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Free Logics are Cut-Free

Abstract. The paper presents a uniform proof-theoretic treatment of several kinds of free logic, including the logics of existence and definedness applied in constructive mathematics and computer science, and called here quasi-free logics. All free and quasi-free logics considered are formalised in the framework of sequent calculus, the latter for the first time. It is shown that in all cases remarkable simplifications of the starting systems are possible due to the special rule dealing with identity and existence predicate. Cut elimination is proved in a constructive way for sequent calculi adequate for all logics under consideration.

Keywords: Sequent calculus, Free logic, Definedness logic, Logic of partial terms, Logic of existence, Cut elimination.

Introduction

Free logics come from different sources, appear under many names, and find multiple applications. In several philosophical contexts and applications to modal logics (see e.g. Bencivenga [3], Garson [8], Priest [24]) they are usually called free logics (the name coined by K. Lambert). In constructive mathematics and applications to computer science they are called logics of existence (Scott [25], Troelstra and van Dalen [29]), logics of definedness (Feferman [6]), or logics of partial terms (Beeson [2]). Despite the many faces, and variety of applications of this branch of logic, the common, and the main, feature is that singular terms are free from existential assumptions, i.e. they are not assumed to denote an existing object (we will call such terms shortly nondenoting), as in the standard classical or intuitionistic logic. Accordingly, both classical and intuitionistic logic may be modified to obtain their free versions. On the other hand, in all logics under consideration quantifiers are assumed to have an existential import.

We provide a systematic treatment of some important free logics in the framework of sequent calculus (SC). Our formalization covers logics which were not so far dealt with in SC, but also provides an improved and simplified account even in cases which were already presented as such calculi. In particular, we provide a uniform formalization of all logics considered (in the language with identity) by means of a simple rule (EI) (existence predicate

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introduction) which is derivable in more standard approaches but is powerful enough to simplify quantifier rules of all systems under consideration. These simplifications are twofold: (a) for all logics discussed the only terms which have to be instantiated when quantifiers are eliminated are variables; (b) for some logics, quantifier rules with no additional existence formulae are sufficient, in contrast to usually applied formalizations. Moreover, all refined SC systems will be proven cut-free in a constructive way. Our analysis will be strictly proof-theoretic in the sense that no reference to a semantic characterization of the discussed logics, except informal remarks, will be needed.

In what follows we consider six different kinds of free logics obtained by using three criteria of division: positive versus negative, inclusive versus noninclusive, and absolutely free versus quasi-free. The former division is well known from the literature on free logics (see, e.g. Bencivenga [3]). In positive logics we admit that atomic formulae with nondenoting terms are evaluated as true or false. In negative logics all such formulae are evaluated as false or, to the same effect, all primitive predicates and functions are strict, i.e. interpreted only on denoting terms. We do not consider here neutral free logics (or in Lehmann's terms [18], a strictly Fregean logic) since they cannot be based on the standard classical or intutionistic basis and the construction of SC requires an adaptation of some nonstandard techniques. The second division concerns the semantic situation; inclusive logics admit empty domains in models, whereas noninclusive ones exclude such a situation. Free logics which are inclusive are called universally free. The last division, on absolutely and quasi-free, is a new one, at least as a terminological proposal (Lehmann used a phrase "somewhat less free version"). It is determined by a semantical treatment of variables. In the case of quasi-free logics valuations of variables are standard functions, i.e. variables are assumed to denote existing objects. In the case of absolutely free logics all terms, including variables, may fail to denote, i.e. valuations are partial functions. All quasi-free logics are also noninclusive.

These six logics in the language without identity will be referred to as:

- PFL (positive FL, universally and absolutely free);
- NFL (negative FL, universally and absolutely free);
- PFL^n (positive noninclusive FL);
- NFL^n (negative noninclusive FL);
- PFL^+ (quasi-free positive FL);
- NFL^+ (quasi-free negative FL).

In the language with identity a subscript = will be added to the respective name (i.e. $\text{PFL}_=$, $\text{NFL}^+_=$, etc.). We will be using also names FL, FL^+ if some result applies to both absolutely free, or both quasi-free logics.

PFL (as well as PFL^n) is the most popular version of FL, known better from philosophical applications and usually grounded on the classical basis (see e.g. Bencivenga [3], Lehmann [18]). The remaining logics are strongly connected with applications in constructive mathematics and computer science, so usually they were rather used in the intuitionistic version. However, the specific axioms and rules may be used also with the classical basis, as was noted by Feferman [6]. Thus $\text{NFL}_=$ is the logic of the existence predicate of Scott [25], whereas $\text{NFL}^+_=$ is the logic of partial terms (LPT) of Beeson [2]. In the former case, Scott examined the intuitionistic version whereas Tennant [27] provided the classical version. LPT is also called the basic system of definedness logics which are treated as a wider family of systems specialised to deal with partial untyped combinatory and lambda calculi (Feferman [6]). Our custom of using a superscript '+' is borrowed from Troelstra and van Dalen [29] who provided an extensive comparison of both negative approaches in the context of constructive mathematics under the names E-logic and E^+-logics respectively. PFL^+ as such was rather not considered so far in isolation. However, in the intuitionistic version it belongs to the wider family of logics with existence predicate which has found an interesting proof-theoretic analysis. Baaz and Iemhoff [1] provided SC for intuitionistic versions of identity-free PFL and PFL^+ with proofs of cut elimination (full for PFL and partial for PFL^+) and interpolation. These results were recently improved by Maffezioli and Orlandelli [19]. $\text{PFL}^+_=$ was not examined so far to the best of our knowledge.

Most of the considered logics have an adequate semantic and axiomatic characterization ($\text{PFL}^+_=$ seems to be an exception). Although the first system which may be treated as a kind of free logic was presented as a natural deduction (ND) system (Jaśkowski [16], see Bencivenga [3]), the number of nonaxiomatic proof systems for several variants of free logic is relatively small. There are ND systems for both kinds of NFL (Tennant [27], Troelstra and van Dalen [29]). Tableau systems can be found in Bencivenga et al. [4], Gumb [10] and in Priest [24]. As for the treatment in terms of SC, Bencivenga [3] contains a formalization of PFL and Gratzl [9] provided SC for $\text{NFL}_=$ where cut is eliminable partially (so called "inessential cuts", in Takeuti's sense, cannot be removed) – both on the classical basis. There is also a fresh work of Pavlović and Gratzl [23] where $\text{PFL}_=$ and $\text{NFL}_=$, also in their noninclusive variant, are examined as cut-free SC. We already mentioned SC for intuitionistic PFL in both variants provided by Baaz and Iemhoff [1], and by Maffezioli and Orlandelli [19]. Some earlier works like that of Trew [28] are concerned mostly with other systems and their peculiar features.

We will provide sequent calculi for all above-mentioned logics in both classical and intuitionistic version. The main results of the paper are the following: (1) A unified proof-theoretic framework will be provided for all logics considered. (2) We will show that all logics with identity may obtain improved SC formalizations on the basis of the additional rule (EI). (3) In the case of quasi-free logics the introduction of (EI) leads to even more drastic simplification of the quantifier rules and elimination of existence predicate. (4) Cut elimination will be proved in a uniform way for most sequent systems presented, including all based on (EI).

In Section 1, we provide the basic technical information on languages, logics and sequent calculi. In Section 2, SC for six free logics in identity-free language will be characterised. The next section deals with the systems for the same logics but with identity. In Section 4, we consider an alternative treatment of these logics based on the application of the special rule (EI) that introduces existence statements into antecedents and uses the quantifier rules with instantiation of terms restricted to variables. Moreover, in Section 5, we will show that quasi-free logics may be characterised by means of quantifier rules of the same kind as those used in the classical and intuitionistic logic, but with the restricted instantiation of terms; this eventually justifies our terminology – quasi-free logics. Section 6 provides a uniform proof of cut elimination for the systems under consideration.

1. Preliminaries

We will be concerned with logics formulated in standard predicate languages with the following logical vocabulary:

- connectives: $\neg, \wedge, \vee, \rightarrow$;
- first-order quantifiers: $\forall, \exists$;
- predicates: E, $=$.

An unary existence predicate E is taken as logical and primitive (with one exception – see Section 6); a binary predicate $=$ is primitive in languages of all systems considered in Sections 3–5.

Our considerations apply to logics formulated in languages with two (possibly empty) sets of primitive extralogical predicate- and function-symbols of any arity n. In the metalanguage they will be denoted by R^n and f^n respectively. Note that metavariable R^n applies also to logical predicates E and $=$.

The category of terms covers variables and complex terms built by means of function symbols. Formulae are built recursively in the standard way. Metavariables $t, t_1, \ldots$ are applied to any terms, φ, ψ, χ denote any formulae, and $\Gamma, \Delta, \Pi, \Sigma$ their finite multisets.

Individual variables will be divided additionally into bound $VAR = \{x, y, z, \ldots\}$ and free variables (parameters) $PAR = \{a, b, c, \ldots\}$. This Gentzen's notational custom facilitates several technical matters but is not essential and all the presented systems and the results holding for them may be easily rephrased for languages where such distinction is not introduced.

