $\mathsf{G}_\diamond^{\triangle}$. ■

In other words, $\mathsf{G}_*^{\triangle}$ can also be seen as the extension of $\mathsf{G}^{\triangle}$ by *arbitrary* involutive negations, i.e.

$$\mathsf{G}_*^{\triangle} = \bigcap_{\diamond} \mathsf{G}_\diamond^{\triangle}$$

where $\diamond$ ranges over all involutive negations on $[0, 1]$.

This equivalence ceases to hold in the presence of constants. For example, $\sim\overline{0.6} \to \overline{0.6}$ is valid in $\mathsf{G}^{\triangle}_{\sim_{\text{Ł}}}(\{\overline{0.6}\})$, but not in $\mathsf{G}^{\triangle}_{\sim_{\text{s}}}(\{\overline{0.6}\})$ (where $\sim_{\text{s}}$ is the nonlinear involutive negation of Example 14). The underlying reason for this is that 0.6 is bigger than the fixed-point of $\sim_{\text{Ł}}$, but smaller than the fixed-point of $\sim_{\text{s}}$. On the other hand, the logics $\mathsf{G}^{\triangle}_{\sim_{\text{Ł}}}(\{\overline{0.6}\})$ and $\mathsf{G}^{\triangle}_{\sim_{\text{s}}}(\{\overline{0.7}\})$ are *essentially* the same, up to a renaming of the constant $\overline{0.6}$ to $\overline{0.7}$.

To see which logics with constants are equivalent, maybe up to a renaming of constants, we introduce some notation. Let $S, T \subseteq [0,1]$ be two sets of constants, and let $\sim_{\text{S}}$ and $\sim_{\text{T}}$ be two involutive negations. Every function f from S to T induces a translation from $\mathsf{G}^{\triangle}_{\sim_{\text{S}}}(S)$-formulas φ to $\mathsf{G}^{\triangle}_{\sim_{\text{T}}}(T)$-formulas φ^f by replacing constants $\bar{s}$ with $\overline{f(s)}$.

Call a function $f : S \to T$ an $(\sim_{\text{S}}, \sim_{\text{T}})$-*isomorphism* if f is an isomorphism of the structures $(S \cup \{\sim_{\text{S}} s \mid s \in S\}, \leq, \sim_{\text{S}})$ and $(T \cup \{\sim_{\text{T}} t \mid t \in T\}, \leq, \sim_{\text{T}})$.

Proposition 24. *Let $S, T \subseteq [0,1]$ be two sets of constants, $\sim_{\text{S}}$ and $\sim_{\text{T}}$ two involutive negations, and let $f : S \to T$ be an $(\sim_{\text{S}}, \sim_{\text{T}})$-isomorphism. Then for every $\mathsf{G}^{\triangle}_{\sim_{\text{S}}}(S)$-formula φ,*

$$\varphi \in \mathsf{G}^{\triangle}_{\sim_{\text{S}}}(S) \Leftrightarrow \varphi^f \in \mathsf{G}^{\triangle}_{\sim_{\text{T}}}(T).$$

Proof. As in Corollary 23, it suffices to show a corresponding statement for sets of quasi-atomic inequalities. Let Γ be a finite set of quasi-atomic inequalities with truth constants in S, and let Γ^f be obtained by replacing each truth constant $\bar{s}$ by $\overline{f(s)}$. Consider the function Ω from $G_*(\Gamma)$ to $G_*(\Gamma^f)$ which maps a constant literal A to $f(A)$ and leaves all other literals unchanged. From the fact that f is an $(\sim_{\text{S}}, \sim_{\text{T}})$-isomorphism, it follows immediately that Ω is a graph isomorphism which also preserves markings (cf. clause (2) in the definition of the constraint graph). Hence Γ is $\sim_{\text{S}}$-consistent iff Γ^f is $\sim_{\text{T}}$-consistent, and by Theorem 22 this implies that Γ is satisfiable over $\mathsf{G}^{\triangle}_{\sim_{\text{S}}}(S)$ iff Γ^f is satisfiable over $\mathsf{G}^{\triangle}_{\sim_{\text{T}}}(T)$. ■

