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The Hahn Embedding Theorem for a Class of Residuated Semigroups

Abstract. Hahn's embedding theorem asserts that linearly ordered abelian groups embed in some lexicographic product of real groups. Hahn's theorem is generalized to a class of residuated semigroups in this paper, namely, to odd involutive commutative residuated chains which possess only finitely many idempotent elements. To this end, the partial lexicographic product construction is introduced to construct new odd involutive commutative residuated lattices from a pair of odd involutive commutative residuated lattices, and a representation theorem for odd involutive commutative residuated chains which possess only finitely many idempotent elements, by means of linearly ordered abelian groups and the partial lexicographic product construction is presented.

Keywords: Involutive residuated lattices, Construction, Representation, Abelian groups, Hahn-type embedding.

1. Introduction

Hahn's celebrated embedding theorem states that every linearly ordered abelian group G can be embedded as an ordered subgroup into the Hahn product $\overrightarrow{\times}^{H}_{\Omega} \mathbb{R}$, where $\mathbb{R}$ is the additive group of real numbers (with its standard order), Ω is the set of archimedean equivalence classes of G (ordered naturally by the dominance relation), $\overrightarrow{\times}^{H}_{\Omega} \mathbb{R}$ is the set of all functions from Ω to $\mathbb{R}$ (alternatively the set of all vectors with real elements and with coordinates taken from Ω) which vanish outside a well-ordered set, endowed with a lexicographical order [12]. Briefly, every linearly ordered abelian group can be represented as a group of real-valued functions on a totally ordered set. By weakening the linearly ordered hypothesis, Conrad, Harvey, and Holland generalized Hahn's theorem for lattice-ordered abelian groups in [6] by showing that any abelian ℓ-group can be represented as a group of real-valued functions on a partially ordered set. By weakening the hypothesis on the existence of the inverse element but keeping the linear order, in this

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paper Hahn's theorem will be generalized to a class of residuated semigroups, namely to linearly ordered odd involutive commutative residuated lattices which possess only finitely many idempotent elements. The generalization will be a by-product of a representation theorem for odd involutive commutative residuated chains which possess only finitely many idempotent elements, by means of the partial lexicographic product construction and using only linearly ordered abelian groups. Thus, the price for not having inverses in our semigroup framework is that our embedding is made into partial lexicographic products, introduced here, rather than lexicographic ones. The building blocks, which are linearly ordered abelian groups, remain the same.

Residuation is a basic concept in mathematics [3] with strong connections to Galois maps [11] and closure operators. Residuated semigroups have been introduced in the 1930s by Ward and Dilworth [22] to investigate ideal theory of commutative rings with unit. Recently the investigation of residuated lattices (that is, residuated monoids on lattices) has become quite intense, initiated by the discovery of the strong connection between residuated lattices and substructural logics [7] via an algebraic logic link [2]. Substructural logics encompass among many others, Classical logic, Intuitionistic logic, Łukasiewicz logic, Abelian logic, Relevance logics, Basic fuzzy logic, Monoidal t-norm logic, Full Lambek calculus, Linear logic, many-valued logics, mathematical fuzzy logics, along with their non-commutative versions. The theory of substructural logics has put all these logics, along with many others, under the same motivational and methodological umbrella, and residuated lattices themselves have been the key component in this remarkable unification. Examples of residuated lattices include Boolean algebras, Heyting algebras [16], MV-algebras [5], BL-algebras [13], and lattice-ordered groups, to name a few, a variety of other algebraic structures can be rendered as residuated lattices. Applications of substructural logics and residuated lattices span across proof theory, algebra, and computer science.

As for the structural description of classes of residuated lattices, non-integral residuated structures, and consequently substructural logics without the weakening rule, are far less understood at present than their integral counterparts. Therefore, some authors try to establish category equivalences between integral and non-integral residuated structures to gain a better understanding of the non-integral case, and in particular, to gain a better understanding of substructural logics without weakening [8,9]. Despite the extensive literature devoted to classes of residuated lattices, there are still very few results that effectively describe their structure, and many of these effective descriptions postulate, besides integrality, the naturally ordered condition,[1] too [1,5,10,13,15,17,18,20]. Linearly ordered odd involutive commutative residuated lattices (another terminology is odd involutive FL_e-chains) are non-integral and not naturally ordered, unless they are trivial. Therefore, from a general point of view, our study is a contribution to the structural description of residuated structures which are neither naturally ordered nor integral.

Definition 1.1. An *FL_e-algebra* is a structure $(X, \wedge, \vee, \circledast, \rightarrow_\circledast, t, f)$ such that $(X, \wedge, \vee)$ is a lattice,[2] $(X, \leq, \circledast, t)$ is a commutative, residuated monoid,[3] and f is an arbitrary constant. Here being *residuated* means that there exists a binary operation $\rightarrow_\circledast$, called the residual operation of $\circledast$, such that

$$x \circledast y \leq z \text{ if and only if } x \rightarrow_\circledast z \geq y.$$

This equivalence is called adjointness condition and $(\circledast, \rightarrow_\circledast)$ is called an adjoint pair. Equivalently, for any $x, z \in X$, the set $\{v \mid x \circledast v \leq z\}$ has its greatest element, and $x \rightarrow_\circledast z$, the so-called residuum of x and z, is defined to be this greatest element:

$$x \rightarrow_\circledast z := \max\{v \mid x \circledast v \leq z\}.$$

This is called the residuation property. Easy consequences of this definition are the exchange property

$$(x \circledast y) \rightarrow_\circledast z = x \rightarrow_\circledast (y \rightarrow_\circledast z),$$

that $\rightarrow_\circledast$ is isotone at its second argument, and that $\circledast$ distributes over arbitrary joins. One defines $x' = x \rightarrow_\circledast f$ and calls an FL_e-algebra *involutive* if $(x')' = x$ holds. We say that the rank of an involutive FL_e-algebra is positive if $t > f$, negative if $t < f$, and 0 if $t = f$. In the zero rank case we also say that the involutive FL_e-algebra is *odd*. Denote the set of *positive* (resp. *negative*) elements of X by $X^+ = \{x \in X : x \geq t\}$ (resp. $X^- = \{x \in X : x \leq t\}$), and call the elements of X^+ different from t *strictly positive*. We call the FL_e-algebra *conic* if all elements of X are comparable with t. If the algebra is linearly ordered we speak about FL_e-*chains*. Algebras will be denoted by bold capital letters, their underlying set by the same regular letter. Commutative residuated lattices are exactly the f-free reducts of FL_e-algebras.

Definition 1.2. The lexicographic product of two linearly ordered sets $\mathbf{A} = (A, \leq_1)$ and $\mathbf{B} = (B, \leq_2)$ is a linearly order set $\mathbf{A} \overrightarrow{\times} \mathbf{B} = (A \times B, \leq)$,

[1] Its dual notion is often called divisibility.

[2] Sometimes the lattice operators are replaced by their induced ordering $\leq$ in the signature, in particular, if an FL_e-*chain* is considered, that is, if the ordering is linear.

[3] We use the word monoid to mean semigroup with unit element.

where $A \times B$ is the Cartesian product of A and B, and $\leq$ is defined by $\langle a_1, b_1\rangle \leq \langle a_2, b_2\rangle$ if and only if $a_1 <_1 a_2$ or $a_1 = a_2$ and $b_1 \leq_1 b_2$.

We shall view such a lexicographic product as an *enlargement*: each element of A is replaced by a whole copy of B. Accordingly, in Section 4 by a *partial* lexicographic product of two linearly ordered sets we will mean a kind of partial enlargement: only *some* elements of the first algebra will be replaced by a whole copy of the second algebra.

In any involutive FL_e-algebra $'$ is an order reversing involution of the underlying set, and $'$ has a single fixed point if the algebra is odd. Hence in odd involutive FL_e-algebras $x \mapsto x'$ is somewhat reminiscent to a reflection operation across a point in geometry, yielding a symmetry of order two with a single fixed point. In this sense $t' = f$ means that the position of the two constants is symmetric in the lattice. Thus, an extreme situation is the integral case, when t is the top element of X and hence f is its bottom element. This case has been deeply studied in the literature. The other extreme situation, when "the two constants are in both the middle of the lattice", (that is, when $t = f$ and so the algebra is odd) is a much less studied scenario. We remark that $t < f$ can also hold in some involutive FL_e-algebras. However, there is no third extremal situation, as it cannot be the case that f is the top element of X and t is its bottom element: For any residuated lattice with bottom element, the bottom element has to be the annihilator of the algebra, but t is its unit element, a contradiction unless the algebra is trivial.

Prominent examples of odd involutive FL_e-algebras are odd Sugihara monoids and lattice-ordered abelian groups. The latter constitutes an algebraic semantics of Abelian Logic [4,19,21] while the former constitutes an algebraic semantics of **IUML***, which is a logic at the intersection of relevance logic and many-valued logic [8]. These two examples represent two extreme situations from another viewpoint: There is a single idempotent element in any lattice-ordered abelian group, whereas all elements are idempotent in any odd Sugihara monoid. The scope of our investigation lies in between these two extremes; we shall assume that the number of idempotent elements of the odd involutive FL_e-chain is finite. For this class a representation theorem along with some corollaries (e.g. the generalization of Hahn's theorem) will be presented in this paper.

2. Odd Involutive $\mathbf{FL}_e$-algebras Versus Partially Ordered Abelian Groups

We start with three preliminary observations, the first of which is being folklore (we present its proof, too, to keep the paper self contained).

PROPOSITION 2.1. *For any involutive FL_e-algebra $(X, \wedge, \vee, \circledast, \rightarrow_\circledast, t, f)$ the following hold true.*

$$x \rightarrow_\circledast y = (x \circledast y')', \tag{2.1}$$

If $t \geq f$ then

$$x \circledast y \leq (x' \circledast y')'. \tag{2.2}$$

If $t \leq f$ and the algebra is conic then $y_1 > y$ implies

$$(x' \circledast y')' \leq x \circledast y_1. \tag{2.3}$$

PROOF. Using that $'$ is an involution one easily obtains $(x \circledast y')' = (x \circledast y') \rightarrow_\circledast f = x \rightarrow_\circledast (y' \rightarrow_\circledast f) = x \rightarrow_\circledast y$. To prove (2.2) one proceeds as follows: $(x \circledast y) \circledast (x' \circledast y') = [x \circledast (x \rightarrow_\circledast f)] \circledast [y \circledast (y \rightarrow_\circledast f)] \leq f \circledast f \leq t \circledast f = f$, hence $x \circledast y \leq (x' \circledast y') \rightarrow_\circledast f$. To show (2.3) we proceed as follows. Since the algebra is involutive $t' = f$ holds. Since the algebra is conic, every element is comparable with f, too. Indeed, for any $a \in X$, if a were not compatible with f then since $'$ is an order reversing involution, a' were not compatible with $f' = t$, which is a contradiction. Hence, by residuation,[4] $y_1 > y = (y')' = y' \rightarrow_\circledast f$ implies $y_1 \circledast y' \not\leq f$, that is, $y_1 \circledast y' > f \geq t$. Therefore, $(x \circledast y_1)' = (x \circledast y_1)' \circledast t \leq (x \circledast y_1)' \circledast y_1 \circledast y' = (y_1 \circledast (y_1 \rightarrow_\circledast x')) \circledast y' \leq x' \circledast y'$ follows by using (2.1). ■

In our investigations a crucial role will be played by the τ function.

DEFINITION 2.2. (τ). For an involutive FL$_e$-algebra $(X, \wedge, \vee, \circledast, \rightarrow_\circledast, t, f)$, for $x \in X$, define $\tau(x)$ to be the greatest element of $Stab_x = \{z \in X \mid z \circledast x = x\}$, the stabilizer set of x. Since t is the unit element of $\circledast$, $Stab_x$ is nonempty. Since $\circledast$ is residuated, the greatest element of $Stab_x$ exists, and it holds true that

$$\tau(x) \circledast x = x \quad \text{and} \quad \tau(x) = x \rightarrow_\circledast x \geq t. \tag{2.4}$$

PROPOSITION 2.3. *For an involutive FL_e-algebra $(X, \wedge, \vee, \circledast, \rightarrow_\circledast, t, f)$ the following holds true.*

[4] Throughout the paper when we say in a proof 'by residuation' or 'by adjointness' we refer to the residuation property and the adjointness property, respectively.

1. *$\tau(x) = \tau(x')$.*
2. *$\tau(\tau(x)) = \tau(x)$.*
3. *$\tau(x \circledast y) \geq \tau(x)$.*
4. *$u \in X^+$ is idempotent if and only if $\tau(u) = u$.*
5. *$Range(\tau) = \{\tau(x) : x \in X\}$ is equal to the set of idempotent elements in X^+.*
6. *If the order is linear then for $x \in X^+$, $\tau(x) \leq x$ holds.*

PROOF.

1. By (2.1), $\tau(x) = x \rightarrow_{\circledast} x = (x \circledast x')' = (x' \circledast x'')' = x' \rightarrow_{\circledast} x' = \tau(x')$.
2. By (2.1), $x \rightarrow_{\circledast} x = \tau(x)$ is equivalent to $x \circledast x' = \tau(x)'$. Hence, $\tau(x) \circledast \tau(x)' = \tau(x) \circledast (x \circledast x') = (\tau(x) \circledast x) \circledast x' = x \circledast x' = \tau(x)'$ follows. Hence, $\tau(\tau(x)) = \tau(x) \rightarrow_{\circledast} \tau(x) = (\tau(x) \circledast \tau(x)')' = \tau(x)'' = \tau(x)$ follows by (2.1).
3. If $u \circledast x = x$ then $u \circledast (x \circledast y) = (u \circledast x) \circledast y = x \circledast y$, that is, $Stab_x \subseteq Stab_{x \circledast y}$.
4. If $u \geq t$ is idempotent then from $u \circledast u = u$, $u \rightarrow_{\circledast} u \geq u$ follows by adjointness. But for any $z > u$, $u \circledast z \geq t \circledast z = z > u$, hence $\tau(u) = u$ follows. On the other hand, $\tau(u) = u$ implies $u \geq t$ by (2.4), and also the idempotency of u since $u \circledast u = u \circledast \tau(u) = u$.
5. If $u > t$ is idempotent then claim (4) shows that u is in the range of τ. If u is in the range of τ, that is $\tau(x) = u$ for some $x \in X$ then claim (2) implies $\tau(u) = u$, hence u is an idempotent in X^+ by claim (4).
6. Since the order is linear, the opposite of the statement is $\tau(x) > x$, but it yields $x \circledast \tau(x) \geq t \circledast \tau(x) = \tau(x) > x$, a contradiction to (2.4). ■

The notion of odd involutive FL$_e$-algebras has been defined with respect to the general notion of residuated lattices by adding further postulates (such as commutativity, an extra constant f, involutivity, and the $t = f$ property). The following theorem relates odd involutive FL$_e$-algebras to (in the setting of residuated lattices, very specific) lattice-ordered abelian groups, thus picturing their precise interrelation. In addition, Theorem 2.4 will serve as the basic step of the induction in the proof of Theorem 11.1, too.

THEOREM 2.4. *For an odd involutive FL$_e$-algebra $\mathbf{X} = (X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$ the following statements are equivalent:*

1. *Each element of X has inverse given by $x^{-1} = x'$, and hence $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered abelian group,*
2. *$\circledast$ is cancellative,*

3. $\tau(x) = t$ *for all* $x \in X$.
4. *The only idempotent element in* X^+ *is* t.

PROOF. First note that by (2.1), claim (1), that is, for all $x \in X$, $x \circledast x' = t$, is equivalent to claim (3). Also claim (1) $\Rightarrow$ claim (2) is straightforward. To see claim (2) $\Rightarrow$ claim (1) we proceed as follows: By residuation, $x \circledast x' \leq f$ holds, therefore by isotonicity of $\rightarrow_\circledast$ at its second argument, $x \rightarrow_\circledast (x \circledast x') \leq x \rightarrow_\circledast f = x'$ follows. By residuation $x \circledast x' \leq x \circledast x'$ is equivalent to $x \rightarrow_\circledast (x \circledast x') \geq x'$, hence we infer $x \rightarrow_\circledast (x \circledast x') = x' = x \rightarrow_\circledast f = x \rightarrow_\circledast t$. By (2.1), $x \circledast (x \circledast x')' = x \circledast t'$ follows, and cancellation by x implies $t = x \circledast x'$, so we are done. Finally, claim (5) in Proposition 2.3 ensures the equivalence of claims (3) and (4). ■

In the light of Theorem 2.4, in the sequel when we (loosely) speak about a subgroup of an odd involutive FL_e-algebra, we always mean a cancellative subalgebra of it. Further, let

$$X_{gr} = \{x \in X \mid x' \text{ is the inverse of } x\}.$$

Evidently, there is a subalgebra $\mathbf{X_{gr}}$ of $\mathbf{X}$ over X_{gr},[5] and $\mathbf{X_{gr}}$ is the largest subgroup of $\mathbf{X}$. We call $\mathbf{X_{gr}}$ the group part of $\mathbf{X}$.

3. Two Illustrative Examples

Unfortunately, there does not exist any easily accessible example in the class of odd involutive FL_e-chains, apart from the cancellative subclass (linearly ordered abelian groups, each with a single idempotent element) or the idempotent subclass (odd Sugihara monoids, all elements are idempotent). Here we start with the two simplest nontrivial examples of odd involutive FL_e-chains over $\mathbb{R}$, both of which have only a single strictly positive idempotent element. These examples will be the inspirational source for the type I-IV partial lexicographic product constructions in Definition 4.2.

Using (2.2) it takes a half-line proof to show that for any odd involutive FL_e-algebra, the residual complement of any negative idempotent element u is also idempotent: $u' \circledast u' \leq (u \circledast u)' = u' = u' \circledast t \leq u' \circledast u'$. The author of the present paper has put a considerable effort into proving the converse of this statement, with no avail. In fact, the converse statement does not hold, as shown by the following counterexample.

[5] This will also follow from Proposition 5.2.

EXAMPLE 3.1. The odd involutive FL_e-chain which is to be defined in this example is $\mathbf{Z} \overrightarrow{\times} \mathbf{R}$ (up to isomorphism), see item B in Definition 4.2 for its formal definition. Here we present an informal account on the motivation and considerations which had led to the discovery of this example (Fig. 1).

Let $I =]-1, 0[$. Consider any order-isomorphic copy $\mathbf{I} = (I, \star, -\frac{1}{2})$ of the linearly ordered abelian group $\mathbf{R} = (\mathbb{R}, +, 0)$.[6] Our plan is to put a series of copies of I in a row (after each other, one for each integer, that is, formally we consider $\mathbb{Z} \times I$) to get $\mathbb{R}$, and to define, based on $\star$, an operation $\circledast$ over $\mathbb{R}$ so that the resulting structure becomes an odd involutive FL_e-chain. The first problem is caused by the fact that whichever involutive structure we start with (in place of $\mathbf{I}$), the above-described repetition of its universe will never yield $\mathbb{R}$. In our example $I =] - 1, 0[$ does not have top and bottom elements, hence by the above-described repetition of it we obtain a set which is order-isomorphic to $\mathbb{R}\backslash\mathbb{Z}$, and not to $\mathbb{R}$. Or if the original structure does have a top element (and hence by involutivity a bottom element, too) then the universe resulted by the above-described repetition will have gaps. To overcome this we extend $\star$ to a new universe $I^\top := I \cup \{0\} =] - 1, 0]$ by letting 0 be an annihilator of $\star$. In other words, we add a new top element to the structure $\mathbf{I}$. The resulting structure, denote it by $\mathbf{I}^\top$, will no longer be an odd involutive FL_e-chain, but it is still an FL_e-chain, albeit obviously not even involutive due to the broken symmetry of its underlying set. Luckily we can immediately eliminate this broken symmetry by putting $\aleph_0$ copies of $I^\top =] - 1, 0]$ in a row, thus obtaining $\mathbb{R}$. Now we can define an operation $\circledast$ over $\mathbb{R}$ by letting for $a, b \in \mathbb{R}$,[7]

$$a \circledast b = \lceil a \rceil + \lceil b \rceil + (a - \lceil a \rceil) \star (b - \lceil b \rceil).$$

(see the graph of an order-isomorphic copy of $\circledast$ in Figure 2, right, for the use of such 3D plots the interested reader is referred to [14]). The slightly cumbersome task of verifying that $(\mathbb{R}, \leq, \circledast, \rightarrow_\circledast, -\frac{1}{2}, -\frac{1}{2})$ is an odd involutive FL_e-chain, is left to the reader. It amounts to verify the associativity of $\circledast$,[8] to compute the residual operation of $\circledast$, to confirm that the residual complement operation is $x' = -x - 1$, and to confirm that $-\frac{1}{2}$ is the unit element and also the fixed point of $'$. Note that the only strictly positive idempotent element of the constructed algebra is 0, while $-\frac{1}{2}$ is its unit element. Note,

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[6] Let, for example, $\star$ be given by $x \star y = f^{-1}(f(x) + f(y))$ where $f : I \to \mathbb{R}$, $f(x) = \tan((x + 0.5) \cdot \pi))$.