$\varphi[t_1/t_2]$ is used for the operation of substitution of an arbitrary term t_2 for all occurrences of a variable t_1 in φ, and similarly $\Gamma[t_1/t_2]$ for a uniform substitution in all formulae in Γ. It is always assumed that the substitution thus represented is correct, i.e. if t_2 is a variable or contains variables, they remain free after substitution.

The six free logics we consider were characterised axiomatically and semantically. For our purposes the former characterization is enough and we briefly recall it here. We present their axiomatic formulations in the language with the existence (or definedness) predicate E. Note that researchers working on mathematical and computer science applications (e.g. Beeson [2], Feferman [6], Gumb [10]) prefer to use not only different terminology and semantics but even different notation. Thus the language is enriched with unary definedness predicate $\downarrow$ applied as a suffix to any term. The informal meaning of $t\downarrow$ is that t is defined or has a denotation. Although Beeson emphasized the difference between the predicate of definedness and existence, with $\downarrow$ expressing the property of terms not of objects, from a technical point of view it is not important and we will be uniformly applying a unary prefix-symbol E as representing either definedness or existence relative to the system under consideration. On the other hand, it is worth emphasizing that the final result of our proof-theoretic analysis can be taken as supporting Beeson's view of significant differences between these two approaches (see Section 5).

PFL may be characterised axiomatically (i.e. as a Hilbert system) by the addition of the following axioms and rules to some standard propositional classical HCL (or intuitionistic – HIL) basis:

$(E\exists I)$ $\varphi[x/t] \wedge Et \rightarrow \exists x\varphi$

$(E\exists E)$ if $\vdash \varphi \wedge Ex \rightarrow \psi$, then $\vdash \exists x\varphi \rightarrow \psi$, where x is not free in ψ

$(E\forall I)$ if $\vdash \psi \wedge Ex \rightarrow \varphi$, then $\vdash \psi \rightarrow \forall x\varphi$, where x is not free in ψ

$(E\forall E)$ $\forall x\varphi \wedge Et \rightarrow \varphi[x/t]$

The resulting system HPFL is universally free, i.e. it is adequate with respect to models with empty domains. If we want to obtain the system for HPFL^n, i.e. the version adequate wrt. nonempty models we must add the axiom of existence:

(E) $\exists x Ex$.

In NFL all predicates are required to be strict in the sense that they are defined only on denoting terms. To obtain the system HNFL we must add a Denotation Principle:

(DP) $R^n t_1 \ldots t_n \rightarrow Et_1 \wedge \ldots \wedge Et_n$

with R^n covering also equality in the case of NFL$_=$. This principle is extended to functions (again all functions are strict in NFL):

(DP') $Ef^n t_1 \ldots t_n \rightarrow Et_1 \wedge \ldots \wedge Et_n$

What is called here FL$^+$ (in both positive and negative version) corresponds to the logic of partial terms LPT (Beeson [2]) or definedness logic (Feferman [6]) or E$^+$ (Troelstra and van Dalen [29]). It seems to be a logic which is halfway to full free logic in the sense that two quantifier principles are like in classical/intuitionistic logic whereas two (universal instantiation and existential generalization) are formulated with existential minor premisses. Thus to obtain HPFL$^+$ we add either to HCL or to HIL $(E\exists E)$ and $(E\forall I)$, but instead of $(E\exists E)$ and $(E\forall I)$ we add classical rules:

$(\exists E)$ if $\vdash \varphi \rightarrow \psi$, then $\vdash \exists x\varphi \rightarrow \psi$, where x is not free in ψ

$(\forall I)$ if $\vdash \psi \rightarrow \varphi$, then $\vdash \psi \rightarrow \forall x\varphi$, where x is not free in ψ

Moreover we need the characteristic set of denotation axioms of the form:

(DA) Ea, for every parameter (free variable).

They syntactically express the fact that variables range over existent objects. In fact, these logics were considered rather as intuitionistic and negative versions, however, Feferman [6] states that the classical basis can also be used and Baaz and Iemhoff [1] deal with the positive version (and on the intuitionistic basis only). For the sake of uniformity we will examine all systems in both their classical and intuitionistic version, even if they were not treated in such a way originally (in particular, in terms of adequate semantics).

$\mathrm{HPFL}_=$ may be characterised by the addition of reflexivity and Leibniz Principle:

(R) $t = t$

(LP) $t_1 = t_2 \wedge \varphi[x/t_1] \rightarrow \varphi[x/t_2]$.

In the same way we obtain $\mathrm{HPFL}^n_=$, $\mathrm{HPFL}^+_=$.

In the case of $\mathrm{HNFL}_=$ we keep (LP) but (R) must be replaced with a kind of restricted reflexivity:

(R') $Et \leftrightarrow t = t$.

The addition of the same axioms is required for $\mathrm{HNFL}^n_=$ and $\mathrm{HNFL}^+_=$. However for the latter we may use also as the axiom of quantified reflexivity:

(QR) $\forall x x = x$.

Let us note that in some formalizations of NFL (e.g. Beeson [2], Scott [25]) also a different version of (LP) is proposed to the effect that $t_1 = t_2$ is replaced with $Et_1 \vee Et_2 \rightarrow t_1 = t_2$ and we have:

(LP') $(Et_1 \vee Et_2 \rightarrow t_1 = t_2) \wedge \varphi[x/t_1] \rightarrow \varphi[x/t_2]$.

This follows from the fact that in NFL a different kind of identity is considered. A difference can be roughly explained semantically without entering into details. FL is often interpreted in models where all terms have a denotation in outer domain OD and exsistent ones in a (inner) subdomain $ID \subseteq OD$. Assumed interpretation of $=$ in PFL is then a diagonal of OD^2, but in NFL it is a diagonal of ID^2. Strictly speaking this latter NFL-identity should be denoted with a different symbol which is often the case. However, in NFL all predicates are strict so it comes to the same and (LP') may be replaced with (LP). Therefore the only essential difference concerns reflexivity which must be weakened. In Section 3 we will justify formally our claim that (LP) is interderivable with (LP') in NFL.

All logics considered will be formalised as sequent calculi with sequents $\Gamma \Rightarrow \Delta$ being ordered pairs of finite multisets of formulae called the antecedent and the succedent, respectively. Moreover, in the intuitionistic version the succedents are restricted to at most one formula. We will use as a propositional basis the calculus G (after Gentzen) which is essentially the calculus G1 of Troelstra nad Schwichtenberg [30]. All necessary structural rules, including cut, weakening and contraction are primitive. The calculus G consists of the following rules:

Definition 1. Rules of G:

$(AX)\ \varphi, \Gamma \Rightarrow \Delta, \varphi$

$(W\Rightarrow)\ \dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$

$(C\Rightarrow)\ \dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$

$(\neg\Rightarrow)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$

$(\wedge\Rightarrow)\ \dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$

$(\vee\Rightarrow)\ \dfrac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$

$(\rightarrow\Rightarrow)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}$

$(Cut)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$

$(\Rightarrow W)\ \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$

$(\Rightarrow C)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$

$(\Rightarrow\neg)\ \dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

$(\Rightarrow\wedge)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

$(\Rightarrow\vee)\ \dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

$(\Rightarrow\rightarrow)\ \dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$

In the case of intutionistic version one of the contraction rules $(\Rightarrow C)$ is dispensable, and $(\Rightarrow \vee)$ must be split into two rules, with either φ or ψ as the only side formula in the succedent.

Let us recall that formulae displayed in the schemata are active, whereas the remaining ones are parametric, or form a context. In particular, all active formulae in the premisses are called side formulae, and the one in the conclusion is the principal formula of respective rule application. Proofs are defined in a standard way as finite trees with nodes labelled by sequents. The height of a proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ is defined as the number of nodes of the longest branch in $\mathcal{D}$. $\vdash_k \Gamma \Rightarrow \Delta$ means that $\Gamma \Rightarrow \Delta$ has a proof of height k.

We prefer to work with G rather than with the calculus G3 which is often preferred as better-behaved for proof search (see e.g. Troelstra and Schwichtenberg [30] or Negri and von Plato [21]; it is also used for FL in Pavlović and Gratzl [23]). However, for our purposes of showing interderivability of several rules in different systems, the calculus with primitive structural rules, including cut, is more convenient. In G3-style calculi we have to prove admissibility of structural rules and invertibility of logical rules to demonstrate their adequacy. Having such a calculus with invertible rules is certainly preferable for proof search and Hintikka-style completeness proofs, but in the study of a variety of systems it is easier to operate with SC having structural rules, cut in particular, as primitive, and prove instead derivability of new rules. The main difference is that admissibility results are in general more complicated and system-dependent, so they should be examined again for every extension, since the addition of new rules can spoil them (although it is possible to make extensions which guarantee the preservation of structural properties – see Negri and von Plato [22]). Proofs of derivability are simpler (just proofs in the system) and automatically hold for all extensions. Moreover, a proof of cut elimination presented in Section 6 is more direct in the sense that we do not need to prove first several auxiliary results on invertibility of logical rules and admissibility of other structural rules which is characteristic in the case of G3-style calculi. The last thing is that in G3, antecedent rules for negation, implication and universal quantifier in the intuitionistic version differ significantly from classical rules to preserve invertibility, whereas in the system G only numerical restriction on succedents matters, and all proofs presented on the classical basis are immediately transformed into intutionistic format.

