The Calculus of Natural Calculation

Abstract. The calculus of Natural Calculation is introduced as an extension of Natural Deduction by proper term rules. Such term rules provide the capacity of dealing directly with terms in the calculus instead of the usual reasoning based on equations, and therefore the capacity of a natural representation of informal mathematical calculations. Basic proof theoretic results are communicated, in particular completeness and soundness of the calculus; normalisation is briefly investigated. The philosophical impact on a proof theoretic account of the notion of meaning is considered.

Keywords: Natural Deduction, Term rules, Proof theory, Proof-theoretic semantics.

1. Introduction

Natural Calculi. In his “*Untersuchungen über das logische Schließen*”¹ Gentzen intends “to set up a formal system which comes as close as possible to actual reasoning,” cf. [11, p. 68]. This is a formal calculus which “lends itself in particular to the formalization of mathematical proofs,” cf. [11, p. 80]. Guided by this motivation, Gentzen presents his calculus of Natural Deduction.² Crucial feature of such natural calculi (in contrast to the ‘logistic’ calculi³) is that the formal proofs do not “start from basic logical propositions [some logical axioms], but rather from *assumptions* [...] to which logical deductions are applied,” cf. [11, p. 75]. The use of assumptions is complemented by the possibility that the “result is then again made independent of the assumptions,” cf. [11, p. 75]. Moreover: reflecting informal argumentations, the inference rules of Natural Deduction introduce and eliminate the logical constants (connectives and quantifiers) occurring as

¹ See [10]; the quotes are taken from the English translation by Szabo, [11].

² Independently of Gentzen and motivated by Łukasiewicz, Jaśkowski [16] presents his own version of Natural Deduction in the same year as Gentzen.

³ The term ‘logistic’ is used here, as by Gentzen, to distinguish natural calculi from such which are not; cf. [11, p. 75].

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main symbols of formulae. The latter motivates the classification of inference rules into introduction and elimination rules, each associated with a logical symbol.

Relevance. One reason to introduce natural calculi is to address philosophical problems related with informal (mathematical) proofs. We mention Robinson's [25] investigation of the *nature* of proofs, the search for a criterion for the *identity* of proofs in the field of general proof theory,⁴ Arana's [2] analysis of the notion of *pure* proofs not using notions extraneous to what is being proved, and Hilbert's 24th (and unpublished) problem about the *simplicity* of proofs.⁵ Even if there remains an unavoidable justificatory gap between the informal proofs and their formal representatives,⁶ it seems wise in such philosophical discussions to minimise this gap by investigating derivations in natural calculi as formal counterparts of informal proofs and to discuss the philosophical problems on formal grounds.

Another field of research closely related to natural calculi is the proof-theoretic semantics project.⁷ One of the central claims of this project is that meaning of language is given by its use.⁸ As a consequence, proof-theoretic semantics aims to establish the meaning of logical constants (connectives and quantifiers) by the introduction (and elimination) rules for these symbols.⁹ Even if we can imagine an adequate analysis of meaning on the base of 'logistic' calculi,¹⁰ it seems again wise to focus on natural calculi, in which the

⁴According to Došen [4], in his discussion of two concurring ideas of a good criterion for the identity of proofs proposed by Prawitz and Lambek, respectively, it was Prawitz [22] (motivated by ideas of Kreisel) who introduced the field of *general proof theory*, in which the identity of proofs and related philosophical questions are investigated.

⁵Hilbert's 24th problem was discovered by Thiele [29]; an overview over this problem and the related topics including a survey of more recent developments is provided by Hipolito and Kahle [14].

⁶As Robinson [25] claims: there cannot be a formal justification that a formalisation of an informal mathematical proof is, indeed, the formalisation of that proof.

⁷As an introduction into the field of proof-theoretic semantics, we suggest [6, 27, 33].

⁸Wittgenstein [35, §43] argues (partially) in the same direction, when he defines meaning ("in a *large* class of cases") as follows: "the meaning of a word is its use in the language." Nevertheless, a sentence later this definition is restricted by a denotational exception: "And the *meaning* of a name is sometimes explained by pointing to its bearer."

⁹The origins of this research field can be tracked back to some succinct comments by Gentzen [11]; the denomination as proof-theoretic semantics is due to a suggestion of Schroeder-Heister, cf. [27].

¹⁰Even if in such calculi the natural inference rules for the connectives and quantifiers are no proper rules: presupposing completeness, we can derive the validity of the natural

meaning of the logical symbols can be associated directly with the inference rules.

Calculations in Proofs. Although there is a philosophical controversy about the “naturalness” of Natural Deduction,¹¹ we consider (for the reasons given by Gentzen) this calculus to be natural—at least with respect to informal mathematical reasoning with statements.¹² But investigating everyday mathematical proofs we have to observe that mathematicians do not only argue with statements, but they also calculate inside their proofs. We consider the traditional representation of equality calculations via formal argumentations with equations (which are, in particular, statements) as unnatural (even in Natural Deduction).¹³

Following Gentzen’s motivation, we provide a natural representation of these informal calculations by extending the calculus of Natural Deduction by some *term rules* for the formal representation of equality calculations (instead of the usual rules for identity). The result of this extension is what we call the calculus of Natural Calculation. Due to the new rules in the calculus, we are able to make *term assumptions* in a derivation to which we can apply (justified) *calculation steps* resulting in new terms. Furthermore, there is an inference rule allowing the *evaluation* of a formal calculation by inferring its result (and by which the proof is made independent of the term assumptions). Due to these proper term rules, calculations (with terms)

Footnote 10 continued

inference rules. This can be used as the base of philosophical considerations in the same way as the natural inference rules themselves.

¹¹Cf. [1, p. 114] for a brief survey of this controversy.

¹²At least, we cannot follow the criticism, as mentioned by Anellis [1, p. 114], complaining that Natural Deduction is too complicated due to too many inference rules. We consider the great number of rules as one of the central aspects turning Natural Deduction natural.

¹³Besides equational calculations, one can easily find other types of calculations in mathematical praxis, as smaller-than calculations or subset calculations. But in contrast to the extension of the calculus by inference rules for the equality symbol, the extensions for the other kinds of symbols depend on specific formal languages and on formal theories formulated in these languages.

The latter corresponds to the fact that the equality symbol is usually counted as a logical symbol and usually present in every formal language. In order to obtain a complete and sound calculus with respect to the standard semantics (with the equality symbol), we focus in these investigations on equality calculations. Alternative calculations can be treated with similar methods as discussed in these investigations; the investigation of such calculations is beyond the possibilities of these investigations and left to future work.

and argumentations (with formulae) are even spatially separated inside a derivation.¹⁴

Changing our perspective on the term rules, we can easily recognise them as elimination and introduction rules for the equality symbol. As a consequence, the treatment of the equality symbol in the calculus of Natural Calculation fits better than the traditional approach into the usual Natural Deduction taxonomy of inference rules for logical symbols.¹⁵

Related Work. Central motivation for the calculus of Natural Calculation is a faithful representation of the informal treatment of terms (in calculations inside of proofs). This reminds of ι -terms (definite descriptions) and ϵ -terms (denoting an arbitrary object satisfying a description) as introduced by Hilbert and Bernays.¹⁶ But in contrast to calculi dealing with such new types of terms (occurring only inside of formulae) via new *formula rules*, we do not introduce new terms at all, but provide *proper term rules* for the treatment of the usual terms of a formal language.

Proper terms occurring in calculi are not a new phenomenon: Jaśkowski [16] uses terms (more precisely, variables) as a second premise for the elimination of the universal quantifier.¹⁷ Similarly, Troelstra and Schwichtenberg [30] provide such modified quantifier rules as a notational variant of the standard rules, where the term premise t is intended to express denotation.¹⁸ In free logics, this second premise is usually notated “ $E(t)$ ”.¹⁹ Occasionally, the notation of the unary relation symbol E is omitted, which results again in a calculus with proper terms. Nevertheless, the terms still represent

¹⁴This separation can be understood as an implementation of what is called by Barendregt [3, p. 205] the Poincaré Principle. Barendregt refers thereby to an observation of Poincaré that verifications of equations seem to be different from proper proofs. Cf. also [20, p. 12].

¹⁵Nevertheless, there remains one interesting aspect distinguishing the equality symbol from the other logical symbols: besides the introduction and the elimination rules, we need another rule for obtaining a complete calculus, namely a rule allowing the substitution of provably equal terms in formulae (or at least in relational atomic formulae).

¹⁶Cf. [12, §8] and [13, §1], respectively.

¹⁷Jaśkowski’s term assumptions mean to “[c]onsider an arbitrary z ” and to determine this way the variable which becomes bound in a subsequent introduction of the universal quantifier; cf. [16, §5].

¹⁸According to Troelstra and Schwichtenberg, such term assumptions t have to be read as a *statement* that the term t denotes (or even as a proof of this statement). These premises are trivially given in classical logic, but “[s]uch extra premises become indispensable [in] logics where terms do not always denote”; cf. [30, p. 40].