[7] $\lceil a \rceil$ stands for the ceiling of a.

[8] A computation reveals that $(a \circledast b) \circledast c = (\lceil a \rceil + \lceil b \rceil) + \lceil c \rceil + (a - \lceil a \rceil) \star (b - \lceil b \rceil) \star (c - \lceil c \rceil)$ thus the associativity of $\star$ ensures the associativity of $\circledast$.

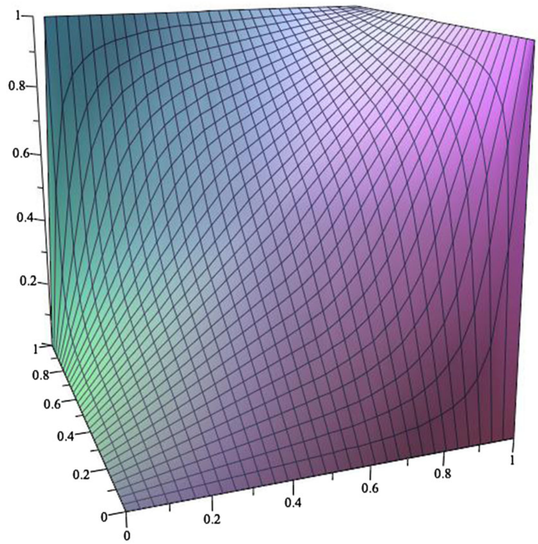

Figure 1. Visualization: the graph of the operation $\star$

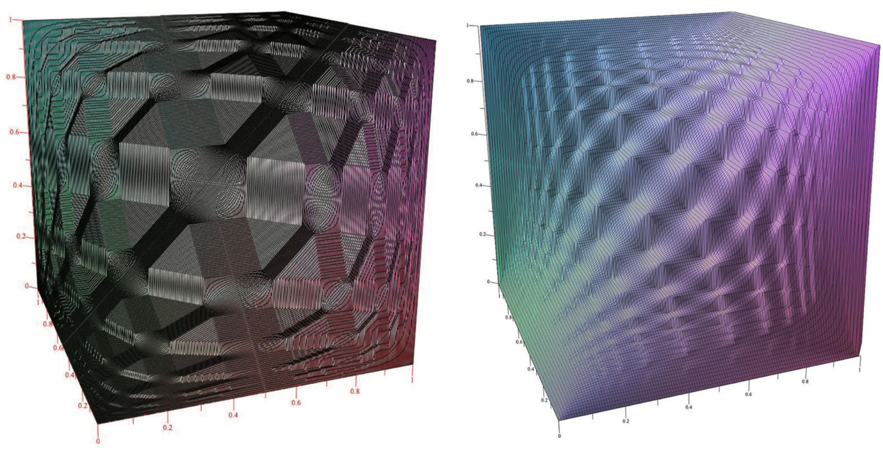

Figure 2. Visualization: $\mathbb{R}_{\mathbb{Z}} \overset{\twoheadrightarrow}{\times} \mathbb{R}$ (left) and $\mathbb{Z} \overset{\twoheadrightarrow}{\times} \mathbb{R}$ (right) shrank into $]0,1[$

however, that the residual complement of 0 is not idempotent: $0' = -1$ and $-1 \circledast -1 = -2$.

In the quest for a deeper understanding of this example, observe that apart from $\star$, the addition operation of $\mathbb{R}$ plays a role in the definition of $\circledast$. Moreover, by using a particular order-isomorphism, we can define $\circledast$ using $\star$ and the addition operation of $\mathbb{Z}$ only, as follows. As said, putting a series of copies of $I^\top$ in a row can, formally, be obtained by considering the lexicographic product $\mathbb{Z} \overrightarrow{\times} I^\top$. Then, $h(a) = (\lceil a \rceil, a - \lceil a \rceil)$ is an order-isomorphism from $\mathbb{R}$ to $\mathbb{Z} \overrightarrow{\times} I^\top$. It is left to the reader to verify that $\circledast$ can equivalently be defined in a *coordinatewise manner* by letting $a \circledast b = h^{-1}[(a_1 + b_1, a_2 \star b_2)]$ where $h(a) = (a_1, a_2)$ and $h(b) = (b_1, b_2)$. By using the natural order-reversing involutions ($a^{\overset{*}{\prime}} = -a$ on $\mathbb{Z}$ and $x^{\overset{\star}{\prime}} = -x - 1$ on I) the reader can also verify that the residual operation of $\circledast$ is given by

$$(a_1, a_2) \to_{\circledast} (b_1, b_2) = ((a_1, a_2) \circledast (b_1, b_2)')',$$

where[9]

$$(x, y)' = \begin{cases} (x^{\overset{*}{\prime}} - 1, 0) & \text{if } y = 0 \\ (x^{\overset{*}{\prime}}, y^{\overset{\star}{\prime}}) & \text{if } y \in I \end{cases}. \tag{3.1}$$

With this notation, the only strictly positive idempotent element of the constructed algebra is $(0, 0)$, while $(0, -\frac{1}{2})$ is its unit element. A moral of this example is that the latter coordinatewise formalism for $\circledast$ enlightens the example by providing a deeper insight into and a clear description of why $\circledast$ makes up for an odd involutive FL_e-chain (why is it associative, residuated, etc.) Compare the example above with the formal definition of $\mathbb{Z} \overset{\twoheadrightarrow}{\times} \mathbb{R}$ in item B in Definition 4.2. Further inspection reveals that in these kind of examples one does not necessarily have to work with groups (like $\mathbb{Z}$ and $\mathbb{R}$ in our example), but also odd involutive FL_e-algebras suffice in both coordinates. But in this case only the *group part* of the first algebra (and no more) can be enlarged. This makes our lexicographic product construction partial in nature, as if the first algebra is not a group then its group part is strictly smaller than the whole algebra. Leaving the rest of the first algebra unchanged will, formally, mean that we consider a Cartesian product there, too, to enable a coordinatewise treatment of the operations, namely, the Cartesian product of the rest of the first algebra with a singleton. This example has inspired the type II partial lexicographic product construction.

Moreover, it turns out that instead of enlarging the *group part* one can enlarge any *subgroup* of the group part. This makes our lexicographic product construction definitively partial in nature, since even if the first algebra is a group (and hence its group part is the whole algebra), an even smaller

[9] Compare (3.1) with the last two rows of (4.2); for the last row of (4.2) see Definition 4.1 for the notation $_\downarrow$.

algebra (a subalgebra of the group part) can be enlarged, by the second algebra extended by a top. This option of *partial* enlargement inspires the type IV partial lexicographic construction defined in item (B) of Definition 4.2.

EXAMPLE 3.2. The odd involutive FL_e-chain which is to be defined in this example is $\mathbb{R}_{\mathbb{Z}} \overset{\rightarrow\!\!\rightarrow}{\times} \mathbb{R}$ (up to isomorphism), see item A in Definition 4.2 for its formal definition.

In order to further exploit the partial nature of our lexicographic product construction, take the subgroup $\mathbf{Z} = (\mathbb{Z}, +, 0)$ of the linearly ordered abelian group of the reals $\mathbf{R} = (\mathbb{R}, +, 0)$. Our plan is to replace each integer inside the reals by a whole copy of the reals (isomorphically, by I from $\mathbf{I} = (I, \star, -\frac{1}{2})$ of Example 3.1), and to define an odd involutive FL_e-algebra over it based on $\star$. Again, if we wish to obtain a universe, which is order isomorphic to $\mathbb{R}$, then replacing each integer inside the reals by $I =]-1, 0[$ is not convenient; the resulting universe will not be topologically connected, as it is easy to see. To overcome this we extend $\star$ to $I^{\top\perp} := I \cup \{-1, 0\} = [-1, 0]$ by letting first 0 be annihilator over $]-1, 0]$, and then -1 be annihilator over $[-1, 0]$. In other words, we add a new top and a new bottom to the structure $\mathbf{I}$, thus obtaining $\mathbf{I}^{\top\perp}$. Now, let the new universe be $\mathbb{Z} \times I^{\top\perp} \cup (\mathbb{R}\backslash\mathbb{Z}) \times \{-1\}$. Again, we define $\circledast$ coordinatewise by letting $(a_1, a_2) \circledast (b_1, b_2) = (a_1 + b_1, a_2 \star b_2)$, and we define $'$, using the natural order-reversing involutions ($a^{\prime_*} = -a$ on $\mathbb{R}$ and $x^{\prime_\star} = -x - 1$ on $I^{\top\perp}$) also coordinatewise by

$$(x, y)' = \begin{cases} (x^{\prime_*}, y^{\prime_\star}) & \text{if } x \in \mathbb{Z} \\ (x^{\prime_*}, -1) & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \end{cases} .$$

Finally, let

$$(a_1, a_2) \to_{\circledast} (b_1, b_2) = ((a_1, a_2) \circledast (b_1, b_2)')'.$$

What we obtain is an odd involutive FL_e-chain, see the graph of an order-isomorphic copy of $\circledast$ in Figure 2, left. Here, too, there is only one strictly positive idempotent element in the constructed algebra, namely $(0, 0)$, while $(0, -\frac{1}{2})$ is its unit element. Note that unlike in the previous example here the residual complement of $(0, 0)$ is idempotent: $(0, 0)' = (0, -1)$ and $(0, -1) \circledast (0, -1) = (0, -1)$. Compare the example above with the formal definition of $\mathbb{R}_{\mathbb{Z}} \overset{\rightarrow\!\!\rightarrow}{\times} \mathbb{R}$ in item A in Definition 4.2. Further inspection reveals that in these kind of examples one does not necessarily have to work with groups (like $\mathbf{R}$ together with its subgroup $\mathbf{Z}$ in the first coordinate, and $\mathbf{R}$ in the second), but any odd involutive FL_e-algebra with any of its subgroup suffices in the first coordinate, and any odd involutive FL_e-algebra suffices in the second one. Summing up, we start with an odd involutive FL_e-algebra and we only enlarge a subgroup of it by an arbitrary odd involutive FL_e-algebra equipped with top and bottom. This example inspired the type I partial lexicographic product construction.

Moreover, it turns out that instead of enlarging a subgroup, one can (1) enlarge only a subgroup of a subgroup by the second algebra equipped with top and bottom, plus (2) enlarge the difference of the two subgroups by only the top and the bottom. This second step is equivalent to enlarging the difference of the two subgroups also by the second algebra equipped with top and bottom, and then *removing* the second algebra, thus leaving only its top and bottom there. Ultimately, at this removal step we take a subalgebra compared to the corresponding algebra which constructed by the type I variant, see Theorem 4.4. The option of *partial* enlargement (when only a subalgebra of a subalgebra is enlarged) in this example inspires the type III partial lexicographic construction defined in item (A) of Definition 4.2.

4. Constructing Involutive FL_e-algebras—Partial Lexicographic Products

We start with a notation.

DEFINITION 4.1. Let $(X, \leq)$ be a poset. For $x \in X$ define

$$x_{\downarrow} = \begin{cases} z & \text{if there exists a unique } z \in X \text{ such that } x \text{ covers } z, \\ x & \text{otherwise.} \end{cases}$$

We define $x_{\uparrow}$ dually. Note that if $'$ is an order-reversing involution of X then it holds true that

$$x'_{\uparrow} = (x_{\downarrow})' \quad \text{and} \quad x'_{\downarrow} = (x_{\uparrow})'. \tag{4.1}$$

We say for $Z \subseteq X$ that Z is discretely embedded into X if for $x \in Z$ it holds true that $x \notin \{x_{\uparrow}, x_{\downarrow}\} \subseteq Z$.

Next, we introduce the construction for the key result of the paper, called *partial lexicographic product* (or partial lexicographic extension) with four slightly different variations in Definition 4.2. Roughly, only a subalgebra is used as a first component of a lexicographic product and the rest of the algebra is left unchanged, hence the adjective 'partial'. This results in an involutive FL_e-algebra, which is odd, too, provided that the second component of the lexicographic product is so, see Theorem 4.4.

In Remark 4.5 we shall refer to some algebraic notions which appear later in the paper at the algebraic decomposition part, thus trying to make a bridge between the different components of the construction part and the decomposition part of the paper.

Definition 4.2. Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be an odd involutive FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \rightarrow_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement ${}^{\prime *}$ and ${}^{\prime \star}$, respectively.

A. Add a new element $\top$ to Y as a top element and annihilator (for $\star$), then add a new element $\bot$ to $Y \cup \{\top\}$ as a bottom element and annihilator. Extend ${}^{\prime\star}$ by $\bot^{\prime\star} = \top$ and $\top^{\prime\star} = \bot$. Let $\mathbf{V} \le \mathbf{Z} \le \mathbf{X_{gr}}$. Let

$$X_{Z_V} \overset{\rightarrow\!\!\!\rightarrow}{\times} Y = (V \times (Y \cup \{\top, \bot\})) \cup ((Z \backslash V) \\ \times\{\top, \bot\}) \cup ((X \backslash Z) \times \{\bot\}),$$

and define $\mathbf{X_{Z_V}} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{Y}$, the *type III partial lexicographic product* of $\mathbf{X}$, $\mathbf{Z}$, $\mathbf{V}$ and $\mathbf{Y}$ as follows:

$$\mathbf{X_{Z_V}} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{Y} = \left(X_{Z_V} \overset{\rightarrow\!\!\!\rightarrow}{\times} Y, \le, \circledast, \rightarrow_\circledast, (t_X, t_Y), (f_X, f_Y)\right),$$

where $\le$ is the restriction of the lexicographical order of $\le_X$ and $\le_{Y\cup\{\top,\bot\}}$ to $X_{Z_V} \overset{\rightarrow\!\!\!\rightarrow}{\times} Y$, $\circledast$ is defined coordinatewise, and the operation $\rightarrow_\circledast$ is given by $(x_1, y_1) \rightarrow_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$, where

$$(x, y)' = \begin{cases} (x^{\prime *}, \bot) & \text{if } x \notin Z \\ (x^{\prime *}, y^{\prime\star}) & \text{if } x \in Z \end{cases}.$$

In the particular case when $\mathbf{V} = \mathbf{Z}$, we use the simpler notation $\mathbf{X_Z} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{Y}$ for $\mathbf{X_{Z_V}} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{Y}$ and call it the *type I partial lexicographic product* of $\mathbf{X}, \mathbf{Z}$, and $\mathbf{Y}$.

B. Assume that X_{gr} is discretely embedded into X. Add a new element $\top$ to Y as a top element and annihilator. Let $\mathbf{V} \le \mathbf{X_{gr}}$. Let

$$X_V \overset{\rightarrow\!\!\!\rightharpoonup}{\times} Y = (X \times \{\top\}) \cup (V \times Y)$$

and define $\mathbf{X_V} \overset{\rightarrow\!\!\!\rightharpoonup}{\times} \mathbf{Y}$, the *type IV partial lexicographic product* of $\mathbf{X}$, $\mathbf{V}$ and $\mathbf{Y}$ as follows:

$$\mathbf{X_V} \overset{\rightarrow\!\!\!\rightharpoonup}{\times} \mathbf{Y} = \left(X_V \overset{\rightarrow\!\!\!\rightharpoonup}{\times} Y, \le, \circledast, \rightarrow_\circledast, (t_X, t_Y), (f_X, f_Y)\right),$$

where $\le$ is the restriction of the lexicographical order of $\le_X$ and $\le_{Y\cup\{\top\}}$ to $X_V \overset{\rightarrow\!\!\!\rightharpoonup}{\times} Y$, $\circledast$ is defined coordinatewise, and the operation $\rightarrow_\circledast$ is given by $(x_1, y_1) \rightarrow_{\circledast} (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$, where $'$ is defined coordinatewise[10] by

$$(x,y)' = \begin{cases} (x^{\overset{*}{\prime}}, \top) & \text{if } x \notin X_{gr} \text{ and } y = \top \\ ((x^{\overset{*}{\prime}})_{\downarrow}, \top) & \text{if } x \in X_{gr} \text{ and } y = \top \\ (x^{\overset{*}{\prime}}, y^{\overset{*}{\prime}}) & \text{if } x \in V \text{ and } y \in Y \end{cases}. \tag{4.2}$$

In the particular case when $\mathbf{V} = \mathbf{X_{gr}}$, we use the simpler notation $\mathbf{X} \overset{\rightharpoonup}{\times} \mathbf{Y}$ for $\mathbf{X_V} \overset{\rightharpoonup}{\times} \mathbf{Y}$ and call it the *type II partial lexicographic product* of $\mathbf{X}$ and $\mathbf{Y}$.

In order to reduce the number of different cases to be considered when checking the residuated nature of some structure (e.g. in the proof of Theorem 4.4) we will rely on the following statement, which generalizes the well-known an equivalent formulation of involutive FL_e-algebras in terms of dualizing constants.

Proposition 4.3. *Let $\mathcal{M} = (M, \leq, \circledast)$ be a structure such that $(M, \leq)$ is a poset and $(M, \circledast)$ is a semigroup. Call $c \in M$ a dualizing element*[11] *of $\mathcal{M}$, if (i) for $x \in M$ there exists $x \rightarrow_{\circledast} c$,*[12] *and (ii) for $x \in M$, $(x \rightarrow_{\circledast} c) \rightarrow_{\circledast} c = x$. If there exists a dualizing element c of $\mathcal{M}$ then $\circledast$ is residuated, and its residual operation is given by $x \rightarrow_{\circledast} y = (x \circledast (y \rightarrow_{\circledast} c)) \rightarrow_{\circledast} c$.*

Proof. Indeed, $z \circledast x \leq y$ is equivalent to $z \circledast x \leq (y \rightarrow_{\circledast} c) \rightarrow_{\circledast} c$. By adjointness it is equivalent to $(z \circledast x) \circledast (y \rightarrow_{\circledast} c) \leq c$. By associativity it is equivalent to $z \circledast (x \circledast (y \rightarrow_{\circledast} c)) \leq c$, which is equivalent to $z \leq (x \circledast (y \rightarrow_{\circledast} c)) \rightarrow_{\circledast} c$ by adjointness. Thus we obtained $x \rightarrow_{\circledast} y = (x \circledast (y \rightarrow_{\circledast} c)) \rightarrow_{\circledast} c$, as stated. ■

Theorem 4.4. *Adapt the notation of Definition 4.2. $\mathbf{X_{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\rightharpoonup}{\times} \mathbf{Y}$ are involutive FL_e-algebras with the same rank as that of $\mathbf{Y}$. In particular, if $\mathbf{Y}$ is odd then so are $\mathbf{X_{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\rightharpoonup}{\times} \mathbf{Y}$. In addition, $\mathbf{X_{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{Y} \leq \mathbf{X_Z} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\rightharpoonup}{\times} \mathbf{Y} \leq \mathbf{X} \overset{\rightharpoonup}{\times} \mathbf{Y}$.*

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[10] Note that intuitively it would make up for a coordinatewise definition, too, in the second line of (4.2) to define it as $(x^{\overset{*}{\prime}}, \bot)$. But $\bot$ is not amongst the set of possible second coordinates. However, since X_{gr} is discretely embedded into X, if $(x^{\overset{*}{\prime}}, \bot)$ would be an element of the algebra then it would be equal to $((x^{\overset{*}{\prime}})_{\downarrow}, \top)$.