This propositional basis, in the classical or intutitonistic version, will be fixed for all systems under consideration. Concerning the rules for quantifiers we display for further reference three sets of rules:

Definition 2. A simple set SQ:

$$(\forall\Rightarrow)\ \frac{\varphi[x/t], \Gamma\Rightarrow \Delta}{\forall x\varphi, \Gamma\Rightarrow \Delta} \qquad (\Rightarrow\forall)\ \frac{\Gamma\Rightarrow \Delta, \varphi[x/a]}{\Gamma\Rightarrow \Delta, \forall x\varphi}$$

$$(\exists\Rightarrow)\ \frac{\varphi[x/a], \Gamma\Rightarrow \Delta}{\exists x\varphi, \Gamma\Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma\Rightarrow \Delta, \varphi[x/t]}{\Gamma\Rightarrow \Delta, \exists x\varphi}$$

Definition 3. An existence set EQ:

$$(\forall E\Rightarrow)\ \frac{\varphi[x/t], \Gamma\Rightarrow \Delta}{Et, \forall x\varphi, \Gamma\Rightarrow \Delta} \qquad (\Rightarrow\forall E)\ \frac{Ea, \Gamma\Rightarrow \Delta, \varphi[x/a]}{\Gamma\Rightarrow \Delta, \forall x\varphi}$$

$$(\exists E\Rightarrow)\ \frac{Ea, \varphi[x/a], \Gamma\Rightarrow \Delta}{\exists x\varphi, \Gamma\Rightarrow \Delta} \qquad (\Rightarrow\exists E)\ \frac{\Gamma\Rightarrow \Delta, \varphi[x/t]}{Et, \Gamma\Rightarrow \Delta, \exists x\varphi}$$

Definition 4. A mixed set MQ:

$$(\forall E\Rightarrow)\ \frac{\varphi[x/t], \Gamma\Rightarrow \Delta}{Et, \forall x\varphi, \Gamma\Rightarrow \Delta} \qquad (\Rightarrow\forall)\ \frac{\Gamma\Rightarrow \Delta, \varphi[x/a]}{\Gamma\Rightarrow \Delta, \forall x\varphi}$$

$$(\exists\Rightarrow)\ \frac{\varphi[x/a], \Gamma\Rightarrow \Delta}{\exists x\varphi, \Gamma\Rightarrow \Delta} \qquad (\Rightarrow\exists E)\ \frac{\Gamma\Rightarrow \Delta, \varphi[x/t]}{Et, \Gamma\Rightarrow \Delta, \exists x\varphi}$$

where in all sets a (the eigenvariable) is not in Γ, Δ, φ and t is any term free for x in all sets of rules.

Definition 5. Restricted quantifier rules

We will consider $(\forall \Rightarrow), (\Rightarrow \exists), (\forall E \Rightarrow), (\Rightarrow \exists E)$ also in restricted form where t may be only an arbitrary (but correctly substitutable) parameter. Such restricted rules will be called $(\forall \Rightarrow'), (\Rightarrow \exists'), (\forall E \Rightarrow'), (\Rightarrow \exists E'),$

and accordingly restricted sets of quantifier rules are called SQR, EQR and MQR.

The first (simple) set is just the set of quantifier rules from Gentzen's LK characterising classical first-order logic and intuitionistic one with empty Δ. It will be of use for us only in the restricted version SQR in Section 5. The remaining sets characterize suitable free logics, in particular MQ (and MQR) is the basic set for PFL$^+$ and NFL$^+$ whereas EQ and EQR characterize the remaining logics. We note here the following fact:

Claim 1. $(\forall E \Rightarrow)$ is derivable by $(\forall \Rightarrow)$; $(\Rightarrow \forall)$ is derivable by $(\Rightarrow \forall E)$. $(\Rightarrow \exists E)$ is derivable by $(\Rightarrow \exists)$; $(\exists \Rightarrow)$ is derivable by $(\exists E \Rightarrow)$.

Proof. requires only the application of $(W \Rightarrow)$. ■

Note that often (see e.g. Bencivenga [3], Baaz and Iemhoff [1]) $(\forall E \Rightarrow)$ and $(\Rightarrow \exists E)$ are introduced as two-premiss rules:

$$(2\forall E\Rightarrow)\ \frac{\Gamma\Rightarrow \Delta, Et \quad \varphi[x/t], \Pi\Rightarrow \Sigma}{\forall x\varphi, \Gamma, \Pi\Rightarrow \Delta, \Sigma} \qquad (2 \Rightarrow\exists E)\ \frac{\Gamma\Rightarrow \Delta, Et \quad \Pi\Rightarrow \Sigma, \varphi[x/t]}{\Gamma, \Pi\Rightarrow \Delta, \Sigma, \exists x\varphi}$$

They are interderivable with their one-premiss variants stated above, but the latter have smaller branching factor and this simplifies proof-figures. They are also better for proof-search and used in G3 but, due to the missing primitive contraction rules, they must be changed a bit to obtain a contraction-absorbing variant:

$$(1\forall E\Rightarrow)\ \frac{\varphi[x/t], \forall x\varphi, Et, \Gamma\Rightarrow \Delta}{\forall x\varphi, Et, \Gamma\Rightarrow \Delta} \qquad (1 \Rightarrow\exists E)\ \frac{Et, \Gamma\Rightarrow \Delta, \exists x\varphi, \varphi[x/t]}{Et, \Gamma\Rightarrow \Delta, \exists x\varphi}$$

and in the intuitionistic variant with no repetition of $\exists x\varphi$ in the premiss of $(1 \Rightarrow \exists E)$ and Δ empty. Such rules were applied by Maffezioli and Orlandelli [19] and Pavlović and Gratzl [23]. In our approach simpler variants are sufficient, since contraction is primitive.

2. The Basic Sequent Calculi for Identity-Free FL

Let us start with GPFL. It may be characterised by means of the existence set of quantifier rules added to G (see definition 1 and 3). Of course, in the case of the intuitionistic version we restrict succedents of rules to at most one formula (active or parametric). We omit the proof of equivalence with H-system; one can find such a proof in Pavlović and Gratzl [23].

To obtain sequent calculi for the remaining five logics let us consider the following rules which can be collectively treated as the rules for elimination of existence predicate in the antecedent:

DEFINITION 6. Existence elimination rules:

$$(EE)\ \frac{Eb, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (EE^r)\ \frac{Ea, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \text{where } a \text{ is not in } \Gamma, \Delta.$$

$$(NEE)\ \frac{Et_i, \Gamma \Rightarrow \Delta}{R^n t_1 \ldots t_n, \Gamma \Rightarrow \Delta} \qquad (NEE')\ \frac{Et_i, \Gamma \Rightarrow \Delta}{Ef^n t_1 \ldots t_n, \Gamma \Rightarrow \Delta}$$

for $i \leq n$, every predicate R^n and function f^n.

Note that (EE) is stronger than (EE^r) (which is called NI in Pavlović and Gratzl [23]), since b is an arbitrary parameter. If we want to obtain GPFL^n, the system excluding empty domains, we must add to GPFL this latter, restricted version of E elimination. The proof of the adequacy with axiomatic formulation is straightforward; in one direction it is sufficient to prove the axiom $\exists x Ex$:

$$\cfrac{(\Rightarrow \exists E)\ \cfrac{Ea \Rightarrow Ea}{Ea, Ea \Rightarrow \exists x Ex}}{(C \Rightarrow)\ \cfrac{Ea \Rightarrow \exists x Ex}{(EE^r)\ \Rightarrow \exists x Ex}}$$

For the other direction it is sufficient to prove derivability of (EE^r) in G with added axiomatic sequent $\Rightarrow \exists x Ex$:

$$\frac{\Rightarrow \exists x Ex \qquad \cfrac{\cfrac{\cfrac{Ea, \Gamma \Rightarrow \Delta}{Ea, Ea, \Gamma \Rightarrow \Delta}\ (W \Rightarrow)}{\exists x Ex, \Gamma \Rightarrow \Delta}\ (\exists E \Rightarrow)}{}}{\Gamma \Rightarrow \Delta}\ (Cut)$$

