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Residuated Structures and Orthomodular Lattices

Abstract. The variety of (pointed) residuated lattices includes a vast proportion of the classes of algebras that are relevant for algebraic logic, e.g., ℓ-groups, Heyting algebras, MV-algebras, or De Morgan monoids. Among the outliers, one counts orthomodular lattices and other varieties of quantum algebras. We suggest a common framework—pointed left-residuated ℓ-groupoids—where residuated structures and quantum structures can all be accommodated. We investigate the lattice of subvarieties of pointed left-residuated ℓ-groupoids, their ideals, and develop a theory of left nuclei. Finally, we extend some parts of the theory of join-completions of residuated ℓ-groupoids to the left-residuated case, giving a new proof of MacLaren's theorem for orthomodular lattices.

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1. Introduction

One of the outstanding achievements of algebraic logic, understood as a research programme at the crossroads of universal algebra and mathematical logic, is the discovery of profound connections between *substructural logics* and *residuated structures*, through which enormous progress has been made in the investigations of both domains [33,43,45,47]. The availability of such a powerful and effective framework has allowed researchers to realise that classes of algebraic structures that were previously thought to have little in common, such as ℓ-groups, Heyting algebras, De Morgan monoids, and MV-algebras, as well as logics with very different origins and motivations, like intuitionistic, linear, relevant, and fuzzy logics, could be subsumed under a common conceptual and methodological umbrella, which ultimately revealed unsuspected similarities between their respective theories. It is by now no exaggeration to say that virtually every alternative to classical logic, and a vast majority of the so-called "algebras of logic", can be incorporated into this schema.

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An exception, however, strikes the eye—quantum logic and their algebras [5, 12, 28, 30] really don't seem to fit in the picture. Quantum structures like orthomodular lattices have, generally speaking, no pair of operations that behaves like a residuated pair. Correspondingly, implications in quantum logics lack the minimal requirements that would entitle them to be regarded as substructural conditionals [28, Ch. 8]. This state of affairs is unfortunate in so far as it hinders the application to quantum logics and quantum algebras of results, tools and techniques that have proved successful in the area of residuated structures and substructural logics.

To overcome this difficulty, all we need to do is to zoom out a little more. Although orthomodular lattices make no instance of residuated lattices, or even of residuated ℓ-groupoids, it has been recently observed that they can be turned into a variety of *left-residuated* ℓ-groupoids having rather strong properties [22]. Similar results can be obtained for other classes of algebras of concern to quantum logicians (see e.g. [8]). It is natural, then, to explore the extent to which the conspicuous amount of information available on residuated ℓ-groupoids can be extended to the left-residuated case, with an eye to encompassing both residuated structures and quantum structures, and consequently both substructural and quantum logics, into a common framework.

By starting an exploration of this path, we hope to provide evidence to the effect that the use of methods from the domain of residuated structures can provide fresh insights that might eventually lead to new approaches to long-standing open problems in quantum logic. For example, it is well-known that orthomodular lattices fail to be closed w.r.t. Dedekind–MacNeille completions, but whether orthomodular lattices admit completions has been an open question for several decades [12]. However, there is a well-rehearsed theory of Dedekind–MacNeille completions of residuated partially ordered groupoids, which uses significant portions of the theory of nuclei on residuated po-groupoids [34, 35], as well as algebraic proof theory techniques [24, 25]. In particular, the identities that axiomatise a given variety $\mathbb{V}$ of residuated ℓ-groupoids can be placed within a hierarchy of complexity classes (the *substructural hierarchy* [24, 25]), somewhat reminiscent of the arithmetical hierarchy in recursion theory. Properties such as whether or not $\mathbb{V}$ is closed under Dedekind–MacNeille completions, or whether or not this closure property is equivalent to $\mathbb{V}$'s admitting completions, depend on the placement of its axioms in this hierarchy. Extending the results in [24, 25, 34, 35] to the left-residuated case would seem to open a promising avenue for making some headway on this problem. In this paper, we intend to lay the foundations for such an approach.

Let us now summarise the discourse of the paper. In Section 2 we dispatch all the necessary preliminaries on residuated structures, quantum structures, and completions of ordered structures. In Section 3 we introduce the variety $\mathbb{PLRG}$ of *pointed left-residuated lattice-ordered groupoids*, which includes both the reducts with a single residual of pointed residuated lattices (in particular, all pointed commutative residuated lattices) and orthomodular groupoids, a term equivalent version of orthomodular lattices. After going over their elementary properties, we study some important subvarieties of $\mathbb{PLRG}$, arriving at an atlas that comprises $\backslash$-free reducts of pointed residuated ℓ-groupoids, hence in particular of pointed residuated lattices and members of their subvarieties, as well as term equivalent incarnations of several classes of algebras of quantum logic, including basic algebras, lattice effect algebras, and orthomodular lattices. Equational bases for some notable subvarieties relative to $\mathbb{PLRG}$ are duly provided. We observe that $\mathbb{PLRG}$ is an arithmetical and 1-ideal determined variety, and we characterise 1-classes of congruences in a large subclass of such. We also develop a theory of *left nuclei* for these structures, modelled after the theory of nuclei for residuated ℓ-groupoids [33,45]. Finally, inspired by the approach in [35], in Section 4 the ideal completion and the Dedekind–MacNeille completion of an orthomodular groupoid $\mathbf{L}$ are obtained as left nucleus-systems on a left residuated ℓ-groupoid over the powerset of $\mathbf{L}$.

2. Preliminaries

For universal algebraic terminology and notation we generally follow [14], except where indicated otherwise. Given an algebra $\mathbf{A}$, with a slight abuse we denote by $\mathrm{Con}\,(\mathbf{A})$ both the lattice of congruences of $\mathbf{A}$ and its universe.

Throughout this paper, an *ordered structure* is a first-order structure where one of the relations is a partial order. If $\mathbf{P}$ is an ordered structure, we denote by $\mathbf{P}^o$ is its order reduct; if $\mathbf{L}$ is a lattice-ordered algebra, its lattice reduct will be referred to as $\mathbf{L}^l$.

If $\mathbf{P}$ is an ordered structure with order reduct $\mathbf{P}^o = (P, \leq)$, we set, for any $X \subseteq P$:

$$U_{\mathbf{P}}(X) = \{a \in P : a \geq x, \text{ for all } x \in X\};$$
$$L_{\mathbf{P}}(X) = \{a \in P : a \leq x, \text{ for all } x \in X\};$$
$$\uparrow_{\mathbf{P}}(X) = \{a \in P : a \geq x, \text{ for some } x \in X\};$$
$$\downarrow_{\mathbf{P}}(X) = \{a \in P : a \leq x, \text{ for some } x \in X\}.$$

Brackets are dropped when X is a singleton. All subscripts and superscripts are likewise omitted, both in these notations and elsewhere, whenever this is not prejudicial to comprehension.

2.1. Residuated Structures

We cannot even remotely hope to give an account of residuated lattices, or of residuated ℓ-groupoids, that makes the present paper self-contained. We just recall some definitions and elementary properties, referring the reader to [33, 43, 45] for extensive treatments of the topic and for properties of these algebras that will be used with no special mention in the sequel.

A binary operation $\cdot$ on a partially ordered set $(P, \leq)$ is said to be *residuated* provided there exist binary operations $\backslash$ and $/$ on P such that for all $a, b, c \in P$,

$$\text{(Res) } a \cdot b \leq c \ \text{ if and only if } \ a \leq c/b \ \text{ iff } \ b \leq a\backslash c.$$

We refer to the operations $\backslash$ and $/$ as the *right residual* and *left residual* of $\cdot$, respectively. We adopt the convention that, in the absence of parentheses, $\cdot$ binds stronger than the other operations symbols. Any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$).

If $\cdot$ is an (associative) operation with two-sided unit 1 and the partial order $\leq$ is a lattice order, the resulting structure $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ is called a *residuated ℓ-groupoid*, resp. a *residuated lattice*. A *pointed residuated ℓ-groupoid*, resp. a *pointed residuated lattice* is an algebra $\mathbf{L} = \big(L, \wedge, \vee, \cdot, \backslash, /, 0, 1\big)$ such that the reduct $(L, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated ℓ-groupoid (a residuated lattice); in other words, nothing is assumed about the additional constant 0. The existence of both residuals for product implies rather strong properties, some of which are listed below, primarily in view of a comparison with the left-residuated structures examined below.

Proposition 1. *The following identities and quasi-identities (and their mirror images) hold in any (pointed) residuated lattice.*

(i) $(x\backslash y) \cdot z \leq x \ \backslash y \cdot z$

(ii) $1\backslash x \approx x$

(iii) $x \leq y \Rightarrow z \cdot x \leq z \cdot y$ *and* $x \leq y \Rightarrow x \cdot z \leq y \cdot z$

(iv) $x \leq y \Rightarrow y\backslash z \leq x\backslash z$ *and* $x \leq y \Rightarrow z\backslash x \leq z\backslash y$

(v) $(x\backslash y) \cdot (y\backslash z) \leq x\backslash z$

(vi) $x \cdot y \ \backslash z \approx y\backslash(x\backslash z)$

(vii) $x \cdot (x \backslash 1) \ \backslash 1 \approx 1$

(viii) $x \backslash (y/z) \approx (x \backslash y)/z$

Proposition 2. *The classes of (pointed) residuated ℓ-groupoids and (pointed) residuated lattices are finitely based varieties in their respective signatures, for the residuation conditions (Res) can be replaced by the following identities (and their mirror images):*

(i) $y \leq x \backslash (x \cdot y \vee z)$

(ii) $x \cdot (y \vee z) \approx x \cdot y \vee x \cdot z$

(iii) $y \cdot (y \backslash x) \leq x.$

2.2. Quantum Structures

In the standard Birkhoff-von Neumann approach to quantum logic, quantum events (or properties) are mathematically represented by projection operators on a complex separable Hilbert space. If $\mathcal{H}$ is a Hilbert space and $\Pi(\mathcal{H})$ is the set of all projection operators on $\mathcal{H}$, the structure

$$(\Pi(\mathcal{H}), \wedge, \vee, ', 0, 1),$$

where 0, resp. 1 are the projections onto the one-element, resp. total subspaces, $(P_X)'$ is the projection onto the subspace $X^{\perp}$ orthogonal to X and $P_X \vee P_Y = P_{(X \cup Y)^{\perp\perp}}$, is a canonical example of an *orthomodular lattice*, a structure which is defined hereafter.

Definition 3. (i) An *ortholattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and $'$ is an orthocomplementation on L, namely an antitone involution s.t. $a \wedge a' = 0$ for all $a \in L$.

(ii) An ortholattice $\mathbf{L}$ is an *orthomodular lattice* if and only if for all $a, b \in L$ such that $a \leq b$, we have that $b = (b \wedge a') \vee a$.

The orthomodular property, for an ortholattice, can be characterised in several equivalent ways. We list some of them in the next theorem, which shows, among other things, that the class of all orthomodular lattices is a variety.

Theorem 4. *Let $\mathbf{L}$ be an ortholattice. The following conditions are equivalent:*

(i) $\mathbf{L}$ *is orthomodular;*

(ii) *for all $a, b \in L$, $a \wedge ((b \wedge a) \vee a') \leq b$;*

(iii) *for all* $a, b \in L$, $a \vee b = ((a \vee b) \wedge b') \vee b$;

(iv) *for all* $a, b \in L$, *if* $a \leq b$ *and* $a' \wedge b = 0$, *then* $a = b$.

Proof. See e.g. [7, Ch. 18]. ■

It would be absolutely pointless to try and offer a précis of the theory of orthomodular lattices in the present context. The reader is referred to [5] for whatever concepts or results are not covered in this short subsection. We will only provide very basic information on the *commuting relation*, of crucial importance for this paper.

Definition 5. Given an orthomodular lattice $\mathbf{L}$ and $a, b \in L$, a is said to *commute* with b in case $(a \wedge b) \vee (a' \wedge b) = b$.

We flag some known facts about the commuting relation that will be put to good use in this paper (items (iii) and (iv) are known as *Kröger's Lemma* [5, Thm. 6.1] and the *Gudder–Schelp Theorem* [4,37], respectively).

Lemma 6. *Let* $\mathbf{L}$ *be an orthomodular lattice, and let* $a, b, c \in L$.

(i) *The commuting relation is reflexive and symmetric.*

(ii) *If* a *commutes with* b, *then it commutes with* $b', b \vee c$ *and* $b \wedge c$.

(iii) *If* b *commutes with* c, *then* $(((a \vee b') \wedge b) \vee c') \wedge c = (a \vee b') \wedge (b \vee c') \wedge c$.

(iv) *If* b *commutes with* c *and* a *commutes with* $b \wedge c$, *then* $a \vee b'$ *commutes with* c.

(v) a *commutes with* b *if and only if* $(a \vee b') \wedge b \leq a$.

The next celebrated result is one of the most useful tools for practicioners of the field:

Theorem 7. (Foulis–Holland) [12, Prop. 2.8] *If* $\mathbf{L}$ *is an orthomodular lattice and* $a, b, c \in L$ *are such that* a *commutes both with* b *and with* c, *then the set* $\{a, b, c\}$ *generates a distributive sublattice of* $\mathbf{L}$.