[11] Dualizing elements are defined in *residuated* structures in the literature, see e.g. [7, Section 3.4.17.].

[12] Here we do not assume that $\circledast$ is residuated. We only postulate that the greatest element of the set $\{z \in M \mid x \circledast z \leq c\}$ exists for all x and a fixed c.

Proof. A. *Proof for the type III extension* $\mathbf{X_{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$.
Clearly, $'$ is an order-reversing involution on $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$. When checking that $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$ is closed under $\circledast$, the only non-trivial cases are (i) when the product of the first coordinates in not in Z, because then the product of the second coordinates has to be $\bot$, and (ii) when the product of the first coordinates in in $Z \setminus V$, because then the product of the second coordinates has to be $\top$ or $\bot$. But if the product of the second coordinates is not $\bot$ then none of the second coordinates can be $\bot$ (since $\bot$ in annihilator), hence both first coordinates must be in Z, as required. Second, if the product of the second coordinates is neither $\top$ nor $\bot$ then none of the second coordinates can be $\top$ or $\bot$ (because of the product table for $\top$ and $\bot$), hence both first coordinates must be in V, as required.

Since $\bot$ is annihilator, $\circledast$ can be expressed as

$$(x_1, y_1) \circledast (x_2, y_2) = \begin{cases} (x_1 * x_2, y_1 \star y_2) & \text{if } x_1, x_2 \in Z \\ (x_1 * x_2, \bot) & \text{otherwise} \end{cases},$$

hence associativity of $\circledast$ and that (t_X, t_Y) is the unit of $\circledast$ are straightforward.

Next, we state that $\circledast$ is residuated. By Proposition 4.3 it suffices to prove that (f_X, f_Y) is a dualizing element of $(X_{Z_V} \overset{\twoheadrightarrow}{\times} Y, \leq, \circledast)$. In more details, that for any $(x, y) \in X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$, $(x, y) \rightarrow_{\circledast} (f_X, f_Y)$ exists, and it equals to $(x, y)'$ (and thus $(x, y) \mapsto (x, y) \rightarrow_{\circledast} (f_X, f_Y)$ is of order two). Equivalently, that for $(x_1, y_1), (x_2, y_2) \in X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$,

$$(x_1, y_1) \circledast (x_2, y_2) \leq (f_X, f_Y) \text{ if and only if } (x_2, y_2) \leq (x_1, y_1)'. \tag{4.3}$$

- Assume that at least one of x_1 and x_2 is not in Z. Due to commutativity of $*$ and $\star$, and the involutivity of $'$, we may safely assume $x_2 \notin Z$. Then $y_2 = \bot$ and thus $(x_1, y_1) \circledast (x_2, y_2) = (x_1 * x_2, \bot)$ is less or equal to (f_X, f_Y) if and only if $x_1 * x_2 \leq_X f_X$, if and only if $x_2 \leq_X {x_1}^{\prime *}$ by residuation, if and only if $(x_2, \bot) \leq ({x_1}^{\prime *}, \ldots) = (x_1, y_1)'$, and we are done.
- Therefore, in the sequel we will assume $x_1, x_2 \in Z$. Then $(x_1, y_1)' = ({x_1}^{\prime *}, {y_1}^{\prime \star})$.
- First let $(x_1 * x_2, y_1 \star y_2) \leq (f_X, f_Y)$. This holds if and only if either $x_1 * x_2 <_X f_X$, or $x_1 * x_2 = f_X$ and $y_1 \star y_2 \leq_Y f_Y$ holds. In the first case, since ${x_1}^{\prime *}$ is the inverse of x_1, $x_2 <_X {x_1}^{\prime *}$ follows, confirming $(x_2, y_2) \leq ({x_1}^{\prime *}, {y_1}^{\prime \star})$, while in the second case $x_2 = {x_1}^{\prime *}$ and by residuation $y_2 \leq_Y {y_1}^{\prime \star}$ follows, hence $(x_2, y_2) \leq ({x_1}^{\prime *}, {y_1}^{\prime \star})$ holds, as stated.

– Second, let $(x_2, y_2) \le ({x_1}^{\overset{*}{\prime}}, {y_1}^{\overset{\star}{\prime}})$. This holds if and only if either $x_2 <_X {x_1}^{\overset{*}{\prime}}$, or $x_2 = {x_1}^{\overset{*}{\prime}}$ and $y_2 \le_Y {y_1}^{\overset{\star}{\prime}}$. In the first case $x_1 * x_2 <_X f_X$ and hence $(x_1 * x_2, y_1 \star y_2) < (f_X, f_Y)$, while in the second case $x_1 * x_2 = x_1 * {x_1}^{\overset{*}{\prime}} = f_X$ holds, and by residuation $y_1 \star y_2 \le_Y f_Y$, hence the proof of (4.3) is concluded.

Summing up, $\circledast$ is residuated and $(x_1, y_1) \rightarrow_{\circledast} (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ holds.

If $\mathbf{Y}$ is odd (resp. positive or negative rank) then ${f_Y}^{\overset{\star}{\prime}} = f_Y$ (resp. ${f_Y}^{\overset{\star}{\prime}} >_Y f_Y$, ${f_Y}^{\overset{\star}{\prime}} <_Y f_Y$) and hence the dualizing element (f_X, f_Y) is a fixed point of $'$ (resp. $(f_X, f_Y)' > (f_X, f_Y)$ or $(f_X, f_Y)' < (f_X, f_Y)$); that is $\mathbf{X}_{\mathbf{Z_V}} \overrightarrow{\times} \mathbf{Y}$ is odd (resp. positive or negative rank), too.

B. *Proof for the type IV extension* $\mathbf{X_V} \overrightarrow{\times} \mathbf{Y}$.
It is straightforward to verify that $'$ is an order-reversing involution on $X_V \overrightarrow{\times} Y$ at the first and third rows of (4.2). At the second row of (4.2), we can use that X_{gr} is discretely embedded into X and so (4.1) holds for $x \in X_{gr}$.

Since $\top$ annihilates all elements of Y, $\circledast$ can be expressed as

$$(x_1, y_1) \circledast (x_2, y_2) = \begin{cases} (x_1 * x_2, y_1 \star y_2) & \text{if } x_1, x_2 \in V \\ (x_1 * x_2, \top) & \text{otherwise} \end{cases}.$$

hence associativity of $\circledast$ and that $((t_X, t_Y))$ is the unit of $\circledast$ are straightforward.

Next, we state that $\circledast$ is residuated. By the claim, it suffices to prove that (f_X, f_Y) is a dualizing element of $(X_V \overrightarrow{\times} Y, \le, \circledast)$. In more details, that for any $(x, y) \in X_V \overrightarrow{\times} Y$, $(x, y) \rightarrow_{\circledast} (f_X, f_Y)$ exists, and it equals to $(x, y)'$ (and thus $(x, y) \mapsto (x, y) \rightarrow_{\circledast} (f_X, f_Y)$ is of order two). Equivalently, that for $(x_1, y_1), (x_2, y_2) \in X_V \overrightarrow{\times} Y$,

$$(x_1, y_1) \circledast (x_2, y_2) \le (f_X, f_Y) \text{ if and only if } (x_2, y_2) \le (x_1, y_1)'. \quad (4.4)$$

– Assume $y_1, y_2 \in Y$. Then $x_1, x_2 \in V$ hold. Then $(x_1 * x_2, y_1 \star y_2) \le (f_X, f_Y)$ holds if and only if $x_1 * x_2 <_X f_X$ holds or if both $x_1 * x_2 = f_X$ and $y_1 \star y_2 \le_Y f_Y$ hold. Since since ${x_1}^{\overset{*}{\prime}}$ is the inverse of x_1, the condition above is equivalent to $x_2 <_X {x_1}^{\overset{*}{\prime}}$ or $x_2 = {x_1}^{\overset{*}{\prime}}$ together with $y_2 \le_Y {y_1}^{\overset{\star}{\prime}}$, which is equivalent to $(x_2, y_2) \le ({x_1}^{\overset{*}{\prime}}, {y_1}^{\overset{\star}{\prime}}) = (x_1, y_1)'$.

– Assume $y_1 \in Y$ and $y_2 = \top$. Then $x_1 \in V$ holds. Since $\top > f_Y$, $(x_1 * x_2, \top) \le (f_X, f_Y)$ holds if and only if $x_1 * x_2 <_X f_X$, which is equivalent to $x_2 <_X {x_1}^{\overset{*}{\prime}}$ since ${x_1}^{\overset{*}{\prime}}$ is the inverse of x_1, which in turn is equivalent to $(x_2, \top) \le ({x_1}'^{*}, {y_1}'^{\star}) = (x_1, y_1)'$ and we are done. The case $y_1 = \top$ and $y_2 \in Y$ is completely analogous.

- Finally, assume $y_1, y_2 = \top$. Then, since $\top > f_Y$, it holds true that

$$(x_1 * x_2, \top) \le (f_X, f_Y) \text{ holds if and only if } x_1 * x_2 <_X f_X.$$

- If $x_1 \in X_{gr}$ then $x_1 * x_2 <_X f_X$ is equivalent to $x_2 <_X {x_1}'^{*}$ since then ${x_1}'^{*}$ is the inverse of x_1. Since X_{gr} is discretely embedded into X and since ${x_1}'^{*} \in X_{gr}$, therefore $x_2 <_X {x_1}'^{*}$ is equivalent to $x_2 \le_X ({x_1}'^{*})_{\downarrow}$, which is equivalent to $(x_2, \top) \le (({x_1}'^{*})_{\downarrow}, \top) = (x_1, \top)'$. The case $x_2 \in X_{gr}$ is completely analogous.
- If $x_1, x_2 \notin X_{gr}$ then $x_1 * x_2$ cannot be equal to f_X, therefore $x_1 * x_2 <_X f_X$ can equivalently be written as $x_1 * x_2 \le_X f_X$, which is equivalent to $x_2 \le_X {x_1}'^{*}$ by residuation. Finally, it is equivalent to $(x_2, \top) \le ({x_1}'^{*}, \top) = (x_1, \top)'$, hence the proof of (4.4) is concluded.

Summing up, $\circledast$ is residuated and $(x_1, y_1) \to_{\circledast} (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ holds.

If $\mathbf{Y}$ is odd (resp. positive or negative rank) then ${f_Y}'^{\star} = f_Y$ (resp. ${f_Y}'^{\star} > f_Y$, ${f_Y}'^{\star} < f_Y$) and hence the dualizing element (f_X, f_Y) is fixed under $'$ (resp. $(f_X, f_Y)' > (f_X, f_Y)$ or $(f_X, f_Y)' < (f_X, f_Y)$); that is $\mathbf{X} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y}$ is odd (resp. positive or negative rank), too.

C. *Proof for* $\mathbf{X_{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{Y} \le \mathbf{X_Z} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y} \le \mathbf{X} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y}$.
Since type I and II partial lexicographic products are particular cases of type III and IV partial lexicographic products, it follows from the previous two claims that $\mathbf{X_V} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ and $\mathbf{X} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y}$ are involutive FL$_e$-algebras, too. We shall verify that $\mathbf{X_{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{Y} \le \mathbf{X_Z} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y} \le \mathbf{X} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y}$. Evidently $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y \subseteq X_Z \overset{\twoheadrightarrow}{\times} Y$ and $X_V \overset{\rightharpoonup\!\!\!\rightarrow}{\times} Y \subseteq X \overset{\rightharpoonup\!\!\!\rightarrow}{\times} Y$ hold. A moment's reflection shows that $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$ (resp. $X_V \overset{\rightharpoonup\!\!\!\rightarrow}{\times} Y$) is closed under the residual complement operation of $\mathbf{X_Z} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ (resp. $\mathbf{X} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y}$), and that $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$ (resp. $X_V \overset{\rightharpoonup\!\!\!\rightarrow}{\times} Y$) is closed under the monoidal operation of $\mathbf{X_Z} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ (resp. $\mathbf{X} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y}$). From the respective definition of the residual complement operations it also easily follows that $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$ (resp. $X_V \overset{\rightharpoonup\!\!\!\rightarrow}{\times} Y$) is closed under the residual operation of $\mathbf{X_Z} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ (resp. $\mathbf{X} \overset{\rightharpoonup\!\!\!\rightarrow}{\times} \mathbf{Y}$). ∎

In the rest of the paper we will not use the partial lexicographic product construction in its full generality. For describing the structure of odd involutive FL$_e$-chains with only finitely-many idempotent elements, first of all, we will apply it to the linearly ordered setting only. Second, it will be sufficient to use FL_e-chains with finitely many positive idempotent elements and linearly ordered abelian groups to play the role of $\mathbf{X}$ and $\mathbf{Y}$, the algebras in the first and in the second coordinate, respectively.

Keeping it in mind, in order to ease the understanding of the rest of the paper, we make the following

Remark 4.5. Concerning Definition 4.2:

- Those subsets of $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$ and of $X_V \overset{\rightharpoonup}{\times} Y$ which are of the form $\{v\} \times Y$ for $v \in V$, will be recovered in an arbitrary odd involutive FL_e-chain $\mathbf{X}$ in Section 6, and will be called *convex components of the group part of* $\mathbf{X}$.
- The inverse operation of the enlargement of each element of V by Y will be done in Section 6 by a homomorphism β which collapses each convex component of the group part of $\mathbf{X}$ into a singleton.
- In case the residual complement of the smallest strictly positive idempotent element of $\mathbf{X}$ is also idempotent then the homomorphism $\gamma \circ \beta$ will collapse each convex component of the group part of $\mathbf{X}$ *together with its infimum and supremum* (in other words, together with its extremals, see below) into a singleton in Section 9.
- Those elements of $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$ which are of the form $(v, \top)$ or $(v, \bot)$ for $v \in V$, and those elements of $X_V \overset{\rightharpoonup}{\times} Y$ which are of the form $(v, \top)$ for $v \in V$ will be recovered in an arbitrary odd involutive FL_e-chain in Section 7, and will be called *extremals*.
- Those elements of $X_{Z_V} \overset{\twoheadrightarrow}{\times} Y$ which are of the form $(z, \top)$ or $(z, \bot)$ for $z \in Z \backslash V$, and those elements of $X_V \overset{\rightharpoonup}{\times} Y$ which are of the form $(z, \top)$ for $z \in X_{gr} \backslash V$ will be recovered in an arbitrary odd involutive FL_e-chain in Section 8, and will be called *pseudo extremals*. We will call these elements *pseudo* extremals, since there is no convex component in the algebra $\mathbf{X_{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\rightharpoonup}{\times} \mathbf{Y}$ for which they would serve as infimum or supremum (i.e., extremal). Their convex component could be there (if it were we would talk about a type I or a type II extension) but it is 'missing' (and thus we have a type III or a type IV extension, only a subalgebra of the related type I or a type II extension, see the last statement in Theorem 4.4).

5. Subuniverses

Sketch of the main theorem (Theorem 11.1) and its proof. For any odd involutive FL_e-chain $\mathbf{X}$, such that there exists its smallest strictly positive idempotent element, we introduce two different decomposition methods in Lemmas 9.4 and 10.2. The algebra will be decomposed by either the first or the second variant depending on whether the residual complement of its smallest strictly positive idempotent element is idempotent or not.

If the residual complement of the smallest strictly positive idempotent element is idempotent then

1. we will define two consecutive homomorphisms β and γ, and will isolate a homomorphic image $\gamma(\beta(\mathbf{X}))$ of $\mathbf{X}$, which has one less positive idempotent elements then $\mathbf{X}$. In this sense $\gamma(\beta(\mathbf{X}))$ will be smaller than the original algebra $\mathbf{X}$.
2. We will also isolate two subgroups of $\gamma(\beta(\mathbf{X}))$, called $\gamma(\beta(\mathbf{X}^{E}_{\tau\geq u}))$ and $\gamma(\beta(\mathbf{X}^{E_c}_{\tau\geq u}))$.
3. Finally we will recover $\mathbf{X}$, up to isomorphism, as the type III partial lexicographic product of $\gamma(\beta(\mathbf{X}))$, $\gamma(\beta(\mathbf{X}^{E}_{\tau\geq u}))$, $\gamma(\beta(\mathbf{X}^{E_c}_{\tau\geq u}))$ and $\mathbf{ker}_\beta$ in Section 9.

If the residual complement of the smallest strictly positive idempotent element is not idempotent then

1. we will isolate a residuated subsemigroup $\mathbf{X}_{\tau\geq u}$ of $\mathbf{X}$, which will be an odd involutive FL_e-algebra, albeit not a subalgebra of $\mathbf{X}$, since its unit element and residual complement operation will differ from those of $\mathbf{X}$. $\mathbf{X}_{\tau\geq u}$ has one less positive idempotent element then $\mathbf{X}$, so here too, $\mathbf{X}_{\tau\geq u}$ will be smaller than the original algebra $\mathbf{X}$.
2. Then we prove that the group-part of $\mathbf{X}_{\tau\geq u}$ is discretely embedded into $\mathbf{X}_{\tau\geq u}$, and we isolate a subgroup $\mathbf{X}^{T_c}_{\tau\geq u}$ of it.
3. Finally we will recover $\mathbf{X}$, up to isomorphism, as the type IV partial lexicographic product of $\mathbf{X}_{\tau\geq u}$, $\mathbf{X}^{T_c}_{\tau\geq u}$ and $\mathbf{ker}_\beta$ in Section 10.

Therefore, if the original odd involutive FL_e-chain $\mathbf{X}$ has finitely many positive idempotents then (since then there exists its *smallest* strictly positive idempotent) we can reconstruct $\mathbf{X}$ from a smaller odd involutive FL_e-chain via enlarging it by a linearly ordered abelian group (either type III or IV in Sections 9 and 10, respectively). By applying this decomposition iteratively to the obtained smaller algebra we end up with an odd involutive FL_e-chain with a single idempotent element, which is a linearly ordered abelian group by Theorem 2.4. Summing up, any odd involutive FL_e-chain which has only finitely many positive idempotent elements will be described by the consecutive application of the partial lexicographic product construction using only linearly ordered abelian groups as building blocks; as many of them as the number of the positive idempotent elements of $\mathbf{X}$.

To this end first we need the classification of elements of $\mathbf{X}$ as described in Definition 5.1.

It is well-known that for any involutive FL_e-chain, if c and c' are idempotent and the interval between them contains the two constants then there is a subalgebra over that interval. However, in order to describe the structure of odd involutive FL_e-chains these subalgebras will not be convenient to consider, they are too small. Rather, it will be wiser to put all elements of X sharing the *same* τ*-value* into one class:

Definition 5.1. For an odd involutive FL_e-chain $(X, \leq, \circledast, \rightarrow_\circledast, t, f)$, for $u \geq t$ and $\square \in \{<, =, \geq\}$ denote

$$X_{\tau \square u} = \{x \in X : \tau(x) \square u\}.$$

The definition above reveals the true nature of τ: note that τ acts as a kind of 'local unit' function: $\tau(z)$ is the unit element for the subset $X_{\tau \geq \tau(z)}$ of X.