To obtain GPFL$^+$ we add MQ, a mixed set of quantifier rules (definition 4) to G, moreover we must add (EE). The latter rule is stronger than (EE^r) which has the side effect that PFL$^+$ is not universally free. This feature is rather not problematic if we remember that logics of this kind are of interest mainly in mathematical applications where the existence of some objects of interest is tacitly assumed (see the remarks in Feferman [6]). (EE) is of course equivalent to the solution applied by Baaz and Iemhoff [1] where, in case of PFL$^+$, axiomatic sequents $\Gamma \Rightarrow Ea$ for arbitrary parameter a are added. The solution based on (EE) was applied by Maffezioli and Orlandelli [19] to obtain full cut elimination in their SC for the same logics, including PFL$^+$. Also in both formulations $(\Rightarrow \forall E)$ and $(\exists E \Rightarrow)$ are replaced with standard quantifier rules $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$. Note that such a simplification is not possible in general for FL since such a system would be incomplete; neither $\Rightarrow \forall x Ex$ nor $\Rightarrow \forall x(\varphi \rightarrow \psi) \rightarrow (\forall x \varphi \rightarrow \forall x \psi)$ is be provable. However, with (EE) at hand these sequents are provable by means of "more classical" rules. It is also easy to show that in the presence of (EE) the existence set is equivalent to this combined set with $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$ replacing $(\Rightarrow \forall E)$ and $(\exists E \Rightarrow)$. By Claim 1 we know that $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$ are derivable in all systems by means of $(W \Rightarrow)$. To show that $(\Rightarrow \forall E)$ and $(\exists E \Rightarrow)$ are derivable in GPFL$^+$ is straightforward. We demonstrate derivability of the former (the latter case is similar):

$$\dfrac{(EE)\ \dfrac{Ea, \Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \varphi[x/a]}}{\Gamma \Rightarrow \Delta, \forall x\varphi}\ (\Rightarrow \forall)$$

If we want to obtain systems for negative variants of these free logics we must add (NEE) and (NEE') to respective GFL described above. The proof of their equivalence to H-systems is immediate. It is sufficient to prove (DP) and (DP') by means of respective rules and demonstrate their derivability by cut with sequents corresponding to denotation principles.

Let us note that for GPFL, GPFL^n, GNFL and GNFL^n the following result holds:

LEMMA 1. *(Substitution) If* $\vdash_k \Gamma \Rightarrow \Delta$, *then* $\vdash_k \Gamma[a/t] \Rightarrow \Delta[a/t]$.

PROOF. By induction on the height of a proof (see e.g. Pavlović and Gratzl [23]). Note that (EE^r) may require similar relettering like $(\exists \Rightarrow)$ and $(\Rightarrow \forall)$. It is important to note that a proof provides the height-preserving admissibility. ■

The reason why this lemma does not hold for GPFL$^+$ and GNFL$^+$ is connected with (EE). Substitution of t for b in the case where the former is not a parameter is not correct. It has important consequences for these two systems, since the substitution lemma is a standard auxiliary result required in the proof of the cut elimination theorem.

Summing up we have the following systems:

DEFINITION 7. Sequent calculi for free logics without identity:

GPFL: G + EQ;

GNFL: GPFL + $\{(NEE), (NEE')\}$;

GPFL^n: GPFL + $\{(EE^r)\}$

GNFL^n: GPFL^n + $\{(NEE), (NEE')\}$;

GPFL$^+$: G + MQ + $\{(EE)\}$;

GNFL$^+$: GPFL$^+$ + (NEE) and (NEE').

3. Enter the Identity

It is well known that the existence predicate E is theoretically dispensable in FL= since it may be defined as $\exists x x = t$ or even as $t = t$ in NFL$_=$. However, for our purposes it is convenient to keep E as primitive and only characterize $=$ by suitable rules. Otherwise, rules for quantifiers involving active existence formulae are more complicated, especially in PFL. Moreover, we have uniform quantifier rules with existence formulae for PFL and NFL.

Principles characterising identity are often treated by addition of suitable axiomatic sequents (see e.g. Takeuti [26]) but such solution may result in restricted cut elimination. This is the case of SC for NFL provided by Gratzl [9] where cuts on identities are not eliminable. For our purposes it is better to use rules:

Definition 8. Rules for free logics with identity:

$$(=I)\ \frac{\varphi[x/t_2], \Gamma \Rightarrow \Delta}{t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta} \qquad (=E)\ \frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(=E^r)\ \frac{b = b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (NEI)\ \frac{t = t, \Gamma \Rightarrow \Delta}{Et, \Gamma \Rightarrow \Delta} \qquad (EI)\ \frac{a = t, \Gamma \Rightarrow \Delta}{Et, \Gamma \Rightarrow \Delta}$$

where φ is atomic in $(=I)$, a is not in Γ, Δ, t in (EI).

$(=I)$ is essentially the rule applied by Negri and von Plato [21] to formalize Leibniz principle. However in the setting of G3, where contraction is not primitive, we need the following contraction-absorbing variant of this rule (similarly as in the case of quantifier rules – see Section 2):

$$(=I')\ \frac{\varphi[x/t_2], t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta}{t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta} \text{ where } \varphi \text{ is atomic}$$

The restriction to atomic formulae is necessary to avoid troubles in the proof of cut elimination. Otherwise, there is a problem with compound cut formulae introduced by means of respective rule to the left premiss of cut, and by means of $(=I)$ to the right premiss of cut. There is no loss of generality however, since the following is provable for all systems considered in this paper.

Lemma 2. *(Leibniz Principle)* $\vdash t_1 = t_2, \varphi[x/t_1] \Rightarrow \varphi[x/t_2]$*, for any formula* φ*.*

Proof. by induction on the complexity of formula φ. The basis is derivable from axiom by $(=I)$. The induction step is proved as in Negri and von Plato [21]. ■

The rule $(= I)$ is universal for all systems considered in this section and the remaining ones. So in what follows let us use an abbreviation $G_=$ for G with added $(= I)$. $(= E), (= E^r)$ and (NEI) will vary for different systems and all of them refer to reflexivity of identity. (EI) is also universal, in the sense that it is correct for all considered free logics. However its remarkable role in uniform formalization of all systems will be considerd in the later sections. Intuitively it says that if some term denotes then there is a parameter with the same denotation. This, rather intuitively obvious, rule is of remarkable technical importance.

Now to obtain $\mathrm{GPFL}_=$, $\mathrm{GPFL}^n_=$ or $\mathrm{GPFL}^+_=$ it is enough to add $(= I)$ and $(= E)$ to respective systems from the preceding section (definition 7). Equivalence to H-systems for these logics follows from lemma 2.

In the case of $\mathrm{GNFL}_=$ we use (NEI) instead of $(= E)$. It must be added together with $(= I)$ to GNFL (as characterised in definition 7). We must also remember that a schematic predicate R^n in the schema of (NEE) can also represent identity. Interderivability with HNFL where (R') is used instead of (R) is obvious. Pavlović and Gratzl [23] use a slightly different but equivalent set of rules and provide a detailed proof of adequacy. The same solution works also for $\mathrm{NFL}^n_=$ and $\mathrm{NFL}^+_=$; it is sufficient to add the same two identity rules to G-systems from the previous section. However it is interesting to note that the last logic may be formalised in a different way and on a weaker basis. We can obtain a formalization of $\mathrm{NFL}^+_=$ from GNFL^+ by adding $(= I)$ and $(= E^r)$ instead of (NEI). Moreover, (EE) may be eliminated in such a system, since it is derivable. Therefore let us distinguish between the two formalizations of $\mathrm{NFL}^+_=$:

$$\mathrm{GNFL1}^+_= \text{ is } G_= + \mathrm{MQ} + \{(NEE), (NEE'), (NEI), (EE)\};$$
$$\mathrm{GNFL2}^+_= \text{ is } G_= + \mathrm{MQ} + \{(NEE), (NEE'), (= E^r)\}.$$

LEMMA 3. *$\mathrm{GNFL1}^+_= \vdash \Gamma \Rightarrow \Delta$ iff $\mathrm{GNFL2}^+_= \vdash \Gamma \Rightarrow \Delta$*

PROOF. by induction on the height of the proof and interderivability of respective rules. In one direction, every application of (EE) in $\mathrm{GNFL1}^+_=$ is replaced with:

$$\dfrac{\dfrac{Eb, \Gamma \Rightarrow \Delta}{b = b, \Gamma \Rightarrow \Delta}\,(NEE)}{\Gamma \Rightarrow \Delta}\,(= E^r)$$

Every application of (NEI) is simulated with:

$$\dfrac{\dfrac{\dfrac{a = a \Rightarrow a = a}{\Rightarrow a = a}\,(= E^r)}{\Rightarrow \forall x x = x}\,(\Rightarrow \forall) \qquad \dfrac{t = t, \Gamma \Rightarrow \Delta}{\forall x x = x, Et, \Gamma \Rightarrow \Delta}\,(\forall E \Rightarrow)}{Et, \Gamma \Rightarrow \Delta}\,(Cut)$$

The converse requires only showing that $(=E^r)$ is derivable in GNFL1$^{\pm}_{=}$:

$$\dfrac{\dfrac{b=b,\Gamma\Rightarrow\Delta}{Eb,\Gamma\Rightarrow\Delta}\,(NEI)}{\Gamma\Rightarrow\Delta}\,(EE)$$

■

It is quite an important observation that we can dispense with (EE) (essential when identity is missing). It shows that no special axioms of the form $\Rightarrow Ea$, as in Baaz and Iemhoff [1], are required if identity is present. The substitution lemma (lemma 1) holds for the same systems with identity as in the preceding section. Again, because of the presence of (EE) it does not hold for GPFL$^{\pm}_{=}$ and GNFL1$^{\pm}_{=}$. It fails also for GNFL2$^{\pm}_{=}$, since $(=E^r)$ generates the same problem with substitution of b by nonparametric t.