Projection operators are not the largest set of operators on a Hilbert space that can be assigned a probability value according to the Born rule. Within the *unsharp approach* to quantum logic [28], *effects* of a space $\mathcal{H}$—bounded linear operators E on $\mathcal{H}$ such that, for any density operator ρ on $\mathcal{H}$, $Tr\,(\rho E) \in [0, 1]$—have been considered more adequate mathematical representatives of the notions of a quantum event and of a quantum property. Several algebraic abstractions of the concrete model of effects on a Hilbert space have been introduced in the literature. Chief among them are *effect algebras* [32,36], partial algebras on which a burgeoning literature is by now available (see e.g. [30, Ch. 1]). They will not be directly considered in what follows, unlike a number of structures that we quickly review hereafter.

DEFINITION 8. [16,17] A *basic algebra* is an algebra $\mathbf{A} = (A, \oplus, ', 0)$ of type $(2, 1, 0)$ satisfying the following identities:

(i) $x \oplus 0 \approx x$;

(ii) $x'' \approx x$;

(iii) $(x' \oplus y)' \oplus y \approx (y' \oplus x)' \oplus x$;

(iv) $(((x \oplus y)' \oplus y)' \oplus z)' \oplus (x \oplus z) \approx 0'$.

It can be proved that *lattice effect algebras* are term equivalent to basic algebras of a certain sort [17]. In fact, given a lattice effect algebra $\mathbf{A} = (A, \wedge, \vee, +, ', 0)$, the algebra $\mathcal{B}(\mathbf{A}) = (A, \oplus, ', 0)$ obtained from $\mathbf{A}$ by setting $x \oplus y = (x \wedge y') + y$ is a basic algebra satisfying the *effect condition*

$$x \leq y' \,\&\, x \oplus y \leq z' \Rightarrow (x \oplus y) \oplus z \approx x \oplus (z \oplus y). \qquad \text{(ec)}$$

In fact, if $a \leq b'$ then $a \oplus b = a + b$ and $a \oplus b \leq c'$ implies

$$(a \oplus b) \oplus c = (a + b) + c = a + (c + b) = a \oplus (c \oplus b),$$

by general properties of lattice effect algebras. Moreover, any basic algebra $\mathbf{B} = (B, \oplus, ', 0)$ satisfying (ec) can be converted into a lattice effect algebra $\mathcal{E}(\mathbf{B}) = (B, \wedge, \vee, +, ', 0)$ by letting $x+y = x \oplus y$ be defined whenever $x \leq y'$. It can be proven that $\mathcal{B}(\mathcal{E}(\mathbf{B})) = \mathbf{B}$ and $\mathcal{E}(\mathcal{B}(\mathbf{A})) = \mathbf{A}$, for any basic algebra $\mathbf{B}$ and any lattice effect algebra $\mathbf{A}$, respectively.

2.3. Completions of Ordered Structures

The next definitions and results, except when credit is explicitly indicated, belong to the folklore of order theory; our presentation of them is heavily indebted to [35]. For further information on completions of ordered structures see [41,53].

DEFINITION 9. Given two ordered structures $\mathbf{P}$ and $\mathbf{L}$ of the same type, we say that $\mathbf{L}$ is a *join-extension* of $\mathbf{P}$, or that $\mathbf{P}$ is *join-dense* in $\mathbf{L}$, if the order of $\mathbf{L}$ restricts to that of $\mathbf{P}$ and, moreover, every element of $\mathbf{L}$ is a join of elements of $\mathbf{P}$. If $\mathbf{L}$ is a join-extension of $\mathbf{P}$ whose partial order is a complete lattice, $\mathbf{L}$ is said to be a *join-completion* of $\mathbf{P}$.

In general, it is not assumed that the algebra reduct of $\mathbf{P}$ is a subalgebra of that of $\mathbf{L}$. The concepts of a *meet-extension* and a *meet-completion* are defined dually.

Ordered structures, of course, include posets as limit cases. We have the following:

Lemma 10. *A join-extension* $\mathbf{L}$ *of a poset* $\mathbf{P}$ *preserves all existing meets in* $\mathbf{P}$. *That is, if* $X \subseteq P$ *and* $\bigwedge^{\mathbf{P}} X$ *exists, then* $\bigwedge^{\mathbf{L}} X$ *exists and* $\bigwedge^{\mathbf{P}} X = \bigwedge^{\mathbf{L}} X$. *Dually, a meet-extension of* $\mathbf{P}$ *preserves all existing joins in* $\mathbf{P}$.

The next result places some constraints on join-completions of a given poset $\mathbf{P}$. If one such join-completion $\mathbf{K}$ is known and $P \subseteq L \subseteq K$, then L is the universe of a join-completion of $\mathbf{P}$ if and only if it arises as the image of a closure operator on $\mathbf{K}$:

Proposition 11. *Let* $\mathbf{P}$ *be a poset, let* $\mathbf{K}$ *be a join-completion of* $\mathbf{P}$, *and let* L *be a subset of* K *that contains* P. *The poset* $\mathbf{L}$, *with respect to the induced partial order from* $\mathbf{K}$, *is a join-completion of* $\mathbf{P}$ *if and only if it is a closure system of* $\mathbf{K}$.

Recall that any poset $\mathbf{P}$ is order-isomorphic to the poset $(\{\downarrow x : x \in P\}, \subseteq)$ of the principal order-ideals of $\mathbf{P}$. The map $f(a) = \downarrow a$ embeds $\mathbf{P}$ into the complete lattice $\mathcal{L}(\mathbf{P})$ of all its order-ideals, which is readily seen to be a join-completion of $\mathbf{P}$, called the *ideal completion* of $\mathbf{P}$. Moreover, each join-extension $\mathbf{L}$ of $\mathbf{P}$ is isomorphic to $(\{\downarrow x \cap P : x \in L\}, \subseteq)$, which is a subposet of $\mathcal{L}(\mathbf{P})$. Taking into account Proposition 11, we obtain that:

Proposition 12. *Let* $\mathbf{P}$ *be a poset.*

(i) $\mathcal{L}(\mathbf{P})$ *is, up to isomorphism, the largest join-completion of* $\mathbf{P}$.

(ii) *The join-completions of a poset* $\mathbf{P}$ *are, up to isomorphism, the closure systems of* $\mathcal{L}(\mathbf{P})$ *containing* P, *with the induced order.*

(iii) $\mathcal{L}(\mathbf{P})$ *is the unique algebraic and dually algebraic distributive lattice whose poset of completely join-prime elements is isomorphic to* $\mathbf{P}$.

The *Dedekind–MacNeille completion* $\mathcal{N}(\mathbf{P})$ of $\mathbf{P}$ is the closure system of $\mathcal{L}(\mathbf{P})$ determined by the closure operator sending any $X \subseteq P$ to $L_{\mathbf{P}}(U_{\mathbf{P}}(X))$. Instead of $\{X \subseteq P : L_{\mathbf{P}}(U_{\mathbf{P}}(X)) = X\}$, we write $\mathcal{N}(P)$.

Proposition 13. *Let* $\mathbf{P}$ *be a poset.*

(i) $\mathcal{N}(\mathbf{P})$ *is, up to isomorphism, the smallest join-completion of* $\mathbf{P}$ [52, Thm. 8.27].

(ii) $\mathcal{N}(\mathbf{P})$ *is the only join- and meet-completion of* $\mathbf{P}$ [3].

(iii) *Any existing meets and joins in* $\mathbf{P}$ *are preserved in* $\mathcal{N}(\mathbf{P})$.

Observe that, in Proposition 13, item (iii) follows from (ii) and Lemma 10. However, the largest join-completion with the property of Proposition 13.(iii) is not $\mathcal{N}(\mathbf{P})$, but the so-called *Crawley completion* $\mathcal{CR}(\mathbf{P})$, consisting of all order-ideals of $\mathbf{P}$ that are closed with respect to any existing joins of their elements [50,51]. In general, the inclusion $\mathcal{N}(\mathbf{P}) \subseteq \mathcal{CR}(\mathbf{P})$ is proper.

Given an ordered structure $\mathbf{Q}$, it is sometimes possible to expand the Dedekind–MacNeille completion $\mathcal{N}(\mathbf{Q}^o)$ of its order reduct $\mathbf{Q}^o$ to an ordered structure $\mathcal{N}(\mathbf{Q})$ of the same type as $\mathbf{Q}$, in such a way that the order-embedding of $\mathbf{Q}^o$ into $\mathcal{N}(\mathbf{Q}^o)$ also preserves the additional operations and relations. This may happen, in particular, if $\mathbf{Q}$ is a (lattice-ordered) algebra—yet, $\mathcal{N}(\mathbf{Q})$ need not satisfy the same identities as $\mathbf{Q}$. As a matter of fact, the Dedekind–MacNeille construction is notoriously inefficient in terms of preserving identities. Classes of ordered structures that are closed under Dedekind–MacNeille completions are few and far between. Boolean algebras and ortholattices are two cases in point.

THEOREM 14. *Let* $\mathbf{L}$ *be an ortholattice.*

(i) *The Dedekind–MacNeille completion* $\mathcal{N}(\mathbf{L}^o)$ *of* $\mathbf{L}^o$ *can be expanded to an ortholattice* $\mathcal{N}(\mathbf{L})$*, by defining the involution as:*

$$X' = \{x' : x \in U(X)\}.$$

(ii) *The Dedekind–MacNeille embedding of* $\mathbf{L}^o$ *into* $\mathcal{N}(\mathbf{L}^o)$ *is also an embedding of* $\mathbf{L}^l$ *into* $\mathcal{N}(\mathbf{L}^l)$ *that preserves the involution, and thus it is an embedding of* $\mathbf{L}$ *into* $\mathcal{N}(\mathbf{L})$.

(iii) [44] $\mathcal{N}(\mathbf{L})$ *is isomorphic to the algebra* $\left(C(\mathbf{L}), \cap, \vee, ^\perp, \{0\}, L\right)$*, where:*

$$\textit{for any } X \subseteq L,\ X^\perp = \{a \in L : a \leq x' \textit{ for all } x \in X\};$$
$$C(\mathbf{L}) = \left\{X \subseteq L : X = X^{\perp\perp}\right\};$$
$$\textit{for any } X, Y \in C(\mathbf{L}),\ X \vee Y = (X \cup Y)^{\perp\perp}.$$

The previous result holds for all ortholattices, hence in particular for orthomodular lattices. Therefore, the Dedekind–MacNeille completion of an orthomodular lattice is always an ortholattice, which however need not be orthomodular. For a counterexample, consider the orthomodular lattice of finite-dimensional or cofinite-dimensional subspaces of a metrically incomplete pre-Hilbert space.

The theory of completions of orthomodular lattices abounds in partial positive results as well as in intriguing open problems. Prominent among the latter is whether an arbitrary orthomodular lattice can always be embedded into a complete one, with no additional request whatsoever on the embedding. We list hereafter a theorem on particular classes of orthomodular lattices that are closed w.r.t. Dedekind–MacNeille completions. Item (i) is due to Harding [40], while item (ii) had been previously proved by Bruns, Greechie, Harding, and Roddy [11].

THEOREM 15. (i) *Let $\mathbb{V}$ be a variety of orthomodular lattices generated by a class with a finite upper bound on the length of their chains. Then $\mathbb{V}$ is closed w.r.t. Dedekind–MacNeille completions.*

(ii) *In particular, $\mathbb{V}$ is closed w.r.t. Dedekind–MacNeille completions if it is generated by a single finite algebra.*

3. Pointed Left-Residuated l-Groupoids

In this section, we start by collecting a few trivia on pointed left-residuated ℓ-groupoids, some of which are simple observations. We then try to expand the boundaries of their structure theory in three directions: the study of the lattice of subvarieties, a characterisation of their ideals, and an investigation into *left nuclei*, certain closure operators on their order reducts that will play a crucial role in the following section.

3.1. Definition and Basic Properties

The next definition is a modification of a definition in [22].

DEFINITION 16. A *left-residuated ℓ-groupoid* is an algebra

$$\mathbf{L} = (L, \wedge, \vee, \cdot, \rightarrow, 1),$$

of type $(2, 2, 2, 2, 0)$, such that:

- $(L, \wedge, \vee)$ is a lattice;
- $(L, \cdot, 1)$ is a left-unital groupoid (a groupoid with a left unit 1);
- the condition $a \cdot b \leq c$ if and only if $a \leq b \rightarrow c$ is satisfied for all $a, b, c \in L$.

A *pointed left-residuated ℓ-groupoid* is an algebra

$$\mathbf{L} = (L, \wedge, \vee, \cdot, \rightarrow, 0, 1),$$

of type $(2, 2, 2, 2, 0, 0)$, such that the reduct $(L, \wedge, \vee, \cdot, \rightarrow, 1)$ is a left-residuated ℓ-groupoid.

Observe that in the latter definition no condition is assumed about the constant 0. Examples of (pointed) left-residuated ℓ-groupoids that immediately spring to mind include \-free reducts (namely, reducts with just one residual) of (pointed) residuated ℓ-groupoids. If product is associative and 1 is a two-sided unit, Definition 16 specialises to \-free reducts of (pointed) residuated lattices [33,45]. Here are two totally unrelated examples, which we owe to Antonino Salibra and Tomasz Kowalski (personal communications), respectively.

EXAMPLE 17. In λ-calculus, a *graph model* [13,48] is a pair (A, i), where A is an infinite set and $i : \wp_{fin}(A) \times A \to A$ is an injective map. For $X, Y \subseteq A$, set:

$$X \cdot Y = \{x \in A : i(Z, x) \in X \text{ for some finite } Z \subseteq Y\};$$
$$X \to Y = \{x \in A : x \notin \mathit{cod}(i)\} \cup \{i(Z, y) : y \in Y\} \cup \{i(W, x) : W \not\subseteq_{fin} X\};$$
$$1 = \{i(X, x) : x \in X\}.$$

Then $(\wp(A), \cap, \cup, \cdot, \to, 1)$ is a left-residuated ℓ-groupoid.