¹⁹For a more detailed survey of free logics, we suggest [19].

statements and are, in particular, not used for calculations (in contrast to the use of terms in our calculus).²⁰

Due to the proper term rules in the calculus of Natural Calculation, subatomic argumentations (the calculations) are incorporated in the proofs. Another attempt to extend the expressiveness of a calculus in this direction is Wieckowski [34] discussing subatomic introduction and elimination rules for atomic formulae (not restricted to equations). But in contrast to our calculus, the terms only appear accompanied by sets of atomic sentences capable of representing elementary informations associated with these terms. Recently, Indrzejczak [15] carried over the idea of extending the calculus by proper term rules allowing calculations to Gentzen's Sequent Calculus.

Such extensions of the calculus by (subatomic) term rules can be understood as a contribution to the more general topic of definitional reflection, as discussed by Schroeder-Heister [26], concerned with inference rules allowing the introduction of atomic formulae.

Overview. After the preliminaries (in Section 2) we provide (in Section 3) a brief analysis of a typical mathematical proof (taken from arithmetics) containing a proper calculation²¹ and introduce subsequently the calculus of Natural Calculation (including the term rules and basic proof theoretic concepts and terminology). Subsequently, we communicate basic proof theoretic properties of the calculus up to completeness and soundness (in Section 4) and consider normalisation in the presence of the term rules (in Section 5). We conclude our investigations (in Section 6) with some comments on the impact of the calculus of Natural Calculation on a proof theoretic approach to the notion of meaning.

²⁰As the terms represent statements, it seems reasonable to call such term occurrences *apparent*, in contradistinction to our term occurrences which are *real*. An interesting limit case is provided by Restall [24]: proper terms t may occur in a derivation as well as existence statements $E(t)$. The terms and the statements are logically equivalent; the introduction of both types of expressions is motivated by subtle philosophical reasons. There are even some term rules present in the calculus (as contraction and cut), but these structural rules are not used for calculations.

²¹In order to avoid misinterpretations: the calculus of Natural Calculation is not coined to deal only with arithmetical calculations, but provides calculation rules for arbitrary situations, in which we deal directly with objects represented by terms of a first order language.

2. Preliminaries

Natural Numbers. Motivated by the von Neumann's [32] definition of the natural numbers as finite ordinals, the set of natural numbers is denoted by ω . $0 \in \omega$ is the least natural number, n' denotes the *successor* of n (this is the number $n + 1$) for all $n \in \omega$.

Primitive Notions of Logic. We use the expression *syntactic entity* to denote all kinds of syntactic entities, as, for example, terms, formulae and derivations. The symbol " $\simeq$ " expresses in the meta-language the syntactic equality of arbitrary syntactic entities.

Formal Languages. A formal language $\mathcal{L}$ is determined by the non-logical symbols (together with their arities).²² In the theoretical parts of our investigations, we presuppose a *paradigmatic* language $\mathcal{L}$ with the following non-logical symbols: some constant symbols c , some n' -ary function symbols f , some m -ary relation symbols R (with $n, m \in \omega$).²³ Some of our examples are formulated in a formal language $\mathcal{L}_{PA}$ of arithmetics with the obvious non-logical symbols.

Besides the non-logical symbols, the following (logical) symbols are available: the falsum ($\perp$), the implication ($\rightarrow$), the universal quantifier ($\forall$) the equality symbol ($=$).²⁴ Furthermore, there are countably many first order variables v_k available as well as the usual auxiliary symbols. The symbols x, y, z (also with subscripts) are meta-variables for the variables. We do not distinguish between parameters occurring always free and quantifiable variables.

We distinguish between a formal language $\mathcal{L}$ and its associated sets; this means that the language $\mathcal{L}$ is not identified with the set of its terms or with the set of its formulae. An exact definition of the terms and formulae is easily extracted from the subsequent definitions of the so called nominal terms and nominal formulae, which are a generalisation of the standard terms and formulae, respectively.

²²The subsequent investigations are formulated with respect to our special account of formal languages; nevertheless, they are easily carried over to the usual variations of this approach. More details about our specific conception of formal languages can be found in the preliminaries of [9].

²³In order to serve as a blueprint for arbitrary concrete formal languages, the paradigmatic language $\mathcal{L}$ remains underdetermined.

²⁴The mentioned logical symbols are sufficient from a classical point of view; for the intuitionistic calculus, the alphabet is extended by the missing logical constants and the relevant definitions are easily generalised with respect to these symbols.

Nominal Forms. So called *nominal terms* and *nominal formulae* are used to represent the *position* of occurrences of terms in terms and of terms in formulae, respectively.²⁵ We provide the formal definitions with respect to a formal language $\mathcal{L}$:

1. *nominal symbols*: The alphabet is extended by countably many *nominal symbols* $*_k$ (with $k \in \omega$). The symbol $*$ is used as an abbreviation for $*_0$ (and not as a meta-variable for arbitrary nominal symbols). V_* is the *set of all nominal symbols*.
2. *nominal terms*: Nominal terms $\mathbf{t}$ are defined according to the same clauses as the standard terms of a formal language, but with the additional clause that every nominal symbol $*_k$ is an (atomic) nominal term; this means, according to the following inductive clauses:

$$*_k \mid x \mid c \mid f(\mathbf{t}_0, \dots, \mathbf{t}_n)$$

T denotes the *set of all nominal terms*; the symbols $\mathbf{t}$, $\mathbf{s}$, $\mathbf{r}$, etc. are used as meta-variables for nominal terms.

4. *nominal formulae*: The nominal formulae $\mathbf{A}$ are defined according to the same clauses as the standard formulae of a formal language, but with respect to nominal terms instead of standard terms; this means, according to the following inductive clauses:

$$\perp \mid (\mathbf{t} = \mathbf{s}) \mid P(\mathbf{t}_0, \dots, \mathbf{t}_{m-1}) \mid (\mathbf{A} \rightarrow \mathbf{B}) \mid (\forall x. \mathbf{A})$$

L denotes the *set of all nominal formulae*; the symbols $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc. are used as meta-variables for nominal formulae.

The expression “*nominal form*” is used as a generic term for all kinds of such generalisations of standard syntactic entities.²⁶ The usual notational conventions are presupposed.

Basic Categorisation. We provide a basic categorisation of the nominal forms as far as needed in these investigations.

1. *standard terms*: A nominal term $\mathbf{t}$ is called a *standard term*, if it is generated without a clause for nominal symbols; otherwise it is called

²⁵The concept of nominal forms is introduced by Schütte [28] to discuss semantic properties of formulae determined by their syntactic structure; additionally, Schütte uses these nominal forms for an adequate description of substitutions; a detailed analysis of the underlying notions of occurrences and their positions is given by Gazzari [9].

²⁶Schütte [28] defines nominal forms slightly differently as arbitrary strings of symbols of the extended alphabet.

a *proper* nominal term. In order to emphasise that a nominal term $\mathbf{t}$ is a standard term, we also use the meta-variables t, s, r , etc. The set of standard terms is denoted by T_0 .

2. *standard formula*: Analogously: a nominal formula $\mathbf{A}$ is called a *standard* formula, if it is generated on the base of standard terms; otherwise, $\mathbf{A}$ is called a *proper* nominal formula. In order to emphasise that a nominal formula is standard, we also use the meta-variables A, B, C , etc. The set of standard formulae is denoted by L_0 .
3. *unary nominal forms*: A proper nominal form, in which only the nominal symbol $*$ (this is the symbol $*_0$) occurs (possibly more than once), is called *unary*.

General Substitution Function. Central tool for dealing with nominal forms is the *general substitution function*, a binary function on nominal forms and arbitrary sequences of nominal terms into the respective set of nominal forms. We provide an intensional definition:

1. *nominal terms*: $\mathbf{t}[\mathbf{t}_0, \dots, \mathbf{t}_n]$ is the result of the simultaneous replacement of all nominal symbols $*_0, \dots, *_n$ in the nominal term $\mathbf{t}$ by the nominal terms $\mathbf{t}_0, \dots, \mathbf{t}_n$.
2. *nominal formulae*: $\mathbf{A}[\mathbf{t}_0, \dots, \mathbf{t}_n]$ is the result of the simultaneous replacement of all nominal symbols $*_0, \dots, *_n$ in the nominal formula $\mathbf{A}$ by the nominal terms $\mathbf{t}_0, \dots, \mathbf{t}_n$.

A recursive definition of the general substitution function is easily provided.

Positions of Occurrences. Nominal forms are capable of representing the *positions* of occurrences of terms in standard terms and formulae; essential idea is to generate a nominal form by *eliminating* the intended occurrences by replacing them by nominal symbols. With the help of the general substitution function, we are able to provide an exact definition:

1. *in terms*: A nominal term $\mathbf{t} \in T$ is a *position* of a standard term $s \in T_0$ in a standard term $t \in T_0$, if $\mathbf{t}[s] \simeq t$.
2. *in formulae*: A nominal formula $\mathbf{A} \in L$ is a *position* of a standard term $s \in T_0$ in a standard formula $A \in L_0$, if $\mathbf{A}[s] \simeq A$.

The position $\mathbf{t}$ (the position $\mathbf{A}$) of an occurrence is unary, but in the limit cases of *no intended occurrences*, where it is the term t (the formula A) itself.