Proposition 5.2. *Let* $\mathbf{X} = (X, \leq, \circledast, \rightarrow_\circledast, t, f)$ *be an odd involutive* FL_e*-chain. Let* $u > t$ *be idempotent.*

1. $X_{\tau<u}$ *and* $X_{\tau=t}$ *are nonempty subuniverses.*
2. $X_{\tau=t} = X_{gr}$ *and hence* $X_{\tau=t}$ *is a nonempty subuniverse of a linearly ordered abelian group. For* $x \in X_{\tau=t}$*, the map* $y \mapsto y \circledast x$ $(y \in X)$ *is strictly increasing.*

The respective subalgebras of $\mathbf{X}$ will be denoted by $\mathbf{X}_{\tau<u}$ and $\mathbf{X}_{\tau=t} = \mathbf{X}_{\mathbf{gr}}$. In the proof of Proposition 5.2 and also later a key role will be played by Lemma 5.4. To prove it, it will be useful to observe first

Proposition 5.3. (Diagonally strict increase) *For any odd involutive* FL_e*-chain* $(X, \leq, \circledast, \rightarrow_\circledast, t, f)$*, if* $x_1 > x$ *and* $y_1 > y$ *then*

$$x_1 \circledast y_1 > x \circledast y.$$

Proof. By (2.2), $(x' \circledast y')' \geq x \circledast y$ holds, hence it suffices to prove $x_1 \circledast y_1 > (x' \circledast y')'$. Assume the opposite, which is $x_1 \circledast y_1 \leq (x' \circledast y') \rightarrow_\circledast f$ since $(X, \leq)$ is a chain. By adjointness from this we obtain $(x' \circledast x_1) \circledast (y' \circledast y_1) =$

$(x_1 \circledast y_1) \circledast (x' \circledast y') \leq f = t$. On the other hand, from $x_1 > x = (x')' = x' \to_{\circledast} f$ it follows by residuation that $x' \circledast x_1 \not\leq f$, which is equivalent to $x' \circledast x_1 > f = t$ since $(X, \leq)$ is a chain. Analogously we obtain $y' \circledast y_1 > t$. Therefore $(x' \circledast x_1) \circledast (y' \circledast y_1) \geq (x' \circledast x_1) \circledast t = x' \circledast x_1 > t$ follows, which is a contradiction. ■

Claims (1) and (3) in Proposition 2.3 show how τ behaves in relation to $'$ and $\circledast$, respectively. If the algebra is odd we can state more.

Lemma 5.4. (τ-lemma) *Let $(X, \leq, \circledast, \to_{\circledast}, t, f)$ be an odd involutive FL_e-chain and A be an expression which contains only the operations $\circledast$, $\to_{\circledast}$ and $'$. For any evaluation e of the variables and constants of A into X, $\tau(e(A))$ equals the maximum of the τ-values of the e-values of the variables and constants of A.*

Proof. First we claim $\tau(x \circledast y) = \max(\tau(x), \tau(y))$ for $x, y \in X$. By claim (3) in Proposition 2.3, $\tau(x \circledast y) \geq \max(\tau(x), \tau(y))$ holds. Assume $\tau(x \circledast y) > \max(\tau(x), \tau(y))$ and let $z \in]\max(\tau(x), \tau(y)), \tau(x \circledast y)] (\neq \emptyset)$ be arbitrary. By (2.4) it holds true that $t \leq \tau(x) < z$, hence the isotonicity of $\circledast$ it holds true that $x = t \circledast x \leq z \circledast x$ and $y = t \circledast y \leq z \circledast y$. Since τ assigns to x the greatest element of the stabilizer set of x, and $z > \max(\tau(x), \tau(y))$, it follows that z does not stabilize x or y. Consequently, $x < z \circledast x$ and $y < z \circledast y$ should hold. Since $t < z$ and $z \leq \tau(x \circledast y)$ yields $z \in Stab_{x \circledast y}$, it follows that $(z \circledast x) \circledast (z \circledast y) = ((x \circledast y) \circledast z) \circledast z = (x \circledast y) \circledast z = x \circledast y$, a contradiction to Proposition 5.3. This settles the claim.

By (2.1), any expression which contains only the connectives $\circledast$, $\to_{\circledast}$ and $'$ can be represented by an equivalent expression using the same variables and constants but containing only $\circledast$ and $'$. An easy induction on the recursive structure of this equivalent expression using the claim above and claim (1) in Proposition 2.3 concludes the proof. ■

Proof of Proposition 5.2. Let $Z \in \{X_{\tau<u}, X_{\tau=t}\}$. Then $t \in Z$, and Lemma 5.4 ensures that Z is closed under $\circledast$ and $\to_{\circledast}$; thus claim (1) is verified. For $x \in X_{\tau=t}$, $x \circledast x' = (x \to_{\circledast} x)' = \tau(x)' = t'' = t$ holds by (2.1) and (2.4), and thus $x \in X_{gr}$, too. For $x \in X_{gr}$, if $x \circledast z = x \circledast t$ then $x^{-1} \circledast x \circledast z = x^{-1} \circledast x \circledast t$, that is $z = t$. Hence $\tau(x) = t$, and thus $x \in X_{\tau=t}$ too. These prove $X_{\tau=t} = X_{gr}$. The rest of claim 2 follows from the cancellation property in groups. ■

6. Collapsing the Convex Components of the Group Part of X into Singletons—The Homomorphism β

Roughly, in algebra under a natural homomorphisms each element is mapped into its class, and the classes are often of the same size. In Definition 6.1 it is not the case: some elements will be mapped into their class (into their respective convex component with respect to $\tau < u$, see below), while on other elements the map will be the identity map. In order to keep notation as simple as possible, these elements will be mapped into sets, too, namely to a singleton containing the element itself.

DEFINITION 6.1. Let $(X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be an odd involutive FL_e-chain with residual complement $'$, $u > t$ idempotent. For $x, y \in X_{\tau<u}$, define $x \sim y$ if $z \in X_{\tau<u}$ holds for any $z \in X$ with $x < z < y$. It is an equivalence relation on $X_{\tau<u}$ since the order is linear. Denote the component of x by $[x]_{\tau<u}$ and call it the convex component of x with respect to $\tau < u$. If u is clear from the context we shall simply write $[x]$.

For $z \in X$ let

$$\beta(z) = \begin{cases} [z]_{\tau<u} & \text{if } z \in X_{\tau<u} \\ \{z\} & \text{if } z \in X_{\tau\geq u} \end{cases}.$$

For $Z \subseteq X$ let $\beta(Z) = \{\beta(z) : z \in Z\}$. For $x, y \in \beta(X)$ let $p \in x, q \in y$ (equivalently, let $p, q \in X$, $x = \beta(p)$ and $y = \beta(q)$), and over $\beta(X)$ define

$$x \leq_\beta y \text{ iff } p \leq q, \tag{6.1}$$

$$x \circledast_\beta y = \beta(p \circledast q), \tag{6.2}$$

$$x \rightarrow_\beta y = \beta(p \rightarrow_{\circledast} q), \tag{6.3}$$

$$x^{'^\beta} = \beta(p'). \tag{6.4}$$

Finally, let $\beta(\mathbf{X}) = (\beta(X), \leq_\beta, \circledast_\beta, \rightarrow_\beta, \beta(t), \beta(f))$.

Although the constructions will be different for the two cases (depending on the idempotency of the residual complement of the smallest strictly positive idempotent element in $\mathbf{X}$), in both cases the proof relies on Lemma 6.2. The map β which makes each convex component of $\mathbf{X_{gr}}$ collapse into a singleton (its convex component with respect to $\tau < u$), and leaves the rest of the algebra unchanged, will be shown to be a homomorphism.

LEMMA 6.2. *Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be an odd involutive FL_e-chain with residual complement $'$, and $u > t$ be idempotent.*

1. *β is a homomorphism from $\mathbf{X}$ to $\beta(\mathbf{X})$, hence $\beta(\mathbf{X})$ is an odd involutive FL_e-chain with involution $'^\beta$,*

2. $\beta(\mathbf{X_{gr}}) := (\beta(X_{gr}), \leq_\beta, \circledast_\beta, \to_\beta, \beta(t), \beta(f))$ *is a subgroup of* $\mathbf{X}$.
3. $[t]_{\tau<u} =]u', u[$ *and it is a nonempty subuniverse of* $\mathbf{X}$*; denote the related subalgebra by* $\mathbf{ker}_\beta$.
4. *If u is the smallest strictly positive idempotent element then* $\mathbf{X}_{\tau<u} = \mathbf{X_{gr}}$, $\mathbf{ker}_\beta$ *is a subgroup of* $\mathbf{X_{gr}}$*, and (qua linearly ordered abelian groups)* $\mathbf{X_{gr}} \cong \beta(\mathbf{X_{gr}}) \overrightarrow{\times} \mathbf{ker}_\beta$.

Proof. (1) Proving that β is a homomorphism amounts to proving that the definitions in (6.1)–(6.4) are independent of the chosen representatives p and q. The definition of $\leq_\beta$ is independent of the choice of p and q, since the components are convex. The ordering $\leq_\beta$ is linear since so is $\leq$. Therefore, (6.1) is well-defined. To prove that (6.2) is well-defined, we prove the following two statements: First we prove that for $p_1, p_2, q \in X_{\tau<u}$, if $p_2 \in [p_1]$ then $p_2 \circledast q \in [p_1 \circledast q]$. Indeed, since X is a chain, we may assume $p_1 < p_2$. If $p_1 \circledast q = p_2 \circledast q$ then we are done, therefore, by isotonicity of $\circledast$ we may assume $p_1 \circledast q < p_2 \circledast q$. Contrary to the claim, suppose that there exists $a \in]p_1 \circledast q, p_2 \circledast q[$ such that $\tau(a) \geq u$. By residuation, $p_1 \circledast q < a$ implies $p_1 \leq q \to_\circledast a$, and since X is a chain, $a < p_2 \circledast q$ is equivalent to $q \to_\circledast a < p_2$. Thus $q \to_\circledast a \in [p_1, p_2[$ holds, hence $\tau(q \to_\circledast a) < u$ follows since $p_2 \in [p_1]$ and $[p_1]$ is convex. However, Lemma 5.4 ensure $\tau(q \to_\circledast a) \geq \tau(a) \geq u$, a contradiction. Second, we prove that for $p \in X_{\tau \geq u}$ and $q_1, q_2 \in X_{\tau<u}$, if $q_2 \in [q_1]$ then $p \circledast q_1 = p \circledast q_2$. Again, we may assume $q_1 < q_2$ since X is a chain. Contrary to the claim suppose $p \circledast q_1 < p \circledast q_2$. By residuation it is equivalent to $p \to_\circledast (p \circledast q_1) < q_2$ since X is a chain. By residuation $q_1 \leq p \to_\circledast (p \circledast q_1)$ holds, too. Thus $p \to_\circledast (p \circledast q_1) \in [q_1, q_2[$ holds, hence $q_2 \in [q_1]$ and the convexity of $[q_1]$ implies $\tau(p \to_\circledast (p \circledast q_1)) \leq u$. However, Lemma 5.4 ensures $\tau(p \to_\circledast (p \circledast q_1)) \geq \tau(p) > u$, a contradiction. Therefore, (6.2) is well-defined. Next, claim (1) in Proposition 2.3 implies that the image of a convex component under ${}'^{\beta}$ is a convex component, hence the definition in (6.4) is independent of the chosen representative. Finally, using (2.1) and that (6.2) and (6.4) are well-defined, for $p_1, p_2, q_1, q_2 \in X$, if $\beta(p_2) = \beta(p_1)$ and $\beta(q_2) = \beta(q_1)$ then it holds true that

$$\beta(p_2 \to_\circledast q_2) = \beta((p_2 \circledast {q_2}')') = \left(\beta(p_2) \circledast_\beta \beta(q_2)^{'^{\beta}}\right)^{'^{\beta}}$$
$$= \left(\beta(p_1) \circledast_\beta \beta(q_1)^{'^{\beta}}\right)^{'^{\beta}} = \beta((p_1 \circledast {q_1}')') = \beta(p_1 \to_\circledast q_1). \quad (6.5)$$

Therefore, (6.3) is well-defined, too. Summing up, β is a homomorphism from $\mathbf{X}$ to $\beta(\mathbf{X})$. Therefore, it readily follows that $\beta(\mathbf{X})$ which was defined as the image of $\mathbf{X}$ under β is an odd involutive FL_e-chain with involution ${}^{\prime\beta}$.
(2) In the light of the previous claim, to verify that $\beta(\mathbf{X_{gr}})$ is a subalgebra of $\beta(\mathbf{X})$ it suffices to note that X_{gr} is a nonempty subuniverse of X by Proposition 5.2. By Theorem 2.4, showing cancellativity of $\beta(\mathbf{X_{gr}})$ is equivalent to showing that each of its elements has inverse; but it is straightforward since each element of $\mathbf{X_{gr}}$ has inverse and a homomorphism preserves this property.
(3) Clearly, $[t]$ is nonempty since $t \in [t]$. By (6.2), $[t] \circledast_\beta [t] = [t \circledast t] = [t]$ holds, hence $[t]_{\tau<u}$ is closed under $\circledast$. By (6.3), $[t] \rightarrow_\beta [t] = [t \rightarrow_\circledast t] = [t]$ holds, hence $[t]_{\tau<u}$ is closed under $\rightarrow_\circledast$, too. For any $z \in]u', u[$, $\tau(z) = \tau(\max(z, z')) \leq \max(z, z') < u$ holds by claims (1) and (6) in Proposition 2.3. On the other hand, since u is idempotent, by claims (1) and (4) in Proposition 2.3 we obtain $\tau(u') = \tau(u) = u$, showing $u', u \notin [t]$, and in turn, $[t] =]u', u[$.
(4) For $x \in X_{\tau<u}$, $\tau(x) > t$ would yield $\tau(x)$ be an idempotent by claim (4) in Proposition 2.3, which is strictly between t and u, contrary to assumption. Hence $X_{\tau<u} = X_{\tau=t}$ holds. Claim (2) in Proposition 5.2 concludes the proof of $X_{\tau<u} = X_{gr}$. Therefore, $\mathbf{X}_{\tau<u}$ is the group part of $\mathbf{X}$. By claim (3), $[t]_{\tau<u}$ is a subuniverse of $\mathbf{X}$, hence it follows from $\mathbf{X}_{\tau<u} = \mathbf{X_{gr}}$ that $[t]_{[\tau<u]}$ is a universe of a linearly ordered group, and thus $\mathbf{ker}_\beta$ is a convex subgroup of $\mathbf{X_{gr}}$. By Hahn's theorem, $\mathbf{X_{gr}}$ and $\mathbf{ker}_\beta$ can be represented as subgroups of Hahn products of additive groups of the reals $\mathbb{R}$. In particular, let

$$\mathbf{X_{gr}} \leq \overset{\rightarrow H}{\times}_{i \in I} \mathbb{R}.$$

Since $\mathbf{ker}_\beta$ is a convex subgroup of $\mathbf{X_{gr}}$, and since convex subgroups are unions of whole archimedean classes, it follows that $\mathbf{ker}_\beta$ is union of whole archimedean classes of $\mathbf{X_{gr}}$. In Hahn products elements of the index set correspond to archimedean classes of the represented group, hence there exists a subset J of I such that

$$\mathbf{ker}_\beta = \mathbf{X_{gr}} \cap \overset{\rightarrow H}{\times}_{i \in I} \hat{\mathbf{X}}_i,$$

where $\hat{\mathbf{X}}_i = \mathbb{R}$ if $i \in J$ and $\hat{\mathbf{X}}_i$ is trivial if $i \in I \backslash J$. Since $\mathbf{ker}_\beta$ is convex, J must be a downset of I. Therefore, $\mathbf{ker}_\beta$ (qua a linearly ordered abelian group) is isomorphic to the restriction of $\mathbf{X_{gr}}$ to its "second half", that is, to its J-coordinates. By the first isomorphism theorem, (qua linearly ordered abelian groups) $\beta(\mathbf{X_{gr}}) \cong \mathbf{X_{gr}}/\mathbf{ker}_\beta$. By the second isomorphism theorem,

$$\mathbf{X_{gr}}/\mathbf{ker}_\beta \cong (\mathbf{X_{gr}} \cdot \overset{\rightarrow H}{\times}_{i \in I} \hat{\mathbf{X}}_i) / \overset{\rightarrow H}{\times}_{i \in I} \hat{\mathbf{X}}_i,$$

thus $\mathbf{X_{gr}}/\mathbf{ker}_\beta$ (qua a linearly ordered abelian group) is isomorphic to the restriction of $\mathbf{X_{gr}}$ to its "first half", that is, to its $I\backslash J$-coordinates:

$$\mathbf{X_{gr}}/\mathbf{ker}_\beta \cong \mathbf{X_{gr}} \cap \overset{\rightarrow H}{\times}_{i\in I} \check{\mathbf{X}}_i,$$

where $\check{\mathbf{X}}_i = \mathbb{R}$ if $i \in I\backslash J$ and $\hat{\mathbf{X}}_i$ is trivial if $i \in J$. Hence (qua linearly ordered abelian groups) $\beta(\mathbf{X_{gr}}) \overset{\rightarrow}{\times} \mathbf{ker}_\beta \cong \mathbf{X_{gr}}$ follows, as stated. ■

7. Extremals

In this section we define and investigate the extremal elements of the convex components of the group part of $\mathbf{X}$.

Definition 7.1. Let $(X, \leq, \circledast, \rightarrow_\circledast, t, f)$ be an odd involutive FL_e-chain and $u > t$ be idempotent. Provisionally, for $v \in X_{\tau<u}$ let

$$\top_{[v]} = \begin{cases} \bigvee_{z\in[v]} z & \text{in case } u \neq t_\uparrow \\ v_\uparrow & \text{in case } u = t_\uparrow \end{cases} \text{ and } \bot_{[v]} = \begin{cases} \bigwedge_{z\in[v]} z & \text{in case } u \neq t_\uparrow \\ v_\downarrow & \text{in case } u = t_\uparrow \end{cases}.$$

Call $\top_{[v]}$ and $\bot_{[v]}$ the top- and the bottom-extremals of the (convex) component of v. Denote

$$X^{T_c}_{\tau\geq u} = \{\top_{[v]} \mid v \in X_{\tau<u}\} \quad \text{(top-extremals of components)},$$
$$X^{B_c}_{\tau\geq u} = \{\bot_{[v]} \mid v \in X_{\tau<u}\} \quad \text{(bottom-extremals of components)},$$
$$X^{E_c}_{\tau\geq u} = X^{T_c}_{\tau\geq u} \cup X^{B_c}_{\tau\geq u} \quad \text{(extremals of components)}.$$

The $\tau \geq u$ subscript in the notations refers to the fact that $X^{T_c}_{\tau\geq u}$, $X^{B_c}_{\tau\geq u}$, and $X^{E_c}_{\tau\geq u}$ are subsets of $X_{\tau\geq u}$. We shall shortly prove it in

Proposition 7.2. *Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_\circledast, t, f)$ be an odd involutive FL_e-chain such that there exists u, the smallest strictly positive idempotent element, and let $v \in X_{\tau<u}$.*

1. *If $u \neq t_\uparrow$ then $[v]$ is infinite, and if $u = t_\uparrow$ then $[v] = \{v\}$. If $u \neq t_\uparrow$ then $\top_{[v]}$ and $\bot_{[v]}$ exist.*
   It holds true that
   $$\top_{[v]} = v \circledast u, \tag{7.1}$$
   $$\bot_{[v]} = v \circledast u', \tag{7.2}$$
   $$\bot_{[v]} = (\top_{[v']})'. \tag{7.3}$$
   If $u = t_\uparrow$ then and $v_\downarrow < v < v_\uparrow$.