EXAMPLE 18. The relevant logic E was introduced by Anderson and Belnap [1] to model the notion of entailment. Its attendant algebras form a variety that is term equivalent to a variety of pointed left-residuated ℓ-groupoids. Namely, let an E-*algebra* be a pointed left-residuated ℓ-groupoid $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, 0, 1)$ satisfying the identities

- $((x \to y) \cdot (z \to x)) \cdot z \leq y$;
- $((x \to y) \cdot (y \to z)) \cdot x \leq z$;
- $x \leq x \cdot x$;
- $(x \to x) \to y \leq y$.

An example of a non-associative E-algebra which is not a reduct of a residuated groupoid and has no right unit is given by the 4-element chain $\bot < 1 < a < \top$, where product is defined as follows:

$\cdot$	$\bot$	1	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
1	$\bot$	1	a	$\top$
a	$\bot$	1	a	$\top$
$\top$	$\bot$	1	$\top$	$\top$

Non-pointed left-residuated ℓ-groupoids will not be further considered in this paper. We employ the same notational shortcuts that are already in force for residuated lattices (Section 2). In particular, remember that products bind stronger than implications, joins or meets.

LEMMA 19. *Let* $\mathbf{L}$ *be a pointed left-residuated* ℓ*-groupoid. Then for all* $X \cup \{a\} \subseteq L$*, whenever* $\bigvee X$ *and* $\bigwedge X$ *esist, we have that:*

(i) $\left(\bigvee X\right) \cdot a = \bigvee_{x \in X} (x \cdot a)$;

(ii) $a \to \bigwedge X = \bigwedge_{x \in X} (a \to x)$.

PROOF. It follows from the fact that for any $a \in L$, the pair $(\varphi_a(x), \psi_a(y))$, where $\varphi_a(x) = x \cdot a$ and $\psi_a(y) = a \to y$, is a residuated pair. ∎

The proof of the following easy lemma is left to the reader. Observe that in a generic pointed left-residuated ℓ-groupoid, product preserves the order only on the right-hand side, unlike in the case of residuated ℓ-groupoids. Therefore, although pointed left-residuated ℓ-groupoids are groupoids and are lattice-ordered, they need not be lattice-ordered groupoids. Similarly, product need not distribute on joins from the left-hand side.

LEMMA 20. *Let* $\mathbf{L}$ *be a pointed left-residuated* ℓ*-groupoid, and let* $a, b, c \in L$*. Then: (i)* $(a \to b) \cdot a \le b$*;* (ii) *if* $a \le b$*, then* $a \cdot c \le b \cdot c$*;* (iii) *if* $a \le b$*, then* $c \to a \le c \to b$*.*

We show that pointed left-residuated ℓ-groupoids form a variety.

LEMMA 21. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, 1, 0)$ *be an algebra of type* $(2, 2, 2, 2, 0, 0)$*. Then* $\mathbf{L}$ *is a pointed left-residuated* ℓ*-groupoid if and only if the following hold:*

(i) $(L, \wedge, \vee)$ *is a lattice;*

(ii) $(L, \cdot, 1)$ *is a left-unital groupoid;*

(iii) *the following identities hold:*

$$x \le y \to (x \cdot y \vee z), \tag{l-res1}$$

$$(x \wedge (y \to z)) \cdot y \le z. \tag{l-res2}$$

PROOF. Suppose that $\mathbf{L}$ is a pointed left-residuated ℓ groupoid. Since $a \cdot b \le a \cdot b \vee c$, we have that $a \le b \to (a \cdot b \vee c)$, while $a \wedge (b \to c) \le b \to c$ implies $(a \wedge (b \to c)) \cdot b \le c$. Conversely, assume that $\mathbf{L}$ satisfies conditions $(1) - (3)$. If $a \cdot b \le c$ then $a \le b \to (a \cdot b \vee c) = b \to c$. The converse can be proved similarly, using (l-res2). ∎

In what follows, we will denote by $\mathbb{PLRG}$ the variety of pointed left-residuated ℓ-groupoids. Next, we examine some important subclasses of $\mathbb{PLRG}$.

DEFINITION 22. A pointed left-residuated ℓ-groupoid $\mathbf{L}$ is said to be:

- *unital*, if 1 is a two-sided unit for $(L, \cdot, 1)$;
- 1-*cyclic*, if $a \cdot b \leq 1$ implies $b \cdot a \leq 1$ for all $a, b \in L$;
- *involutive*, if $(a \to 0) \to 0 = a$ for all $a \in L$;
- *strongly involutive*, if $a \to b = (b \to 0) \cdot a \to 0$ for all $a, b \in L$;
- *antitone*, if $a \leq b$ implies $b \to 0 \leq a \to 0$ for all $a, b \in L$;
- *left-monotonic*, if $a \leq b$ implies $c \cdot a \leq c \cdot b$ for all $a, b, c \in L$;
- *left-distributive*, if $a \cdot (b \vee c) = a \cdot b \vee a \cdot c$ for all $a, b, c \in L$;
- *integral*, if 1 is the top element of $(L, \wedge, \vee)$;
- 0-*bounded*, if 0 is the bottom element of $(L, \wedge, \vee)$;
- *strongly idempotent*, if $a \cdot (a \vee b) = a$ for all $a, b \in L$;
- *divisible*, if $(a \to (a \wedge b)) \cdot a = a \wedge b$ for all $a, b \in L$;
- *Sasakian*, if $a \cdot b = (a \vee (b \to 0)) \wedge b$ for all $a, b \in L$;
- $\to$-*Sasakian*, if $a \to b = (b \wedge a) \vee (a \to 0)$ for all $a, b \in L$.

By Lemma 19, if $\mathbf{L}$ is integral, then it is divisible if and only if for all $a, b \in L$ we have that $(a \to b) \cdot a = a \wedge b$. Since we will not consider divisibility for non-integral members of $\mathbb{PLRG}$, hereafter by divisibility we will mean the property that $(a \to b) \cdot a = a \wedge b$ for all $a, b \in L$.

Strong idempotency is a rather powerful assumption. Indeed, we show that it implies several of the properties in the previous list.

LEMMA 23. *Every strongly idempotent member of* $\mathbb{PLRG}$ $\mathbf{L}$ *is:* (i) *integral;* (ii) *unital;* (iii) *divisible.*

PROOF. (i) By strong idempotency, $1 = 1 \cdot (1 \vee a) = 1 \vee a$.

(ii) By (i) and strong idempotency, $a \cdot 1 = a \cdot (a \vee 1) = a$.

(iii) Let $a, b, c \in L$. It will suffice to prove the nontrivial inequality $a \wedge b \leq (a \to b) \cdot a$, since the converse inequality easily follows from integrality and Lemma 20.(i)–(ii). Observe that $a \wedge b = (a \wedge b) \cdot ((a \wedge b) \vee a) = (a \wedge b) \cdot a$, by strong idempotency. Thus $(a \wedge b) \cdot a = a \wedge b \leq b$, whence $a \wedge b \leq a \to b$. By Lemma 20.(ii), $a \wedge b = (a \wedge b) \cdot a \leq (a \to b) \cdot a$. ■

We now recall a construction that will be crucial in the following sections. Let $\mathbf{L}$ =$(L, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ be a pointed left-residuated ℓ-groupoid. It is well-known (see e.g. [34, Lm. B1]) that the structure

$$\mathcal{P}(\mathbf{L}) = \left(\wp(L), \cap, \cup, \cdot^{\mathcal{P}(L)}, \rightarrow^{\mathcal{P}(L)}, \{0\}, \{1\}\right),$$

where for $X, Y \subseteq L$,

$$X \cdot^{\mathcal{P}(L)} Y = \{a \cdot b : a \in X, b \in Y\},$$
$$X \rightarrow^{\mathcal{P}(L)} Y = \left\{c : \{c\} \cdot^{\mathcal{P}(L)} X \subseteq Y\right\},$$

is also a pointed left-residuated ℓ-groupoid. We codify this fact and a few easy additional properties in the next

Lemma 24. *If $\mathbf{L}$ is a pointed left-residuated ℓ-groupoid, then $\mathcal{P}(\mathbf{L})$ is a left-monotonic, antitonic pointed left-residuated ℓ-groupoid. Unitality is preserved in the construction.*

3.2. Some Notable Subvarieties

In this subsection we undertake a preliminary study of the lattice of subvarieties of $\mathbb{PLRG}$. The resulting atlas includes, perhaps for the first time, the main varieties of residuated algebras and of algebras of quantum logic in a common framework. The first subvariety we focus on is a term equivalent counterpart of basic algebras.

Definition 25. A pointed left-residuated ℓ-groupoid $\mathbf{L} = (L, \wedge, \vee, \cdot, \rightarrow, 1, 0)$ is a *basic groupoid* if it is unital, divisible and strongly involutive.

Given a basic algebra $\mathbf{A}$, if we put $x \cdot y = (x' \oplus y')'$, $x \rightarrow y = y \oplus x'$, and $1 = 0'$, then $\mathcal{L}(\mathbf{A}) = (A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ is a basic groupoid [8]. Conversely, any basic groupoid can be converted into a basic algebra $\mathcal{B}(\mathbf{L}) = (L, \oplus, ', 0)$ by setting $x \oplus y = (x' \cdot y')'$. Moreover, $\mathcal{B}(\mathcal{L}(\mathbf{A})) = \mathbf{A}$ and $\mathcal{L}(\mathcal{B}(\mathbf{L})) = \mathbf{L}$ [21]. In view of the above considerations, we have the following:

Lemma 26. *Basic groupoids are term equivalent to basic algebras.*

We denote by $\mathbb{B}$ the subvariety of $\mathbb{PLRG}$ corresponding to basic groupoids. We also observe that a basic algebra $\mathbf{B}$ satisfies (ec) from Section 2.2 if and only if $\mathcal{L}(\mathbf{B})$ satisfies the following *effect groupoid quasiequation*:

$$x \leq y \;\&\; y \rightarrow x \leq z \Rightarrow z \rightarrow (y \rightarrow x) \approx z \cdot y \rightarrow x. \tag{eg}$$

Indeed, for all $a, b \in \mathcal{L}(B)$, we have that $a \oplus b = a \oplus b'' = b' \rightarrow a$. Therefore, under the indicated assumptions, if $a, b, c \in \mathcal{L}(B)$,

$$c \rightarrow (b \rightarrow a) = (a \oplus b') \oplus c' = a \oplus (c' \oplus b') = c \cdot a \rightarrow b.$$

Conversely, assume that $a \leq b'$ and $a \oplus b \leq c'$ in $\mathbf{B}$. Then one has $b' \to a \leq c'$ in $\mathcal{L}(\mathbf{B})$. By (eg), it follows that $c' \to (b' \to a) = c' \cdot b' \to a$. Translating both sides of the preceding equality, one has $(a \oplus b) \oplus c = a \oplus (c \oplus b)$.
Finally, note that (eg) is equivalent to the following identity (hereafter denoted by (eg) as well):

$$(x \cdot (x' \vee y)) \cdot ((x' \vee y) \vee z \to x') \approx x \cdot (((x' \vee y) \vee z \to x') \cdot (x' \vee y)).$$

Therefore, we conclude that lattice effect algebras are term equivalent to the subvariety $\mathbb{LEA}$ of basic groupoids axiomatised relative to $\mathbb{B}$ by (eg).

Since any orthomodular lattice induces a lattice effect algebra, it seems reasonable to surmise that a handy characterisation of this class in the context of left-residuated ℓ-groupoids can be achieved. This result was first obtained by Chajda and Länger [22].

Definition 27. A pointed left-residuated ℓ-groupoid $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, 1, 0)$ is an *orthomodular groupoid* if it is involutive, antitone, strongly idempotent, Sasakian and $\to$-Sasakian.

Let $\mathbf{A} = (A, \wedge, \vee, ', 0, 1)$ be an orthomodular lattice. The algebra $f(\mathbf{A}) = (A, \wedge, \vee, \cdot, \to, 0, 1)$, where

$$x \cdot y = (x \vee y') \wedge y \text{ (Sasaki projection);}$$
$$x \to y = (y \wedge x) \vee x' \text{ (Sasaki hook),}$$

is an orthomodular groupoid. Conversely, given an orthomodular groupoid $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, 0, 1)$, the algebra $g(\mathbf{L}) = (L, \wedge, \vee, ', 0, 1)$, where $x' = x \to 0$, is an orthomodular lattice. Further, the correspondences f and g are mutually inverse. Thus, we have:

Theorem 28. [22] *Orthomodular groupoids are term equivalent to orthomodular lattices.*

The variety of orthomodular groupoids will be denoted by $\mathbb{OG}$. In what follows, Theorem 28 will be used without special mention to speed up computations. In other words, we will prove our arithmetical results in whatever signature (ortholattices or pointed left-residuated ℓ-groupoids) we find most convenient.

Orthomodular groupoids, in general, fail left-monotonicity. It is easy to see that $\mathbf{L} \in \mathbb{OG}$ is left-monotonic if and only if $g(\mathbf{L})$ is Boolean (cf. [12]). If $g(\mathbf{L})$ is Boolean, then $\mathbf{L}$ satisfies $x \cdot y \approx x \wedge y$, whence left-monotonicity follows. Conversely, if $\mathbf{L}$ is left-monotonic, then by integrality and unitality $a \leq 1$ implies $b \cdot a \leq b \cdot 1 = b$ and the latter inequality implies that $g(\mathbf{L})$ is Boolean. A restricted form of left-monotonicity, though, holds in $\mathbb{OG}$. This is but one of a bunch of useful observations that we collect in the next lemma.