Finally let us comment on the alternative characterization of identity used sometimes in negative logics. In Section 2 we mentioned that in NFL Leibniz Principle (LP) for identity is expressed sometimes by means of a slightly more complicated principle (LP') which in G may be formulated as an axiomatic sequent:

$$Et_1\vee Et_2\rightarrow t_1=t_2,\varphi[x/t_1]\Rightarrow\varphi[x/t_2]$$

However, if all predicates are strict it is equivalent to ordinary (LP). This may be shown easily in the framework of our systems:

Lemma 4. (LP) *is interderivable with* (LP') *in GNFL*$_{=}$ *(and its extensions)*

Proof. Provability of (LP') is harder and goes by induction on the complexity of $\varphi[x/t]$. In the basis we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Et_1\Rightarrow Et_1}{Et_1\Rightarrow Et_1,Et_2}\,(\Rightarrow W)}{Et_1\Rightarrow Et_1\vee Et_2}\,(\Rightarrow\vee)\qquad \dfrac{\varphi[x/t_2]\Rightarrow\varphi[x/t_2]}{t_1=t_2,\varphi[x/t_1]\Rightarrow\varphi[x/t_2]}\,(=I)}{Et_1,Et_1\vee Et_2\rightarrow t_1=t_2,\varphi[x/t_1]\Rightarrow\varphi[x/t_2]}\,(\rightarrow\Rightarrow)}{\varphi[x/t_1],Et_1\vee Et_2\rightarrow t_1=t_2,\varphi[x/t_1]\Rightarrow\varphi[x/t_2]}\,(NEE)}{Et_1\vee Et_2\rightarrow t_1=t_2,\varphi[x/t_1]\Rightarrow\varphi[x/t_2]}\,(C\Rightarrow)$$

Inductive step is proved like lemma 2. Provability of (LP) from (LP') does not even require (NEE):

$$\dfrac{\dfrac{\dfrac{t_1=t_2\Rightarrow t_1=t_2}{Et_1\vee Et_2,t_1=t_2\Rightarrow t_1=t_2}\,(W\Rightarrow)}{t_1=t_2\Rightarrow Et_1\vee Et_2\rightarrow t_1=t_2}\,(\Rightarrow\rightarrow)\qquad Et_1\vee Et_2\rightarrow t_1=t_2,\varphi[x/t_1]\Rightarrow\varphi[x/t_2]}{t_1=t_2,\varphi[x/t_1]\Rightarrow\varphi[x/t_2]}\,(Cut)$$

■

Thus we are entitled to use the same rule $(= I)$ for identity in all cases.

Summing up we have the following systems:

Definition 9. Sequent calculi for free logics with identity:

GPFL$_=$: G$_=$ + EQ + $\{(= E)\}$;

GPFL$^n_=$: GPFL$_=$ + $\{(EE^r)\}$;

GPFL$^+_=$: G$_=$ + MQ + $\{(= E), (EE)\}$;

GNFL$_=$: G$_=$ + EQ + $\{(NEE), (NEE'), (NEI)\}$;

GNFL$^n_=$: GNFL$_=$ + $\{(EE^r)\}$;

GNFL1$^\pm$: G$_=$ + MQ + $\{(NEE), (NEE'), (NEI), (EE)\}$;

GNFL2$^\pm$: G$_=$ + MQ + $\{(NEE), (NEE'), (= E^r)\}$;

where G$_=$ = G + $\{(= I)\}$.

We could also schematize our systems in a simpler way by emphasizing the fact that they are just the same systems which were considered in the previous section (see definition 7) but with different sets of identity rules added. Let SI = $\{(= I), (= E)\}$, NI = $\{(= I), (NEI)\}$ and RI = $\{(= I), (= E^r)\}$, then:

GPFL$_=$ = GPFL + SI;

GPFL$^n_=$ = GPFL^n + SI;

GPFL$^+_=$ = GPFL$^+$ + SI;

GNFL$_=$ = GNFL + NI;

GNFL$^n_=$ = GNFL^n + NI;

GNFL1$^\pm$ = GNFL$^+$ + NI;

GNFL2$^\pm$ = GNFL$^+$ + RI (but without (EE)).

Note that we cannot obtain something like GPFL2$^\pm$ where (EE) is dispensable, since (NEE) is required to prove its derivability.

4. Enter (EI)

We did not consider the role of (EI) so far. This rule is derivable in all systems we have introduced in the previous section; we state this as:

Lemma 5. *(EI) is derivable in GPFL$_=$, GPFL$^n_=$, GPFL$^+_=$, GNFL$_=$, GNFL$^n_=$, GNFL1$^\pm$ and GNFL2$^\pm$.*

PROOF. The following derivation justifies its derivability in $\mathrm{GPFL}_=$ and $\mathrm{GPFL}^n_=$:

$$
(Cut)\,\dfrac{(\Rightarrow \exists E)\,\dfrac{(=E)\,\dfrac{t=t\Rightarrow t=t}{\Rightarrow t=t}}{Et\Rightarrow \exists x x=t}\qquad (\exists E\Rightarrow)\,\dfrac{(W\Rightarrow)\,\dfrac{a=t,\Gamma\Rightarrow\Delta}{Ea,a=t,\Gamma\Rightarrow\Delta}}{\exists x x=t,\Gamma\Rightarrow\Delta}}{Et,\Gamma\Rightarrow\Delta}
$$

This proof works also for $\mathrm{GPFL}^{\pm}_=$ (in this case $(W\Rightarrow)$ is not required in the right branch since $(\exists\Rightarrow)$ is the right rule). It must be only slightly modified to obtain a derivability of (EI) in $\mathrm{GNFL}_=$, $\mathrm{GNFL}^n_=$ and $\mathrm{GNFL1}^{\pm}_=$; in the left branch we must use (NEI) instead of $(=E)$ and then we need also to use contraction to eliminate one occurrence of Et. In the case of $\mathrm{GNFL2}^{\pm}_=$ a derivation is considerably more complicated:

$$
(Cut)\,\dfrac{(C\Rightarrow)\,\dfrac{(\Rightarrow\exists)\,\dfrac{(Cut)\,\dfrac{(\Rightarrow\forall)\,\dfrac{(=E^r)\,\dfrac{a=a\Rightarrow a=a}{\Rightarrow a=a}}{\Rightarrow\forall x x=x}\qquad (\forall\Rightarrow)\,\dfrac{t=t\Rightarrow t=t}{\forall x x=x, Et\Rightarrow t=t}}{Et\Rightarrow t=t}}{Et,Et\Rightarrow\exists x x=t}}{Et\Rightarrow\exists x x=t}\qquad (\exists\Rightarrow)\,\dfrac{a=t,\Gamma\Rightarrow\Delta}{\exists x x=t,\Gamma\Rightarrow\Delta}}{Et,\Gamma\Rightarrow\Delta}
$$

■

Incidentally note that the left subtree shows how $Et\Rightarrow t=t$ is provable in $\mathrm{GNFL2}^{\pm}_=$ (in $\mathrm{GNFL1}^{\pm}_=$ the proof is trivial by (NEI)).

So far we have noticed that substitution lemma fails for our formalizations of FL^+ and blamed rules $(EE),(=E^r)$ for that. However, as we shall see, using (EI) can help. Consider versions of $(\forall E\Rightarrow)$ and $(\Rightarrow\exists E)$ with instantiation terms t restricted to parameters only (see definition 5). It appears that in the presence of (EI) these rules are sufficiently strong to derive their unrestricted versions. Let us call such restricted versions $\mathrm{GrPFL}_=$, $\mathrm{GrPFL}^n_=$, $\mathrm{GrPFL}^{\pm}_=$, $\mathrm{GrNFL}_=$, $\mathrm{GrNFL}^n_=$, $\mathrm{GrNFL1}^{\pm}_=$ and $\mathrm{GrNFL2}^{\pm}_=$ respectively. The following holds:

LEMMA 6. *In all restricted systems (i.e. with (EI) and restricted quantifier rules $(\forall E\Rightarrow')$ and $(\Rightarrow\exists E')$) their nonrestricted versions are derivable.*

PROOF. We demonstrate the case of $(\forall E\Rightarrow)$:

$$
(C\Rightarrow)\,\dfrac{(EI)\,\dfrac{(C\Rightarrow)\,\dfrac{(=I)\,\dfrac{(\forall E\Rightarrow')\,\dfrac{(Cut)\,\dfrac{\varphi[x/a],a=t\Rightarrow\varphi[x/t]\qquad \varphi[x/t],\Pi\Rightarrow\Sigma}{\varphi[x/a],a=t,\Pi\Rightarrow\Sigma}}{\forall x\varphi, Ea, a=t,\Pi\Rightarrow\Sigma}}{\forall x\varphi, Et, a=t, a=t,\Pi\Rightarrow\Sigma}}{\forall x\varphi, Et, a=t,\Pi\Rightarrow\Sigma}}{\forall x\varphi, Et, Et,\Pi\Rightarrow\Sigma}}{\forall x\varphi, Et,\Pi\Rightarrow\Sigma}
$$

where the leftmost leaf is provable by lemma 2. The case of $(\Rightarrow \exists E)$ is similar. By inspection of applied rules it is obvious that this proof holds in restricted systems for all logics under consideration. ■

THEOREM 1. *Restricted systems are equivalent to their unrestricted counterparts from Section 3.*

PROOF. It follows from Lemmas 5 and 6. ■

We also obtain for all restricted systems:

LEMMA 7. *(Restricted Substitution) If* $\vdash_k \Gamma \Rightarrow \Delta$, *then* $\vdash_k \Gamma[a/b] \Rightarrow \Delta[a/b]$.