2. $\perp_{[v]}, \top_{[v]} \in X_{\tau \geq u}$, $\perp_{[v]}, \top_{[v]} \notin [v]$.
3. *The following product table holds true for* $\circledast$*:*
4. *If* u' *is idempotent then* $X_{\tau \geq u}^{T_c} \cap X_{\tau \geq u}^{B_c} = \emptyset$.

Proof. (1) By claim (3) in Lemma 6.2, $[t] =]u', u[$ is a universe of a subgroup of $\mathbf{X}$. It is trivial if $u = t_\uparrow$, hence then $[t] = \{t\}$. Non-trivial linearly ordered abelian groups are known to be infinite and having no greatest or least element. $[t]$ is non-trivial if $u \neq t_\uparrow$, so $]u', u[$ has no greatest or least element; this verifies $|[t]| = \infty$ and the existence of $\top_{[t]}$ and $\perp_{[t]}$ along with $\top_{[t]} = u$, $\perp_{[t]} = u'$.

Note that $[v]$ is equal to the coset $v \circledast [t]$ since $\mathbf{X}_{\tau < u}$ is a group. Indeed, by (6.2) $v \circledast [t] \subseteq [v] \circledast_\beta [t] \subseteq [v \circledast t] = [v]$ holds, whereas for any $w \in [v]$, by (6.2) $v^{-1} \circledast w \in [v^{-1} \circledast w] = [v^{-1}] \circledast_\beta [w] = [v^{-1}] \circledast_\beta [v] = [v^{-1} \circledast v] = [t]$ holds, and hence $w = v \circledast (v^{-1} \circledast w) \in v \circledast [t]$.

Since for $z \in [t]$, the mapping $z \mapsto v \circledast z$ is a bijection between the two cosets $[t]$ and $[v]$, it follows that $[v]$ has the same cardinality as $[t]$.

Assume $u \neq t_\uparrow$. Using the infinite distributivity of $\vee$ over $\circledast$, $v \circledast u = v \circledast \bigvee_{w \in [t]} w = v \circledast \bigvee_{w \in [t]} v^{-1} \circledast v \circledast w = v \circledast v^{-1} \circledast \bigvee_{w \in [t]} v \circledast w = \bigvee_{w \in [t]} v \circledast w$ follows. Since the mapping above is a bijection, it is equal to $\bigvee_{v \circledast w \in [v]} v \circledast w = \bigvee_{z \in [v]} z$, thus confirming the existence of $\top_{[v]}$ along with (7.1) in the case of $u \neq t_\uparrow$.

Next we prove, the existence of $\perp_{[v]}$ if $u \neq t_\uparrow$. Because of (6.4), $(\top_{[v']})' = (\bigvee_{z \in [v']} z)'$ is equal to $(\bigvee_{z' \in [v]} z)'$. Therefore $\bigvee_{z' \in [v]} z$ exists and since $'$ is an order reversing involution of X, it follows that $(\bigvee_{z' \in [v]} z)'$ is equal to $\bigwedge_{z' \in [v]} z' = \perp_{[v]}$.

The previous argument also shows (7.3) if $u \neq t_\uparrow$. (7.3) readily follows from (4.1) if $u = t_\uparrow$.

Next we prove (7.1) and (7.2) when $u = t_\uparrow$. By monotonicity of $\circledast$, $v \circledast u \geq v \circledast t = v$ holds, but there cannot hold equality, since $\tau(v) < u$, but $\tau(v \circledast u) \geq \tau(u) = u$ by Lemma 5.4. Hence, $v < v \circledast u$ follows. An analogous argument shows $v \circledast u' < v$. Now, if there would exists a, b such that $v \circledast u' < a < v < b < v \circledast u$ then by multiplying it by v' (the inverse of v) it would yield $u' < v' \circledast a < t < v' \circledast b < u$, contradicting $u = t_\uparrow$. Summing up, it holds true that $\perp_{[v]} = v_\downarrow = v \circledast u' < v$ and $\top_{[v]} = v_\uparrow = v \circledast u > v$.

It remains to prove (7.2) in the $u = t_\uparrow$ case. By (7.3), (7.1), and (2.2) $\perp_{[v]} = (\top_{[v']})' = (v' \circledast u)' \geq v \circledast u'$. By (6.2), $v \circledast u' = w \circledast u'$ for any $w > v$, $w \in [v]$ (such a w exists since $[v]$ has no greatest element). Finally by (2.3), $w \circledast u' \geq (v' \circledast u)' = (\top_{[v']})'$. It follows that there is equality everywhere, hence (7.2) follows.

Table 1. (Extremals): If $v, w \in X_{\tau<u}$, $a_{\downarrow} = a \in X_{\tau\geq u}\backslash X_{\tau\geq u}^{T_c}$, and $y \in X_{\tau\geq u}$ then the following holds true

$\circledast$	$\bot_{[w]}$	w	$\top_{[w]}$	y
$\bot_{[v]}$		$\bot_{[v\circledast w]}$	$\bot_{[v\circledast w]}$	
v	$\bot_{[v\circledast w]}$	$v \circledast w$	$\top_{[v\circledast w]}$	
$\top_{[v]}$	$\bot_{[v\circledast w]}$	$\top_{[v\circledast w]}$	$\top_{[v\circledast w]}$	$v \circledast y$
a	$a \circledast w$	$a \circledast w$	$a \circledast w$	

(2) By (7.1), Lemma 5.4, and claim (4) in Proposition 2.3, $\tau(\top_{[v]}) = \tau(v \circledast u) = \tau(u) = u$, and (7.3) yields $\tau(\bot_{[v]}) = u$, thus confirming $\bot_{[v]}, \top_{[v]} \in X_{\tau\geq u}$, and in turn, $\bot_{[v]}, \top_{[v]} \notin [v]$.
(3) Grey items in Table 1 are either straightforward or follow from the black items thereof by commutativity. Therefore, we prove only the black items.

- Using (7.1), $v \circledast \top_{[w]} = v \circledast w \circledast u = \top_{[v\circledast w]}$ confirms Table $1_{(2,3)}$.
- A similar proof using the idempotency of u ensures Table $1_{(3,3)}$.
- By (7.1), $\top_{[v]} \circledast y = (v \circledast u) \circledast y = v \circledast (u \circledast y) = v \circledast y$, where the latest equality is implied by $y \in X_{\tau\geq u}$; this proves Table $1_{(3,4)}$.
- By (7.2), $v \circledast \bot_{[w]} = v \circledast (w \circledast u') = (v \circledast w) \circledast u' = \bot_{[v\circledast w]}$ holds as stated in Table $1_{(2,1)}$.
- Next, by (7.1) and (7.2) $\bot_{[v]} \circledast \top_{[w]} = (v \circledast u) \circledast (w \circledast u') = (v \circledast w) \circledast (u \circledast u') = (v \circledast w) \circledast u' = \bot_{[v\circledast w]}$, where $u \circledast u' = u'$ follows from $u \circledast u' = (u \rightarrow_{\circledast} u)' = \tau(u)' = u'$, so we are done with the proof of Table $1_{(1,3)}$.
- There exists $Z_2 \subset X\backslash\{a'\}$ such that $a' = \bigwedge Z_2$, since $a'_{\uparrow} = a'$. We may safely assume $Z_2 \subset X_{\tau\geq u}\backslash\{a'\}$ since a' is not the greatest lower bound of any component of $X_{\tau<u}$. Therefore, by (2.3), Table $1_{(3,4)}$, and (2.2), respectively, for $z \in Z_2$ it holds true that $a \circledast \bot_{[w]} \geq (z \circledast \top_{[w']})' = (z \circledast w')' \geq z' \circledast w$. Hence, using that $\circledast$ is residuated, $a \circledast w = (\bigvee_{z\in Z_2} z') \circledast w = \bigvee_{z\in Z_2} (z' \circledast w) \leq a \circledast \bot_{[w]}$ follows. On the other hand, by the isotonicity of $\circledast$, $a \circledast \bot_{[w]} \leq a \circledast w$ holds, too. This proves Table $1_{(4,1)}$.

  (4) First we claim that if u' is idempotent then $X_{\tau\geq u}^{T_c}$ and $X_{\tau\geq u}^{B_c}$ are either disjoint or coincide. Indeed, if there exists $c, d \in X_{\tau<u}$ such that $\top_{[c]} = \bot_{[d]}$ then $c \circledast c' = t$ follows from claim (2) in Proposition 5.2, and by Table $1_{(2,3)}$ and Table $1_{(2,1)}$, for any $v \in X_{\tau<u}$, $\top_{[v]} = \top_{[c\circledast c'\circledast v]} = \top_{[c]} \circledast c' \circledast v = \bot_{[d]} \circledast c' \circledast v = \bot_{[d\circledast c'\circledast v]}$ and $\bot_{[v]} = \bot_{[d\circledast d'\circledast v]} = \bot_{[d]} \circledast d' \circledast v = \top_{[c]} \circledast d' \circledast v = \top_{[c\circledast d'\circledast v]}$ hold, thus the proof of the claim is concluded.
  Therefore, it suffices to prove that $\bot_{[t]} \notin X_{\tau\geq u}^{T_c}$. If there would exist

$v \in X_{\tau<u}$ such that $\bot_{[t]} = \top_{[v]}$ then by Table 1$_{(1,3)}$, $\top_{[v]} \circledast \bot_{[t]} = \bot_{[v]}$ would follow, a contradiction to the idempotency of $\bot_{[t]}$. ■

8. Gaps—Motto: "Not All Gaps Created Equal"

In order to treat the case when the order density of $\mathbf{X}$ is not assumed (c.f. Table 1$_{(4,1)}$, where a kind of density condition must have been assumed for a), first we classify gaps in $X_{\tau\geq u}$. It turns out that some gaps behave like extremal elements of "missing" components of $\mathbf{X}$, those will be called pseudo extremals.

Definition 8.1. Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_\circledast, t, f)$ be an odd involutive FL$_e$-chain with residual complement $'$, such that there exists u, the smallest strictly positive idempotent. Let

$$\begin{array}{lll}
X_{\tau\geq u}^{Gap} = \{x \in X_{\tau\geq u} \mid x_\downarrow < x\} & & \text{(upper elements of gaps in } X_{\tau\geq u}) \\
X_{\tau\geq u}^{G_2} = \{x \in X_{\tau\geq u}^{Gap} \mid x \circledast u' = x\} & & \\
X_{\tau\geq u}^{T_{ps}} = \{x \in X_{\tau\geq u}^{Gap} \mid x \circledast u' = x_\downarrow\} & & \text{(pseudo top-extremals)} \\
X_{\tau\geq u}^{B_{ps}} = \{x_\downarrow \in X \mid x \in X_{\tau\geq u}^{T_{ps}}\} & & \text{(pseudo bottom-extremals)} \\
X_{\tau\geq u}^{E_{ps}} = X_{\tau\geq u}^{T_{ps}} \cup X_{\tau\geq u}^{B_{ps}} & & \text{(pseudo extremals)} \\
X_{\tau\geq u}^{T} = X_{\tau\geq u}^{T_c} \cup X_{\tau\geq u}^{T_{ps}} & & \text{(top-extremals)} \\
X_{\tau\geq u}^{B} = X_{\tau\geq u}^{B_c} \cup X_{\tau\geq u}^{B_{ps}} & & \text{(bottom-extremals)} \\
X_{\tau\geq u}^{E} = X_{\tau\geq u}^{T} \cup X_{\tau\geq u}^{B} & & \text{(extremals)}
\end{array}$$

For $x \in X_{\tau\geq u}$ denote $x_\Downarrow$ the predecessor of x inside $X_{\tau\geq u}$, that is let

$$x_\Downarrow = \begin{cases} z & \text{if there exists } X_{\tau\geq u} \ni z < x \text{ such that there is} \\ & \text{no element in } X_{\tau\geq u} \text{ strictly between } z \text{ and } x, \\ x & \text{if for any } X_{\tau\geq u} \ni z < x \text{ there exists } v \in X_{\tau\geq u} \\ & \text{such that } z < v < x \text{ holds.} \end{cases}$$

Clearly,

$$x_\Downarrow = \begin{cases} x_\downarrow & \text{if } x \in X_{\tau\geq u} \setminus X_{\tau\geq u}^{T_c} \\ \bot_{[v]} & \text{if } x = \top_{[v]} \in X_{\tau\geq u}^{T_c} \end{cases}.$$

Define $x_\Uparrow$ dually.

PROPOSITION 8.2. *Let* $\mathbf{X} = (X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ *be an odd involutive FL$_e$-chain with residual complement* $'$*, such that there exists* u*, the smallest strictly positive idempotent.*

1. $X^{B_{ps}}_{\tau\geq u} \subseteq X_{\tau\geq u}$, $X^{T_{ps}}_{\tau\geq u} \cap X^{T_c}_{\tau\geq u} = \emptyset$, $X^{B_{ps}}_{\tau\geq u} \cap X^{B_c}_{\tau\geq u} = \emptyset$, *and* $X^{G_2}_{\tau\geq u} \cap X^{T}_{\tau\geq u} = \emptyset$.
2. $X^{Gap}_{\tau\geq u} \setminus X^{T_c}_{\tau\geq u} = X^{T_{ps}}_{\tau\geq u} \cup X^{G_2}_{\tau\geq u}$.
3. *For* $v \in X_{\tau<u}$ *and* $x \in X_{\tau\geq u}$,

$$x \circledast \bot_{[v]} = \begin{cases} (x \circledast v)_{\Downarrow} (< x \circledast v \in X^{T_c}_{\tau\geq u}) & \text{if } x \in X^{T_c}_{\tau\geq u} \\ (x \circledast v)_{\downarrow} (< x \circledast v \in X^{T_{ps}}_{\tau\geq u}) & \text{if } x \in X^{T_{ps}}_{\tau\geq u} \\ x \circledast v & \text{if } x \in X_{\tau\geq u} \setminus X^{T}_{\tau\geq u} \end{cases}. \quad (8.1)$$

4. *If* $x \in X^{T_{ps}}_{\tau\geq u}$ *then* ${x_{\downarrow}}' \in X^{T_{ps}}_{\tau\geq u}$*, too.*
5. *If* $x, y \in X^{T_{ps}}_{\tau\geq u}$ *then* $(x \circledast y) \circledast u' \neq x \circledast y$.
6. *For* $x \in X^{T_{ps}}_{\tau\geq u}$, $\tau(x) = u$.
7. *The following product table holds for* $\circledast$*:*

PROOF. (1) By Lemma 5.4 and claims (1) and (3) in Proposition 2.3, for $x \in X^{T_{ps}}_{\tau\geq u}$, $\tau(x_{\downarrow}) = \tau(x \circledast u') \geq \tau(u') = u$ holds, ensuring $X^{B_{ps}}_{\tau\geq u} \subseteq X_{\tau\geq u}$. Second, if $X^{T_{ps}}_{\tau\geq u} \ni x = \top_{[v]} \in X^{T_c}_{\tau\geq u}$ then $x > x_{\downarrow} \in X_{\tau\geq u}$ holds by the previous item, hence $\top_{[v]} > (\top_{[v]})_{\downarrow}$. It follows that $(\top_{[v]})_{\downarrow}$ is in $X_{\tau<u}$ since $\top_{[v]}$ is a top-extremal of a convex component in $X_{\tau<u}$, a contradiction to $x_{\downarrow} \in X_{\tau\geq u}$. A completely analogous argument proves $X^{B_{ps}}_{\tau\geq u} \cap X^{B_c}_{\tau\geq u} = \emptyset$: If $x \in X^{B_{ps}}_{\tau\geq u}$ then $x < x_{\uparrow} \in X^{T_{ps}}_{\tau\geq u} \subseteq X_{\tau\geq u}$. But if $x \in X^{B_c}_{\tau\geq u}$ then $x < x_{\uparrow} \in X_{\tau<u}$, a contradiction. Finally, obviously, $X^{G_2}_{\tau\geq u} \cap X^{T_{ps}}_{\tau\geq u} = \emptyset$, hence it suffices to prove $X^{G_2}_{\tau\geq u} \cap X^{T_c}_{\tau\geq u} = \emptyset$. If $x \in X^{G_2}_{\tau\geq u}$ then $x \circledast u' = x$, whereas if $x \in X^{T_c}_{\tau\geq u}$ then by Table $1_{(3,1)}$, $x \circledast u' = \bot_{[v]}$ holds. But $\bot_{[v]} \neq x$.
(2) First we claim $x \circledast \bot_{[v]} \geq x_{\downarrow} \circledast v$ for $x \in X^{Gap}_{\tau\geq u} \setminus X^{T_c}_{\tau\geq u}$. Indeed, $x \circledast \bot_{[v]} \geq ({x_{\downarrow}}' \circledast \top_{[v']})'$ follows from (2.3) since $x > x_{\downarrow}$. Now $x \neq \top_{[v]}$, and it implies $x_{\downarrow} \in X_{\zeta\geq u}$, that is, ${x_{\downarrow}}' \in X_{\zeta\geq u}$ by Lemma 5.4, and hence $({x_{\downarrow}}' \circledast \top_{[v']})' = ({x_{\downarrow}}' \circledast v')'$ follows by Table $1_{(3,4)}$. By (2.2), $({x_{\downarrow}}' \circledast v')' \geq x_{\downarrow} \circledast v$, and we are done. Setting $v = t$ in the claim proves claim (2).
(3) For $x \in X_{\tau\geq u}$, either $x \in X^{T_c}_{\tau\geq u}$ holds, or $x \notin X^{T_c}_{\tau\geq u}$ and $x_{\downarrow} = x$ holds, or $x \notin X^{T_c}_{\tau\geq u}$ and $x_{\downarrow} < x$ holds. In the first case, by Table $1_{(1,3)}$ and Table $1_{(2,3)}$, we obtain $\top_{[w]} \circledast \bot_{[v]} = \bot_{[w \circledast v]} = (\top_{[w \circledast v]})_{\Downarrow} = (\top_{[w]} \circledast v)_{\Downarrow}$, as required. In the

second case, $x \circledast \bot_{[v]} = x \circledast v$ holds by Table 1$_{(4,1)}$, as required in the third row of (8.1).