EXAMPLE (POSITION OF OCCURRENCES). We provide some example positions in the standard term $t \simeq 0 + (0 + 0)$; in order to refer informally to the intended positions, we underline them.

1. $* + (0 + 0)$ represents the (simple) position $\underline{0} + (0 + 0)$ of 0 in t .
2. $* + (0 + *)$ represents the (multiple) position $\underline{0} + (0 + \underline{0})$ of 0 in t .
3. $0 + *$ represents the (simple) position $0 + \underline{(0 + 0)}$ of $0 + 0$ in t .

Auxiliary Methods. We introduce some auxiliary methods.

1. *sets of variables and nominal symbols*: $V_x(\mathbf{t})$ is the set of variables occurring in a nominal term $\mathbf{t}$, $V_*(\mathbf{t})$ the set of nominal symbols occurring in a nominal term $\mathbf{t}$, and $V_*(\mathbf{A})$ the set of nominal symbols occurring in a nominal formula $\mathbf{A}$. (All these sets are defined easily by recursion.)
2. *dominance*: We say that the nominal symbol $*$ is under the *dominance* of a quantifier $\forall x$, if that nominal symbol occurs inside the scope of this quantifier. The set of the respective variables x is defined recursively:

- (a) *A atomic*: $\mathbf{dom}$
- (b) *implication*: $\mathbf{dom}(\mathbf{A} \rightarrow \mathbf{B}) = \mathbf{dom}(\mathbf{A}) \cup \mathbf{dom}(\mathbf{B})$
- (c) *quantification*:

$$\mathbf{dom}(\forall x.\mathbf{A}) = \begin{cases} \mathbf{dom}(\mathbf{A}) \cup \{x\} & \text{if } * \in V_*(\mathbf{A}) \\ \mathbf{dom}(\mathbf{A}) & \text{otherwise} \end{cases}$$

3. *freeness*: A unary nominal formula $\mathbf{A}$ is called *free* for a standard term t , if no variable of t becomes bound by an application of the general substitution function on $\mathbf{A}$ and t . More formally, $\mathbf{A}$ is *free* for t , if:

$$\mathbf{dom}(\mathbf{A}) \cap V_x(t) = \emptyset$$

Natural Deduction. We presuppose the calculus of Natural Deduction as given by Gentzen [11]. The following inference rules are called the *traditional* rules for the equality symbol $=$:

1. *equivalence relation*:

$$\frac{}{t = t} \quad ; \quad \frac{t = s}{s = t} \quad ; \quad \frac{t = s \quad s = r}{t = r}$$

2. *substitutivity*:

$$\frac{t = s}{\mathbf{t}[t] = \mathbf{t}[s]} \quad ; \quad \frac{t = s \quad \mathbf{A}[t]}{\mathbf{A}[s]}$$

We provide the necessary restrictions:

- (a) *arbitrary occurrences*: Arbitrary many (but more than zero many) occurrences of t can be replaced by the term s in the standard term $r \simeq \mathfrak{t}[t]$ (in the standard formula $A \simeq \mathbf{A}[t]$, respectively). Formally: $\mathfrak{t}$ and $\mathbf{A}$ both are unary nominal forms.
- (b) *bound variables*: No bound occurrence of a variable may be affected by an application of the inference rules. More formally: the nominal formula $\mathbf{A}$ is free for both terms t and s .²⁷

Provability is defined, as usual, via the existence of suitable derivations. Provability in the pure calculus without rules for equality is denoted by $\vdash$, in its extension by the traditional rules for equality by $\vdash_{=}$.

Canonical Properties of Identity. The following axioms are called in these investigations the *canonical properties* of identity.²⁸

1. *reflexivity and symmetry*: $\forall x. x = x \quad ; \quad \forall xy. x = y \rightarrow y = x$
2. *transitivity*: $\forall xyz. x = y \wedge y = z \rightarrow x = z$
3. *compatibility - function symbols*: For all n' -ary function symbols f :

$$\forall x_0 \dots x_n y_0 \dots y_n. (\bigwedge_{k < n'} x_k = y_k) \rightarrow f(\vec{x}) = f(\vec{y})$$

4. *compatibility - relation symbols*: For all m -ary relation symbols R :

$$\forall x_0 \dots x_{m-1} y_0 \dots y_{m-1}. (\bigwedge_{k < m} x_k = y_k) \wedge R(\vec{x}) \rightarrow R(\vec{y})$$

The set of these axioms is denoted by $T_{=}$. The canonical properties of the identity relation and the (traditional) identity rules are equivalent. For all sets Γ of formulae and all formulae A :

$$\Gamma \vdash_{=} A \quad \Leftrightarrow \quad \Gamma + T_{=} \vdash A$$

3. Introduction of the Calculus

We introduce in this section the calculus of Natural Calculation, with focus on the proper term rules for a faithful representation of calculations (inside proofs) together with the relevant basic proof theoretic terminology.

²⁷If we would distinguish between quantifiable variables (occurring only bound in a derivation) and parameters (always occurring free), then this condition would be satisfied trivially, as we have to demand in this case that only parameters may occur as variables in the terms t , s , r and $\mathfrak{t}[t]$ (in all rules).

²⁸The definition depends on the non-logical symbols of the formal language $\mathcal{L}$.

3.1. Analysis: Informal Calculations

As a motivation, we first discuss a typical example of a calculation inside a mathematical proof.

A Calculation inside a Proof. The arithmetical statement that

- $n + 0 = 0 + n$ for all natural numbers $n \in \omega$

is proved via an induction. In the induction base, the equation $0 + 0 = 0 + 0$ has to be proved, which is trivial. In the induction step, the postulated equation has to be proved with respect to n' under the presupposition of this equation with respect to n . In order to do so, one *calculates* as follows (the justifications notated above the equality symbols are explained below):

$$n' + 0 \stackrel{(A_1)}{=} n' \stackrel{(A_1)}{=} (n + 0)' \stackrel{(IH)}{=} (0 + n)' \stackrel{(A_2)}{=} 0 + n'$$

Once having provided the calculation, one continues to argue as follows: By the transitivity of the identity relation, we conclude that $n' + 0 = 0 + n'$. The postulated statement follows now by induction principle.

Analysis (Informal Calculation). We analyse the informal calculation and identify natural aspects, which will be present in the calculus of Natural Calculation.

1. *justification*: Each *calculation step* is *justified* by an argument, namely by the induction hypothesis (IH) and by the following two axioms (definitional equations) of addition in arithmetics:

$$(A_1) \quad n + 0 = n \text{ for all } n \in \omega.$$

$$(A_2) \quad n + m' = (n + m)' \text{ for all } n, m \in \omega.$$

Instead of such statements also an auxiliary calculation can serve as a justification (not presented in our example).

2. *orientation of justification*: There are two different ways, in which an equation is used to justify a calculation step.

- The equations (A_1) and (IH) are used *as stated* to justify the first and the third calculation step, respectively.
- This is different with respect to the remaining calculation steps. In the second step, we would need actually the equation $n = n + 0$ as a justification, but only (A_1) is given, which is the symmetric variant of $n = n + 0$.

To use (A_1) as a justification for the second calculation step means to read (A_1) from the right to the left. The same is true with respect to (A_2) as the justification of the fourth calculation step.

We distinguish both kinds of justifications by calling the former *positive* and the latter *negative*.

3. *substructural calculations*: The calculation steps are, in the general case, *substructural* transformations of the terms with which we calculate. By that we mean that we do not replace the full term, but only some (occurrences of) a subterm inside of the full term.²⁹ From another point of view: the terms in the justificatory equation do not have to be the full term, to which we apply the calculation step, and the full resulting term, respectively, but they can be a subterm of the former term and a term provably equal to that subterm.
4. *sequential calculation*: The calculation is sequential: one starts with a term, then calculates the next term, and so on, until one reaches the last term of the calculation.³⁰
5. *evaluation*: We draw the attention on a subtle aspect. The first statement in the informal proof following immediately the calculation is a résumé: the calculation is evaluated and, with reference to transitivity, the *result* of the calculation is stated.

Analysis (Traditional Representation). There are some unnatural aspects of the usual representation of such an informal calculation in Natural Deduction via the traditional inference rules for equality.

1. *dealing with equations*: Instead of dealing directly with terms (as in the informal calculation), one is forced to deal with equations.
2. *variability*: There is a great variability of formalising the calculation; different derivations containing different equations can be used for such a representation. The sequential character of a calculation is lost.

²⁹Our classification of rules and inference steps as “substructural” refers to the formal *structure* of terms (and also of the formulae) and to the fact that substructural rules can act inside of this structure. Our terminology follows Jaśkowski’s usage of the term “structural rule” meaning rules “which refer to the external appearance of expressions”; cf. [16, p. 5].

This usage of “substructural” is obviously different from its predominant use in the field of proof theory to denote “substructural logics”, in which the structural rules of Gentzen’s Sequent Calculus are restricted. There the intended structure is the structure of sequents and derivations rather than of terms and formulae (see the collection “Substructural Logics” [5] edited by Došen and Schroeder-Heister).

³⁰Due to possible auxiliary calculations, we abstain from calling this property “linearity”.