PROOF. The same as for lemma 1. (EI) may require relettering to avoid conflict of the fresh parameter with substituted parameter. Since we admit only substitution of parameters for parameters, (EE) and $(= E^r)$ are unproblematic and the lemma holds for restricted systems for FL^+. ■

Although this result is too weak for systems examined in Sections 2 and 3 it is sufficient for restricted systems of this section, since only parameters are instantiated when quantifier rules are applied. It makes possible a proof of cut elimination in Section 6 for all logics. Moreover, this is also important for their possible applications to logics in the languages extended with operators (description-, abstraction-operator). In such logics terms generated by the application of unrestricted quantifier rules may be very complex, thus breaking the subformula property and destroying the possibility of a constructive cut elimination proof. In restricted systems this problem is solved which opens the possibility for constructing well-behaved rules for operators.

Let us make a summary of restricted systems:

DEFINITION 10. Restricted sequent calculi for free logics with identity:

$\mathrm{GrPFL}_=$: $\mathrm{Gr}_=$ + EQR + $\{(= E)\}$;

$\mathrm{GrPFL}^n_=$: $\mathrm{GrPFL}_=$ + $\{(EE^r)\}$;

$\mathrm{GrPFL}^+_=$: $\mathrm{Gr}_=$ + MQR + $\{(= E), (EE)\}$;

$\mathrm{GrNFL}_=$: $\mathrm{Gr}_=$ + EQR + $\{(NEE), (NEE'), (NEI)\}$;

$\mathrm{GrNFL}^n_=$: $\mathrm{GrNFL}_=$ + $\{(EE^r)\}$;

$\mathrm{GrNFL1}^+_=$: $\mathrm{Gr}_=$ + MQR + $\{(NEE), (NEE'), (NEI), (EE)\}$;

$\mathrm{GrNFL2}^+_=$: $\mathrm{Gr}_=$ + MQR + $\{(NEE), (NEE'), (= E^r)\}$;

where $\mathrm{Gr}_=$: G + $\{(= I), (EI)\}$ and EQR, MQR are restricted sets of quantifier rules (see definition 5).

5. Simplified Systems for Quasi-Free Logics

The application of (EI) permits restricted quantifier rules in all systems but this is not the end. What is even more surprising with (EI), is that for $\text{PFL}^{\pm}_{=}$ and $\text{NFL}^{\pm}_{=}$ we can simplify quantifier rules even further. Instead of using MQR we can take SQR with $(\forall \Rightarrow'), (\Rightarrow \exists')$ which are standard (for classical and intuitionistic logic) quantifier rules with instantiation terms restricted to parameters (see definition 2 and 5). Note first that:

Lemma 8. $(\forall \Rightarrow'), (\Rightarrow \exists')$ *are derivable in all restricted systems for* $PFL^{\pm}_{=}$ *and* $NFL^{\pm}_{=}$.

Proof. We show it for $(\forall \Rightarrow')$. In $\text{GrNFL2}^{\pm}_{=}$:

$$\dfrac{(Cut)\ \dfrac{(= E^r)\ \dfrac{(NEE)\ \dfrac{Ea \Rightarrow Ea}{a = a \Rightarrow Ea}}{\Rightarrow Ea} \qquad \dfrac{\varphi[x/a], \Gamma \Rightarrow \Delta}{Ea, \forall x\varphi, \Gamma \Rightarrow \Delta}\ (\forall E \Rightarrow')}{\forall x\varphi, \Gamma \Rightarrow \Delta}}{}$$

in $\text{GrNFL1}^{\pm}_{=}$ or $\text{GrPFL}^{\pm}_{=}$, the proof is even simpler, since in the leftmost branch we need only one application of (EE). ∎

Note that this result holds even for the systems for FL^{+} without identity and (EI). Moreover, if we use these rules in $\text{FL}^{\pm}_{=}$ instead of $(\forall E \Rightarrow')$ and $(\Rightarrow \exists E')$ the resulting two formalizations are equivalent. This follows from the more general result:

Lemma 9. *In the presence of* (EI), $(\forall E \Rightarrow)$ *is derivable by* $(\forall \Rightarrow')$ *and* $(\Rightarrow \exists E)$ *by* $(\Rightarrow \exists')$.

Proof. We demonstrate this for $(\forall E \Rightarrow)$; the proof for $(\Rightarrow \exists E)$ is similar. Let us consider the case with t which is not a parameter:

$$(Cut)\ \dfrac{\varphi[x/a], a = t \Rightarrow \varphi[x/t] \qquad \varphi[x/t], \Pi \Rightarrow \Sigma}{(EI)\ \dfrac{(\forall \Rightarrow')\ \dfrac{\varphi[x/a], a = t, \Pi \Rightarrow \Sigma}{\forall x\varphi, a = t, \Pi \Rightarrow \Sigma}}{\forall x\varphi, Et, \Pi \Rightarrow \Sigma}}$$

where the leftmost leaf is derivable by lemma 2.

Of course in case t is a parameter the proof is even simpler, since the application of (EI) and lemma 2 is dispensable. ∎

This result holds in general for all considered systems (since (EI) is derivable in all systems). Moreover, in case of $\text{FL}^{\pm}_{=}$ (in all variants and formulations) it yields that $(\forall \Rightarrow')$ and $(\Rightarrow \exists')$ are interderivable with $(\forall E \Rightarrow')$ and

$(\Rightarrow \exists E')$. Let us call such restricted systems for $\mathrm{FL}^{\pm}_{=}$ (with SQR replacing MQR) simplified systems and denote as GsL for a logic L (e.g. $\mathrm{GsPFL}^{\pm}_{=}$). From lemma 8 and 9 follows:

THEOREM 2. *Simplified systems* $GsPFL^{\pm}_{=}$*,* $GsNFL1^{\pm}_{=}$ *and* $GsNFL2^{\pm}_{=}$ *are equivalent to their restricted counterparts* $GrPFL^{\pm}_{=}$*,* $GrNFL1^{\pm}_{=}$ *and* $GrNFL2^{\pm}_{=}$*.*

It is an interesting feature of quasi-free logics that they may be formalised by means of quantifier rules totally freed from existence assumptions. Formulae of this kind are necessary only for (EI) (and $(NEE), (NEE')$ in $\mathrm{GsNFL2}^{\pm}_{=}$; for $\mathrm{GsNFL1}^{\pm}_{=}$ additionally in (NEI)). It shows also that definedness logic although treated as a kind of free logic is in some sense very different. More general quantifier rules are not necessary and even the definedness predicate is dispensable, since it may be defined. Let us examine this last possiblity for $\mathrm{GsNFL2}^{\pm}_{=}$. In general, the fact that E is definable does not lead to satisfying results if we want to obtain cut-free SC. However, in the case of negative quasi-free logics the situation is different. Recall that Et may be defined as $t = t$ in $\mathrm{NFL}_{=}$. One part is in fact present as (NEI) in $\mathrm{GsNFL1}^{\pm}_{=}$ and the corresponding sequent was proved incidentally in $\mathrm{GNFL2}^{\pm}_{=}$ in Section 4; the other one is immediate by (NEE). Let us consider now the E-free language and the following rules:

DEFINITION 11. Existence-free variants:

$$(Str)\ \frac{a = t_i, \Gamma \Rightarrow \Delta}{R^n t_1 \dots t_n \Gamma \Rightarrow \Delta} \quad (NEE_{=})\ \frac{t_i = t_i, \Gamma \Rightarrow \Delta}{R^n t_1 \dots t_n \Gamma \Rightarrow \Delta} \quad (EI')\ \frac{a = t, \Gamma \Rightarrow \Delta}{t = t, \Gamma \Rightarrow \Delta}$$

where $t, t_i, i \leq n$ is any term, a is not in Γ, Δ, t, t_i.