Example 25. To illustrate this result, let $\sim_{\text{s}}$ be the involutive negations considered in Example 14, and let S contain $0, 1$, the fixed point $\tau_{\sim_{\text{s}}} = 0.618...$ and some point to the right of $\tau_{\sim_{\text{s}}}$, say 0.7. Then a $(\sim_{\text{s}}, \sim_{\text{Ł}})$-isomorphim f can easily be constructed by mapping 0 and 1 onto themselves, $\tau_{\sim_{\text{s}}}$ to $\tau_{\sim_{\text{Ł}}} = 1/2$ and 0.7 to some point to the right of $\tau_{\sim_{\text{Ł}}}$, like 0.9. By the proposition, $\phi \in \mathsf{G}^{\triangle}_{\sim_{\text{s}}}(\{0, \tau_{\sim_{\text{s}}}, 0.7, 1\}) \Leftrightarrow \phi^f \in \mathsf{G}^{\triangle}_{\sim_{\text{Ł}}}(\{0, 0.5, 0.9, 1\})$.

Example 26. Using Proposition 24, we can conclude that there are – up to renaming of constants – only three different logics $\mathsf{G}^{\triangle}_{*}(\{0, s, 1\})$ using three constants: The logic where $s < \tau_*$, the logic where $s = \tau_*$, and the logic

$$\frac{D \bigvee C \lhd A \bigvee B \lhd C \qquad D \bigvee C \lhd B}{D \bigvee C \lhd A \wedge B}\ (\lhd\wedge) \qquad \frac{D \bigvee A \lhd C \bigvee B \lhd C}{D \bigvee A \wedge B \lhd C}\ (\lhd\wedge)$$

$$\frac{D \bigvee A \lhd C \qquad D \bigvee B \lhd C}{D \bigvee A \vee B \lhd C}\ (\vee\lhd) \qquad \frac{D \bigvee C \lhd A \bigvee C \lhd B}{D \bigvee C \lhd A \vee B}\ (\lhd\vee)$$

$$\frac{D \bigvee \top \leq C \bigvee B < A \qquad D \bigvee \top \leq C \bigvee B \leq C}{D \bigvee C \leq A \to B}\ (\to\leq) \qquad \frac{D \bigvee B < A \qquad D \bigvee B < C}{D \bigvee A \to B < C}\ (\to<)$$

$$\frac{D \bigvee C < B \bigvee A \leq B \qquad D \bigvee C < B \bigvee C < \top}{D \bigvee C < A \to B}\ (<\to) \qquad \frac{D \bigvee A \leq B \bigvee C \leq B}{D \bigvee C \leq A \to B}\ (\leq\to)$$

$$\frac{D \bigvee A < \top \qquad D \bigvee \bot < C}{D \bigvee \triangle A < C}\ (\triangle <) \qquad \frac{D \bigvee \top \leq A \bigvee C \leq \bot}{D \bigvee C \leq \triangle A}\ (\triangle \leq)$$

$$\frac{D \bigvee \top \leq A \bigvee C \leq \bot}{D \bigvee C \leq \triangle A}\ (\leq \triangle) \qquad \frac{D \bigvee \top \leq A \qquad D \bigvee C < \top}{D \bigvee C < \triangle A}\ (< \triangle)$$

$$\frac{D \bigvee B \lhd \sim A}{D \bigvee A \lhd \sim B}\ (\lhd\sim) \qquad \frac{D \bigvee \sim B \lhd A}{D \bigvee \sim A \lhd B}\ (\sim\lhd)$$

$$\frac{D \bigvee B \lhd A}{D \bigvee \sim A \lhd \sim B}\ (\sim\lhd\sim)$$

Figure 4. The calculus $\mathsf{DWS}(\mathsf{G}_*^\triangle)$ of disjunctive winning strategies

where $s > \tau_*$. In general, there are as many logics in n constants as there are orderings of the set $\{s_1, \ldots, s_{n-2}, \sim s_1, \ldots, \sim s_{n-2}\}$ satisfying $s_i \neq s_j$ for $i \neq j$ and which can be realized by an involutive negation on $[0, 1]$.

3.4. A Proof-System for Involutive Negations

Similarly to Section 2, we can think of the disjunctive winning strategies as a calculus that fits into the framework of sequent-of-relations if we use | instead of $\bigvee$. We will call this generalized calculus $\mathsf{DWS}(\mathsf{G}_*^\triangle(S))$. By Theorem 20, the calculus is sound and complete for $\mathsf{G}_*^\triangle(S)$. $\mathsf{DWS}(\mathsf{G}_*^\triangle(S))$ differs from $\mathsf{DWS}(\mathsf{G}^\triangle)$ by having the three new disjunctive rules for negation, and by allowing all valid quasi-atomic states (which now contains constants from S and involutive negation) as axioms. The full calculus is pictured in Figure 4.