Lemma 29. *Let* $\mathbf{L}$ *be an orthomodular groupoid, and let* $a, b, c \in L$. *Then:* (i) $a \cdot (a \to b) \leq b$; (ii) $a \cdot b \leq b$; (iii) *if* a *commutes with* b, *then* $a \cdot b \leq c$ *if and only if* $b \leq a \to c$; (iv) *if* a *commutes with* b *and* $b \leq c$, *then* $a \cdot b \leq a \cdot c$; (v) $a \to a \cdot b = a \to b = a \to (a \wedge b)$; (vi) $b \to a \cdot b = a \vee (b \to 0)$.

Proof. We confine ourselves to (v) and (vi).

(v) Since $a \cdot b \leq b$ by (ii), by Lemma 20.(iii) $a \to a \cdot b \leq a \to b$. Since by Lemma 23.(iii) $(a \to b) \cdot a \leq a \wedge b$, we also have that $a \to b \leq a \to (a \wedge b)$. Finally, $a \wedge b \leq (a \vee (b \to 0)) \wedge b = a \cdot b$, so by Lemma 20.(iii) $a \to (a \wedge b) \leq a \to a \cdot b$.

(vi) $b \to a \cdot b = (a \cdot b \wedge b) \vee (b \to 0) = ((a \vee (b \to 0)) \wedge b) \vee (b \to 0) = a \vee (b \to 0)$. ■

Actually, Chajda and Länger's axiomatisation of $\mathbb{OG}$ can be streamlined in different ways. We collect these alternative sets of postulates in the following

Theorem 30. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, 1, 0)$ *be a pointed left-residuated* ℓ-*groupoid. The following are equivalent:*

(i) $\mathbf{L}$ *is orthomodular;*

(ii) $\mathbf{L}$ *is strongly idempotent and strongly involutive;*

(iii) $\mathbf{L}$ *is unital, Sasakian and strongly involutive.*

Proof. (i) implies (ii). We only need to prove that $\mathbf{L}$ is strongly involutive. This is immediate if we use Theorem 28; however, we include here a direct proof. For $a, b \in L$, we use involutivity and antitonicity, plus the fact that $\mathbf{L}$ is Sasakian and $\to$-Sasakian, and argue as follows:

$$\begin{aligned}
((a \to 0) \cdot b) \to 0 &= (((a \to 0) \vee (b \to 0)) \wedge b) \to 0 \\
&= (((a \to 0) \to 0) \wedge ((b \to 0) \to 0)) \vee (b \to 0) \\
&= (a \wedge b) \vee (b \to 0) \\
&= b \to a.
\end{aligned}$$

(ii) implies (iii). By Lemma 23.(ii) every strongly idempotent $\mathbf{L} \in \mathbb{PLRG}$ is unital, whence in particular $a \cdot 1 \leq a$ implies $a \leq 1 \to a$. Thus, by strong involutivity, $a = 1 \to a = ((a \to 0) \cdot 1) \to 0 = (a \to 0) \to 0$. Moreover, suppose that $a \leq b = (b \to 0) \to 0$, whence $a \cdot (b \to 0) \to 0 \geq 1$. Applying strong involutivity, $(b \to 0) \to (a \to 0) \geq 1$, whence $b \to 0 \leq a \to 0$. Thus, strong involutivity and unitality jointly imply involutivity and antitonicity. By strong idempotency, $((a \to 0) \wedge b) \cdot b = (a \to 0) \wedge b \leq a \to 0$, whence $(a \to 0) \wedge b \leq b \to (a \to 0)$ and thus by strong involutivity, involutivity and antitonicity $a \cdot b = (b \to (a \to 0)) \to 0 \leq ((a \to 0) \wedge b) \to 0 = a \vee (b \to 0)$. Moreover, by Lemma 23.(i) and Lemma 20.(ii) we have $a \cdot b \leq b$ and so $a \cdot b \leq (a \vee (b \to 0)) \wedge b$. Conversely, note that $(a \vee (b \to 0)) \wedge b \leq a \vee (b \to 0)$; thus, by strong idempotency, Lemma 19, Lemma 20.(ii)–(iii) and Lemma 23,

$$\begin{aligned}(a \vee (b \to 0)) \wedge b &= ((a \vee (b \to 0)) \wedge b) \cdot b \leq (a \vee (b \to 0)) \cdot b \\ &= a \cdot b \vee ((b \to 0) \cdot b) \leq a \cdot b \vee 0 = a \cdot b.\end{aligned}$$

(iii) implies (i). We must prove that **L** is strongly idempotent and $\to$-Sasakian; this will suffice for our claim, by the proof of the previous item. We proved on that occasion that strong involutivity and unitality jointly imply involutivity and antitonicity. Thus $a \to b = (b \to 0) \cdot a \to 0 = (((b \to 0) \vee (a \to 0)) \wedge a) \to 0$, which boils down to $a \to b = (b \wedge a) \vee (a \to 0)$. Hence **L** is $\to$-Sasakian. By lattice absorption,

$$\begin{aligned}a \wedge b &= (a \wedge b) \wedge ((b \to 0) \vee (a \wedge b)) \\ &= a \wedge b \wedge (b \to a).\end{aligned}$$

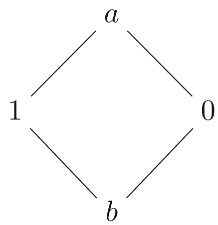
So $a \wedge b \leq b \to a$, hence $(a \wedge b) \cdot b \leq a$. Also, $(a \wedge b) \cdot b = ((a \wedge b) \vee (b \to 0)) \wedge b \leq b$. Thus, $(a \wedge b) \cdot b \leq a \wedge b$. Conversely, $a \wedge b \leq (a \wedge b) \vee (b \to 0)$ and $a \wedge b \leq b$, whence $a \wedge b \leq ((a \wedge b) \vee (b \to 0)) \wedge b = (a \wedge b) \cdot b$. Summing up, $(a \wedge b) \cdot b = a \wedge b$, and strong idempotency is obtained replacing b by $a \vee b$. ■

It is easily seen that left-monotonicity is equivalent to the identity $z \cdot x \leq z \cdot (x \vee y)$. Thus, left-monotonic members of $\mathbb{PLRG}$ form a variety, hereafter denoted by $\mathbb{PLRG}_{lm}$. In light of the above considerations, since for any $\mathbf{L} \in \mathbb{OG}$, if **L** is not Boolean, then $\mathbf{L} \in \mathbb{B} \backslash \mathbb{PLRG}_{lm}$, we conclude that $\mathbb{B} \nsubseteq \mathbb{PLRG}_{lm}$. Furthermore, any non-divisible pointed commutative residuated lattice is witness to the fact that $\mathbb{PLRG}_{lm} \nsubseteq \mathbb{B}$. Left-distributive pointed left-residuated ℓ-groupoids form a variety $\mathbb{PLRG}_{ld}$, which is a subvariety of $\mathbb{PLRG}_{lm}$. The inclusion is obvious. That such an inclusion is proper is shown by the next

Example 31. Let us consider the pointed left-residuated ℓ-groupoid

$$\mathbf{L} = (\{0, 1, a, b\}, \wedge, \vee, \cdot, \to, 0, 1)$$

with the Hasse diagram



whose operations are defined according to the following Cayley tables:

$\cdot$	0	1	a	b
0	0	0	a	0
1	0	1	a	b
a	0	a	a	0
b	b	b	b	b

$\to$	0	1	a	b
0	a	0	b	a
1	b	1	b	1
a	a	a	a	a
b	b	b	b	1

A routine check shows that $\mathbf{L}$ is left-monotonic. However, one has that $0 \cdot (0 \vee 1) = 0 \cdot a = a \neq 0 = 0 \vee 0 = 0 \cdot 0 \vee 0 \cdot 1$.

By generalising to the non-integral and non-associative case Ono and Komori's [46] embedding technique, one can characterise the $\backslash$-free subreducts of unital pointed residuated ℓ-groupoids (in the interests of space, proof is omitted):

Proposition 32. *Let $\mathbf{L}$ be a unital pointed left-residuated ℓ-groupoid. Then $\mathbf{L}$ is the $\backslash$-free subreduct of a unital pointed residuated ℓ-groupoid if and only if it is left-distributive.*

Another well-understood subvariety of $\mathbb{PLRG}_{lm}$ is represented by *pointed commutative residuated ℓ-groupoids*, which satisfy the identity $x \cdot y \approx y \cdot x$. This subvariety, here referred to as $\mathbb{PCRG}$, has been the object of extensive investigations from both the algebraic and the proof-theoretical perspective (see e.g. [23,29,34]). This applies, in particular, to its most prominent subvarieties, such as the varieties $\mathbb{PCRL}$ of pointed commutative residuated lattices, $\mathbb{MV}$ of MV-algebras and $\mathbb{H}$ of Heyting algebras.

It is well-known that a pointed residuated lattice is (term equivalent to) a Heyting algebra if and only if it is 0-bounded and satisfies the identity $x \cdot y \approx x \wedge y$ [43]. Equally succint axiomatisations of $\mathbb{H}$ are available in the context of $\mathbb{PLRG}$.

Proposition 33. *Let $\mathbf{L} \in \mathbb{PLRG}$. The following are equivalent:*

(i) $\mathbf{L}$ *is a Heyting algebra;*

(ii) **L** *is* 0*-bounded and satisfies the identity* $x \cdot y \approx x \wedge y$*;*

(iii) **L** *is unital, distributive and Sasakian;*

(iv) **L** *is unital, left-monotonic and Sasakian.*

PROOF. (i) implies (ii). This implication is clear from the preceding remarks.

(ii) implies (iii). Unitality is obvious. Let $a, b, c \in L$. Using the assumption and Lemma 19,

$$(a \vee b) \wedge c = (a \vee b) \cdot c = a \cdot c \vee b \cdot c = (a \wedge c) \vee (b \wedge c)\,.$$

Moreover, our assumptions imply that $0 \leq (b \to 0) \wedge b = (b \to 0) \cdot b \leq 0$, hence $(b \to 0) \wedge b = 0$. Therefore, since distributivity has already been proved,

$$a \cdot b = a \wedge b = (a \wedge b) \vee ((b \to 0) \wedge b) = (a \vee (b \to 0)) \wedge b.$$

(iii) implies (iv). Under the indicated hypotheses, **L** is integral and 0-bounded. Indeed, for all $a \in L$,

$$\begin{aligned} 1 &= 1 \vee (1 \wedge (a \vee (1 \to 0))) = 1 \vee a \cdot 1 = 1 \vee a, \\ a &= a \cdot 1 = (a \vee (1 \to 0)) \wedge 1 = (a \vee 0) \wedge 1 = a \vee 0. \end{aligned}$$

Consequently, observe that for all x,

$$\begin{aligned} 0 &\leq (x \to 0) \wedge x = ((x \to 0) \wedge x) \vee ((x \to 0) \wedge x) \\ &= ((x \to 0) \vee (x \to 0)) \wedge x = (x \to 0) \cdot x \leq 0. \end{aligned}$$

Now, suppose that $b \leq c$. Then $a \wedge b \leq a \wedge c$, and thus

$$\begin{aligned} a \cdot b &= (a \vee (b \to 0)) \wedge b = (a \wedge b) \vee ((b \to 0) \wedge b) \\ &\leq (a \wedge c) \vee ((c \to 0) \wedge c) = (a \vee (c \to 0)) \wedge c = a \cdot c. \end{aligned}$$

(iv) implies (i). By the proof of the previous item (which does not depend on distributivity in the relevant part), **L** is 0-bounded and integral. Fix $a, b \in L$. Since **L** is Sasakian, we have that $a \wedge b \leq a \cdot b$, and since **L** is left-monotonic and integral, $a \cdot b \leq a$ and $a \cdot b \leq b$. Hence $a \cdot b = a \wedge b$. It follows that **L** is right-residuated and product is associative and commutative, hence **L** is a commutative pointed residuated lattice. Therefore **L** is a Heyting algebra. ■

It is well-known that MV-algebras are exactly associative basic algebras [18,20]. Also, a basic algebra **B** is associative if and only if the pointed left-residuated ℓ-groupoid $\mathcal{L}(\mathbf{B})$ is such. Indeed, if **B** is associative, then it is an MV-algebra and therefore $\mathcal{L}(\mathbf{B})$ is a commutative, divisible, integral, distributive, involutive residuated lattice. Moreover, if $\mathcal{L}(\mathbf{B})$ is associative, then in **B** one has

$$(x \oplus (y \oplus z))' = x' \cdot (y \oplus z)' = x' \cdot (y' \cdot z') = (x' \cdot y') \cdot z' = ((x \oplus y) \oplus z)'.$$

Applying $'$ to both sides of the preceding identity, associativity follows since $'$ is an involution. Moreover, regarded as a residuated lattice, any MV-algebra $\mathbf{L}$ satisfies the identity

$$x \to (y \to z) \approx x \cdot y \to z.$$

Therefore, $\mathbf{L}$ trivially satisfies (eg) and we conclude that $\mathbb{MV}$, the term equivalent counterpart of MV-algebras in $\mathbb{PLRG}$, is included in $\mathbb{LEA}$ (see also [31]). By Theorem 30 and Lemma 23.(iii), orthomodular groupoids are a proper subvariety of $\mathbb{B}$. Moreover, using the term equivalences in Lemma 26 and Theorem 28, it is easily seen that any orthomodular groupoid is an effect groupoid (cf. e.g. [28, Ch. 5]). Therefore, we conclude that $\mathbb{OG} \subseteq \mathbb{LEA}$. Summing up, the variety $\mathbb{MV} \vee \mathbb{OG}$ is a subvariety of $\mathbb{LEA}$. Now, within the lattice of subvarieties of basic algebras, the varietal join of MV-algebras and orthomodular lattices is strictly smaller than the variety of effect basic algebras [39]. In [39], a relative equational basis for this join is given via the identity

$$x \oslash (x \oplus y) \wedge z \wedge z' = 0,$$

where $x \oslash y = (x' \oplus y)'$. We now give an analogous result in the context of $\mathbb{PLRG}$.