$(NEE_{=})$ and (EI') are just E-free versions of (NEE) and (EI) whereas (Str) (for strict) is a new and more general rule, as we will show. One may easily see that (Str) is derivable by these two rules. On the other hand, (EI') is a special case of (Str), whereas $(NEE_{=})$ is derivable:

$$\dfrac{(Str)\ \dfrac{(\Rightarrow \exists)\ \dfrac{a = t_i \Rightarrow a = t_i}{a = t_i \Rightarrow \exists x x = t_i}}{R^n t_1 \dots t_n \Rightarrow \exists x x = t_i} \qquad \dfrac{\dfrac{\dfrac{t_i = t_i, \Gamma \Rightarrow \Delta}{a = t_i, a = a, \Gamma \Rightarrow \Delta}\ (= I)}{a = t_i, \Gamma \Rightarrow \Delta}\ (= E^r)}{\exists x x = t_i, \Gamma \Rightarrow \Delta}\ (\exists \Rightarrow)}{R^n t_1 \dots t_n, \Gamma \Rightarrow \Delta}\ (Cut)$$

Thus we can obtain a formalization of $\mathrm{NFL}^{\pm}_{=}$ in the E-free language by means of (Str) replacing both $(NEE_{=})$ and (EI'). This does not work however if we have strict functions, since E is indispensable in (NEE'). To overcome the problem we must introduce generalised versions of (Str)

and $(NEE_=)$. We will define inductively the auxiliary notion of k-depth occurrence of t in atomic φ which will be symbolised $\varphi[t]^k$.

Let $\varphi := R^n t_1 \dots t_n$, we say that:

- t has 0-depth occurrence in φ, if $t := t_i, i \leq n$;
- t has $k+1$-depth occurrence in φ, if there is some $f^n t_1 \dots t_n$ which has k-depth occurrence in φ and $t := t_i, i \leq n$.

Now we can redefine (Str) and $(NEE_=)$:

Definition 12. Generalised existence-free variants:

$$(Str')\ \frac{a = t, \Gamma \Rightarrow \Delta}{\varphi[t]^k \Gamma \Rightarrow \Delta} \quad (NEE'_=)\ \frac{t = t, \Gamma \Rightarrow \Delta}{\varphi[t]^k \Gamma \Rightarrow \Delta}$$

where t is any term having k-depth occurrence ($k \geq 0$) in (atomic) φ, a is not in Γ, Δ, φ.

The proof of interderivability of (Str') with (EI') and $(NEE'_=)$ is the same as for the preceding set (which is sufficient for the case of languages without functional terms). In this way we obtain also derivability of (NEE') and two simple SC characterizations of $\mathrm{NFL}^{\pm}_{=}$ in existence-free language. We call them $\mathrm{GsNFL3}^{\pm}_{=}$ and $\mathrm{GsNFL4}^{\pm}_{=}$. In fact, we could use also these results to obtain a simplified axiomatization of LPT with no use of the definedness predicate. It is enough to take an axiomatization of the standard classical or intutionistic pure (i.e. with variables only) first-order logic, for identity add (LP) (admitting all kind of terms) and $\forall x x = x$, and finally add a rule:

(STR): if $\vdash a = t \rightarrow \psi$, then $\vdash \varphi[t]^k \rightarrow \psi$,

where a is not in φ, ψ, t is any term, and φ is atomic.

No special axioms of the form Ea or denotation principles for strict predicates and functions are required. Indeed this is very simple and economical axiomatization of Feferman's definedness logic. One may observe that this result provides also another justification for Beeson's remarks concerning differences between definedness and existence predicate, mentioned in Section 1.

Let us summarize the systems of this section:

Definition 13. Simplified sequent calculi for quasi-free logics with identity:

$\mathrm{GsPFL}^{\pm}_{=}$: $\mathrm{Gr}_= + \mathrm{SQR} + \{(=E), (EE)\}$;

$\mathrm{GsNFL1}^{\pm}_{=}$: $\mathrm{Gr}_= + \mathrm{SQR} + \{(NEE), (NEE'), (NEI), (EE)\}$;

$\mathrm{GsNFL2}^{\pm}_{=}$: $\mathrm{Gr}_= + \mathrm{SQR} + \{(NEE), (NEE'), (=E^r)\}$;

$\mathrm{GsNFL3}^{\pm}_{=}$: $\mathrm{Gs}_= + \mathrm{SQR} + \{(NEE'_=), (=E^r)\}$;

GsNFL4$^{\pm}_{=}$: $G_{=}$ + SQR + $\{(Str'), (= E^r)\}$;

where $Gr_{=}$: G + $\{(= I), (EI)\}$, $Gs_{=}$: G + $\{(= I), (EI')\}$ and $G_{=}$: G + $\{(= I)\}$

6. Cut Elimination

We provide a general schema of proof of cut elimination for almost all systems described in Sections 2–5. More specifically, it covers all restricted and simplified systems introduced in Sections 4–5, and all systems for absolutely free logics presented in Sections 2–3. It does not apply to systems for quasi-free logics from Sections 2–3, for which the substitution lemma does not hold.

The general strategy of the proof was originally applied for hypersequent calculi by Metcalfe et al. [20] and later extensively used in this framework (see e.g. Ciabattoni et al. [5], Indrzejczak [11], [14], Kurokawa [17]). However it is also applicable to standard sequent calculi (see Indrzejczak [12], [13]) and allows for an elegant proof which helps to avoid many complexities inherent in other methods of proving cut elimination. In particular, we avoid well known problems with contraction, since two auxiliary lemma deal with this problem in advance.

We assume that all proofs are regular in the sense that every parameter a which is fresh by side condition on the respective rule must be fresh in the entire proof, not only on the branch where the application of this rule takes place. There is no loss of generality since every proof may be systematically transformed into regular proof by the restricted substitution lemma (lemma 7 in Section 4).

Let us define the notions of cut-degree and proof-degree:

1. Cut-degree is the complexity of cut-formula φ, i.e. the number of connectives and operators occurring in φ and is denoted as $d\varphi$;
2. Proof-degree ($d\mathcal{D}$) is the maximal cut-degree in $\mathcal{D}$.

The proof of cut elimination theorem is based on two lemmata which successively make a reduction: first on the height of the right, and then on the height of the left premiss of cut. φ^k, Γ^k denote $k > 0$ occurrences of φ, Γ, respectively.

Lemma 10. *(Right reduction) Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathcal{D}_2 \vdash \varphi^k, \Pi \Rightarrow \Sigma$ *with* $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$*, and* φ *principal in* $\Gamma \Rightarrow \Delta, \varphi$*, then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$ *and* $d\mathcal{D} < d\varphi$*.*

Proof. It goes by induction on the height of $\mathcal{D}_2$. The basis is trivial, since $\Gamma \Rightarrow \Delta, \varphi$ is identical with $\Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$. The induction step requires examination of all cases of possible derivations of $\varphi^k, \Pi \Rightarrow \Sigma$, and the role of the cut-formula in the transition. In cases where all occurrences of φ are parametric we simply apply the induction hypotheses to the premisses of $\varphi^k, \Pi \Rightarrow \Sigma$ and then apply the respective rule – it is essentially due to the context independence of almost all rules and regularity of proofs which prevents violation of side conditions on eigenvariables. If one of the occurrences of φ in the premiss(es) is a side formula of the last rule we must additionally apply weakening to restore the missing formula before the application of the relevant rule.

In cases where one occurrence of φ in $\varphi^k, \Pi \Rightarrow \Sigma$ is principal we make use of the fact that φ in the left premiss is also principal (note that for the cases of contraction and weakening it is trivial). Note that due to condition that φ is principal in the left premiss it must be compound, since all rules introducing atomic formulae as principal are working only in the antecedents. Hence all cases where one occurrence of atomic φ in the right premiss would be introduced by means of such rules are not considered in the proof of this lemma. The only exception is with $\Gamma \Rightarrow \Delta, \varphi$ being an axiom with principal atomic φ, but it does not make any harm.

We analyse the case of $\forall x\varphi$ with rules taken from the existence set. Hence $\mathcal{D}_1$ finishes with:

$$(\Rightarrow \forall E)\ \frac{Ea, \Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

where a is fresh; and $\mathcal{D}_2$ ends with:

$$(\forall E \Rightarrow)\ \frac{\varphi[x/t], \forall x\varphi^{k-1}, \Pi \Rightarrow \Sigma}{\forall x\varphi^k, Et, \Pi \Rightarrow \Sigma}$$

where $k > 0$.

The expected result is $Et, \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$. We construct a new proof:

$$\frac{Et, \Gamma \Rightarrow \Delta, \varphi[x/t] \qquad \varphi[x/t], \Gamma^{k-1}, \Pi \Rightarrow \Delta^{k-1}, \Sigma}{Et, \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma}\ (Cut)$$

where the left leaf is obtained by substitution lemma (lemma 1) on the premiss of $(\Rightarrow \forall E)$ in our original proof, and the right one by the induction hypothesis applied to the premiss of $(\forall E \Rightarrow)$ application in the original proof. All cuts are of lower degree hence the new proof satisfies the condition of the lemma. Proofs of other cases are similar. ■

Note that this case (and the dual for $\exists x\varphi$) shows why the proof fails for G-systems with unrestricted quantifier rules for quasi-free logics; lemma 1 cannot be applied here. On the other hand, restricted and simplified G-systems for these logics have no problem here, since t must be a parameter and the restricted substitution lemma (lemma 7) justifies the replacement.