We have remarked earlier that in order to accept the disjunctive winning strategies as proof system in the usual sense, it is pertinent to establish that we can efficiently check whether the leaves of a strategy for **P** (or the leaves of a disjunctive strategy in a semantic game) are indeed winning. As a lower limit, checking axioms must be simpler than checking validity of claims. Now G is known to be coNP-complete, and hence any conservative extension of it, such as $\mathsf{G}_*^{\triangle}(S)$, is at least coNP-hard. Checking axioms, on the other hand, turns out to be decidable in polynomial time:

Corollary 27. *It can be checked in polynomial time whether a quasi-atomic disjunctive state is winning in* $\mathsf{G}_*^{\triangle}(S)$.

Proof. Let D be a quasi-atomic disjunctive state and Γ its associated set of inequalities. By Theorem 22, it suffices to check whether Γ is $*$-consistent. This can be done in polynomial time using an algorithm for the reachability problem as a subroutine. ∎

Remark 28. Another method of efficiently deciding quasi-atomic winning states in $\mathsf{G}_*^{\triangle}(S)$ is offered by Proposition 24. The idea is to fix an $(*, \sim_{\text{Ł}})$-isomorphism from S to some suitably chosen sets of constants S' (such an isomorphism can always be found following the construction in the second half of the proof of Theorem 22 or in Example 25). By Proposition 24, the problem of verifying quasi-atomic winning states in $\mathsf{G}_*^{\triangle}(S)$ can then be reduced to $\mathsf{G}_{\sim_{\text{Ł}}}^{\triangle}(S')$, where linear programming methods are available.

Remark 29. Yet another proof of this corollary can be given by interpreting the satisfiability problem of a finite set of inequalities Γ as a constraint satisfaction problem (CSP) over a finite domain D. Call $\otimes : D^3 \to D$ a *majority operation* if $\otimes(a, a, b) = \otimes(a, b, a) = \otimes(b, a, a)$ for all $a, b \in D$. It has been shown in [17] that a CSP Γ over a finite domain D is solvable in polynomial time if it is closed under some majority operation. It can be shown that the median is such a majority operation fitting our setting. However, some work is required to explicitly reduce the original domain $[0, 1]$ of possible truth values of an interpretation to a finite set D.

Having formulated our game-theoretic approach as a sequent-of-relations calculus, the connections to SeqGZL in [10] now become evident. In fact, SeqGZL has exactly the same logical rules as $\mathsf{DWS}(\mathsf{G}_{\sim_{\text{Ł}}}^{\triangle}([0, 1]))$, except for implication. The apparent difference of SeqGZL is that it includes structural rules, rules for constants and 'trivial' axioms (A1) and (A2) instead of having all valid basic sequent-of-relations (analogous to the quasi-atomic states) as axioms:

Axioms

$$\frac{}{A \leq A}\ (A1) \qquad \frac{}{\bar{s} \lhd \bar{t}}\ (A2), \text{ where } s \lhd t$$

Rules for constants

$$\frac{D \mid A \lhd \overline{0.5}}{D \mid A \lhd \sim A}\ (\lhd\tfrac{1}{2}) \qquad \frac{D \mid \overline{0.5} \lhd A}{D \mid \sim A \lhd A}\ (\tfrac{1}{2}\lhd)$$

$$\frac{D \mid A \lhd \bar{0}}{D \mid A \lhd B}\ (\lhd 0) \qquad \frac{D \mid \bar{1} \lhd A}{D \mid B \lhd A}\ (1\lhd)$$

$$\frac{D \mid \overline{\sim_{Ł} t} \lhd A}{D \mid \sim t \lhd A}\ (\sim c\lhd) \qquad \frac{D \mid A \lhd \overline{\sim_{Ł} t}}{D \mid A \lhd \sim t}\ (\lhd \sim c)$$

Structural rules

$$\frac{D \mid A \lhd B \mid A \lhd B}{D \mid A \lhd B}\ (\text{EC}) \qquad \frac{D \mid A \leq B \qquad D \mid C \lhd_1 D}{D \mid C \lhd_1 B \mid A \lhd_2 D}\ (\text{com})$$

$$\frac{D}{D \mid A \lhd B}\ (\text{EW}) \qquad \frac{D \mid A < B \qquad D \mid B \leq A}{D}\ (\text{cut})$$

Given this similarity, one might wonder whether the additional rules of SeqGZL are needed precisely to derive the valid quasi-atomic states. Indeed, this follows from the invertibility of the logical rules – which are precisely our disjunctive rules – together with the following lemma:

LEMMA 30. *All valid quasi-atomic states are derivable in* SeqGZL.