Lemma 34. *Let* $\mathbf{L} \in \mathbb{PLRG}$*. The following are equivalent:*

(i) $\mathbf{L} \in \mathbb{MV} \vee \mathbb{OG}$*;*

(ii) $\mathbf{L}$ *is a basic groupoid that satisfies* (eg) *and the following identity:*

$$x \cdot ((x \to 0) \cdot y) \wedge (z \wedge (z \to 0)) = 0. \qquad \text{(mv+og)}$$

Proof. If $\mathbf{L} \in \mathbb{MV} \vee \mathbb{OG}$, then $\mathcal{B}(\mathbf{L})$ is an effect basic algebra satisfying $x \oslash (x \oplus y) \wedge z \wedge z' = 0$ (see [39]). Therefore, $\mathbf{L}$ is an effect groupoid satisfying mv+og. Conversely, under the indicated assumptions, $\mathcal{B}(\mathbf{L})$ is an effect basic algebra and the translation of (mv+og) yields that $x \oslash (x \oplus y) \wedge z \wedge z' = 0$ is satisfied. We conclude that $\mathbf{L} \in \mathbb{MV} \vee \mathbb{OG}$. ■

In the investigations on probability theory and the theory of expert systems, a prominent role is played by non-associative fuzzy logics [9,26]. A case in point is the weakly implicative logic $\mathcal{L}_{CBA}$, which is algebraisable with the variety of commutative basic algebras as an equivalent algebraic semantics [9]. In our framework, this variety corresponds to the commutative subvariety $\mathbb{CB}$ of the variety $\mathbb{B}$ of basic groupoids. Moreover, $\mathbb{CB} = \mathbb{PCRG} \cap \mathbb{B}$. Observe further that $\mathbb{MV} \subset \mathbb{CB}$ (cf. [15]) and $\mathbb{CB} \not\subseteq \mathbb{LEA}$. In fact, assume

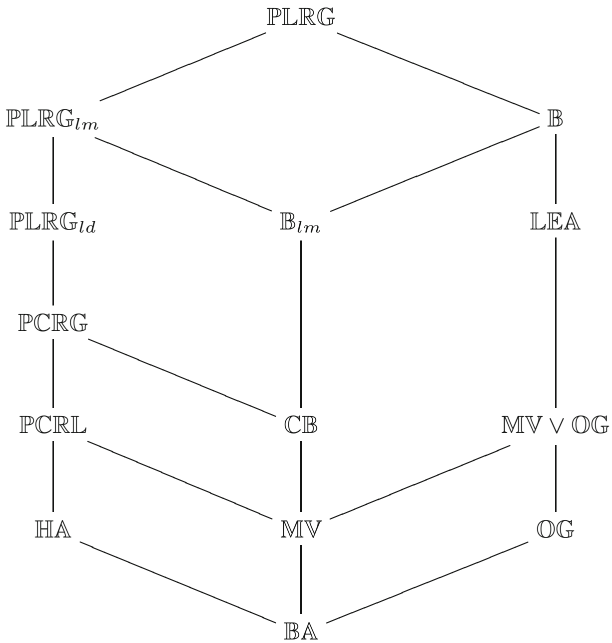


Figure 1. The lattice of subvarieties of $\mathbb{PLRG}$

by way of contradiction that $\mathbb{CB} \subseteq \mathbb{LEA}$ and consider $\mathbf{L} \in \mathbb{CB}$. Thus, there exists a commutative lattice effect algebra $\mathbf{A}$ such that $\mathcal{L}(\mathbf{A}) = \mathbf{L}$. By [17, Corollary 4.7], $\mathbf{A}$ is an MV-algebra and $\mathbf{L} = \mathcal{L}(\mathbf{A}) \in \mathbb{MV}$. Since $\mathbf{L}$ was arbitrary, we would have that $\mathbb{CB} \subset \mathbb{MV}$, which contradicts the above observations. Finally, let us briefly discuss the variety $\mathbb{B}_{lm}$ of *left-monotonic* basic groupoids, which is term equivalent to the variety of monotone basic algebras (see [10]). Remarkably enough, it has been shown in [10, Theorem 4.7] that every finite monotone basic algebra is an MV-algebra. So, any finite member of $\mathbb{B}_{lm}$ is in $\mathbb{CB}$. However, [21, Example 10] proves that there are infinite monotone and non-commutative basic algebras. So we conclude that $\mathbb{CB} \subsetneq \mathbb{B}_{lm}$.

Figure 1 provides a synopsis of the main subvarieties of $\mathbb{PLRG}$ that we have encountered in our investigation.

3.3. Ideals and Congruences

Both residuated lattices and orthomodular lattices have very strong congruence properties and a well-behaved ideal theory. Thus, it makes sense to investigate the extent to which these features carry on to their common generalisation of pointed left-residuated ℓ-groupoids. It turns out that the outcome is rather satisfactory. For a start, $\mathbb{PLRG}$ is both arithmetical and 1-ideal-determined, as we now proceed to show.

Recall that a necessary and sufficient condition for a variety $\mathbb{V}$ to be congruence permutable is the existence of a *Maltsev term* for $\mathbb{V}$, namely, a ternary term $p(x,y,z)$ s.t. $\mathbb{V} \vDash p(x,x,y) \approx y, p(x,y,y) \approx x$ [14, Thm. II.12.2]. Also, recall that a necessary and sufficient condition for a pointed variety $\mathbb{V}$ (with a constant 1 in its type) to be 1-regular is the existence of *Csákány terms* for $\mathbb{V}$, viz. binary terms $b_1(x,y), \ldots, b_n(x,y)$ s.t. $\mathbb{V} \vDash b_1(x,y) \approx 1 \,\&...\&\, b_n(x,y) \approx 1 \iff x \approx y$ [27]. The reader is further reminded that for a pointed variety $\mathbb{V}$ (with a constant 1 in its type) the property of 1-subtractivity is equivalent to 1-permutability (thus obviously implied by congruence permutability: [38, Cor. 1.9]). A pointed variety is 1-*ideal-determined* in case it is both 1-subtractive and 1-regular; if $\mathbf{A}$ is a member of a 1-ideal-determined variety $\mathbb{V}$, the lattice $\mathcal{I}(\mathbf{A})$ of Ursini $\mathbb{V}$-ideals of $\mathbf{A}$ is isomorphic to the lattice $\mathrm{Con}(\mathbf{A})$ of congruences of $\mathbf{A}$ [2, Thm. 2.6].

Lemma 35. $\mathbb{PLRG}$ *is an arithmetical and* 1*-ideal-determined variety.*

Proof. Clearly, $\mathbb{PLRG}$ is congruence distributive since it is a variety of lattice-ordered algebras. For congruence permutability, observe that, setting $x \leftrightarrow y = (x \to y) \wedge (y \to x) \wedge 1$, the term

$$p(x,y,z) = ((x \leftrightarrow y) \cdot z) \vee ((y \leftrightarrow z) \cdot x)$$

is a Maltsev term for $\mathbb{PLRG}$. The term $x \leftrightarrow y$, moreover, is easily seen to be a Csákány term for $\mathbb{PLRG}$. Therefore $\mathbb{PLRG}$ is 1-regular and congruence permutable, whence also 1-ideal-determined. ■

Analogues of Lemma 35 have been provided for several subvarieties of $\mathbb{PLRG}$. In particular, Chajda and Radelecki [23] come very close to the generality attained in this lemma, since the only additional assumption they make on their algebras are 0-boundedness and integrality.

When studying a 1-ideal determined variety $\mathbb{V}$, the obvious question arises as to whether the Ursini $\mathbb{V}$-ideals of members of $\mathbb{V}$ can be given a handy description, possibly by specifying a finite basis of ideal terms. It is well-known that this problem has been successfully addressed for several subvarieties of $\mathbb{PLRG}$. In particular, if $\mathbb{V}$ is either the class of the $\backslash$-free reducts of pointed 1-cyclic residuated lattices or the class of pointed commutative residuated lattices, and $\mathbf{A} \in \mathbb{V}$, $H \subseteq A$ is a $\mathbb{V}$-ideal of $\mathbf{A}$ if and only if it is a convex (normal) subuniverse of the 0-free reduct of $\mathbf{A}$ [6,42]. Other characterisations of ideals in individual varieties that are term equivalent to subvarieties of

$\mathbb{PLRG}$ are included e.g. in [12,19,49]. We observe that occasionally, e.g. for orthomodular lattices [12, Prop. 4.7], we have extremely elegant streamlined descriptions.

Although we were unable to specify a finite basis of ideal terms for $\mathbb{PLRG}$, ideals can be described in at least two cases, jointly encompassing all the subvarieties of interest mentioned in this paper: left-monotonic pointed left-residuated ℓ-groupoids and basic groupoids. For a start observe that, mimicking the strategy in [6], the following lemma can be proved without difficulty:

LEMMA 36. *Let* $\mathbf{L} \in \mathbb{PLRG}$ *and* $\theta \in \mathrm{Con}\,(\mathbf{L})$. *The following conditions are equivalent:*

(i) $a\ \theta\ b$;

(ii) $[(a \to b) \wedge 1]\ \ \theta\ 1$ *and* $[(b \to a) \wedge 1]\ \ \theta\ 1$.

Next, we define a putative notion of an ideal, with an eye to identifying the subvarieties $\mathbb{V}$ of $\mathbb{PLRG}$ where it actually describes $\mathbb{V}$-ideals.

DEFINITION 37. Let $\mathbf{L} \in \mathbb{PLRG}$ and let $H \subseteq L$ be a convex subuniverse of the 0-free reduct of $\mathbf{L}$. H is said to be *strongly normal* if:

(i) for all $a, b, c \in L$, $(a \to b) \wedge 1$ and $(b \to a) \wedge 1 \in H$ imply $(c \cdot a \to c \cdot b) \wedge 1$ and $(a \cdot c \to b \cdot c) \wedge 1 \in H$;

(ii) for all $a, b, c \in L$, $(a \to b) \wedge 1$ and $(b \to a) \wedge 1 \in H$ imply $((c \to a) \to (c \to b)) \wedge 1$ and $((a \to c) \to (b \to c)) \wedge 1 \in H$.

Observe that, if H is a convex strongly normal subuniverse of the 0-free reduct of $\mathbf{L}$ and $a \in H$, then for every $c \in L$ we have that $\lambda_c(a) = (c \to a \cdot c) \wedge 1 \in H$. We also prove that convex strongly normal subuniverses admit a simplified characterisation for orthomodular groupoids. Recall that, if $\mathbf{L}$ is an orthomodular lattice, then $H \subseteq L$ is an $\mathbb{OML}$-ideal of $\mathbf{L}$ if and only if it is a lattice filter of $\mathbf{L}$ such that for all $a \in H, b \in L$, we have that $(a \wedge b') \vee b \in H$. Correspondingly:

PROPOSITION 38. *Let* $\mathbf{L} \in \mathbb{OG}$ *and let* $\emptyset \neq H \subseteq L$. *Then* H *is a convex strongly normal subuniverse of the* 0*-free reduct of* $\mathbf{L}$ *if and only if it is a lattice filter of* $\mathbf{L}$ *such that for all* $a \in H, b \in L$, *we have that* $b \to a \in H$.

PROOF. If H is a convex strongly normal subuniverse of $\mathbf{L}$ and $a \in H, b \in L$, then $(a \to 1) \wedge 1, (1 \to a) \wedge 1 \in H$ and thus, using Lemma 29.(v), we obtain that $b \to ba = b \to a \in H$.

Conversely, let H be a lattice filter of $\mathbf{L}$ such that for all $a \in H, b \in L$, we have that $b \to a \in H$. Then H contains 1 and is closed w.r.t. meets and joins, so in particular if $a, b \in H$ then $a \wedge b \in H$ and $a \wedge b \leq (a \vee b') \wedge b = a \cdot b \in H$, thus $a \to b = (b \wedge a) \vee a' \geq b \wedge a$, and so $a \to b \in H$. Hence H is a convex subuniverse of the 0-free reduct of $\mathbf{L}$. Also, if $a \to b, b \to a \in H$, then since H is an $\mathbb{OML}$-ideal of $\mathbf{L}$, it follows that

$$\{(a, b) : a \to b, b \to a \in H\}$$

is an equivalence that respects $\wedge, \vee$ and $'$, hence also product and implication, which are definable in terms of these operations. So conditions (1) and (2) in Definition 37 are satisfied. ∎

We now show that the 1-class of a congruence is always a convex strongly normal subuniverse of the 0-free reduct of $\mathbf{L}$.