Lemma 11. *(Left reduction) Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi^k$ *and* $\mathcal{D}_2 \vdash \varphi, \Pi \Rightarrow \Sigma$ *with* $d\mathcal{D}_1, d\mathcal{D}_2 < d\varphi$*, then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma, \Pi^k \Rightarrow \Delta, \Sigma^k$ *and* $d\mathcal{D} < d\varphi$*.*

Proof. This is proved by induction on the height of $\mathcal{D}_1$ but with some important differences. First note that we do not require φ to be principal in $\varphi, \Pi \Rightarrow \Sigma$ so it includes the case with φ atomic. In all these cases we just apply the induction hypothesis. This guarantees that even if an atomic cut formula was introduced in the right premiss by one of the rules $(NEE), (EI), (NEI), (=I), (Str)$, and the like, the reduction of the height is done only on the left premiss, and we always obtain the expected result. Now, in cases where one occurrence of φ in $\Gamma \Rightarrow \Delta, \varphi^k$ is principal we first apply the induction hypothesis to eliminate all other $k-1$ occurrences of φ in premisses and then we apply the respective rule. Since the only new occurrence of φ is principal we can make use of the right reduction lemma again and obtain the result, possibly after some applications of structural rules. ■

Now we are ready to prove the cut elimination theorem:

Theorem 3. *Every proof in all G-systems specified above can be transformed into cut-free proof.*

Proof. by double induction: primary on $d\mathcal{D}$ and subsidiary on the number of maximal cuts (in the basis and in the inductive step of the primary induction). We always take the topmost maximal cut and apply lemma 9 to it. By successive repetition of this procedure we diminish either the degree of a proof or the number of maximal cuts in it until we obtain a proof with $d = 0$. ■

7. Concluding Remarks

We have presented several sequent calculi for six free logics, both in the classical and the intuitionistic version. In particular, we introduced a variety of systems for quasi-free logics, which were not dealt with in this framework so far, and showed in this case how to overcome the problems with cut elimination. All systems discussed for the six free logics considered in the last section are not only cut-free but satisfy also a kind of the subformula property to the effect that for every provable sequent, its proof may be built from subformulae of the root, identities and existence statements (in case of GsNFL3$^{\pm}_{=}$ and GsNFL4$^{\pm}_{=}$ only identities). The former property permits for further standard applications like e.g. proofs of interpolation theorems which was in fact performed for PFL^{+} (Baaz and Iemhoff [1], Maffezioli and Orlandelli [19]). The latter property may be helpful in providing some practical proof search procedures. In fact all these systems may be easily transformed into G3-style calculi and then into tableau systems (see e.g. Priest [24]), in case of quasi-free logics on the classical basis almost identical to systems for classical logic. In this paper we focused however on the theoretical basis; possible refinements and applications to automated deduction require further study.

In fact, independently of the research provided in this paper, Pavlović and Gratzl [23] presented recently a study of SC for free logics. Their systems are based on G3 but equivalent to ours presented in Section 3. For those systems they proved the cut admissibility theorem in Dragalin-style (see Negri and von Plato [21] for details). However, they do not provide SC for quasi-free logics and restrict their considerations to languages with variables as the only terms. Thus the problem of finding a solution for arbitrary terms is not dealt with in that paper.

The most significant result of this paper is the construction of systems which use restricted rules for quantifiers. This is important not only for quasi-free logics and construction of cut-free systems for them but, as we mentioned in Section 4, opens the room for decent, cut-free systems with rules for operators making complex terms in all kinds of free logics. Indrzejczak [12] provided such a result for sequent calculus for first-order modal logics based on PFL with definite descriptions. The results presented in this paper open the prospects for extending this type of analysis to other theories of definite descriptions (see e.g. Bencivenga [3], Fefermann [6], Scott [25]), as well as to other kinds of complex terms, including indefinite descriptions, lambda terms (Feferman [6]) or class-forming abstraction operators (Tennant [27]). The application of quasi-free logics as the basis seems to be particularly promising in the modal contexts using richer languages. One may find an approach of this kind in Fitting and Mendelsohn [7] where negative quasi-free logic is the basis for modal extensions dealing with denoting/nondenoting and rigid/nonrigid terms, including definite descriptions. Recently Indrzejczak [15] extended such an approach to hybrid modal logics providing cut-free SC for them. The results presented in this paper provide a firm ground for further extensions of this kind.

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References

[1] Baaz, M., and R. Iemhoff, Gentzen Calculi for the existence predicate *Studia Logica* 82(1):7–23, 2006.

[2] Beeson, M., *Foundations of Constructive Mathematics*, Springer, Berlin, 1985

[3] Bencivenga, E., Free Logics, in D. Gabbay, and F. Guenthner, (eds.), *Handbook of Philosophical Logic, Vol. III*, Reidel Publishing Company, Dordrecht, 1986, pp. 373–426.

[4] Bencivenga, E., K. Lambert, and B. C. van Fraassen, *Logic, Bivalence and Denotation*, Ridgeview, Atascadero, 1991.

[5] Ciabattoni, A., G. Metcalfe, and F. Montagna, Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions, *Fuzzy Sets and Systems* 161(3):369–389, 2010.

[6] Feferman, S., Definedness, *Erkenntnis* 43:295–320, 1995.

[7] Fitting, M., and R. L. Mendelsohn, *First-Order Modal Logic*, Kluwer, Dordrecht 1998.

[8] Garson, J.W., *Modal Logic for Philosophers*, Cambridge University Press, Cambridge 2006.

[9] Gratzl, N., A sequent calculus for a negative free logic, *Studia Logica* 96:331–348, 2010.

[10] Gumb, R., An extended joint consistency theorem for a nonconstructive logic of partial terms with definite descriptions, *Studia Logica* 69(2):279–292, 2001.

[11] Indrzejczak, A., Eliminability of cut in hypersequent calculi for some modal logics of linear frames, *Information Processing Letters* 115/2:75–81, 2015.

[12] Indrzejczak, A., Cut-Free Modal Theory of Definite Descriptions, in G. Bezhanishvili, G. D'Agostino, G. Metcalfe, and T. Studer, (eds.), *Advances in Modal Logic 12*, College Publications, 2018, pp. 387–406.

[13] Indrzejczak, A., Rule-generation theorem and its applications, *Bulletin of the Section of Logic* 47(4):265–281, 2018.

[14] Indrzejczak, A., Cut elimination in hypersequent calculus for some logics of linear time, *Review of Symbolic Logic* 12(4):806–822, 2019.

[15] Indrzejczak, A., Existence, Definedness and Definite Descriptions in Hybrid Modal Logic, in N. Olivetti, R. Verbrugge, and S. Negri, (eds.), *Advances in Modal Logic 13*, College Publications, 2020, pp. 349–368.

[16] Jaśkowski, S., On the rules of suppositions in formal logic, *Studia Logica* 1:5–32, 1934.

[17] Kurokawa, H., Hypersequent calculi for modal logics extending S4, in Y. Nakano, K. Satoh, and D. Bekki, (eds.), *New Frontiers in Artificial Intelligence*, Springer, 2014, pp. 51–68.

[18] Lehmann, S., More Free Logic, in D. Gabbay, and F. Guenthner, (eds.), *Handbook of Philosophical Logic, Vol V*, 2nd edition, Springer, 2002, pp. 197–259.

[19] Maffezioli, P., and E. Orlandelli, Full cut elimination and interpolation for intuitionistic logic with existence predicate, *Bulletin of the Section of Logic* 48(2):137–158, 2019.

[20] Metcalfe, G., N. Olivetti, and D. Gabbay, *Proof Theory for Fuzzy Logics*, Springer, Berlin, 2008.

[21] Negri, S., and J. von Plato, *Structural Proof Theory*, Cambridge University Press, Cambridge, 2001.

[22] Negri, S., and J. von Plato, *Proof Analysis*, Cambridge University Press, Cambridge, 2011.

[23] Pavlović, E., and N. Gratzl, A more unified approach to free logics, to appear in *Journal of Philosophical Logic*. https://doi.org/10.1007/s10992-020-09564-7.

[24] Priest, G., *An Introduction to Non-classical Logic*, Cambridge University Press, Cambridge, 2001.

[25] Scott, D., Identity and Existence in Intuitionistic Logic, in M. Fourman, C. Mulvey, and D. Scott, (eds.), *Applications of Sheaves*, Springer, 1979, pp. 660–696.

[26] Takeuti, G., *Proof Theory*, North-Holland, Amsterdam, 1987.

[27] Tennant, N., *Natural Logic*, Edinburgh University Press, 1978.

[28] Trew, A., Nonstandard theories of quantification and identity, *Journal of Symbolic Logic* 35:267–294, 1970.

[29] Troelstra, A. S., and D. van Dalen, *Constructivism in Mathematics*, vol. I, North-Holland, 1988.

[30] Troelstra, A. S., and H. Schwichtenberg *Basic Proof Theory*, Oxford University Press, Oxford, 1996.

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