3. *artificial equations*: Depending on the chosen representation, artificial equations may occur, which are not (explicitly) present in the calculation. In particular, it may occur that we have to start the representation with the trivial equation $t = t$ which is never claimed inside a calculation.³¹
4. *evaluation*: There is no feature in the calculus with the traditional rules to represent the evaluation of a calculation (as there is no real representation of the calculations themselves).

3.2. Term Rules

We provide the term rules for a direct treatment of terms in the calculus allowing thereby a faithful representation of informal calculations inside of formal proofs.

DEFINITION 1 (TERM RULES). The calculus of Natural Deduction (as given by Gentzen [11] and without the traditional rules for equality) is extended by the following term rules:

1. *positive / negative calculations steps (elimination of the equality)*:

$$\frac{\mathbf{r}[t] \quad t = s}{\mathbf{r}[s]} (E=) \quad ; \quad \frac{s = t \quad \mathbf{r}[t]}{\mathbf{r}[s]} (E=)$$

We demand: the nominal term $\mathbf{r}$ is unary.

The equation $t = s$ ($s = t$, respectively) is called the *direct justification* of the respective inference step, the full derivation containing the direct justification as conclusion is called the *justification* of the respective inference step.

2. *evaluation of calculations (introduction of the equality)*:

$$\frac{[t] \quad s}{t = s} (I=)$$

The discharge of the term t in the assumptions is mandatory.

³¹The latter aspect reminds of the analogous observation with respect to formulae in calculi different from Natural Deduction. In Hilbert style calculi, the argumentation is based on the statement of tautologies; the sequent calculus is based on the statement of trivial sequents of the form $A \vdash A$. We never observed a real mathematical proof, in which the argumentation started with the exposition of some tautologies or in which the provability of a statement out of this statement was claimed. Avoiding such phenomena turns Natural Deduction into a natural calculus.

3. *positive/negative auxiliary calculation (term)*:

$$\frac{\mathbf{r}[t] \quad \begin{array}{c} [t] \\ s \end{array}}{\mathbf{r}[s]} \text{ (Aux)} \quad ; \quad \frac{\begin{array}{c} [s] \\ t \end{array} \quad \mathbf{r}[t]}{\mathbf{r}[s]} \text{ (Aux)}$$

We demand: the nominal term $\mathbf{r}$ is unary. The discharge of the term t (of the term s , respectively) in the assumptions is mandatory.

The calculation from t to s (from s to t , respectively) is called the *direct justification* of the respective inference step; the full derivation containing the direct justification is called the *justification* of the respective inference step.³²

4. *positive/negative auxiliary calculation (formula)*:

$$\frac{\mathbf{A}[t] \quad \begin{array}{c} [t] \\ s \end{array}}{\mathbf{A}[s]} \text{ (Aux)} \quad ; \quad \frac{\begin{array}{c} [s] \\ t \end{array} \quad \mathbf{A}[t]}{\mathbf{A}[s]} \text{ (Aux)}$$

We demand: the nominal formula $\mathbf{A}$ is unary and free for t and s . The discharge of the term t (of the term s , respectively) in the assumptions is mandatory.

The (direct) justifications are given as in the case of auxiliary calculations with respect to terms.

REMARK (TERM RULES).

1. *alternative terminology*: The calculation steps and the evaluation of calculation rule are provided with an alternative denomination as elimination and introduction of the equality, respectively. The former way of naming the rules puts the focus on the terms and their transformations, the latter on the equality symbol (and thereby on the treatment of atomic formulae). The chosen point of view also determines, which premises of the inference rules should be counted as main premise.
2. *redundancies*: Subsequently, we will see that negative auxiliary calculations steps (with respect to terms and formulae) are redundant, as well as the positive auxiliary calculation steps with respect to terms. Additionally, the positive auxiliary calculation steps with respect to formulae are only needed, if there are relation symbols available in the underlying formal language; and in this case, we could even restrict them to atomic relational expressions $R(\vec{t})$.

³²The notion of a calculation is formally introduced below. Essentially: a calculation begins with an open term assumption and contains (besides the terms) only the direct justifications of the calculation steps applied in that calculation.

We provided all of these redundant rules for obtaining a natural calculus, in which we are able to represent all kinds of informal calculation steps.

3. *immediate introduction*: Equations of the form $t = t$ can be derived via an immediate application of the introduction of equality rule as follows:

$$\frac{[t]^1}{t = t} \text{ (I=:1)}$$

This immediate introduction may seem artificial (and therefore unnatural), but it is unavoidable, if we do not want to have axioms of the form $t = t$. This is similar to the immediate introduction of the implication, which corresponds to axioms of the form $A \rightarrow A$.

3.3. Derivations, Calculations and their Kernel

A *derivation* is a tree of terms and formulae composed according to the usual inference rules for formulae as well as for the new term rules. A *calculation* is a derivation in which only calculation rules are applied (and not even the evaluation rule).

DEFINITION 2 (DERIVATIONS AND CALCULATIONS).

1. *derivations, (open) assumptions and conclusion*: Allowing *term assumptions* t as a second kind of atomic derivation (besides the usual formula assumptions), the *derivations* $\mathbf{D}$ are defined as expected on the base of the standard rules of Natural Deduction (as given by Gentzen [11]) together with the term rules (as introduced above), but without any additional identity rule.

In parallel, the set $\mathbf{hyp}(\mathbf{D})$ of *open term and formula assumptions* of a derivation $\mathbf{D}$ and its *conclusion* $\mathcal{C}(\mathbf{D})$ (possibly a term) are defined as expected.

2. *formal calculation*: A derivation $\mathbf{D}$ is called a (*formal*) *calculation*, if $\mathbf{D}$ is generated only on the base of calculation steps and auxiliary calculations (term) and if $\mathbf{D}$ has at least one term assumption and arbitrary many formula assumptions (which are equations).

REMARK (DERIVATIONS AND CALCULATIONS).

1. *calculations*: We can characterise a calculation as a derivation $\mathbf{D}$ having a term conclusion and in which formulae only occur as assumptions.
2. *term assumptions and term conclusions*: As the discharge of term assumptions is mandatory for all term rules allowing such a discharge, the following both statements hold for all derivations $\mathbf{D}$:

- (a) There is at most one open term assumption in $\mathbf{D}$. In particular, the intersection $\mathbf{hyp}(\mathbf{D}) \cap \mathbf{T}_0$ of the set of open assumption of $\mathbf{D}$ and of the set of all standard terms is empty or a singleton.
- (b) The conclusion $\mathcal{C}(\mathbf{D})$ is a term, if and only if there is an open term assumption.

Special Terminology. We introduce some special terminology coined for the treatment of calculations $\mathbf{D}$.

- 1. *initial term*: The open term assumption $t \in \mathbf{hyp}(\mathbf{D})$ is also called the *initial* term of the calculation $\mathbf{D}$ and is denoted by $\mathbf{init}(\mathbf{D})$. (simple recursive definition)
- 2. *final term*: The conclusion $\mathcal{C}(\mathbf{D})$ is also called the *final* term of the calculation $\mathbf{D}$ and is denoted by $\mathbf{fin}(\mathbf{D})$. (simple recursive definition)
- 3. *result*: The equation $\mathbf{init}(\mathbf{D}) = \mathbf{fin}(\mathbf{D})$ is called the *result* of the calculation $\mathbf{D}$ and is denoted by $\mathcal{R}(\mathbf{D})$.
- 4. *closed calculation*: A derivation $\mathbf{D}$ is called a *closed calculation*, if $\mathbf{D}$ is the result of an application of the evaluation rule on a formal calculation (in which the result of that calculation is inferred).

A central proof theoretic method is the replacement of derivations by proof theoretically equivalent derivations. In order to discuss such a replacement of a calculation occurring inside of a derivation, it is convenient to introduce the *kernel* of a derivation.

DEFINITION 3 (KERNEL). Let $\mathbf{D}$ be an arbitrary derivation.

- 1. *term conclusion*: If $\mathbf{D}$ has a term conclusion, then the *kernel* of $\mathbf{D}$ is that calculation $\mathbf{C}$ occurring inside of $\mathbf{D}$ which starts at the open term assumption of $\mathbf{D}$ and which ends in the term conclusion of $\mathbf{D}$.
- 2. *formula conclusion*: If $\mathbf{D}$ has a formula conclusion, then the kernel is the empty tree.

(Using empty trees, a recursive definition is easily given.)

REMARK (KERNEL).

- 1. *intuition*: The kernel of a derivation $\mathbf{D}$ with term conclusion is easily obtained by replacing (bottom up) all derivations above an equation (occurring as a justification of a calculation step) by the equation itself. In general, the kernel of a derivation $\mathbf{D}$ is not a subderivation (in the usual sense).

- EXAMPLE (DERIVATIONS, CALCULATIONS AND THEIR KERNELS). The following calculation is the formal representation of the informal calculation analysed in the motivating arithmetical example. The nominal terms used in the inference steps are provided on the left side of the inference rules; the proper terms in the derivation are highlighted for readability.