Lemma 39. *If $\mathbf{L} \in \mathbb{PLRG}$ and $\theta \in \mathrm{Con}\,(\mathbf{L})$, then $1/\theta = \{a \in A : a\ \theta\ 1\}$ is a convex strongly normal subuniverse of the 0-free reduct of $\mathbf{L}$.*

Proof. $1/\theta$ is a subuniverse of the 0-free reduct of $\mathbf{L}$ because 1 is idempotent with respect to all the binary operations of $\mathbf{L}$. Moreover, members of $\mathrm{Con}\,(\mathbf{L})$ are in particular lattice congruences, whence convexity follows. Finally, let $(a \to b) \wedge 1 \in 1/\theta$ and $(b \to a) \wedge 1 \in 1/\theta$. By Lemma 36, $a\theta b$. Thus, for any $c \in L$, $c \cdot a \theta c \cdot b$ and $a \cdot c \theta b \cdot c$. Applying Lemma 36 backwards, $(c \cdot a \to c \cdot b) \wedge 1 \in 1/\theta$ and $(a \cdot c \to b \cdot c) \wedge 1 \in 1/\theta$, which suffices for (1) in Definition 37. Item (2) is established similarly. ∎

As usual, associating a suitable convex strongly normal subuniverse to a given congruence is slightly trickier. If $\mathbf{L} \in \mathbb{PLRG}$ and $H \subseteq L$, let $\theta_H = \{(a, b) : (a \to b) \wedge 1 \in H \text{ and } (b \to a) \wedge 1 \in H\}$. Following closely [6] again, we obtain that:

Lemma 40. $\theta_H = \{(a, b) : \exists h \in H, h \cdot a \leq b$ *and* $h \cdot b \leq a\}$.

Lemma 41. *Let $\mathbf{L} \in \mathbb{PLRG}$ be either left-monotonic or a basic groupoid, and let H be a convex strongly normal subuniverse of the 0-free reduct of $\mathbf{L}$. Then θ_H is a congruence on $\mathbf{L}$.*

Proof. Since $\mathbb{PLRG}$ and its subvarieties are congruence permutable, to achieve our conclusion it suffices to show that θ_H is a tolerance on $\mathbf{L}$. θ_H is clearly reflexive and symmetric, and by items (1) and (2) in Definition 37 it is compatible with product and implication on both sides. To show compatibility with the lattice operations, we distinguish cases.

Let $\mathbf{L}$ be a basic groupoid, $a, b, c \in L$, and $(a, b) \in \theta_H$. Thus, $(a \to b) \wedge 1, (b \to a) \wedge 1 \in H$. By (2) in Definition 37, $((c \to a) \to (c \to b)) \wedge 1 \in H$

and $((c \to b) \to (c \to a)) \wedge 1 \in H$, whence by (1) in the same Definition and by divisibility,

$$((a \wedge c) \to (b \wedge c)) \wedge 1 = ((c \to a) \cdot c \to (c \to b) \cdot c) \wedge 1 \in H,$$

and similarly $((b \wedge c) \to (a \wedge c)) \wedge 1 \in H$. Compatibility with join follows from the involutivity condition.

On the other hand, let $\mathbf{L}$ be left-monotonic, $a, b, c \in L$, and $(a, b) \in \theta_H$. Therefore $(a \to b) \wedge 1, (b \to a) \wedge 1 \in H$. Since $(a \to b) \wedge 1 \leq 1$ and $(a \to b) \wedge 1 \leq a \to b$, we obtain on the one hand that $((a \to b) \wedge 1) \cdot c \leq c$, and on the other that $((a \to b) \wedge 1) \cdot a \leq b$. By left-monotonicity,

$$((a \to b) \wedge 1) \cdot (a \wedge c) \leq ((a \to b) \wedge 1) \cdot a \wedge ((a \to b) \wedge 1) \cdot c \leq b \wedge c.$$

Hence $(a \to b) \wedge 1 \leq (a \wedge c) \to (b \wedge c)$, whereby $(a \to b) \wedge 1 \leq ((a \wedge c) \to (b \wedge c)) \wedge 1 \leq 1$, and we obtain $((a \wedge c) \to (b \wedge c)) \wedge 1 \in H$ by convexity. The remaining compatibility condition for meet, as well as the conditions for join, are analogous. ■

THEOREM 42. *Let* $\mathbf{L} \in \mathbb{PLRG}$ *be either a* 1*-cyclic left-monotonic or a basic groupoid. The lattice* $\mathrm{CN}(\mathbf{L})$ *of convex strongly normal subuniverses of* $\mathbf{L}$ *is isomorphic to* $\mathrm{Con}(\mathbf{L})$*. The isomorphism is given by the mutually inverse maps* $H \mapsto \theta_H$ *and* $\theta \mapsto 1/\theta$*.*

PROOF. Before we start, we remind the reader that basic groupoids are integral and therefore 1-cyclic.

If $H \in \mathrm{CN}(\mathbf{L})$ and $\theta \in \mathrm{Con}(\mathbf{L})$, then θ_H is a congruence by Lemma 41 and $1/\theta$ is a convex strongly normal subuniverse by Lemma 39. It is clear that the maps $H \mapsto \theta_H$ and $\theta \mapsto 1/\theta$ are isotone. It remains to be shown that they are mutually inverse, since it will then follow that they are lattice homomorphisms.

Given $\theta \in \mathrm{Con}(\mathbf{L})$,

$$\begin{aligned} a\ \theta\ b &\Leftrightarrow (a \to b) \wedge 1\ \theta\ 1 \text{ and } (b \to a) \wedge 1\ \theta\ 1 \\ &\Leftrightarrow (a \to b) \wedge 1\ \in 1/\theta \text{ and } (b \to a) \wedge 1\ \in 1/\theta \\ &\Leftrightarrow a\ \theta_{1/\theta}\ b \end{aligned}$$

For the other direction, for any $H \in \mathrm{CN}(\mathbf{L})$ we must show that $H = 1/\theta_H$. However, if $a \in H$, then $(1 \to a) \wedge 1, (a \to 1) \wedge 1 \in H$, so $a \in 1/\theta_H$. Conversely, if $a \in 1/\theta_H$, in light of Lemma 40 there exists some $h \in H$ such that $h \cdot a \leq 1$ and $h \cdot 1 \leq a$. By 1-cyclicity $a \cdot h \leq 1$ and thus $a \leq h \to 1$, whence by the convexity of H, $h \cdot 1 \leq a \leq h \to 1$ implies $a \in H$. ■

3.4. Left Nuclei

Just like closure operators are the bread and butter of the working order theorist, nuclei (closure operators that interact smoothly with the product operation) are a fundamental tool for the practicioner of residuated structures. The theory of nuclei for residuated lattices is rich and far-reaching (see [33, 45] for comprehensive accounts); much of it transfers to the non-associative case with only a modest detriment. When we drop one residual, however, one needs more extensive adjustments to get started. In this subsection, we try to recover some important parts of the theory of nuclei of residuated ℓ-groupoids in the left-residuated case. Henceforth, to simplify notation, we mostly omit dots in product terms, replacing them by mere juxtaposition.

Let $\mathbf{L} \in \mathbb{PLRG}$, and let $\gamma : L \to L$. For future reference, we list below some conditions that γ may satisfy for all $a, b \in L$:

N1 $\gamma(a)\gamma(b) \leq \gamma(ab)$;

N2 $\gamma(a)\gamma(b) \leq \gamma(a\gamma(b))$;

N3 $a \to \gamma(b) = \gamma(a \to \gamma(b))$;

N4 $a \to \gamma(b) \leq \gamma(a) \to \gamma(b)$.

Clearly, N1 implies N2 whenever γ is idempotent.

Definition 43. Let $\mathbf{L} \in \mathbb{PLRG}$. A closure operator γ on $\mathbf{L}^l$ is a *left nucleus* on $\mathbf{L}$ if and only if it satisfies N3; it is a *nucleus* on $\mathbf{L}$ if it satisfies N1 and N3.

Proposition 44. *Let $\mathbf{L} \in \mathbb{PLRG}$ and let γ be a closure operator on $\mathbf{L}^l$.*

(i) *If γ is a left nucleus, it satisfies N2.*

(ii) *If γ is a left nucleus and satisfies N4, it is a nucleus.*

(iii) *If $\mathbf{L} \in \mathbb{OG}$, N1 and N3 are equivalent; hence, γ is a nucleus on $\mathbf{L}$ if and only if it satisfies any of these equivalent conditions.*

Proof. (i) We have that:

$$\begin{array}{ll} a\gamma(b) \leq \gamma(a\gamma(b)) & \gamma \text{ is enlarging} \\ \Rightarrow a \leq \gamma(b) \to \gamma(a\gamma(b)) & \text{left resid.} \\ \Rightarrow \gamma(a) \leq \gamma(\gamma(b) \to \gamma(a\gamma(b))) & \gamma \text{ is monotonic} \\ \Rightarrow \gamma(a) \leq \gamma(b) \to \gamma(a\gamma(b)) & \text{N3} \\ \Rightarrow \gamma(a)\gamma(b) \leq \gamma(a\gamma(b)) & \text{left resid.} \end{array}$$

(ii) We have that:

$$
\begin{array}{ll}
\quad ab \leq \gamma(ab) & \gamma \text{ is enlarging} \\
\Rightarrow a \leq b \to \gamma(ab) & \text{left resid.} \\
\Rightarrow a \leq \gamma(b) \to \gamma(ab) & \text{N4} \\
\Rightarrow \gamma(a) \leq \gamma(\gamma(b) \to \gamma(ab)) & \gamma \text{ is monotonic} \\
\Rightarrow \gamma(a) \leq \gamma(b) \to \gamma(ab) & \text{N3} \\
\Rightarrow \gamma(a)\gamma(b) \leq \gamma(ab) & \text{left resid.}
\end{array}
$$

(iii) We first show that N3 implies N4, whence by (ii) N3 implies N1. To that effect, observe that b commutes with $\gamma(b)$ and that $\gamma(a) \wedge b$ commutes with $b = b \wedge \gamma(b)$, whence by Lemma 6.(iv) $b \to \gamma(a) = (\gamma(a) \wedge b) \vee b'$ commutes with $\gamma(b)$. Thus:

$$
\begin{array}{ll}
\quad b(b \to \gamma(a)) \leq \gamma(a) & \text{Lm. 29.(i)} \\
\Rightarrow b \leq (b \to \gamma(a)) \to \gamma(a) & \text{left resid.} \\
\Rightarrow b \leq \gamma(b \to \gamma(a)) \to \gamma(a) & \text{N3} \\
\Rightarrow \gamma(b) \leq \gamma(\gamma(b \to \gamma(a)) \to \gamma(a)) & \gamma \text{ is monotonic} \\
\Rightarrow \gamma(b) \leq \gamma(b \to \gamma(a)) \to \gamma(a) & \text{N3} \\
\Rightarrow \gamma(b) \leq (b \to \gamma(a)) \to \gamma(a) & \text{N3} \\
\Rightarrow \gamma(b)(b \to \gamma(a)) \leq \gamma(a) & \text{left resid.} \\
\Rightarrow b \to \gamma(a) \leq \gamma(b) \to \gamma(a) & \text{Lm. 29.(iii)}
\end{array}
$$

Next, we show that N1 implies N3. It clearly suffices to establish $\gamma(a \to \gamma(b)) \leq a \to \gamma(b)$. Using properties of nuclei and Lemma 29.(iii):

$$\gamma(b \to \gamma(a))\, b \leq \gamma(b \to \gamma(a))\,\gamma(b) \leq \gamma((b \to \gamma(a))\, b) \leq \gamma(\gamma(a)) = \gamma(a),$$

whence, by left-residuation, $\gamma(b \to \gamma(a)) \leq b \to \gamma(a)$. This application of the Lemma is justified because $b' \leq b \to \gamma(a) \leq \gamma(b \to \gamma(a))$, whence b' commutes with $\gamma(b \to \gamma(a))$ and so does b. ∎

The proof of Proposition 44 immediately implies the following:

Corollary 45. *If* $\mathbf{L} \in \mathbb{PLRG}_{lm}$*, and* γ *is a closure operator on* $\mathbf{L}^l$*, N1 implies N3; hence,* γ *is a nucleus on* $\mathbf{L}$ *if and only if it satisfies N1.*

The next theorem generalises a standard result in the theory of residuated lattices: for γ a nucleus, γ-closed elements are the universe of a residuated lattice under appropriately redefined operations. Here, something analogous happens:

Theorem 46. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, 0, 1)$ *be a pointed left-residuated* ℓ*-groupoid, and let* γ *be a (left) nucleus on it.*

(i) *The structure*

$$\mathbf{L}_\gamma = (\gamma[L], \wedge, \vee_\gamma, \cdot_\gamma, \rightarrow, \gamma(0), \gamma(1)),$$

where $a \vee_\gamma b = \gamma(a \vee b)$ *and* $a \cdot_\gamma b = \gamma(ab)$*, is a pointed left-residuated* ℓ*-groupoid, called a* (left) nucleus-system *of* $\mathbf{L}$*.*

(ii) *Integrality,* 0*-boundedness and left-monotonicity are preserved in the construction.*

(iii) *If* $\mathbf{L}$ *is left-monotonic, antitonicity holds in* $\mathbf{L}_\gamma$*.*

(iv) *If* $\mathbf{L}$ *is left-monotonic, the construction preserves the inequation* $x \leq x(x \vee y)$*, while if* γ *is a nucleus, the construction preserves the inequation* $x(x \vee y) \leq x$*.*