PROOF. See Theorem 6.6 in [10]. ■

Unlike in the case of (*ec*) and (*ew*), due to their non-invertibility there is no obvious interpretation of the rules ($\lhd 0$), ($1\lhd$) and (*com*) in the setting of disjunctive winning strategies. But clearly, since $\mathsf{DWS}(\mathsf{G}_*^{\triangle}(S))$ is complete already without these rules, their inclusion would not increase the strategic power of **P**.

4. Summary and Conclusion

We have investigated Gödel logic, one of the fundamental fuzzy logics, and some of its extensions from a game semantic perspective.

We presented a game for reducing truth degree comparison claims $F < G$ or $F \leq G$, i.e., claims about the relative order of arbitrary $\mathsf{G}^{\triangle}$-formulas, to atomic comparison claims. In Section 3, we have extended this game to claims involving $\mathsf{G}_*^{\triangle}(S)$-formulas, i.e., those that contain involutive negation and constants different from 0 and 1. In Section 2.2, we demonstrated that moving from single states to disjunctions of states yields a characterization of validity in $\mathsf{G}^{\triangle}$ in terms of 'disjunctive winning strategies'. This general scheme shows how semantic games can be systematically lifted to provability games, where the latter operate on the level of validity rather than the level of truth in a model. Moreover, disjunctions of states can be viewed as sequents-of-relations in the sense of [5,6]. Hence, disjunctive winning strategies provide an interpretation of proofs in this calculus.

The same approach has also been applied to the extended version of a game for $\mathsf{G}_*^{\triangle}(S)$ in Section 3.2. We introduced involutive negations and additional constants. We observed that the choice of a specific truth function for involutive negation matters only in the presence of additional constants. In order to justly call our provability game a calculus, we obtained in Section 3.3 a simple syntactic criterion allowing us to tell whether a given quasi-atomic disjunctive state is winning. Finally, we have compared our provability game to the calculus SeqGZL for Gödel logic with involutive negation and constants (but lacking implication) introduced in [10].

A number of topics for further research arise from our game based take on Gödel logic. First of all, one may ask whether our approach extends to the first-order level. However, in both directions of a potential lifting, several technical and conceptual issues arise. For example, the naive game rule for $C \lhd \forall x.A(x)$ can induce an infinite choice of **O** which results in an infinitely branching calculus. On the other hand, the disjunctive rule inspired by other first-order calculi

$$\frac{D \bigvee C \lhd A(y)}{D \bigvee C \lhd \forall x.A(x)}$$

is non-invertible and it is not clear how a convincing game-theoretic interpretation of this rule might look like. In this light, it is interesting to see whether not only sequents-of-relations and SeqGZL but also the arguably better known hypersequent calculus **HLC** of Avron [1] can be systematically related to a truth degree comparison game for Gödel logic. The latter allows a straightforward extension to the first-order level, which solving at least technical aspects of the question. Another direction of research is to generalize our scheme and apply it to other fuzzy logics. The questions that arise are always two: Whether semantic games can easily be lifted to provability games and whether the latter may be seen as efficient analytic calculi. While it has already been shown in [14] that a similar approach relates Giles's game for Łukasiewicz logic to a corresponding hypersequent calculus, it remains open whether and how this method can be extended to yet further fuzzy logics. Finally, we want to point out that a game based approach to fuzzy logics may open the route to more sophisticated models of reasoning under vagueness than cannot be achieved by sticking with truth functional logics. For example, it is natural to ask what happens if the players of a game have only imperfect information about their opponent's moves. For classical logic this leads to Independence Friendly (IF) logic of Hintikka and Sandu [18]. Given the fact that vagueness may be seen as a phenomenon involving a lack of full share of (precise) information between speaker and hearer of vague statements, it seems attractive to explore the impact of imperfect information on truth degree comparison games.

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