[illegible]

1. *derivation and kernel*: The presented calculation is a derivation, as calculations are specific derivations. As the formula assumptions of a calculation are atomic derivations, the calculation is already equal to its kernel.
2. *evaluating the calculation*: Inferring in the next inference step the formula $x' + 0 = 0 + x'$ (introduction of the equality symbol) transforms this calculation into a closed calculation.
3. *proper derivation*: Replacing the provable assumptions by their derivations from the axioms of PA transforms the presented calculation into a derivation, which is not a calculation anymore. In particular, the presented calculation is a part of this derivation, which is not even a subderivation of the resulting derivation. The kernel of this derivation is the presented calculation (even if in the derivations of the provable assumptions occur some more calculations).

³³In order to describe formally such a transformation of a derivation, we would need to develop a suitable theory of occurrences and to define the kernel as an occurrence of a calculation in a derivation. Such an effort is beyond the needs of these investigations; we will use the notion of a kernel only informally.

We provide another example to illustrate the notion of a non-trivial kernel. In the following example derivation **D**, the kernel **C** is highlighted:

$$\frac{\frac{[z]^3 \quad \mathbf{z} = \mathbf{y} \quad (E=)}{\mathbf{y}} \quad \frac{\frac{\frac{x = y \quad [y]^2 \quad (E=)}{x} \quad \frac{[x]^1 \quad (Aux:1)}{x} \quad \frac{x \quad (I=:2)}{\mathbf{y} = \mathbf{x}} \quad \frac{\mathbf{f}(\mathbf{x}) \quad (E=)}{\mathbf{f}(\mathbf{y})} \quad (Aux:3)}{\mathbf{f}(\mathbf{z})}$$

We provide separately the kernel **C** of the example derivation:

$$\frac{\frac{[z]^1 \quad \mathbf{z} = \mathbf{y}}{\mathbf{y}} \quad \frac{\frac{y = x \quad \mathbf{f}(x)}{\mathbf{f}(y)} \quad (1)}{\mathbf{f}(z)}$$

Observe that the kernel of **D** is not a subderivation of **D**, also the open assumptions of **D** and of its kernel **C** are different. If we replace the kernel **C** in **D** by a calculation **C'** having the same equations as open assumptions as **C**, then we obtain again a derivation.

Inferring the result $f(x) = f(z)$ of the kernel in another inference step transforms the presented derivation **D** (which is not a calculation) into a derivation with empty kernel (as the conclusion is a formula and not a term).

REMARK (ANALYSIS OF THE CALCULUS). We communicate some more philosophical aspects of the calculus of Natural Calculation.

1. *informal reading*: In contrast to the truth values of formulae, the denotation of a term is an object of the universe. As a consequence, the informal interpretation of term rules has to be essentially different from the usual interpretations of formula rules.

A term assumption t can be read as “*chose t as a starting point for a calculation*”.³⁴ A term occurrence s in the subsequent calculation depends on the undischarged term assumption t , insofar the term s is “calculated from this starting point”.³⁵ When evaluating the calculation, this is to infer the result of the calculation, we may (and have to) “forget” this dependence. This is reflected by the mandatory discharge of the term assumption t .

³⁴Observe that the suggested phrase has no truth value.

³⁵This dependence on a term assumption t does not imply *necessarily* that the term s has the same denotation as t . In an extensions of the calculus of Natural Calculation (not discussed in this paper, but left to future work) which allows, for example, strict smaller-than calculations, the denotation of s has to be explicitly different from the denotation of the starting point t .

The discharge of the term assumptions has to be mandatory: if we would not discharge a term assumption when evaluating the respective calculation, then we would come into the inconvenient situation that the next calculation would have two, possibly different starting terms.

2. *naturalness*: It is easily seen that the natural aspects of informal calculations (as discussed in the analysis of our example proof) are implemented in the calculus: there is a proper rule representing the evaluation of calculations (the introduction of the equality rule); the other rules have justifications which can be used positively and negatively (reflected by the position of the justifications in the premises); furthermore, these rules are substructural, as it is possible to transform occurrences of subterms in the calculation steps. Finally, the calculations are sequential: due to the mandatory discharge of term assumptions, there is always a uniquely determined open term assumption in a calculation, even if the calculation is a proper tree due to auxiliary calculation steps. This means a calculation is a calculation from a term to a term, and such a calculation can always be extended by another calculation step.

3.4. Derivability and Calculability

The notion of derivability in the calculus of Natural Calculation is introduced as expected; we complement the definition by providing the analogous concept of calculability coined especially for terms.³⁶

DEFINITION 4 (DERIVABILITY).

1. *derivability*: We say that a formula A is *derivable* from a set Γ of formulae in the calculus of Natural Calculation (formally, $\Gamma \Vdash A$), if there is a derivation $\mathbf{D}$ such that $\mathbf{hyp}(\mathbf{D}) \subseteq \Gamma$ and $\mathcal{C}(\mathbf{D}) \simeq A$.
2. *equivalent derivations*: We call two derivations (*proof theoretically*) *equivalent*, if they have the same open assumptions and the same conclusion.
3. *calculability*: A term s is *calculable* from a term t under the assumption of a set Γ of formulae, if there is a derivation $\mathbf{D}$ such that $t \in \mathbf{hyp}(\mathbf{D})$, $\mathbf{hyp}(\mathbf{D}) \subseteq \Gamma \cup \{t\}$ and $\mathcal{C}(\mathbf{D}) \simeq s$.

EXAMPLE (DERIVABILITY). The canonical properties of identity are derivable in the calculus of Natural Calculation (formally, $\Vdash A$ for all $A \in T_-$).

³⁶Subsequently, we will not use the concept of calculability.

Exemplarily, we provide the respective derivations for reflexivity, symmetry and transitivity; the terms in the derivations are highlighted for readability.

1. *reflexivity and symmetry*:

$$\frac{\frac{[\mathbf{x}]^1}{x = x} \text{ (1)}}{\forall x. x = x} \quad ; \quad \frac{\frac{\frac{[x = y]^2 \quad [\mathbf{y}]^1}{\mathbf{x}}}{y = x} \text{ (1)}}{\frac{x = y \rightarrow y = x}{\forall xy. x = y \rightarrow y = x}} \text{ (2)}$$

2. *transitivity*:

$$\frac{\frac{[\mathbf{x}]^1 \quad \frac{[x = y \wedge y = z]^2}{x = y}}{\mathbf{y}} \quad \frac{[x = y \wedge y = z]^2}{\mathbf{y} = z}}{\frac{\frac{\mathbf{z}}{x = z} \text{ (1)}}{x = y \wedge y = z \rightarrow x = z} \text{ (2)}}{\forall xyz. x = y \wedge y = z \rightarrow x = z}$$

4. Basic Properties of the Calculus

We discuss some basic proof theoretic properties of the calculus of Natural Calculation up to completeness and soundness.

4.1. Dual Calculations

The *dual calculations* $\mathbf{D}^\delta$ of a calculation $\mathbf{D}$ is a calculation in which the same direct justifications are assumed as in $\mathbf{D}$ (but with their positions alternated), and in which the initial and final terms are exchanged. The latter means that the results of a calculation and of its dual calculation are symmetric. We provide the formal definition of dual calculations.

DEFINITION 5 (DUAL CALCULATION). The dual calculation $\mathbf{D}^\delta$ of a calculation $\mathbf{D}$ is defined recursively on the structure of $\mathbf{D}$ as follows.

1. *term assumption*: $\mathbf{D} \simeq t \quad \rightsquigarrow \quad \mathbf{D}^\delta \simeq t.$
2. *elimination of equality*:

$$\mathbf{D} \simeq \frac{\frac{\mathbf{D}_0}{\mathbf{t}[t]} \quad t = s}{\mathbf{t}[s]} \quad \rightsquigarrow \quad \mathbf{D}^\delta \simeq \frac{t = s \quad \mathbf{t}[s]}{\frac{\mathbf{t}[t]}{\mathbf{D}_0^\delta}}$$

And:

$$\mathbf{D} \simeq \frac{s = t \quad \mathbf{D}_0}{\mathfrak{t}[s]} \rightsquigarrow \mathbf{D}^\delta \simeq \frac{\mathfrak{t}[s] \quad s = t}{\mathfrak{t}[t]} \mathbf{D}_0^\delta$$

3. *auxiliary calculation - term*:

$$\mathbf{D} \simeq \frac{\mathbf{D}_0 \quad \mathbf{D}_1}{\mathfrak{t}[t]} \quad \rightsquigarrow \quad \mathbf{D}^\delta \simeq \frac{\mathbf{D}_1}{s} \quad \frac{[t] \quad \mathfrak{t}[s]}{\mathfrak{t}[t]} \mathbf{D}_0^\delta$$

And:

$$\mathbf{D} \simeq \frac{\mathbf{D}_1}{t} \quad \frac{[s] \quad \mathbf{D}_0}{\mathfrak{t}[t]} \quad \rightsquigarrow \quad \mathbf{D}^\delta \simeq \frac{\mathfrak{t}[s] \quad \mathbf{D}_1}{\mathfrak{t}[t]} \quad \frac{[s]}{t} \mathbf{D}_0^\delta$$

REMARK (DUAL DERIVATIONS). The result of replacing the kernel of a derivation $\mathbf{D}$ by the dual calculation of that kernel is called analogously *dual derivation* and denoted by $\mathbf{D}^\delta$. In the limit case, where the kernel is empty, as the derivation has no term conclusion, we have $\mathbf{D} \simeq \mathbf{D}^\delta$.