In the third case, either $x \in X^{T_{ps}}_{\tau \geq u}$ or $x \in X^{G_2}_{\tau \geq u}$ holds by item (2). Hence, since $X^{G_2}_{\tau \geq u} \cap X^{T}_{\tau \geq u} = \emptyset$ by claim (1), to prove (8.1) it remains to prove

$$x \circledast \bot_{[v]} = \begin{cases} (x \circledast v)_{\downarrow} (< x \circledast v \in X^{T_{ps}}_{\tau \geq u}) & \text{if } x \in X^{T_{ps}}_{\tau \geq u} \\ x \circledast v & \text{if } x \in X^{G_2}_{\tau \geq u} \end{cases}.$$

By (7.2), $x \circledast \bot_{[v]} = x \circledast v \circledast u'$ holds. If $x \in X^{G_2}_{\tau \geq u}$ then $x \circledast v \circledast u'$ is equal to $x \circledast v$ and thus the proof of the second row is concluded. If $x \in X^{T_{ps}}_{\tau \geq u}$ then $x \circledast v \circledast u'$ cannot be equal to $x \circledast v$ since – referring to claim (2) in Proposition 5.2 – by cancelling v we would obtain $x_{\downarrow} = x \circledast u' = x$, a contradiction. Therefore, $(x \circledast v) \circledast u' < x \circledast v$ holds. Finally, the assumption that there exists $c \in X$ such that $(x \circledast v) \circledast u' < c < x \circledast v$ would lead, again cancelling by v, to $x_{\downarrow} = x \circledast u' < c \circledast v' < x$, a contradiction. Hence, $(x \circledast v) \circledast u' = (x \circledast v)_{\downarrow} < x \circledast v$. The proof of (8.1) is concluded.
(4) Let $x \in X^{T_{ps}}_{\tau \geq u}$. Then $x_{\downarrow} \in X^{B_{ps}}_{\tau \geq u} \subseteq X_{\tau \geq u}$ and $x_{\downarrow} \notin X^{B_c}_{\tau \geq u}$ follow by claim (1). Hence $x_{\downarrow}' \notin X^{T_c}_{\tau \geq u}$ follows from (7.3). By Lemma 5.4, $x_{\downarrow}' \in X_{\tau \geq u}$ follows from $x_{\downarrow} \in X_{\tau \geq u}$, thus $x_{\downarrow}' \in X_{\tau \geq u} \setminus X^{T_c}_{\tau \geq u}$ holds. Hence by (8.1), $u' \circledast x_{\downarrow}' \geq (x_{\downarrow}')_{\downarrow}$ holds true. On the other hand, $x \circledast u' = x_{\downarrow}$ implies $u' \rightarrow_{\circledast} x_{\downarrow} \geq x$ and hence by (2.1), $x_{\downarrow} < x$ and (4.1), $u' \circledast x_{\downarrow}' \leq x' = (x_{\downarrow}')_{\downarrow} < x_{\downarrow}'$ holds. Summing up, we obtained $x_{\downarrow}' \circledast u' = (x_{\downarrow}')_{\downarrow}$ and $(x_{\downarrow}')_{\downarrow} < x_{\downarrow}'$, as required.
(5) From the opposite of claim (5), $x \circledast y = ((x \circledast y) \circledast u') \circledast u' = (x \circledast u') \circledast (y \circledast u') = x_{\downarrow} \circledast y_{\downarrow}$ would follow, contradicting Proposition 5.3.
(6) $\tau(x) \geq u$ is straightforward. It holds true that $x \circledast (\tau(x) \circledast u') = (x \circledast \tau(x)) \circledast u' = x \circledast u' = x_{\downarrow} < x$. Hence, by monotonicity of $\circledast$, $\tau(x) \circledast u' < t$ follows, which implies $\tau(x) \leq (u')' = u$ by residuation.
(7) Grey items in Table 2 are either straightforward, or can readily be seen by the commutativity of $\circledast$, or are inherited from Table 1, so it suffices to prove the items in black.

- Table 2$_{(4,1)}$ and Table 2$_{(1,6)}$ follow from (8.1).
- As for Table 2$_{(2,4)}$, Table 2$_{(3,4)}$, Table 2$_{(4,4)}$, Table 2$_{(6,4)}$, by Lemma 5.4, the products $v \circledast s$, $\top_{[v]} \circledast s$, $z \circledast s$, $x \circledast s$, respectively are in $X_{\tau \geq u}$. Assume by contradiction that any of these products is in $X^{T}_{\tau \geq u}$, that is, $? \circledast s \in X^{T}_{\tau \geq u}$ for $? \in \{v, \top_{[v]}, z, x\}$. Then, by (8.1), $u' \circledast (? \circledast s) = (? \circledast s)_{\Downarrow} < ? \circledast s$ holds, whereas $? \circledast (u' \circledast s) = ? \circledast s$ holds by Table 2$_{(1,4)}$, a contradiction.
- As for Table 2$_{(2,6)}$, we need to prove that $(v \circledast y)_{\downarrow} < v \circledast y$ and $(v \circledast y) \circledast u' = (v \circledast y)_{\downarrow}$. Indeed, $(v \circledast y) \circledast u' = v \circledast y$ and cancellation by v (see

Table 2. (Gaps): If $v, w \in X_{\tau<u}$, $x, y \in X^{T_{ps}}_{\tau\geq u}$, $z, s \in X_{\tau\geq u} \backslash X^{T}_{\tau\geq u}$ then the following holds true

$\circledast$	$\bot_{[w]}$	w	$\top_{[w]}$	s	$y_{\downarrow}$	y
$\bot_{[v]}$		$\bot_{[v\circledast w]}$	$\bot_{[v\circledast w]}$	$v \circledast s$		$(v \circledast y)_{\downarrow}(< v \circledast y \in X^{T_{ps}}_{\tau\geq u})$
v	$\bot_{[v\circledast w]}$	$v \circledast w$	$\top_{[v\circledast w]}$	$\in X_{\tau\geq u} \backslash X^{T}_{\tau\geq u}$		$v \circledast y \in X^{T_{ps}}_{\tau\geq u}$
$\top_{[v]}$	$\bot_{[v\circledast w]}$	$\top_{[v\circledast w]}$	$\top_{[v\circledast w]}$	$\in X_{\tau\geq u} \backslash X^{T}_{\tau\geq u}$		$v \circledast y$
z	$z \circledast w$	$z \circledast w$	$z \circledast w$	$\in X_{\tau\geq u} \backslash X^{T}_{\tau\geq u}$	$z \circledast y$	$z \circledast y$
$x_{\downarrow}$				$x \circledast s$		
x	$(x \circledast w)_{\downarrow}$	$x \circledast w \in X^{T_{ps}}_{\tau\geq u}$	$x \circledast w$	$\in X_{\tau\geq u} \backslash X^{T}_{\tau\geq u}$		

claim (2) in Proposition 5.2) would yield $y_{\downarrow} = y \circledast u' = y$, a contradiction, so $(v \circledast y) \circledast u' < v \circledast y$ follows. If there were c such that $(v \circledast y) \circledast u' < c < v \circledast y$ then a cancellation by v would yield $y_{\downarrow} = y \circledast u' < v' \circledast c < y$, a contradiction. Summing up, $(v \circledast y) \circledast u' = (v \circledast y)_{\downarrow} < v \circledast y$, and hence the proof of Table $2_{(2,6)}$ is concluded.

- Finally, $z \circledast y_{\downarrow} = z \circledast (y \circledast u') = (z \circledast u') \circledast y$, and by (8.1) it is equal to $z \circledast y$ as stated in Table $2_{(4,5)}$. ∎

9. One-Step Decomposition—when u' is Idempotent

Definition 9.1. Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be an odd involutive FL_e-chain with residual complement $'$, such that there exists u, the smallest strictly positive idempotent. Assume u' is idempotent. Let

$$X^{E_{ps}}_{[\tau\geq u]} = \{\{\{p\}, \{p_{\downarrow}\}\} \mid p \in X^{T_{ps}}_{\tau\geq u}\},$$
$$X^{E_{c}}_{[\tau\geq u]} = \{\{\{\top_{[v]}\}, \{\bot_{[v]}\}\} \mid v \in X_{\tau<u}\},$$
$$X^{E}_{[\tau\geq u]} = X^{E_{c}}_{[\tau\geq u]} \cup X^{E_{ps}}_{[\tau\geq u]}.$$

Let γ be defined on $\beta(X)$ by $\gamma(x) =$

$$\begin{cases} \{[v], \{\top_{[v]}\}, \{\bot_{[v]}\}\} & \text{if } x \in \{[v], \{\top_{[v]}\}, \{\bot_{[v]}\}\} \text{ for some } v \in X_{\tau<u} \\ \{\{p\}, \{p_{\downarrow}\}\} & \text{if } x \in \{\{p\}, \{p_{\downarrow}\}\} \in X^{E_{ps}}_{[\tau\geq u]} \\ \{x\} & \text{if } x = \{p\}, p \in X_{\tau\geq u} \backslash X^{E}_{\tau\geq u} \end{cases} \tag{9.1}$$

By letting $x \in a, y \in b$ for $a, b \in \gamma(\beta(X)) = \{\gamma(z) : z \in \beta(X)\}$, over $\gamma(\beta(X))$ define

$$a \le_\gamma b \text{ iff } x \le_\beta y, \tag{9.2}$$
$$a \circledast_\gamma b = \gamma(x \circledast_\beta y), \tag{9.3}$$
$$a \to_\gamma b = \gamma(x \to_{\circledast_\beta} y), \tag{9.4}$$
$$a^{\prime^\gamma} = \gamma(x^{\prime^\beta}). \tag{9.5}$$

Proposition 9.2. *γ, defined in Definition 9.1, is well-defined.*

Proof. We need to prove the following statement: If $a, b \in X^E_{[\tau \ge u]}$, $a \ne b$ then $a \cap b = \emptyset$. Indeed, the case $a, b \in X^{E_c}_{[\tau \ge u]}$, $a \ne b$ follows from claim 4 in Proposition 7.2. Next, if $p \in X^{T_{ps}}_{\tau \ge u}$ then $p_\downarrow \in X^{G_2}_{\tau \ge u}$, since $p_\downarrow \circledast u' = (p \circledast u') \circledast u' = p \circledast (u' \circledast u') = p \circledast u' = p_\downarrow$. Therefore, the case $a, b \in X^{E_{ps}}_{[\tau \ge u]}$, $a \ne b$ follows, since $X^{T_{ps}}_{\tau \ge u}$ and $X^{G_2}_{\tau \ge u}$ are disjoint by claim (1) in Proposition 8.2. Hence it remains to prove $\{\{\top_{[v]}\}, \{\bot_{[v]}\}\} \cap \{\{p\}, \{p_\downarrow\}\} = \emptyset$ for $v \in X_{\tau < u}$ and $p \in X^{T_{ps}}_{\tau \ge u}$. The impossibility of $p = \top_{[v]}$ or $p_\downarrow = \bot_{[v]}$ is clear from $|[v]| \ne \emptyset$. As for $p \ne \bot_{[v]}$, by contradiction, $\bot_{[v]} \circledast u' = (\bot_{[v]})_\downarrow$ would lead, by Table $1_{(2,1)}$, to $\bot_{[v]} = v \circledast \bot_{[t]} = v \circledast u' = v \circledast (u' \circledast u') = (v \circledast u') \circledast u' = \bot_{[v]} \circledast u' = (\bot_{[v]})_\downarrow$, a contradiction to $p > p_\downarrow$. Finally, $p_\downarrow = \top_{[v]}$ would imply $\top_{[v]} = p_\downarrow = p \circledast u' = p \circledast (u' \circledast u') = (p \circledast u') \circledast u' = p_\downarrow \circledast u' = \top_{[v]} \circledast u' = \bot_{[v]}$ by Table $1_{(3,1)}$, which is a contradiction. ■

Proposition 9.3. *Let $\mathbf{X} = (X, \le, \circledast, \to_\circledast, t, f)$ be an odd involutive FL_e-chain with residual complement $'$, such that there exists u, the smallest strictly positive idempotent element. Assume u' is idempotent. The following product table holds for $\circledast$ (in the bottom-right 2×2 entries the formula on the left applies if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau \ge u}$, whereas the formula on the right applies if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau < u}$ or $x \circledast y = (x \circledast y)_\downarrow$.*

Proof. Grey items in Table 3 are either inherited from Table 2, or straightforward, or are readily seen by the commutativity of $\circledast$. Hence it suffices to prove the items in black.

- By claim (4) in Proposition 7.2, $\bot_{[v]} \notin X^T_{\tau \ge u}$ holds, and thus $\bot_{[v]} \circledast \bot_{[w]} = \bot_{[v]} \circledast w = \bot_{[v \circledast w]}$ follows from (8.1) and Table $1_{(1,2)}$, respectively, thus confirming Table $3_{(1,1)}$.
- Next, $\bot_{[v]} \circledast y_\downarrow = \bot_{[v]} \circledast (y \circledast u') = (\bot_{[v]} \circledast \bot_{[t]}) \circledast y$, which is equal to $\bot_{[v]} \circledast y$ by Table $3_{(1,1)}$, and hence the second row of (8.1) concludes the proof of Table $3_{(1,5)}$.
- Next, $\top_{[v]} \circledast y_\downarrow = \top_{[v]} \circledast (y \circledast u') = (\top_{[v]} \circledast y) \circledast u' = (\top_{[v]} \circledast y) \circledast (u' \circledast u') = (\top_{[v]} \circledast u') \circledast (u' \circledast y)$. By Table $1_{(3,1)}$ it is equal to $\bot_{[v]} \circledast$

Table 3. (When u' is idempotent): If $v, w \in X_{\tau<u}$, $x, y \in X_{\tau\geq u}^{T_{ps}}$, $z, s \in X_{\tau\geq u}\backslash X_{\tau\geq u}^{T}$ then the following holds true

$\circledast$	$\bot_{[w]}$	w	$\top_{[w]}$	s	$y_\downarrow$	y
$\bot_{[v]}$	$\bot_{[v\circledast w]}$	$\bot_{[v\circledast w]}$	$\bot_{[v\circledast w]}$	$v \circledast s$	$(v \circledast y)_\downarrow$	$(v \circledast y)_\downarrow$
v	$\bot_{[v\circledast w]}$	$v \circledast w$	$\top_{[v\circledast w]}$	$\in X_{\tau\geq u}\backslash X_{\tau\geq u}^{T}$	$(v \circledast y)_\downarrow$	$v \circledast y \in X_{\tau\geq u}^{T_{ps}}$
$\top_{[v]}$	$\bot_{[v\circledast w]}$	$\top_{[v\circledast w]}$	$\top_{[v\circledast w]}$	$\in X_{\tau\geq u}\backslash X_{\tau\geq u}^{T}$	$(v \circledast y)_\downarrow$	$v \circledast y$
z	$z \circledast w$	$z \circledast w$	$z \circledast w$	$\in X_{\tau\geq u}\backslash X_{\tau\geq u}^{T}$	$z \circledast y$	$z \circledast y$
$x_\downarrow$	$(x \circledast w)_\downarrow$	$(x \circledast w)_\downarrow$	$(x \circledast w)_\downarrow$	$x \circledast s$	$(x \circledast y)_\downarrow \mid \bot_{[r]}$	$(x \circledast y)_\downarrow \mid \bot_{[r]}$
x	$(x \circledast w)_\downarrow$	$x \circledast w \in X_{\tau\geq u}^{T_{ps}}$	$x \circledast w$	$\in X_{\tau\geq u}\backslash X_{\tau\geq u}^{T}$	$(x \circledast y)_\downarrow \mid \bot_{[r]}$	$x \circledast y \in X_{\tau\geq u}^{T_{ps}} \mid \top_{[r]}$

$y_\downarrow$ and by Table 3$_{(1,5)}$ the latest is equal to $(v \circledast y)_\downarrow$, as stated in Table 3$_{(3,5)}$.

- By monotonicity of $\circledast$, Table 3$_{(1,5)}$ and Table 3$_{(3,5)}$ ensure Table 3$_{(2,5)}$.
- Finally, we prove the remaining 2×2 entries at the bottom-right of Table 3.
  As for Table 3$_{(6,6)}$, by claim (5) in Proposition 8.2 and by (8.1) it follows that $x \circledast y$ is either in $X_{\tau\geq u}^{T_c}$ or in $X_{\tau\geq u}^{T_{ps}}$. Clearly, $x \circledast y \in X_{\tau\geq u}^{T_{ps}}$ if and only if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau\geq u}$, and $x \circledast y \in X_{\tau\geq u}^{T_c}$ if and only if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau<u}$ or $x \circledast y = (x \circledast y)_\downarrow$ holds. As for Table 3$_{(5,5)}$, Table 3$_{(5,6)}$, and Table 3$_{(6,5)}$, $x_\downarrow \circledast y_\downarrow = (x \circledast u') \circledast (y \circledast u') = (x \circledast y) \circledast (u' \circledast u') = (x \circledast y) \circledast u' = (x \circledast u') \circledast y = x \circledast (y \circledast u') = x_\downarrow \circledast y = x \circledast y_\downarrow$, which is equal to $(x \circledast y)_\downarrow$ if $x \circledast y \in X_{\tau\geq u}^{T_{ps}}$, and is equal to $\bot_{[r]}$ if $x \circledast y = \top_{[r]} \in X_{\tau\geq u}^{T_c}$. ■

Lemma 9.4. *Let* $\mathbf{X} = (X, \leq, \circledast, \rightarrow_\circledast, t, f)$ *be an odd involutive* FL_e*-chain with residual complement* $'$*, such that there exists* u*, the smallest strictly positive idempotent and* u' *is idempotent.*

1. $\mathbf{Y} = (\gamma(\beta(X)), \leq_\gamma, \circledast_\gamma, \rightarrow_\gamma, \gamma(\beta(t)), \gamma(\beta(f)))$ *is an odd involutive* FL_e*-chain with involution* $'^{\gamma}$*. The set of positive idempotent elements of* $\mathbf{Y}$ *is order isomorphic to the set of positive idempotent elements of* $\mathbf{X}$ *deprived of* t*.*
2. *Over* $\gamma(\beta(X_{\tau\geq u}^{E}))$ *there is a subgroup* $\mathbf{Z}$ *of* $\mathbf{Y}$*, and over* $\gamma(\beta(X_{\tau\geq u}^{E_c}))$ *there is a subgroup* $\mathbf{V}$ *of* $\mathbf{Z}$*.*
3. $\mathbf{X} \cong \mathbf{Y}_{\mathbf{Z}_\mathbf{V}} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{ker}_\beta$.

Proof.

1. Because of claim (1) in Lemma 6.2, it suffices to verify that γ is a homomorphism from $\beta(X)$ to $\gamma(\beta(X))$. The definition of $\leq_\gamma$ is independent of the choice of x and y, since the sets $\{[v], \{\top_{[v]}\}, \{\bot_{[v]}\}\}$ (for $v \in X_{\tau<u}$) and $\{\{p\}, \{p_\downarrow\}\}$ (for $p \in X_{\tau \geq u}^{T_{ps}}$) are convex in $\beta(X)$. The ordering $\leq_\beta$ is linear since so is $\leq$. Therefore, (9.2) is well-defined. Preservation of the monoidal operation can readily be checked in Table 3, and preservation of the residual complement can readily be seen from (6.4), (7.3) and claim (4) in Proposition 8.2. In complete analogy to the way we proved the preservation of the implication under β in (6.5), we can verify the preservation of the implication under γ.
   If $c \geq u$ is an idempotent element of $\mathbf{X}$ then $\gamma(\beta(c))$ is also idempotent and $\gamma(\beta(c)) \geq \gamma(\beta(u)) = \gamma(\beta(t))$ (thus also positive) since $\gamma \circ \beta$ is a homomorphism. Assume $c, d \geq u$ are idempotent elements of $\mathbf{X}$ and $c < d$. Then $c, d \in X_{\tau \geq u}$ holds because of claim (4) in Proposition 2.3. Therefore, referring to the definition of β and γ, and claim (2) in Proposition 7.2, $\gamma(\beta(c)) = \gamma(\beta(d))$ could hold only if either $c = \bot_{[v]}$ and $d = \top_{[v]}$ for some $v \in X_{\tau<u}$, or $c = d_\downarrow$ for some $d \in X_{\tau \geq u}^{T_{ps}}$. The latter case readily leads to a contradiction, since by claim (6) in Proposition 8.2 $\tau(d) = u$, whereas claim (4) in Proposition 2.3 shows $\tau(d) = d > c \geq u$. In the former case, by Table $3_{(3,3)}$, the idempotency of d implies $\top_{[v \circledast v]} = \top_{[v]}$, that is, $[v \circledast v] = [v]$, that is, $\beta(v \circledast v) = \beta(v)$ that is, $\beta(v) \circledast_\beta \beta(v) = \beta(v)$. Since $\mathbf{X}_{\tau<u} = \mathbf{X_{gr}}$ holds by claim (4) in Lemma 6.2, it holds that v' is the inverse of v, and hence $\beta(v')$ is the inverse of $\beta(v)$ in $\beta(\mathbf{X})$. Therefore, $\beta(v) = (\beta(v') \circledast_\beta \beta(v)) \circledast_\beta \beta(v) = \beta(v') \circledast_\beta (\beta(v) \circledast_\beta \beta(v)) = \beta(v') \circledast_\beta \beta(v) = \beta(v' \circledast v) = \beta(t)$ follows, that is, $[v] = [t]$. But it contradicts to $c = \bot_{[t]} \geq u$. These conclude the proof of the second statement.