Proof. (i) $(\gamma[L], \wedge, \vee_\gamma)$ is a lattice by properties of closure operators. If $a \in \gamma[L]$, by left-unitality, Lemma 20.(ii) and the properties of left nuclei, $\gamma(1) \cdot_\gamma a = \gamma(\gamma(1)a) \leq \gamma(\gamma(1a)) = a$ and $1 \leq \gamma(1)$ implies $a \leq \gamma(1)a \leq \gamma(\gamma(1)a)$, whence $\gamma(1) \cdot_\gamma a = a$. Moreover, $\gamma[L]$ is closed with respect to implications by N3. It remains to be shown that $\mathbf{L}_\gamma$ is left-residuated. However, observe that, whenever $c = \gamma(c)$, $a \cdot_\gamma b \leq c$ if and only if $ab \leq c$. In fact, if $a \cdot_\gamma b = \gamma(ab) \leq c$, then $ab \leq \gamma(ab) \leq c$, while if $ab \leq c$, then $\gamma(ab) \leq \gamma(c) = c$. (ii) For integrality, simply observe that if $a \leq 1$ for all $a \in L$, then $1 \leq \gamma(1) \leq 1$ and our conclusion clearly follows. For 0-boundedness, if $0 \leq a$ for all $a \in L$, then by monotonicity of γ, $\gamma(0) \leq b$ for all $b \in \gamma[L]$. For left-monotonicity, if $a \leq b$, then for any c we have that $ca \leq cb$ and thus $c \cdot_\gamma a = \gamma(ca) \leq \gamma(cb) = c \cdot_\gamma b$. (iii) If $\mathbf{L}$ is left-monotonic, then $a \leq b$ implies $b \rightarrow c \leq a \rightarrow c$ for all c. In fact, if $a \leq b$, then $(b \rightarrow c)a \leq (b \rightarrow c)b \leq c$, whence the conclusion follows. So in particular $b \rightarrow \gamma(0) \leq a \rightarrow \gamma(0)$ whenever $a, b \in \gamma[L]$. (iv) If $\mathbf{L}$ is left-monotonic, then $a \leq a(a \vee b) \leq a\gamma(a \vee b) \leq \gamma(a\gamma(a \vee b)) = a \cdot_\gamma (a \vee_\gamma b)$. If γ is a nucleus and $a \in \gamma[L]$, then $a\gamma(a \vee b) \leq \gamma(a(a \vee b)) \leq \gamma(a) = a$, whereby $a \cdot_\gamma (a \vee_\gamma b) = \gamma(a\gamma(a \vee b)) \leq \gamma(a) = a$. ■

4. Completions

In this section, by and large inspired by the thought-provoking paper [35] from which several of the following results are adapted, we obtain some of the known results on completions of quantum structures as a spin-off from an investigation into pointed left-residuated ℓ-groupoids. To do so, we put to good use Theorem 28. The vantage point yielded by this result is remarkable. Indeed, (join-)completions of *residuated* ℓ-groupoids have been thoroughly studied in the literature on residuated lattices and on substructural logics [24,25,33–35]. These enquiries make a crucial recourse to a strengthening of Proposition 11 to the effect that join-completions of residuated ℓ-groupoids can be obtained as nucleus-systems of their ideal completions [35, Thm. 3.5]. Using in the same guise the theory of left nuclei we have just developed, we can obtain a new proof of Theorem 14 for orthomodular lattices.

A more ambitious continuation of this project would be as follows. The papers [24,25] build a fascinating hierarchy of complexity classes of formulas in the language of residuated ℓ-groupoids, which is used to determine which varieties of residuated ℓ-groupoids are closed with respect to Dedekind–MacNeille completions, as well as for which varieties of residuated ℓ-groupoids closure with respect to Dedekind–MacNeille completions is equivalent to the property of admitting completions (see e.g. [24, Thm. 6.3]). Extending these results to the left-residuated case might offer an interpretation of the deep reason behind the fact that orthomodular lattices are not closed under completions by cuts; possibly, it might also suggest a new strategy to prove, or disprove, that $\mathbb{OML}$ admits completions.

Let $\mathbf{L}$ be a pointed left-residuated ℓ-groupoid. We are now going to construct a pointed left-residuated ℓ-groupoid $\mathcal{L}(\mathbf{L})$ as a left nucleus-system of $\mathcal{P}(\mathbf{L})$ (see Section 3.1) in such a way that $(\mathcal{L}(\mathbf{L}))^l$ is nothing but the ideal completion of $\mathbf{L}^l$.

Lemma 47. *If* $\mathbf{L}\in \mathbb{PLRG}$*, then the map* $\gamma_\downarrow$*, defined by* $\gamma_\downarrow(X) = \downarrow X$*, is a left nucleus on* $\mathcal{P}(\mathbf{L})$*.*

Proof. Clearly, $\gamma_\downarrow$ is a closure operator on $\mathcal{P}(\mathbf{L})$. As to N3, observe that, for $X, Y \subseteq L$,

$$\gamma_\downarrow(X \to \gamma_\downarrow(Y)) = \{c : \exists d\, (c \leq d\, \&\, \forall a\, (a \in X \Rightarrow \exists b\, (b \in Y\, \&\, da \leq b)))\}.$$

Thus, let $c \leq d$ and fix an $a \in X$. Then there is $b \in Y$ such that $da \leq b$. Since $c \leq d$, by Lemma 20.(ii) $ca \leq da \leq b$, whence $c \in X \to \gamma_\downarrow(Y)$. The converse inclusion follows from the fact that γ is enlarging. ∎

The previous lemma justifies the next definition, where we use the notation of Theorem 46:

Definition 48. If $\mathbf{L}\in \mathbb{PLRG}$, its *ideal completion* is the structure $\mathcal{L}(\mathbf{L}) = \mathcal{P}(\mathbf{L})_{\gamma_\downarrow}$.

By Proposition 12, $(\mathcal{L}(\mathbf{L}))^o$ is the largest join-completion of $\mathbf{L}^o$. The next results list some properties of $\mathcal{L}(\mathbf{L})$.

Theorem 49. (i) *If* $\mathbf{L}\in \mathbb{PLRG}$*,* $\mathcal{L}(\mathbf{L})$ *is a left-monotonic and antitonic pointed left-residuated* ℓ*-groupoid.*

(ii) *If* $\mathbf{L}$ *is integral, then so is* $\mathcal{L}(\mathbf{L})$.

(iii) *If* $\mathbf{L}$ *is* 0*-bounded, then so is* $\mathcal{L}(\mathbf{L})$.

(iv) *If* $\mathbf{L}$ *is orthomodular, then* $\mathcal{L}(\mathbf{L})$ *satisfies the quasi-identity* $xy \approx 0 \Rightarrow yx \approx 0$ *and the identity* $xx' \approx 0$.

Proof. (i) $\mathcal{L}(\mathbf{L}) \in \mathbb{PLRG}$ by Lemma 47, Lemma 24 and Theorem 46.(i). It is left-monotonic by Lemma 24 and Theorem 46.(ii). Finally, it is antitonic by Lemma 24 and Theorem 46.(iii).

(ii) If $\mathbf{L}$ is integral, then $\gamma_{\downarrow}(\{1\}) = L$, which is the top element in $\mathcal{L}(\mathbf{L})$.

(iii) If $\mathbf{L}$ is 0-bounded, then every element in $\mathcal{L}(\mathbf{L})$ contains 0, whence $\{0\} = \gamma_{\downarrow}(\{0\})$ is the bottom element in $\mathcal{L}(\mathbf{L})$.

(iv) Let $\gamma_{\downarrow}(XY) = \{0\}$. For any x in X and any y in Y, $xy \leq xy$ and thus $xy = 0$, whence $yx = 0$ as $\mathbf{L} \in \mathbb{OG}$. Then $Y \cdot_{\gamma_{\downarrow}} X \subseteq \{0\}$, while the converse inclusion is clear. Applying this quasi-identity to the fact that $(X \to \{0\})\, X = \{0\}$, we obtain that the identity $xx' \approx 0$ is satisfied in $\mathcal{L}(\mathbf{L})$. ■

Theorem 49 is of limited value in the general case. In fact, an arbitrary $\mathbf{L} \in \mathbb{PLRG}$ need not embed into its ideal completion: the map $\varphi(x) = \gamma_{\downarrow}(\{x\})$, which is an order-embedding of $\mathbf{L}^{o}$ into $(\mathcal{L}(\mathbf{L}))^{o}$, does not necessarily preserve products or implications.[1] However:

Proposition 50. *If* $\mathbf{L}$ *is an antitonic, integral and* 0*-bounded pointed left-residuated* ℓ*-groupoid, then the order-embedding of* $\mathbf{L}^{o}$ *into* $(\mathcal{L}(\mathbf{L}))^{o}$ *preserves* $0, 1$, *and* $'$.

Proof. If $\mathbf{L}$ is integral and 0-bounded, $\varphi(0) = \{0\} = 0^{\mathcal{L}(\mathbf{L})}$ and $\varphi(1) = L = 1^{\mathcal{L}(\mathbf{L})}$. We now check that φ preserves $'$. However:

$$\begin{aligned} \varphi\left(x \to^{\mathbf{L}} 0\right) \quad &= \{a \in L : a \leq x \to 0\}; \\ \varphi(x) \to^{\mathcal{L}(\mathbf{L})} \varphi(0) &= \{a \in L : \forall b\, (b \leq x \Rightarrow ab = 0)\}. \end{aligned}$$

If $a \leq x \to 0$ and $b \leq x$, then by antitonicity $x \to 0 \leq b \to 0$, whereby $a \leq b \to 0$ and $ab = 0$. If for all $b \leq x$ we have that $ab = 0$, then this holds in particular for x itself, whence $ax = 0$ and thus $a \leq x \to 0$. ■

[1] A simple counterexample is given by the (ℓ-groupoid corresponding to the) modular ortholattice $\mathbf{MO}_2$. In fact, if a, b are two atoms of such, neither of which is the orthocomplement of the other, $b = ab \in \varphi(a)\, \varphi(1)$, but $b \notin \varphi(a1) = \varphi(a) = \{a, 0\}$.

It may be worth to observe that, when $\mathbf{L}$ is Sasakian, $\mathcal{L}(\mathbf{L})$ need not be such by any means.[2] Therefore, Proposition 50 does not imply that under this additional assumption $\mathbf{L}$ embeds into $\mathcal{L}(\mathbf{L})$.

Next, we introduce two properties of elements of a pointed left-residuated ℓ-groupoid $\mathbf{L}$ that will play a decisive role in the subsequent results. Hereafter, for any fixed $d \in L$, let the map γ_d be defined for all $a \in L$ by the condition

$$\gamma_d(a) = (a \to d) \to d.$$

Definition 51. Let $\mathbf{L} \in \mathbb{PLRG}$. An element $d \in L$ is *right-residuated* if and only if, for all $a, b \in L$,

$$ab \leq d \text{ if and only if } a \leq b \to d \text{ if and only if } b \leq a \to d.$$

Definition 52. Let $\mathbf{L} \in \mathbb{PLRG}$. An element $d \in L$ is *weakly associative and commutative (wac)* if and only if it satisfies the following conditions for all $a, b \in L$:

W1 $a((ab \to d)b) = (ab)(ab \to d)$;
W2 $\gamma_d(a)((ab \to d)b) = (\gamma_d(a)(ab \to d))b$;
W3 $\gamma_d(a)(\gamma_d(b)(ab \to d)) = (\gamma_d(a)(ab \to d))\gamma_d(b)$;
W4 $\gamma_d(a)(\gamma_d(b)(ab \to d)) = (\gamma_d(a)\gamma_d(b))(ab \to d)$.

Using Lemma 6 and Theorem 7, we observe that in any $\mathbf{L} \in \mathbb{OG}$, for any $a, b, c \in L$, $a(bc) = (ab)c$ and $bc = cb$ whenever b and c commute. Therefore 0 is wac in any orthomodular groupoid. The proof of the next lemma is left to the reader.

Lemma 53. *Let* $\mathbf{L} \in \mathbb{PLRG}$*, and let d be a right-residuated and wac element of L. Then, for all $a, b \in L$:* (i) *if $a \leq b$, then $b \to d \leq a \to d$; (ii) $a(a \to d) \leq d$; (iii)* $\bigvee X \to d = \bigwedge_{x \in X} (x \to d)$.

If $\mathbf{L}$ is a residuated lattice, the map γ_d is a nucleus on $\mathbf{L}$ whenever d is a cyclic element of L (namely, an element for which the two residuals always yield a common value). Besides the cyclicity of d, which is a restricted form of commutativity, two further aspects play a key role in this property: the

[2] For a counterexample, we look again at the (ℓ-groupoid corresponding to the) modular ortholattice $\mathbf{MO}_2$. Keeping the notation from the preceding footnote, observe that:

$$\downarrow a \cdot \downarrow 1 = \{a, a', b, b', 0\}$$
$$(\downarrow a \vee (\downarrow 1)') \wedge \downarrow 1 = \{a, 0\}.$$

associativity of product, and the presence of both residuals for the same operation. However, if restricted forms of associativity, commutativity and right-residuation are built into equalities involving d, an analogous result is available in a larger context.