A recursive definition of the dual derivations is easily obtained from the definition above: the calculations have to be understood as derivations, the equations occurring as direct justifications have to be replaced by derivations having these equations as conclusions; in the missing clauses, the dual derivation is equal to the derivation itself.

OBSERVATION (NEGATIVE AUXILIARY CALCULATIONS). As a consequence of the existence of dual calculations, the negative auxiliary calculations (with respect to terms and formulae) are redundant. (We can replace any negative auxiliary calculation by its dual calculation at positive position.)

4.2. $\mathfrak{t}$ -Variants of Calculations

For every unary nominal term $\mathfrak{t}$, the $\mathfrak{t}$ -variant $\mathfrak{t}(\mathbf{D})$ of a calculation $\mathbf{D}$ is a calculation, in which all terms t occurring in the main path of the calculation $\mathbf{D}$ are replaced by the term $\mathfrak{t}[t]$. This is done again without changing the direct justifications of $\mathbf{D}$. We provide the formal definition.

DEFINITION 6 (**t**-VARIANT). Let $\mathbf{t}$ be an arbitrary, but unary nominal term. The $\mathbf{t}$ -variant $\mathbf{t}(\mathbf{D})$ of a calculation $\mathbf{D}$ is defined recursively on the structure of $\mathbf{D}$ as follows:

1. *term assumption*: $\mathbf{D} \simeq t \rightsquigarrow \mathbf{t}(\mathbf{D}) \simeq \mathbf{t}[t]$
2. *elimination of equality*:

$$\mathbf{D} \simeq \frac{\mathbf{D}_0 \quad \mathbf{r}[t] \quad t = s}{\mathbf{r}[s]} \rightsquigarrow \mathbf{t}(\mathbf{D}) \simeq \frac{\mathbf{t}(\mathbf{D}_0) \quad \mathbf{t}[\mathbf{r}[t]] \quad t = s}{\mathbf{t}[\mathbf{r}[s]]}$$

And:

$$\mathbf{D} \simeq \frac{s = t \quad \mathbf{D}_0 \quad \mathbf{r}[t]}{\mathbf{r}[s]} \rightsquigarrow \mathbf{t}(\mathbf{D}) \simeq \frac{s = t \quad \mathbf{t}(\mathbf{D}_0) \quad \mathbf{t}[\mathbf{r}[t]]}{\mathbf{t}[\mathbf{r}[s]]}$$

3. *auxiliary calculation - term*:

$$\mathbf{D} \simeq \frac{\mathbf{D}_0 \quad \mathbf{r}[t] \quad \frac{\mathbf{D}_1 \quad [t]}{s}}{\mathbf{r}[s]} \rightsquigarrow \mathbf{t}(\mathbf{D}) \simeq \frac{\mathbf{t}(\mathbf{D}_0) \quad \mathbf{t}[\mathbf{r}[t]] \quad \frac{\mathbf{t}(\mathbf{D}_1) \quad [t]}{s}}{\mathbf{t}[\mathbf{r}[s]]}$$

REMARK (**t**-VARIANTS).

1. *well-defined*: The $\mathbf{t}$ -variants of calculations are all well-defined: First, we mention that we do not have to consider the negative auxiliary calculations, as we can assume that they are replaced by their dual calculations in positive position. Second, the $\mathbf{t}$ -variants are, indeed, generated according to the term rules, as $\mathbf{t}[\mathbf{r}][t] \simeq \mathbf{t}[\mathbf{r}[t]]$ for all unary nominal terms $\mathbf{t}$ and $\mathbf{r}$, and all standard terms t .
2. *t-variants of derivations*: As in the case of dual derivations, the notion of a $\mathbf{t}$ -variant $\mathbf{t}(\mathbf{D})$ of a derivation $\mathbf{D}$, which is the derivation in which the kernel is replaced by its $\mathbf{t}$ -variant, is easily defined.

OBSERVATION (LINEAR CALCULATIONS). Every calculation $\mathbf{D}$ can be transformed into a linear calculation (without any applications of auxiliary calculations with respect to terms). In order to eliminate an auxiliary calculation step, we may apply the following conversion:

$$\frac{\mathbf{D}_0 \quad \mathbf{r}[t] \quad \frac{\mathbf{D}_1 \quad [t]}{s}}{\mathbf{r}[s]} \rightsquigarrow \frac{\mathbf{D}_0 \quad \mathbf{r}[t] \quad \mathbf{r}(\mathbf{D}_1)}{\mathbf{r}[s]}$$

Observe that we do not change the direct justifications by an application of this conversion, and the number of auxiliary calculation steps decreases by one. As a consequence, positive auxiliary steps (terms) are redundant.³⁷

OBSERVATION (AUXILIARY CALCULATIONS - FORMULAE). A second consequence of the existence of the $\mathbf{t}$ -variants is that we can restrict applications of the auxiliary calculation steps (formulae) to the replacement of complete terms occurring in the formula. By this restriction the inference rule becomes substructural with respect to formulae (as terms occurring inside of the formulae are replaced), but not with respect to terms.

4.3. Completeness and Soundness

The calculus of Natural Calculation is sound and complete (with respect to the standard calculus extended by standard identity rules). The proof of soundness can be simplified (slightly) by the presupposition that all calculations are linear.

THEOREM 1 (COMPLETENESS AND SOUNDNESS). *For all sets Γ of formulae and all formulae A :*

$$\Gamma \vdash = A \quad \Longleftrightarrow \quad \Gamma \Vdash A$$

PROOF (SKETCH)

1. *completeness*: We can replace every application of the traditional identity rules by a derivation in the calculus of Natural Calculation (similar to the derivations used for the canonical properties). As a consequence, the calculus of Natural Calculation is complete.
2. *soundness*: First, it is proved by induction over the length of linear calculations that the result of such a calculation is derivable from the direct justifications of this calculation in the standard calculus (extended by standard identity rules). This means for all linear calculations $\mathbf{D}$:

$$\mathbf{hyp}(\mathbf{D}) \vdash = \mathcal{R}(\mathbf{D})$$

Without loss of generality, we may assume that every calculation in a derivation $\mathbf{D}$ of Natural Calculation is linear, and therefore, we may replace the calculations by standard derivations. As a consequence, the calculus of Natural Calculation is sound.

■

³⁷This does not hold with respect to formulae: the positive auxiliary calculation steps (formulae) are unavoidable.

The proof of the theorem demands no application of the classical *reductio ad absurdum* rule; therefore, the theorem holds with respect to the classical as well as to the intuitionistic calculus, and, as a consequence, with respect to their respective semantics.

5. Normalisation

Following, in principle, Prawitz's approach [21] to normalisation, we discuss this topic here with focus on the new term rules and their influence on normalisation. We show that applications of the auxiliary calculation rule (formula) can be restricted to relational atomic formulae,³⁸ we provide a new type of cut formulae, namely cut formulae with respect to the equality symbol, complementing those introduced by Prawitz, we state the existence of normal derivations and discuss their structure.³⁹

5.1. Trivialisation of Auxiliary Calculations

Applications of the auxiliary calculation rule (formula) can be *trivialised*. By that we mean that applications of this rule can be restricted to relational atomic formulae.⁴⁰ In order to show this result, some proof conversions are introduced: if the auxiliary calculation (formula) rule is applied to a complex formula, then this application can be reduced to applications on the direct subformulae, and if it is applied to equations, then this application can be transformed into justified calculation steps on the equated terms.

DEFINITION 7 (TRIVIALISATION).

1. *equations*:

$$\frac{\mathbf{D}_0 \quad \mathbf{D}_1}{\mathbf{t}[a] = \mathbf{s}[a] \quad b} \rightsquigarrow \frac{\frac{[b] \quad \mathbf{D}_1^\delta}{\mathbf{t}[b]} \quad \mathbf{D}_0 \quad \mathbf{D}_1}{\frac{\mathbf{t}[a] = \mathbf{s}[a] \quad b}{\mathbf{s}[a]} \quad \frac{\mathbf{s}[b]}{\mathbf{t}[b] = \mathbf{s}[b]}^{(1)}} \quad (\dagger)$$

³⁸This is analogous to the result that the *reductio ad absurdum* and the *ex falso quodlibet* rules can be restricted to atomic formulae; cf. [21].

³⁹In order to avoid misunderstandings: we only discuss here normalisation on the level of formulae. An investigation of the normalisation of calculations is an independent and non-trivial task and left to future work.

⁴⁰This guarantees that auxiliary calculation steps (formulae) do not appear in an elimination or an introduction part of a normal derivation, but only in the middle parts. This is analogous to the treatment of both *falsum* rules in [21].

The symbols a and b denote arbitrary terms (and not necessarily constant symbols); observe that we apply at $(\dagger)$ a justified calculation step instead of an auxiliary calculation step (formula).