2. To prove the first statement, by Theorem 2.4, it suffices to prove that any element of $\gamma(\beta(X_{\tau \geq u}^{E}))$ has an inverse in $\gamma(\beta(X_{\tau \geq u}^{E}))$. For $v \in X_{\tau<u}$, $\{[v], \{\top_{[v]}\}, \{\bot_{[v]}\}\} \circledast_\gamma \{[v'], \{\top_{[v']}\}, \{\bot_{[v']}\}\} = \gamma([v] \circledast_\beta [v']) = \gamma(\beta(v \circledast v')) = \gamma(\beta(t))$, and for $p \in X_{\tau \geq u}^{T_{ps}}$,
$$\begin{aligned}\{\{p\}, \{p_\downarrow\}\} \circledast_\gamma \{\{p'\}, \{p'_\downarrow\}\} &= \gamma(\{p\} \circledast_\beta \{p'\}\}) = \gamma(\beta(p \circledast p')) \\ &= \gamma(\beta((p \to_\circledast p)')) = \gamma(\beta(\tau(p)')).\end{aligned}$$
   By claim (6) in Proposition 8.2, it is equal to $\gamma(\beta(u')) = \gamma(\beta(t))$. The second statement is straightforward since $X_{\tau \geq u}^{E_c} \subseteq X_{\tau \geq u}^{E}$, and $\gamma \circ \beta$ is a homomorphism.

3. Referring to Lemma 6.2, let $\delta = (\beta|_{X_{gr}}, \mu)$ denote the isomorphism from $\mathbf{X_{gr}}$ to $\beta(\mathbf{X_{gr}}) \overrightarrow{\times} \mathbf{ker}_\beta$. Define a mapping $\alpha : X \to Y_{Z_V} \overrightarrow{\overrightarrow{\times}} ker_\beta$ by

$$\alpha(x) = \begin{cases} (\gamma(\beta(x)), \mu(x)) & \text{if } x \in X_{\tau<u} \\ (\gamma(\beta(x)), \top) & \text{if } x \in X_{\tau\geq u}^T \\ (\gamma(\beta(x)), \bot) & \text{if } x \in X_{\tau\geq u} \backslash X_{\tau\geq u}^T \end{cases},$$

or in a more detailed form,

$$\alpha(x) = \begin{cases} (\gamma(\beta(x)), \bot) & \text{if } x \in X_{\tau\geq u}^{B_c} \\ (\gamma(\beta(x)), \mu(x)) & \text{if } x \in X_{\tau<u} \\ (\gamma(\beta(x)), \top) & \text{if } x \in X_{\tau\geq u}^{T_c} \\ (\gamma(\beta(x)), \bot) & \text{if } x \in X_{\tau\geq u}^{B_{ps}} \\ (\gamma(\beta(x)), \top) & \text{if } x \in X_{\tau\geq u}^{T_{ps}} \\ (\gamma(\beta(x)), \bot) & \text{if } x \in X_{\tau\geq u} \backslash X_{\tau\geq u}^E \end{cases}. \tag{9.6}$$

Then α is an isomorphism from $\mathbf{X}$ to $\mathbf{Y_{Z_V}} \overrightarrow{\overrightarrow{\times}} \mathbf{ker}_\beta$. Indeed, intuitively, what $\gamma \circ \beta$ does is that it makes each component together with its extremals in X collapse into a singleton, and also makes each coherent pair of pseudo extremals, that is, a gap $\{x_\downarrow, x\}$ ($x_\downarrow < x \in X_{\tau\geq u}^{T_{ps}}$) collapse into a singleton. On the other hand, the construction $\mathbf{Y_{Z_V}} \overrightarrow{\overrightarrow{\times}} \mathbf{ker}_\beta$ does the exact opposite, namely, it replaces any image of such a component by a whole component equipped with a top and a bottom (extremals), and it replaces each image of a pair of coherent pseudo extremals by a top and a bottom (pseudo extremals), just like α does in (9.6); see the related definitions, in claim (2) and Definitions 6.1, 9.1, and 4.2/A. Thus, α is an order-isomorphism from X to $Y_{Z_V} \overrightarrow{\overrightarrow{\times}} ker_\beta$, the universe of $\mathbf{X} \cong \mathbf{Y_{Z_V}} \overrightarrow{\overrightarrow{\times}} \mathbf{ker}_\beta$. A straightforward verification using Definition 4.2/A and Table 3 shows the preservation of the monoidal operation under α. Moreover, (6.4), (7.3) and claim (4) in Proposition 8.2 shows the preservation of the residual complement, under α. In complete analogy to the way we proved the preservation of the implication under β in (6.5), we can verify the preservation of the implication under α, too. Finally, it is clear that α maps the unit element of $\mathbf{X}$ to the unit element of $\mathbf{Y_{Z_V}} \overrightarrow{\overrightarrow{\times}} \mathbf{ker}_\beta$. ■

Table 4. (When u' is not idempotent): If $v, w \in X_{\tau<u}$, and $x, y \in X_{\tau\geq u}^{T_{ps}}$ then it holds true that

$\circledast$	$\top_{[w]}$	y
$\top_{[v]}$	$\top_{[v\circledast w]}$	$v \circledast y \in X_{\tau\geq u}^{T_{ps}}$
x	$x \circledast w \in X_{\tau\geq u}^{T_{ps}}$	$\in X_{\tau\geq u}^{T_{ps}} \mid \in X_{\tau\geq u}^{T_c}$

10. One-Step Decomposition—When u' is Not Idempotent

In order to handle the case when u' is not idempotent, we shall consider a residuated subsemigroup $\mathbf{X}_{\tau\geq u}$ of $\mathbf{X}$, which is an odd involutive FL$_e$-algebra, albeit not a subalgebra of $\mathbf{X}$, since its unit element u and residual complement ${}'^{u}$ will differ from those of $\mathbf{X}$. Then we prove that the group-part of $\mathbf{X}_{\tau\geq u}$ (namely $\mathbf{X}_{\tau\geq u}^{T}$) is discretely embedded into $\mathbf{X}_{\tau\geq u}$, that $\mathbf{X}_{\tau\geq u}^{T_c} \leq \mathbf{X}_{\tau\geq u}^{T}$, and finally we will recover $\mathbf{X}$, up to isomorphism, as a type IV partial lexicographic product of $\mathbf{X}_{\tau\geq u}$, $\mathbf{X}_{\tau\geq u}^{T_c}$ and $\mathbf{ker}_\beta$.

Proposition 10.1. *Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be an odd involutive FL$_e$-chain with residual complement $'$, such that there exists u, the smallest strictly positive idempotent. Assume u' is not idempotent.*

1. *The following product table holds for $\circledast$ (in the bottom-right entry the formula on the left applies if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau\geq u}$, whereas the formula on the right applies if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau<u}$ or $x \circledast y = (x \circledast y)_\downarrow$.*
2. *$X_{\tau\geq u}^{T}$ is closed under $'$.*
3. *$X_{\tau\geq u}^{T}$ is discretely embedded into $X_{\tau\geq u}$.*

Proof.

(1) Grey items in Table 4 are either readily seen by symmetry of $\circledast$ or are inherited from Table 2, so it suffices to prove the items in black.

- By Lemma 5.4, $x \circledast y \in X_{\tau\geq u}$ holds. Since by claim (5) in Proposition 8.2, $(x \circledast y) \circledast u' \neq x \circledast y$ holds, it follows from (8.1) that either $x \circledast y \in X_{\tau\geq u}^{T_c}$ or $x \circledast y \in X_{\tau\geq u}^{T_{ps}}$. Clearly, the former holds if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau<u}$. If $x \circledast y = (x \circledast y)_\downarrow$ then $x \circledast y \in X_{\tau\geq u}^{T_{ps}}$ cannot hold, thus $x \circledast y \in X_{\tau\geq u}^{T_c}$ holds. Finally, if $x \circledast y > (x \circledast y)_\downarrow \in X_{\tau\geq u}$ then $x \circledast y \in X_{\tau\geq u}^{T_c}$ cannot hold, thus $x \circledast y \in X_{\tau\geq u}^{T_{ps}}$ holds. This proves Table $4_{(2,2)}$.

- To prove Table 4$_{(2,1)}$, that is, $x \circledast \top_{[w]} \in X^{T_{ps}}_{\tau \geq u}$ we proceed as follows. First we show that $x \circledast \top_{[w]}$ cannot be in $X^{T_c}_{\tau \geq u}$. Indeed, if $x \circledast \top_{[w]} = \top_{[v]}$ then by Table 1$_{(3,4)}$ and Table 1$_{(3,1)}$, $x_\downarrow \circledast w = (x \circledast u') \circledast w = (x \circledast w) \circledast u' = (x \circledast \top_{[w]}) \circledast u' = \top_{[v]} \circledast u' = \bot_{[v]}$. Multiplying both sides by w' we obtain, using Table 1$_{(2,1)}$, $x_\downarrow = w' \circledast \bot_{[v]} = \bot_{[w' \circledast v]}$, a contradiction to claim (1) in Proposition 8.2.
  Next, we show $(x \circledast \top_{[w]})_\downarrow < x \circledast \top_{[w]}$. Assume the opposite, that is, $(x \circledast \top_{[w]})_\downarrow = x \circledast \top_{[w]}$. Since $x \circledast \top_{[w]}$ is not in $X^{T_c}_{\tau \geq u}$, by Table 1$_{(4,1)}$, $(x \circledast \top_{[w]}) \circledast u' = x \circledast \top_{[w]}$ follows. But it leads to $x_\downarrow \circledast \bot_{[w]} = (x \circledast u') \circledast (\top_{[w]} \circledast u') = ((x \circledast \top_{[w]}) \circledast u') \circledast u' = x \circledast \top_{[w]}$, a contradiction to Proposition 5.3.
  Note that we have just proved $(x \circledast \top_{[w]}) \circledast u' \neq x \circledast \top_{[w]}$, too. Hence, using that $x \circledast \top_{[w]} \in X_{\tau \geq u}$ holds by Lemma 5.4, by (8.1), $x \circledast \top_{[w]} \in X^{T_{ps}}_{\tau \geq u}$ must hold, since the other two lines of (8.1) cannot apply.

(2) *Claim* For any odd involutive FL$_e$-chain $(X, \leq, \circledast, \rightarrow_\circledast, t, f)$, $x, y, a, b \in X$ and $a < b$, if $x \circledast [a, b] = y$ [13] then $y' \circledast]a, b] = x'$. Indeed, let $u \in]a, b]$. By (2.1) it suffices to prove $u \rightarrow_\circledast y = x$. Now, $x \circledast u = y$ holds, and if, by contradiction, for $x_1 > x$, $x_1 \circledast u = y$ would hold, than together with $x \circledast a = y$ it would contradict to Proposition 5.3. This settles the claim. Let $x \in X^T_{\tau \geq u}$. Then $x' \in X_{\tau \geq u}$ by Lemma 5.4. If $x' \notin X^T_{\tau \geq u}$ then by (8.1), $x' \circledast u' = x'$ would hold. It would imply $x' = (x' \circledast u') \circledast u' = x' \circledast (u' \circledast u') = x' \circledast (u'_\downarrow)$, that is, $x' \circledast [u'_\downarrow, u'] = x'$, which, by the claim would yield $x \circledast u' = x$, contradicting to $x \in X^T_{\tau \geq u}$ by (8.1).

(3) We prove that $X^T_{\tau \geq u}$ is closed under $_\Downarrow$ and $_\Uparrow$, and neither $_\Downarrow$ nor $_\Uparrow$ have fixed point in $X^T_{\tau \geq u}$. By (7.3), for $w \in X_{\tau < u}$, $(\top_{[w]})_\Downarrow = \bot_{[w]} = (\top_{[w']})'$ holds by (7.3), and by claim (2) it is in $X^T_{\tau \geq u}$. Therefore, it remains to prove $x_\downarrow \in X^T_{\tau \geq u}$ if $x \in X^{T_{ps}}_{\tau \geq u}$. The opposite, that is, $x_\downarrow \in X_{\tau \geq u} \setminus X^T_{\tau \geq u}$ would imply $x_\downarrow \circledast u' = x_\downarrow$ by (8.1), and it would lead to $x \circledast u' = x_\downarrow = x_\downarrow \circledast u' = (x_\downarrow \circledast u') \circledast u' = x_\downarrow \circledast (u' \circledast u')$, a contradiction to Proposition 5.3. It is straightforward that $_\Downarrow$ has no fixed point in $X^T_{\tau \geq u}$. The statements on $_\Uparrow$ follow from (4.1). ∎

Lemma 10.2. *Let* $\mathbf{X} = (X, \leq, \circledast, \rightarrow_\circledast, t, f)$ *be an odd involutive FL$_e$-chain with residual complement* $'$, *such that there exists* u, *the smallest strictly positive idempotent, and* u' *is not idempotent.*

[13] We mean $x \circledast z = y$ for all $z \in [a, b]$.

1. $\mathbf{X}_{\tau\geq u}=(X_{\tau\geq u},\leq,\circledast,\rightarrow_{\circledast},u,u)$ *is an odd involutive FL_e-chain with residual complement*
$$
{}^{\prime^{\mu}}\ :\ x\mapsto\begin{cases}x' & \text{if } x\in X_{\tau\geq u}\setminus X_{\tau\geq u}^{T}\\ (x_{\Downarrow})'=x'_{\Uparrow} & \text{if } x\in X_{\tau\geq u}^{T}\end{cases}. \qquad (10.1)
$$
The set of positive idempotent elements of $\mathbf{X}_{\tau\geq u}$ equals the set of positive idempotent elements of $\mathbf{X}$ deprived of t.

2. $\mathbf{X}_{\tau\geq u}^{T}=(X_{\tau\geq u}^{T},\leq,\circledast,\rightarrow_{\circledast},u,u)$ *is the group part of $\mathbf{X}_{\tau\geq u}$, and $\mathbf{X}_{\tau\geq u}^{T_c}=(X_{\tau\geq u}^{T_c},\leq,\circledast,\rightarrow_{\circledast},u,u)$ is a subalgebra of $\mathbf{X}_{\tau\geq u}^{T}$.*

3. $\mathbf{X}\cong(\mathbf{X}_{\tau\geq u})_{\left(\mathbf{X}_{\tau\geq u}^{T_c}\right)}\overset{\rightarrow}{\times}\mathbf{ker}_{\beta}$.

Proof. (1) The set $X_{\tau\geq u}$ is nonempty since by claim (4) in Proposition 2.3, u is an element in it. $X_{\tau\geq u}$ is closed under $\circledast$ and $'$ by Lemma 5.4, and hence under $\rightarrow_{\circledast}$, too, by (2.1). Clearly, u is the unit element of $\circledast$ over $X_{\tau\geq u}$. The residual complement operation in $\mathbf{X}_{\tau\geq u}$ is ${}^{\prime^{\mu}}$, since by (2.1) and (8.1), for $x\in X_{\tau\geq u}$ it holds true that $x\rightarrow_{\circledast}u=(x\circledast u')'=(x\circledast\bot_{[t]})'=x^{\prime^{\mu}}$. Next we prove that $\mathbf{X}_{\tau\geq u}$ is involutive, that is, that ${}^{\prime^{\mu}}$ is an order reversing involution of $X_{\tau\geq u}$. Indeed, if $x\in X_{\tau\geq u}\setminus X_{\tau\geq u}^{T}$ then $x^{\prime^{\mu}}=x'\in X_{\tau\geq u}\setminus X_{\tau\geq u}^{T}$ by Lemma 5.4 and claim (2), and hence $\left(x^{\prime^{\mu}}\right)^{\prime^{\mu}}=x$. If $x\in X_{\tau\geq u}^{T}$ then $x^{\prime^{\mu}}=x'_{\Uparrow}\in X_{\tau\geq u}^{T}$ holds by claims (2) and (3) in Proposition 10.1, and using claim (3) in Proposition 10.1, it follows that $\left(x^{\prime^{\mu}}\right)^{\prime^{\mu}}=(x'_{\Uparrow})'_{\Uparrow}=x$, so we are done. Next, $u^{\prime^{\mu}}=(u_{\Downarrow})'=(u')'=u$, so $\mathbf{X}_{\tau\geq u}$ is odd. Finally, for any idempotent element c such that $c\geq u$, also $c\in\mathbf{X}_{\tau\geq u}$ holds because of claim (4) in Proposition 2.3. Since u is the smallest strictly positive idempotent element in X, the same claim shows that the only positive idempotent element of $\mathbf{X}$ which is missing from $\mathbf{X}_{\tau<u}$ is t.

(2) *Claim* ${}^{\prime^{\mu}}$ is inverse operation on $X_{\tau\geq u}^{T}$. Indeed, first let $x\in X_{\tau\geq u}^{T_c}$. By (7.3) and Table $1_{(3,3)}$, $x\circledast x^{\prime^{\mu}}=\top_{[v]}\circledast(\top_{[v]})^{\prime^{\mu}}=\top_{[v]}\circledast((\top_{[v]})_{\Downarrow})'=\top_{[v]}\circledast(\bot_{[v]})'=\top_{[v]}\circledast\top_{[v']}=\top_{[v\circledast v']}=\top_{[t]}=u$. Second, let $x\in X_{\tau\geq u}^{T_{ps}}$. Then $x\circledast x^{\prime^{\mu}}=x\circledast(x_{\Downarrow})'=x\circledast(x_{\downarrow})'=x\circledast(x\circledast u')'=x\circledast(x\rightarrow_{\circledast}u)\leq u$. On the other hand, since $x>x_{\downarrow}$, by residuation and by using that X is a chain, $x\circledast x^{\prime^{\mu}}=x\circledast(x_{\downarrow})'>t$ holds, and it implies $x\circledast x^{\prime^{\mu}}\geq u$, since $x\circledast x^{\prime^{\mu}}\in X_{\tau\geq u}$ by Lemma 5.4, and u is the smallest element of $X_{\tau\geq u}$, which is greater than t. This settles the claim.

Therefore, $\mathbf{X}_{\tau\geq u}^{T}\subseteq(\mathbf{X}_{\tau\geq u})_{\mathbf{gr}}$ holds. If, by contradiction, there would exists $p\in(\mathbf{X}_{\tau\geq u})_{\mathbf{gr}}\setminus\mathbf{X}_{\tau\geq u}^{T}$ then its inverse is $p^{\prime^{\mu}}$ by Theorem 2.4, and by the claim,

p'^{μ} would also be in $(\mathbf{X}_{\tau\geq u})_{\mathbf{gr}}\backslash\mathbf{X}^{T}_{\tau\geq u}$. Thus $p \circledast p'^{\mu} = u \in X^{T}_{\tau\geq u}$ follows, a contradiction to Table $2_{(4,4)}$.

As for the second statement, Table $4_{(1,1)}$ shows that $X^{T_c}_{\tau\geq u}$ is closed under $\circledast$. (7.3) and the second line of (10.1) show that $X^{T_c}_{\tau\geq u}$ is closed under $'^{\mu}$. Since $\mathbf{X}_{\tau\geq u}$ is an odd involutive FL_e-chain with residual complement $'^{\mu}$, it follows that $x \rightarrow_{\circledast} y = (x \circledast y'^{\mu})'^{\mu}$, and hence $X^{T_c}_{\tau\geq u}$ is closed under $\rightarrow_{\circledast}$, too.