LEMMA 54. *Let either* $\mathbf{L} \in \mathbb{PLRG}_{lm}$ *or* $\mathbf{L} \in \mathbb{OG}$*, and let* d *be a right-residuated and wac element of* L*. Then the map* γ_d *is a nucleus on* $\mathbf{L}$*.*

PROOF. The proof that γ_d is a closure operator is again left to the reader (see e.g. [33, Lm. 2.8 and Lm. 3.35]). Observe that $L_{\gamma_d} = \{x \to d : x \in L\}$. We show that $\gamma_d(a)\gamma_d(b) \leq \gamma_d(ab)$, for all $a, b \in L$. This will be enough in the cases mentioned in our statement, in light of Proposition 44 and Corollary 45. By Lemma 53.(ii), $(ab)(ab \to d) \leq d$. By W1, it follows that $a((ab \to d)b) \leq d$, whence $(ab \to d)b \leq a \to d$ as d is right-residuated. By Lemma 53.(i), $\gamma_d(a) \leq (ab \to d)b \to d$. Using left-residuation, we obtain $\gamma_d(a)((ab \to d)b) \leq d$, and, by W2, $(\gamma_d(a)(ab \to d))b \leq d$. So $\gamma_d(a)$ $(ab \to d) \leq b \to d$, and by Lemma 53.(i), $\gamma_d(b) \leq \gamma_d(a)(ab \to d) \to d$, whereby

$$(\gamma_d(a)(ab \to d))\gamma_d(b) \leq d,$$

since d is right-residuated. By W3, $\gamma_d(a)(\gamma_d(b)(ab \to d)) \leq d$, and thus by W4, $(\gamma_d(a)\gamma_d(b))(ab \to d) \leq d$. Finally, this entails $\gamma_d(a)\gamma_d(b) \leq \gamma_d(ab)$. ∎

THEOREM 55. *Let either* $\mathbf{L} \in \mathbb{PLRG}_{lm}$ *or* $\mathbf{L} \in \mathbb{OG}$*, and let* γ *be a nucleus on* $\mathbf{L}$*. If* $d \in L$ *is wac and* $b \to d \leq a \to d$*, for all* $a, b \in L$ *such that* $a \leq b$*, then condition B) below implies condition A):*

A) $\mathbf{L}_\gamma$ *is d-involutive, i.e.* $(a \to d) \to d = a$ *for all* $a \in L_\gamma$*;*

B) d *is right-residuated in* L *and* $\gamma = \gamma_d$*.*

If $\mathbf{L}$ *is orthomodular, the two conditions are equivalent.*

PROOF. B) implies A). Suppose that d is right-residuated. Then γ_d is a nucleus by Lemma 54. Furthermore, $d = 1d \leq d$ implies $d \leq 1 \to d$, and $1 \to d = 1(1 \to d) \leq d$, whence $d = 1 \to d \in L_{\gamma_d}$. Finally, $\mathbf{L}_{\gamma_d}$ is d-involutive, because d is right-residuated in $L \supseteq L_{\gamma_d}$ and $\gamma_d(a) = a$ for all $a \in L_{\gamma_d}$. A) implies B). Suppose that $\mathbf{L}$ is orthomodular and that $\mathbf{L}_\gamma$ is d-involutive.

Claim 1: for all $a \in L$, $\gamma(a) \to d = a \to d$. One inequality is straightforward from our assumption that $b \to d \leq a \to d$, for all $a, b \in L$ such that

$a \leq b$. Conversely, taking into account that $d \in L_\gamma$:

$$\begin{array}{ll}
a\,(a \to d) \leq d & \text{Lm. 29.(i)} \\
\Rightarrow a \leq (a \to d) \to d & \text{left resid.} \\
\Rightarrow a \leq \gamma\,(a \to d) \to d & \text{N3} \\
\Rightarrow \gamma\,(a) \leq \gamma\,(\gamma\,(a \to d) \to d) & \gamma \text{ is monotonic} \\
\Rightarrow \gamma\,(a) \leq \gamma\,(a \to d) \to d & \text{N3} \\
\Rightarrow \gamma\,(a) \leq (a \to d) \to d & \text{N3}
\end{array}$$

Thus:

$$\begin{array}{rll}
a \to d & = \gamma\,(a \to d) & \text{N3} \\
& = (\gamma\,(a \to d) \to d) \to d & \mathbf{L}_\gamma \text{ invol.} \\
& = ((a \to d) \to d) \to d & \text{N3} \\
& \leq \gamma\,(a) \to d & \text{Assumption on } d
\end{array}$$

Claim 1 is therefore settled.

Claim 2: d is right-residuated in L_γ. If $\gamma\,(a)\,\gamma\,(b) \leq d$, then $\gamma\,(a) \leq \gamma\,(b) \to d$, whence $\gamma\,(b) = \gamma_d\,(\gamma\,(b)) \leq \gamma\,(a) \to d$ because $\mathbf{L}_\gamma$ is d-involutive. If $\gamma\,(b) \leq \gamma\,(a) \to d$, then $\gamma\,(a) = \gamma_d\,(\gamma\,(a)) \leq \gamma\,(b) \to d$, and thus $\gamma\,(a)\,\gamma\,(b) \leq d$.

Next, we show that d is right-residuated in L. Suppose that $ab \leq d$. Then $a \leq b \to d$, whence, by Claim 1, $a \leq \gamma\,(b) \to d$. It follows that $\gamma\,(a) \leq \gamma\,(\gamma\,(b) \to d) = \gamma\,(b) \to d$, by N3. Because of Claim 2, $\gamma\,(b)\,\gamma\,(a) \leq d$ and thus, applying again Claim 1, $b \leq \gamma\,(b) \leq \gamma\,(a) \to d = a \to d$. For the other direction, if $b \leq a \to d$, then by Claim 1 $b \leq \gamma\,(a) \to d$, whence monotonicity of γ and N3 yield $\gamma\,(b) \leq \gamma\,(\gamma\,(a) \to d) = \gamma\,(a) \to d$. Therefore, by Claim 2, $\gamma\,(a)\,\gamma\,(b) \leq d$ and thus $a \leq \gamma\,(a) \leq \gamma\,(b) \to d = b \to d$. A final application of left-residuation entails that $ab \leq d$.

Therefore, by Lemma 54, γ_d is a nucleus on $\mathbf{L}$. So, taking into account Claim 1 and the fact that $\mathbf{L}_\gamma$ is d-involutive,

$$\gamma_d\,(a) = (a \to d) \to d = (\gamma\,(a) \to d) \to d = \gamma\,(a)\,.$$

■

The next theorem shows under what conditions the Dedekind–MacNeille completion of an orthomodular groupoid $\mathbf{M}$ can be obtained as a nucleus-system under the operator γ_0 of a join-completion of $\mathbf{M}$ which is a pointed left-residuated ℓ-groupoid. Subsequently, we prove that the ideal completion $\mathcal{L}\,(\mathbf{M})$ of $\mathbf{M}$ satisfies these conditions. The crucial fact, here, is that the application of the nucleus γ_0 allows us to recover the involutivity equation which is generally lost in passing from $\mathbf{M}$ to $\mathcal{L}\,(\mathbf{M})$.

Theorem 56. *Let* $\mathbf{M} \in \mathbb{OG}$*, and let* $\mathbf{L}$ *be a left-monotonic pointed left-residuated* ℓ*-groupoid such that* $\mathbf{L}^o$ *is a join-completion of* $\mathbf{M}^o$*; suppose further that the term operation* $'^{\mathbf{L}}$ *extends the corresponding operation* $'^{\mathbf{M}}$ *and that* 0 *is right-residuated and wac in* $\mathbf{L}$*. Then* $(\mathbf{L}\gamma_0)^l = \mathcal{N}(\mathbf{M}^l)$*, i.e.,* $\mathbf{L}\gamma_0$ *is the Dedekind–MacNeille completion of* $\mathbf{M}$*.*

Proof. By Lemma 54, γ_0 is a nucleus. We now have to show that $(\mathbf{L}\gamma_0)^l$ is a join-and-meet completion of $\mathbf{M}^l$, whence the result follows by Proposition 13.(ii). Actually, Theorem 55 implies that $\mathbf{L}\gamma_0$ is involutive; thus the map sending an $a \in L\gamma_0$ to $a \to 0$ is an involution of $\mathbf{L}\gamma_0$, which means that it is enough to show that $(\mathbf{L}\gamma_0)^l$ is a meet completion of $\mathbf{M}^l$. Thus, let $a \in L\gamma_0$, whence there is $b \in L$ s.t. $a = b \to 0$. Since $\mathbf{L}^l$ is a join-completion of $\mathbf{M}^l$, $b = \bigvee^{\mathbf{L}} X$ for some $X \subseteq M$ and then

$$a = b \to 0 = \left(\bigvee^{\mathbf{L}} X\right) \to 0 = \bigwedge_{x \in X} \left(x \to^{\mathbf{L}} 0\right) = \bigwedge_{x \in X} \left(x \to^{\mathbf{M}} 0\right),$$

since $'^{\mathbf{L}}$ extends $'^{\mathbf{M}}$. Our claim follows. ∎

Corollary 57. *If* $\mathbf{L} \in \mathbb{OG}$*, then*

$$\left(\mathcal{L}(\mathbf{L})_{\gamma_{\{0\}}}\right)^l = \left(\left(\mathcal{P}(\mathbf{L})_{\gamma_\downarrow}\right)_{\gamma_{\{0\}}}\right)^l = \mathcal{N}\left(\mathbf{L}^l\right).$$

Proof. The result follows from Theorem 56 provided we can prove the following three claims:

(i) $\{0\}$ is right-residuated in $\mathcal{L}(\mathbf{L})$.

(ii) $\{0\}$ is wac in $\mathcal{L}(\mathbf{L})$.

(iii) The term operation $'^{\mathcal{L}(\mathbf{L})}$ extends $'^{\mathbf{L}}$.

As regards Claim (i), by Theorem 49.(iv), $X \cdot^{\mathcal{L}(\mathbf{L})} Y \subseteq \{0\}$ if and only if $Y \cdot^{\mathcal{L}(\mathbf{L})} X \subseteq \{0\}$ if and only if $Y \subseteq X \to^{\mathcal{L}(\mathbf{L})} \{0\}$. For Claim (ii) we have to check all cases of Definition 52, one of which we compute explicitly, i.e. W1. Let us show that for any downsets $X, Y \subseteq L$,

$$\begin{aligned} X \cdot^{\mathcal{L}(\mathbf{L})} &\left(\left(X \cdot^{\mathcal{L}(\mathbf{L})} Y \to^{\mathcal{L}(\mathbf{L})} \{0\}\right) \cdot^{\mathcal{L}(\mathbf{L})} Y\right) \\ &= \left(X \cdot^{\mathcal{L}(\mathbf{L})} Y\right)\left(X \cdot^{\mathcal{L}(\mathbf{L})} Y \to^{\mathcal{L}(\mathbf{L})} \{0\}\right). \end{aligned}$$

By Theorem 49.(iv), the right-hand side of this equality equals $\{0\}$, whence it suffices to prove that so does its left-hand side. In fact:

$$\begin{aligned}
&X \cdot^{\mathcal{L}(\mathbf{L})} \left(\left(X \cdot^{\mathcal{L}(\mathbf{L})} Y \rightarrow^{\mathcal{L}(\mathbf{L})} \{0\}\right) \cdot^{\mathcal{L}(\mathbf{L})} Y\right) \\
&\quad = \gamma_{\downarrow}\left(X \cdot^{\mathcal{P}(\mathbf{L})} \gamma_{\downarrow}\left(\left(\gamma_{\downarrow}\left(X \cdot^{\mathcal{P}(\mathbf{L})} Y\right) \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\}\right) \cdot^{\mathcal{P}(\mathbf{L})} Y\right)\right) \\
&\quad = \left\{a \in L : \exists xyzw \begin{pmatrix} x \in X \,\&\, y \in Y \,\&\, w \in \gamma_{\downarrow}(XY) \rightarrow \{0\} \\ \&\, z \leq wy \,\&\, a \leq xz \end{pmatrix}\right\}.
\end{aligned}$$

In the remainder of the proof, we will drop unnecessary superscripts. Now, let a have the indicated properties. Then $w \leq (xy)'$, whence $z \leq wy \leq (xy)' y = x' \wedge y$, and thus $z \leq x'$, whereby z commutes with x, by Lemma 6.(ii). An application of Lemma 29.(iii) yields $a \leq xz \leq x(x' \wedge y) = 0$. Claim (iii) follows from Proposition 50. ■

Observe that the $(\wedge, \vee, ', 0, 1)$-term reduct of the algebra we have just constructed is the same as the algebra in Theorem 14.(iii), whence Corollary 57 counts as a new proof of MacLaren's theorem in the orthomodular case. Indeed,

$$\begin{aligned}
X^{\prime \mathcal{L}(\mathbf{L})_{\gamma_{\{0\}}}} &= X \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\} \\
&= \{a : \forall x\,(x \in X \Rightarrow ax = 0)\} \\
&= \{a : \forall x\,(x \in X \Rightarrow a \leq x')\} = X^{\perp}; \\
\gamma_{\{0\}}\left(\gamma_{\downarrow}(X)\right) &= \left(\downarrow X \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\}\right) \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\} = X^{\perp\perp}; \\
X \vee^{\mathcal{L}(\mathbf{L})_{\gamma_{\{0\}}}} Y &= \left(\downarrow (X \cup Y) \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\}\right) \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\} \\
&= (X \cup Y)^{\perp\perp}; \\
0^{\mathcal{L}(\mathbf{L})_{\gamma_{\{0\}}}} &= \left(\{0\} \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\}\right) \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\} = \{0\}; \\
1^{\mathcal{L}(\mathbf{L})_{\gamma_{\{0\}}}} &= \left(L \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\}\right) \rightarrow^{\mathcal{P}(\mathbf{L})} \{0\} = L.
\end{aligned}$$

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