2. *implication*:

$$\frac{\frac{\mathbf{D}_0 \quad \frac{\mathbf{D}_1 \quad [t] \quad s}{\mathbf{A}[t] \rightarrow \mathbf{B}[t]}}{\mathbf{A}[s] \rightarrow \mathbf{B}[s]}}{\quad} \rightsquigarrow \frac{\frac{\mathbf{D}_0 \quad \frac{\frac{\frac{[s] \quad \mathbf{D}_1^\delta}{\mathbf{A}[s]} \quad t}{\mathbf{A}[t]} \quad [t]}{\mathbf{A}[t] \rightarrow \mathbf{B}[t]}}{\mathbf{B}[t]} \quad \mathbf{D}_1 \quad s}{\frac{\mathbf{B}[s]}{\mathbf{A}[s] \rightarrow \mathbf{B}[s]}} \quad (1)$$

3. *universal quantifier*:

$$\frac{\frac{\mathbf{D}_0 \quad \frac{\mathbf{D}_1 \quad [t] \quad s}{\forall x. \mathbf{A}[x, t]}}{\forall x. \mathbf{A}[x, s]}}{\quad} \rightsquigarrow \frac{\frac{\frac{\mathbf{D}_0 \quad \frac{\forall x. \mathbf{A}[x, t]}{\mathbf{A}[y, t]} \quad [t]}{\mathbf{A}[y, s]} \quad \mathbf{D}_1 \quad s}{\forall x. \mathbf{A}[x, s]}}$$

We demand that y is a fresh variable neither occurring in $\mathbf{D}_0$ nor in $\mathbf{D}_1$.

REMARK (TRIVIALISATION).

1. *well-defined*: The conversion for trivialising the auxiliary calculation steps are all well-defined. Essential argument in the case of the conversion (universal quantifier), is that neither x nor y occur in the terms t and s . As a consequence, the substitutions due to the rules for the universal quantifier do not affect the substitution due to the auxiliary calculation step.
2. *limit cases*: There are some limit cases not mentioned in the definition. For example: in the conversion (implication), the substitution due to the auxiliary calculation is represented by a unary nominal formula $\mathbf{A} \rightarrow \mathbf{B}$. As $\mathbf{A} \rightarrow \mathbf{B}$ is unary (by the demands of the inference rule), it is excluded that both $\mathbf{A}$ and $\mathbf{B}$ are standard formulae. But it is still possible that one of both, for example $\mathbf{A}$, is standard. In this case, trivially $\mathbf{A}[t] \simeq \mathbf{A}[s]$, and we may not use $\mathbf{A}[s]$ as a premise for an application of the (positive) auxiliary calculation step in the converted derivation; the conversion given above has to be simplified in the obvious way. Such limit cases are also possible with respect to the conversion (equation); the resulting simplifications are implicitly subsumed in the definition of the respective conversions.

3. *equivalent derivations*: It is immediate by the definition of the conversions that the converted derivations are equivalent to the original derivations, even in the implicit limit cases.

PROPOSITION 1 (TRIVIAL AUXILIARY CALCULATIONS) *Every derivation $\mathbf{D}$ can be transformed into an equivalent derivation $\mathbf{F}$ in which the positive auxiliary calculation rule is only applied to atomic relational formulae $P(\vec{t})$.*

PROOF (SKETCH) The proof is easily given by an induction over a suitable rank of derivations, which can be given as follows: the rank of equations is 1, that of relational expressions $P(\vec{t})$ and of $\perp$ equals 0. The rank of a complex formula is the maximum of the ranks of its direct subformulae increased by 1. The rank of a derivation is an ordered pair $\langle n, m \rangle$, where n is the maximal rank of a formula such that this formula is a premise of a (positive) auxiliary calculation step, and m is the number of such formulae with maximal rank. The pairs $\langle n, m \rangle$ are ordered lexicographically based on the canonical well order of the natural numbers.⁴¹ If a (positive) application of the auxiliary calculation (formula) on a formula with maximal rank is converted by a suitable conversion, then the rank of the resulting derivation is lowered. A derivation with minimal rank $\langle 0, 0 \rangle$ is as desired. ■

5.2. Normal Derivations

Due to the introduction and elimination rules for the equality symbol, a new kind of *cut formulae* is identified; the suitable corresponding proof conversion for their elimination are easily found.

DEFINITION 8 (CUT FORMULAE/CUT ELIMINATION).

1. *cut formula*: An occurrence of an equation $A \simeq t = s$ in a derivation $\mathbf{D}$ is called a *cut formula* (or a *maximal formula*), if A is the result of an introduction of the equality symbol and the premise of the corresponding elimination. Schematically, if:

$$\frac{\mathbf{D}_1 \quad \frac{[t] \quad \mathbf{D}_0}{s} \quad (I)}{r[t]} \quad (E) \quad ; \quad \frac{\frac{[t] \quad \mathbf{D}_0}{s} \quad (I) \quad \mathbf{D}_1}{r[s]} \quad (E)$$

⁴¹A similar rank function for derivations is implicitly used by Prawitz [21, p. 40] in his proof of weak normalisation for classical logic; that rank function is made explicit, for example, in the logic textbook [31, p. 193] by van Dalen. The difference between that rank function and the rank function used above is due to the different kinds of conversions.

The derivations $\mathbf{D}_0$ and $\mathbf{D}_1$ are arbitrary derivations (with term conclusion) and not necessarily calculations.

2. *cut elimination*: Cut formulae with respect to the equality symbol (as defined above) are eliminated by a conversion into the following derivations, respectively:

$$\frac{\mathbf{D}_1 \quad \frac{[t] \quad \mathbf{D}_0}{s}}{r[s]} \quad ; \quad \frac{\frac{[t] \quad \mathbf{D}_0}{s} \quad \mathbf{D}_1}{r[t]}$$

The kernel of the resulting derivation is not linear, but easily linearised.

We provide the definition of a normal derivation:

DEFINITION 9 (NORMAL DERIVATION). A derivation $\mathbf{D}$ is in *normal form*, if there are no cut formulae present in $\mathbf{D}$, neither with respect to the equality symbol nor with respect to the other logical constants.

Normalisation as discussed by Prawitz [21] can be carried over to the calculus of Natural Calculation.

PROPOSITION 2 (NORMAL FORM). *Every derivation $\mathbf{D}$ can be transformed into a normal derivation $\mathbf{F}$ having the same or less open assumptions as $\mathbf{D}$ and the same conclusion. More formally, $\mathbf{F}$ satisfies the following conditions:*

$$\mathbf{hyp}(\mathbf{F}) \subseteq \mathbf{hyp}(\mathbf{D}) \quad ; \quad \mathcal{C}(\mathbf{F}) \simeq \mathcal{C}(\mathbf{D})$$

PROOF (SKETCH) The same presuppositions as in [21] have to be assumed with respect to language, to the falsum rules and to the use of variables in the derivations; furthermore, it is presupposed that the auxiliary calculation rule (formula) is only applied to relational atomic formulae (proposition 5.2), that the redundant term rules are not applied at all and that the calculations are linear (see the observations in Section 4.1 and 4.2). Then the proof succeeds with the methods of [21]. ■

5.3. The Structure of Normal Derivations

Normalisation in the calculus of Natural Calculation is compatible with standard normalisation, but there are some differences: the simple division of a path into an elimination part E , a minimal part M and an introduction part I has to be generalised in the presence of calculations.

In a first step, we investigate the *generalised normal paths* in the case that a calculation is closed by an evaluation step; subsequently, we discuss

REMARK (NORMALISATION).

1. *even more normalisation*: In the second type of generalised normal paths, several subsequent applications of the auxiliary calculation rule (formula) are possible. Without loss of generality, we can demand some restrictions such that these applications become uniquely determined: the inference step has to affect the full term, this term has to be replaced by a different term. Each term of the relational atomic formula can only be affected once by an application of this rule and the applications have to be from the left to the right.
2. *subformula property*: As the subformula property is formulated modulo the terms occurring in the formulae, the subformula property as formulated by Prawitz holds also in the calculus of Natural Calculation.
3. *inside the calculations*: As mentioned before, the inner structure of calculations is not investigated in this paper. Whether and how the calculations themselves can be normalised is independent of the normalisation of the formula parts of a derivation (as discussed here) and is left to future work.

5.4. Complexity of Normalisation

We comment briefly the complexity of the derivations under normalisation with respect to the conversions introduced in these investigations.

1. *complexity measure*: We presuppose that the complexity of a derivation is given by the number of terms and formulae occurring in that derivation.
2. *linearisation*: Transforming a calculation into its dual calculation does not change its size. The same holds for a transformation into a $\mathbf{t}$ -variant. (Observe that the complexity of the terms themselves increases, if we replace the terms in a calculation by their $\mathbf{t}$ -variants.)
Therefore, the linearisation of a calculation reduces the number of terms in a calculation in the typical case by two; if the auxiliary calculation consists only of one term, then the complexity only decreases by one.
3. *trivialisation*: Applications of the conversions for the trivialisation of auxiliary calculations (formulae) may result in an exponential growth of complexity. The trivialisation with respect to the equality duplicates, essentially, the justification of the inference step (one copy of the justification and one copy of the dual justification of the same size); additionally, the derivation is extended by four terms. Analogously with respect to the implication; there the additional extension is by four formulae. The

trivialisation with respect to the universal quantifier has no duplication of justifications and only an additional increase by two formulae.