(3) By the previous claim and by claim (3) in Proposition 10.1,

$$(\mathbf{X}_{\tau\geq u})_{(\mathbf{X}^{T_c}_{\tau\geq u})} \overrightarrow{\overrightarrow{\times}}\, \mathbf{ker}_{\beta}$$

is well-defined. Referring to Lemma 6.2, let $\delta = (\beta|_{X_{gr}}, \mu)$ denote the isomorphism from $\mathbf{X_{gr}}$ to $\beta(\mathbf{X_{gr}}) \overrightarrow{\times}\, \mathbf{ker}_{\beta}$. Define a mapping $\alpha \ : \ X \rightarrow Z_V \overrightarrow{\overrightarrow{\times}}\, ker_{\beta}$ by

$$\alpha(x) = \begin{cases} (\top_{\beta(x)}, \mu(x)) & \text{if } x \in X_{\tau<u} \\ (x, u) & \text{if } x \in X_{\tau\geq u} \end{cases}.$$

α is clearly injective and surjective.

- Denote the monoidal operation of $(\mathbf{X}_{\tau\geq u})_{(\mathbf{X}^{T_c}_{\tau\geq u})} \overrightarrow{\overrightarrow{\times}}\, \mathbf{ker}_{\beta}$ by $\diamond = (\circledast, \circledast)$.
  - If $x, y \in X_{\tau<u}$ then $x \circledast y \in X_{\tau<u}$ holds by Lemma 5.4, and hence by Table $4_{(1,1)}$ and by using that μ is an isomorphism, $\alpha(x) \diamond \alpha(y) = (\top_{\beta(x)}, \mu(x)) \diamond (\top_{\beta(y)}, \mu(y)) = (\top_{\beta(x)} \circledast \top_{\beta(y)}, \mu(x) \circledast \mu(y)) = (\top_{\beta(x\circledast y)}, \mu(x \circledast y)) = \alpha(x \circledast y)$.

  - If $x, y \in X_{\tau\geq u}$ then $x \circledast y \in X_{\tau\geq u}$ holds by Lemma 5.4, and using the idempotency of u, $\alpha(x \circledast y) = (x \circledast y, u) = (x, u) \diamond (y, u) = \alpha(x) \diamond \alpha(y)$ follows.
  - If $x \in X_{\tau<u}$ and $y \in X_{\tau\geq u}$ then $x \circledast y \in X_{\tau\geq u}$ holds by Lemma 5.4, and hence, using Table $1_{(3,4)}$ and that u annihilates all elements in $]u', u[$ (see Table $1_{(2,3)}$), $\alpha(x \circledast y) = (x \circledast y, u) = (\top_{\beta(x)} \circledast y, \mu(x) \circledast u) = (\top_{\beta(x)}, \mu(x)) \diamond (y, u) = \alpha(x) \diamond \alpha(y)$ holds.

  Summing up, we have verified $\alpha(x \circledast y) = \alpha(x) \diamond \alpha(y)$ for all $x, y \in X$.
- Denote the residual complement of $(\mathbf{X}_{\tau\geq u})_{(\mathbf{X}^{T_c}_{\tau\geq u})} \overrightarrow{\overrightarrow{\times}}\, \mathbf{ker}_{\beta}$ by $'^{\diamond}$. Referring to (4.2) and (10.1), respectively, $'^{\diamond}$ can be written in the following form

$$(x,y)^{\prime^{\diamond}} = \begin{cases} (x^{\prime^{\mu}}, u) & \text{if } x \notin (X_{\tau\geq u})_{gr} \text{ and } y = u \\ ((x^{\prime^{\mu}})_{\Downarrow}, u) & \text{if } x \in (X_{\tau\geq u})_{gr} \text{ and } y = u \\ (x^{\prime^{\mu}}, y') & \text{if } x \in X_{\tau\geq u}^{T_c} \text{ and } y \in]u', u[\end{cases}$$
$$= \begin{cases} (x', u) & \text{if } x \in X_{\tau\geq u} \setminus X_{\tau\geq u}^{T} \text{ and } y = u \\ (x', u) & \text{if } x \in X_{\tau\geq u}^{T} \text{ and } y = u \\ (x^{\prime^{\mu}}, y') & \text{if } x \in X_{\tau\geq u}^{T_c} \text{ and } y \in]u', u[\end{cases},$$

that is,

$$(x,y)^{\prime^{\diamond}} = \begin{cases} (x', u) & \text{if } y = u \\ (x^{\prime^{\mu}}, y') & \text{if } y \in]u', u[\end{cases}.$$

– If $x \in X_{\tau<u}$ then by Lemma 5.4, $x' \in X_{\tau<u}$, too, and by using (7.3), (10.1) and that μ is an isomorphism, it holds true that

$$\alpha(x') = (\top_{\beta(x')}, \mu(x')) = ({\perp_{\beta(x)}}', \mu(x)') = (((\top_{\beta(x)})_{\Downarrow})', \mu(x)')$$
$$= (\top_{\beta(x)}^{\prime^{\mu}}, \mu(x)') = (\top_{\beta(x)}, \mu(x))^{\prime^{\diamond}} = \alpha(x)^{\prime^{\diamond}}.$$

– If $x \in X_{\tau\geq u}$ then by Lemma 5.4, $x' \in X_{\tau\geq u}$, too, and hence $\alpha(x') = (x', u) = (x, u)^{\prime^{\diamond}} = \alpha(x)^{\prime^{\diamond}}$.

Summing up, we have verified $\alpha(x') = \alpha(x)^{\prime^{\diamond}}$ for all $x \in X$.

- Since α preserves multiplication and residual complements, and since both $\mathbf{X}$ and $(\mathbf{X}_{\tau\geq u})_{(\mathbf{X}_{\tau\geq u}^{T_c})} \overset{\twoheadrightarrow}{\times} \mathbf{ker}_{\beta}$ are involutive, it follows by (2.1) that α preserves residual operation, too, that is, $\alpha(x \rightarrow_{\circledast} y) = \alpha(x) \rightarrow_{\diamond} \alpha(y)$.
- Also, $\alpha(t) = (u, t)$ holds, so α preserves the unit element, too.

Therefore, α is an isomorphism from $(\mathbf{X}_{\tau\geq u})_{(\mathbf{X}_{\tau\geq u}^{T_c})} \overset{\twoheadrightarrow}{\times} \mathbf{ker}_{\beta}$, as stated. ∎

11. Group Representation

The main theorem of the paper asserts that up to isomorphism, any odd involutive FL_e-chain which has only finitely many positive idempotent elements can be built by iterating finitely many times the type III and type IV partial lexicographic product constructions using only linearly ordered abelian groups, as building blocks.

Theorem 11.1. *If $\mathbf{X}$ is an odd involutive FL_e-chain, which has only $n \in \mathbf{N}$, $n \geq 1$ positive idempotent elements then there exist linearly ordered abelian groups $\mathbf{G}_i$ $(i \in \{1, 2, \ldots, n\})$, $\mathbf{V}_1 \leq \mathbf{Z}_1 \leq \mathbf{G}_1$, $\mathbf{V}_i \leq \mathbf{Z}_i \leq \mathbf{V}_{i-1} \overset{\rightarrow}{\times} \mathbf{G}_i$*

$(i \in \{2, \dots, n-1\})$, *and a binary sequence* $\iota \in \{III, IV\}^{\{2,\dots,n\}}$ *such that* $\mathbf{X} \cong \mathbf{X}_n$, *where* $\mathbf{X}_1 := \mathbf{G}_1$ *and for* $i \in \{2, \dots, n\}$,

$$\mathbf{X}_i := \begin{cases} \mathbf{X}_{i-1}{}_{\mathbf{Z}_{i-1}}{}_{\mathbf{V}_{i-1}} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{G}_i & \text{if } \iota_i = III \\ \mathbf{X}_{i-1}{}_{\mathbf{V}_{i-1}} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{G}_i & \text{if } \iota_i = IV \end{cases}. \tag{11.1}$$

Proof. Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_\circledast, t, f)$. Induction by n, the number of idempotent elements in X^+. If $n = 1$ then the only idempotent in X^+ is t, hence Theorem 2.4 implies that $(X, \leq, \circledast, t)$ is a linearly ordered abelian group $\mathbf{G}_1$ and we are done. Assume that the theorem holds up to $k-1$ (for some $2 \leq k < n$), and let $\mathbf{X}$ be an odd involutive FL_e-chain which has k positive idempotent elements. Since the number of idempotents in X^+ is finite, there exists u, the smallest idempotent above t.

If u' is idempotent then by Lemma 9.4 (by denoting $\alpha = \gamma \circ \beta$)

$$\mathbf{X} \cong \alpha(\mathbf{X})_{\alpha\left(\mathbf{X}^{E}_{\tau \geq u}\right)_{\alpha\left(\mathbf{X}^{E_c}_{\tau \geq u}\right)}} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{ker}_\beta$$

holds, where $\alpha\left(\mathbf{X}^{E_c}_{\tau \geq u}\right) \leq \alpha\left(\mathbf{X}^{E}_{\tau \geq u}\right)$ are subgroups of the odd involutive FL_e-chain $\alpha(\mathbf{X})$, and $\mathbf{ker}_\beta$ is a linearly ordered abelian group. Therefore, if u' is idempotent then set

$$\mathbf{X}_{k-1} = \alpha(\mathbf{X}), \mathbf{Z}_{k-1} = \alpha\left(\mathbf{X}^{E}_{\tau \geq u}\right), \mathbf{V}_{k-1} = \alpha\left(\mathbf{X}^{E_c}_{\tau \geq u}\right), \mathbf{G}_k = \mathbf{ker}_\beta, \text{ and } \iota_k = III.$$

If u' is not idempotent then by Lemma 10.2

$$\mathbf{X} \cong (\mathbf{X}_{\tau \geq u})_{\left(\mathbf{X}^{T_c}_{\tau \geq u}\right)} \overset{\rightarrow\!\!\!\rightarrow}{\times} \mathbf{ker}_\beta$$

holds, where $\mathbf{X}^{T_c}_{\tau \geq u}$ is a subgroup of the odd involutive FL_e-chain $\mathbf{X}_{\tau \geq u}$, and $\mathbf{ker}_\beta$ is a linearly ordered abelian group. Therefore, if u' is not idempotent then set

$$\mathbf{X}_{k-1} = \mathbf{X}_{\tau \geq u}, \mathbf{V}_{k-1} = \mathbf{X}^{T_c}_{\tau \geq u}, \mathbf{G}_k = \mathbf{ker}_\beta, \text{ and } \iota_k = IV.$$

By Lemmas 9.4 and 10.2 the number of positive idempotent elements of $\mathbf{X}_{k-1}$ (be it equal to either $\alpha(\mathbf{X})$ or $\mathbf{X}_{\tau \geq u}$) is one less than that of $\mathbf{X}$. Therefore, by the induction hypothesis, the theorem holds for $\mathbf{X}_{k-1}$, that is, there exist linearly ordered abelian groups $\mathbf{G}_i$ $(i \in \{1, 2, \dots, k-1\})$, $\mathbf{V}_1 \leq \mathbf{Z}_1 \leq \mathbf{G}_1$, $\mathbf{V}_i \leq \mathbf{Z}_i \leq \mathbf{V}_{i-1} \overset{\rightarrow}{\times} \mathbf{G}_i$ $(i \in \{2, \dots, k-2\})$, and a binary sequence $\iota \in \{III, IV\}^{\{2,\dots,k-1\}}$ such that $\mathbf{X}_1 := \mathbf{G}_1$ and for $i \in \{2, \dots, k-1\}$, (11.1) holds. Finally, we need to check that $\mathbf{Z}_{k-1} \leq \mathbf{G}_{k-1}$ if $k = 2$, and $\mathbf{Z}_{k-1} \leq \mathbf{V}_{k-1} \overset{\rightarrow}{\times} \mathbf{G}_{k-1}$ if $k > 2$. But it holds, since $\mathbf{Z}_{k-1} \leq (\mathbf{X}_{k-1})_{\mathbf{gr}}$ for $k \geq 2$, and since the monoidal operation in a partial lexicographic product is defined coordinatewise, in both cases $(\mathbf{X}_{k-1})_{\mathbf{gr}} = \mathbf{V}_{k-2} \overset{\rightarrow}{\times} \mathbf{G}_{k-1}$ if $k > 2$, or $(\mathbf{X}_{k-1})_{\mathbf{gr}} = \mathbf{G}_{k-1}$ if $k = 2$. ∎

Remark 11.2. Note that by Lemmas 9.4 and 10.2, and claim (4) in Theorem 2.4, linearly ordered abelian groups are exactly the indecomposable algebras with respect to the type III and type IV partial lexicographic product constructions.

Remark 11.3. Denote the one-element odd involutive FL_e-algebra by $\mathbf{1}$. If the algebra in Theorem 11.1 is bounded then in its group representation $\mathbf{G}_1 = \mathbf{1}$, since all other linearly ordered groups are infinite and unbounded, and both type III and type IV constructions preserve boundedness of the first component.

Remark 11.4. (*A unified treatment*) The reader might think that entirely different things are going in Sections 9 and 10, that is, depending on whether the residual complement of the smallest strictly positive idempotent element u is idempotent or not. It is not the case. In both cases a particular nuclear retract[14] plays a key role: Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be an odd involutive FL_e-chain with residual complement $'$, and u be the (existing) smallest strictly positive idempotent element of X. Let $\varphi : X \to X$ be defined by

$$\varphi(x) = ((x \circledast u')' \circledast u')'.$$

The interested reader might wish to verify using what has already been proven in this paper that

1. φ is a nucleus of $\mathbf{X}$.
2. The related nuclear retract $\mathbf{X}_\varphi$ over $X_\varphi = \{\varphi(x) : x \in X\}$ is isomorphic to $\mathbf{Y}$ of Lemma 9.4 if u' is idempotent, and (not only isomorphic, but also) equal to $\mathbf{X}_{\tau \geq u}$ of Lemma 10.2 if u' is not idempotent.
3. There is a subgroup[15] $\mathbf{X}_1$ of $\mathbf{X}_\varphi$ over $X_1 = \{x \in X_\varphi : x \circledast u' < x\}$.
4. There is a subgroup $\mathbf{X}_2$ of $\mathbf{X}_1$ over $X_2 = \varphi(X \setminus (X_\varphi \cup (X_\varphi)'))$.
5. There is a subalgebra $\mathbf{u}$ of $\mathbf{X}$ over $]u', u[$, and there is a subalgebra $\mathbf{ker}_\varphi$ of the residuated semigroup reduct of $\mathbf{X}$ over $\ker_\varphi = \{x \in X : \varphi(x) = u\}$. $\ker_\varphi = [u', u]$ if u' is idempotent, and $\ker_\varphi =]u', u]$ if u' is not idempotent[16].

[14]For notions which are not defined here, the reader may consult e.g. [7].

[15]We mean cancellative subalgebra, see Theorem 2.4.

[16]$\mathbf{ker}_\varphi$ is isomorphic to $\mathbf{u}^{\top\perp}$ if u' is idempotent, and isomorphic to $\mathbf{u}^{\top}$ if u' is not idempotent, where $\mathbf{u}^{\top\perp}$ and $\mathbf{u}^{\top}$ mean $\mathbf{u}$ equipped with a new top and bottom element or $\mathbf{u}$ equipped with a new top element, respectively, just like in items A and B in Definition 4.2.

One can recover $\mathbf{X}$ as follows:

1. Enlarge[17] the subset X_2 of X_φ by $\ker_\varphi$[18].
2. Enlarge the subset $X_1 \backslash X_2$ by $\ker_\varphi \setminus]u', u[$.
3. Enlarge the subset $X_\varphi \backslash X_1$ of X_φ by $\{\varphi(u')'\}$.

Equip the obtained set by the lexicographic ordering, and define the monoidal operation on it coordinatewise. It has then (u, t) as its unit element, it is residuated and thus it determines a unique residual operation. The here-constructed algebra will be isomorphic to $\mathbf{X}$. We believe that this unified treatment, although more elegant, does not make the discussion any shorter or any more transparent.

We say that an odd involutive FL_e-chain $\mathbf{X}$ is represented as a *finite* (or more precisely, n-ary) partial lexicographic product of linearly ordered abelian groups $\mathbf{G}_1 \dots, \mathbf{G}_n$, if $\mathbf{X}$ arises via finitely many iterations of the type III and type IV constructions using linearly ordered abelian groups $\mathbf{G}_1 \dots, \mathbf{G}_n$ in the way it is described in Theorem 11.1. By using this terminology we may rephrase Theorem 11.1: Every odd involutive FL_e-chain, which has only finitely many positive idempotent elements can be represented as a finite partial lexicographic product of linearly ordered abelian groups. By Hahn's theorem one can embed the linearly ordered abelian groups into some lexicographic products of real groups. Therefore, a side result of Theorem 11.1 is the generalization of Hahn's embedding theorem from the class of linearly ordered abelian groups to the class of odd involutive FL_e-chains which have only finitely many positive idempotent elements:

Corollary 11.5. *Odd involutive FL_e-chains which have exactly $n \in \mathbb{N}$ positive idempotent elements embed in some n-ary partial lexicographic product of lexicographic products of real groups.*

Finally, we remark that ordinary lexicographic products can be used instead of partial lexicographic products if the less ambitious goal of embedding only the monoidal reduct is aimed at.

Corollary 11.6. *The monoid reduct of any odd involutive FL_e-chain which has only finitely many idempotent elements embeds in the finite lexicographic product of $\mathbf{G}_1^{\top\perp} \overrightarrow{\times} \mathbf{G}_2^{\top\perp} \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n^{\top\perp}$, where $\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_n$ are the linearly ordered abelian groups of its group representation.*

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[17] Throughout this example enlargement is meant in the sense of Definition 1.2.

[18] Here we can even save the amendment of the second algebra by top or top and bottom elements, compared to the definition of the partial lexicographic products.

Proof. By Theorem 4.4, $\mathbf{X}_{\mathbf{Z_V}} \overset{\twoheadrightarrow}{\times} \mathbf{G} \leq \mathbf{X}_{\mathbf{Z}} \overset{\twoheadrightarrow}{\times} \mathbf{G}$ and $\mathbf{X}_{\mathbf{V}} \overset{\rightarrowtail}{\times} \mathbf{G} \leq \mathbf{X} \overset{\rightarrowtail}{\times} \mathbf{G}$. hold. Observe that the monoidal reduct of $\mathbf{X} \overset{\rightarrowtail}{\times} \mathbf{G}$ embeds into the monoidal reduct of $\mathbf{X}_{(\mathbf{X_{gr}})} \overset{\twoheadrightarrow}{\times} \mathbf{G}$, which in turn embeds into the monoidal reduct of $\mathbf{X}^{\top\perp} \overset{\rightarrow}{\times} \mathbf{G}^{\top\perp}$. Now let an odd involutive FL_e-chain, which has only finitely many idempotent elements, be given. Take its group representation. Guided by its consecutive iterative steps, in each step consider $(\mathbf{X}_{i-1})^{\top\perp} \overset{\rightarrow}{\times} {\mathbf{G}_i}^{\top\perp}$ instead of $\mathbf{X}_{i-1_{\mathbf{Z}_{i-1_{\mathbf{V}_{i-1}}}}} \overset{\twoheadrightarrow}{\times} \mathbf{G}_i$ or $\mathbf{X}_{i-1_{\mathbf{V}_{i-1}}} \overset{\rightarrowtail}{\times} \mathbf{G}_i$. In the end, this results in the original algebra being embedded into the lexicographic product $\mathbf{G}_1^{\top\perp} \overset{\rightarrow}{\times} \mathbf{G}_2^{\top\perp} \overset{\rightarrow}{\times} \cdots \overset{\rightarrow}{\times} \mathbf{G}_n^{\top\perp}$, where $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n$ are linearly ordered abelian groups. ∎

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