4. *elimination of equations*: Elimination of an equation as a cut formula decreases the complexity of the derivation by one. Linearising the result can result, as discussed above, in a further decrease of complexity.

We emphasise that the normalisation of derivations in the calculus of Natural Calculation also requires the application of the well-known conversions for the logical constants as given by Prawitz [21].

6. Philosophical Comments

The main goal of our investigation was to provide an extension of Gentzen's calculus of Natural Deduction by proper term rules for the equality symbol allowing a faithful representation of informal calculations occurring inside of mathematical proofs. This is complemented subsequently by a brief consideration of the impact of these inference rules on philosophical theories of meaning (of the equality symbol) in the spirit of proof-theoretic semantics.⁴²

Lacking denotation as a candidate for meaning (in the proof theoretic account), it is a tempting idea to identify meaning with essential properties, which are, in the case of the equality symbol, the canonical properties describing the identity relation as a universal congruence relation. This idea is supported by Kahle, when he writes: "From a technical point of view, one could say that the proof-theoretic semantics of the *equality relation* is given by the axioms involving this relation.," cf. [17, p. 159].

In this conception of meaning, meaning is not only determined by language use (this is, by inference rules), but it is an expressible meaning (expressible in a first order language), which, additionally, is formally provable (in the calculus). We investigate this proof theoretic approach to meaning in more detail.

⁴²In the contemporary discussion of meaning (of logical constants) in the field of proof-theoretic semantics, harmony of inference rules is a central aspect; cf. [6, 26]. This motivates Read [23] and Klev [18], for example, to investigate inference rules for the identity and to figure out which configuration of inference rules fits best into the paradigm of harmony.

Our approach is different: the communicated inference rules are determined by their actual use in mathematical praxis; we analyse these rules and consider their impact on theories of meaning in the spirit of proof-theoretic semantics. A discussion of harmony in general and also with respect to our rules is left to the future.

Correspondence to the Calculus. We can identify special features of the calculus of Natural Calculation corresponding to special aspects (these are specific canonical properties) of such a meaning of the equality symbol:⁴³

1. *reflexivity*: Reflexivity is established by the fact that the introduction rule can be applied immediately on a term (without any calculation steps).
2. *transitivity*: Transitivity is established by the fact that the introduction rule can be applied after several (after at least two) calculation steps.⁴⁴
3. *symmetry*: Symmetry is established by the fact that the elimination rule has a positive and negative version and that these versions are restricted by the same class of nominal terms.
4. *compatibility - function symbols*: Compatibility with respect to function symbols is established by the fact that the elimination rules are restricted to arbitrary nominal terms (in which the nominal symbols may occur as proper subterms), and not only to the nominal symbol $*$.⁴⁵
5. *compatibility - relation symbols*: Compatibility with respect to relation symbols is established by the rules for auxiliary calculations with respect to formulae.

It is worth mentioning that the auxiliary calculation rule with respect to terms seems to have no influence on the canonical properties of identity, but only on the calculus itself: without this rule, every calculation would be linear; as a consequence, cut elimination would have to be combined necessarily with the conversions used for linearisation.

Distributed Meaning. Our conception of meaning is a distributed meaning. There is no single rule determining the full meaning of the equality symbol, neither an introduction rule nor an elimination rule. As seen above, specific aspects of the meaning of the equality symbol correspond with specific features of the calculus found in different rules. Therefore, it is possible to manipulate separately these features of the calculus and modify this way the

⁴³This correspondence is not the reason for the meaning of the equality symbol, but a good argument for the choice of inference rules.

⁴⁴This is not a trivial property of the introduction rule: if we would formulate rules for the not-equal relation, then we would have to restrict the introduction rule to calculations in which exactly once the respective elimination rule was applied.

⁴⁵By this fact, we may replace proper subterms and do not have to replace the complete term in an application of the elimination rules.

properties (this is the meaning) of the symbol characterised by the inference rules.⁴⁶

As a consequence, neither is the introduction rule for the equality symbol nor the elimination rules (nor the auxiliary calculation rules) a *definition* of that symbol. Putting things further, it seems clear that neither is the elimination rule the result of applying a (simple) function on the introduction rule, nor vice versa. To obtain such a function, we have to add an additional argument ensuring that the function works as desired. The latter means that such a function depends on the semantic role of the symbol characterised by the inference rules.

Independence of the Calculus. The meaning of the equality symbol is independent of the chosen calculus, as the canonical properties are derivable in every complete and sound calculus (even with respect to intuitionistic calculi). Moreover, the canonical properties can be determined even in (denotational) semantics.⁴⁷ We consider this as a desirable aspect of our account to meaning, as it is plausible that meaning is independent of the mechanisms used to determine meaning.⁴⁸ At least, it would be odd to assert that meaning is different in different calculi, that, for example, the Hilbert calculus meaning of the equality symbol is different from its Natural Deduction meaning and so on.

Nevertheless, natural calculi still have a special role in this context, as the inference rules (and possibly other features of the calculus) are directly related with the meaning.

Dependence on the Context. Another observation with respect to our proof theoretic account of meaning is that in a formal language without any relation symbols, all auxiliary calculations are redundant; in particular, we need no auxiliary calculations with respect to formulae. The latter means that introduction and elimination of the equality symbol are sufficient to determine the meaning of the equality symbol in such a context. Correspondingly, there are no compatibility axioms for relation symbols in the

⁴⁶In particular, we are able to provide analogous term rules for the (strict) smaller-than relation, which is reflexive (anti-reflexive), anti-symmetric and transitive, or the not-equal relation, which is symmetric, but anti-reflexive and not transitive.

⁴⁷The latter does not mean that our proof theoretic meaning is the traditional meaning of denotational semantics. On the contrary, meaning in the traditional denotational semantics depends on $\mathfrak{L}$ -structures, in which the non-logical symbols are interpreted. But as there is no proof theoretic counterpart for specific structures, a proof theoretic meaning cannot agree with the traditional denotational conception of meaning.

⁴⁸This reminds on the Fregean [7] distinction between sense and meaning.

set of canonical properties. This means that the meaning of the equality symbol depends on the language in which this symbol is used.

We consider such a dependence on the context as unproblematic. At least, there is a similar dependence on the context in a denotational semantics, where the meaning of the equality symbol is understood as the identity relation on the universe of a structure (the denotation of the equality symbol); the identity relation depends on the structure, in which the equality symbol is interpreted: if the universes of two structures are different, then so are the respective identity relations on the universes.

We have to draw attention to another phenomenon: in the language $\mathfrak{L}_{\text{PA}}$ of arithmetics, for example, there are sentences about the equality symbol not provable from the canonical properties, but from the axioms of PA. Having accepted the dependence of meaning on the context of language use, it seems plausible that in the context of arithmetics, the meaning of the equality symbol is changed due to the derivability of these new sentences.

This view is supported by the fact that we may transform the axioms of PA into inference rules. Investigate, for example, the following both versions of an inference rule (formulated as term rules in Natural Calculation style) corresponding to the axiom $\forall xy. S(x) = S(y) \rightarrow x = y$ of PA:

$$\frac{S(t) = S(s) \quad t}{s} \quad \text{or even} \quad \frac{S(t) = S(s) \quad \mathbf{r}[t]}{\mathbf{r}[s]}$$

It seems difficult to find a reason, why the term rules introduced in Natural Calculation should determine the meaning of the equality symbol, but not this especially arithmetical rule.⁴⁹

As a consequence of this dependence on the context, there is reason to doubt the traditional distinction between logical and non-logical symbols. It seems reasonable to distinguish between axioms provable without assumptions in a complete and sound calculus establishing the logical kernel of meaning and axioms provable in extensions of such calculi establishing non-logical aspects of the meaning of symbols under discussion.⁵⁰

⁴⁹Analogously, we could argue that the induction rule in arithmetics, this is the rule version of the axiom schema of induction of PA, alters the meaning of the universal quantifier in the context of arithmetics. If meaning is given by inference rules, it is admissible to ask: what is the syntactic entity of which the meaning is determined by the induction rule? It seems that the universal quantifier (in the context of arithmetics) is the answer.

⁵⁰On the long run, we have to find a better characterisation of complete and sound calculi, as the latter is not a proof theoretic concept, but a concept of standard denotational semantics.

This conception of meaning as context dependent axiomatisation is complemented by Kahle’s observation that the *sense* of a specific equation should depend on the axiomatic background in which this equation is investigated, cf. [17, p. 159].

Expressible Meaning. Central aspect of our proof theoretic approach to meaning is that it is an *expressible* meaning: the meaning of the equality symbol is given by sentences of a formal language. Recalling that the equality symbol connects two terms, we can understand the canonical properties of the equality symbol as axioms formulated in the metalanguage.

We suggest to consider, whether there is an analogous conception of expressible meaning for the connectives and quantifiers, expressed in a respective metalanguage. Good candidates for metalingual expressions could be sequents describing the essential properties of the logical constants or sentences of second order logic axiomatising these constants.

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⁵¹The slides of the talk given at the conference “Advances in Proof-Theoretic Semantics” in Tübingen can be found in the online proceedings [8].

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