Janusz Czelakowski

Deontology of Compound Actions

Abstract. This paper, being a companion to the book [2] elaborates the deontology of sequential and compound actions based on relational models and formal constructs borrowed from formal linguistics. The semantic constructions presented in this paper emulate to some extent the content of [3] but are more involved. Although the present work should be regarded as a sequel of [3] it is self-contained and may be read independently. The issue of permission and obligation of actions is presented in the form of a logical system $\models$. This system is semantically defined by providing its intended models in which the role of actions of various types (atomic, sequential and compound ones) is accentuated. Since the consequence relation $\models$ is not finitary, other semantically defined variants of $\models$ are defined. The focus is on the finitary system $\models_f$ in which only finite compound actions are admissible. An adequate axiom system for $\models_f$ it is defined. The strong completeness theorem is the central result. The role of the canonical model in the proof of the completeness theorem is emphasized.

Keywords: Frame, Model, Atomic action, Sequential action, Compound action, Permission, Prohibition, Obligation.

Mathematics Subject Classification: 03B50, 03B60, 03B80.

Obligation should be action guiding
Tamminga [16]

Introduction and Overview

The category of a situation is central in the ontology of action. Generally speaking, actions transform situations into new situations. From the mathematical viewpoint, situations are modelled as complex set-theoretic entities encompassing such factors as states of affairs, spatio-temporal coordinates, agents, the way the agents cooperate etc. The undertaken actions and their succession may also be components of situations. It is not necessary to present here a detailed account of situation theory. In the simplified description we shall present, three categories of pertinent objects are isolated: states of affairs (simply: states), atomic actions, and compound actions and,

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to a modest extent, agents of actions. Situations are reducible here to states; we therefore abstract from other components of situations.

Atomic actions resemble black boxes. Each atomic action from a state being the input leads to a state being the output of the performed action. Thus there are states that form the input of the atomic action and the states that make up the output of the action. A given atomic action undertaken in a given input yields, if performable, a state belonging in the input. Each atomic action is therefore identified with a binary relation on the set of states. This is in accordance with the paradigm adopted in dynamic logic; see [6].

In turn, compound actions are defined as sets of finite sequences of atomic actions. From the formal linguistic perspective, compound actions may be regarded as formal languages over the alphabet formed by the set of atomic actions. The actions encountered in the everyday situations are compound. We mention a few: baking a bread, manufacturing a car, making everyday morning routine etc. Each of these actions can be performed in various ways depending on the choice of a sequence of atomic actions the given compound action encompasses. To each compound action one assigns a binary relation—the resultant relation of the compound action. The resultant relation abstracts from the way a given compound action is performed—the initial states and the final state matter here; the intermediary states and atomic action that make up the compound action are disregarded from the perspective of the resultant relation. Thus the input and output of the resultant relation are relevant; the other factors are omitted. The resultant relation of a compound action is also a binary relation on the set of states. But this resultant relation need not belong to the preselected set of atomic actions. For example, while making the morning routine, the compound action which may last an hour, we may distinguish other subactions as shaving, washing, dressing up etc. Each of these is also compound; they may be performed in various combinations. As to the resultant relation we distinguish here one initial state in which I am not washed, not shaved and not dressed. There is also one final state in which the things are the other way round: I am wasked, shaved and dressed, and ready to make breakfast. Each of the mentioned complex subactions: washing, shaving, dressing etc. can be performed in unlimited number of ways.

In this paper obligations, permissions and prohibitions concern actions; thereby actions, and not state of affairs, are deontologically loaded. The idea that actions are primary bearers of deontic values is not new. This issue, in the context of propositional deontic logic, is discussed e.g. in [14] and [10]; see also *Final remarks below.*

An *obligation* is a proposition that renders the course of action that an agent is required to take. This notion is formally defined in this paper. As is well known, obligations depend on various normative contexts, such as legal or moral ones. There are also other obligations, such as obligations of etiquette, social obligations, religious and political obligations etc. Obligations are expressed as propositional requirements which must be fulfilled.

The paper is concerned with a bunch of issues centred around the deontological problem of obligation of compound actions. In the monograph [2] a certain logically coherent conception of obligation of *atomic actions* is presented. In a more elaborated form this conception is expressed in the form of two simple logical systems. The first system $\boldsymbol{DL}$ is based on two specific deontological axioms: the closure principle for *atomic* actions, $P\alpha \leftrightarrow \neg F\alpha$ (any atomic action α is permitted if and only if it is not forbidden), and $O\alpha \rightarrow P\alpha$ (any atomic action α which is obligatory is permitted).[1] The underlying logic is CPC. In the system $\boldsymbol{DL}^{+}$ the closure principle is annulled and replaced by the weaker axiom $P\alpha \rightarrow \neg F\alpha$ (any permitted atomic action α is not forbidden) while Kant's Principle is retained. The completeness theorems for these two systems are provided. The semantics for $\boldsymbol{DL}$ and $\boldsymbol{DL}^{+}$ is based on elementary action systems as proper semantic units. Obligation of an atomic action A in a given state u means that A is performable in u and all options of departing from u by means of performing actions *other* than A are blocked (cf. Leo Apostel's remark: "an act is obligatory, if it is the only act such that there is no other act equally good or better"; [1], p. 75).

The above monograph left the problem of the deontology of *compound* actions for further scrutiny. Any compound action is viewed as set of finite sequences of atomic actions. The basic difficulty consists in the apprehension of obligation of a *compound* action in a given state u. In this work another approach to this problem is presented. This approach expounds the idea of obligation as a commitment to attainability of definite goals. This is the *task-oriented* deontology: a compound action is compulsory in virtue of the fact that it is designed as a set of sequences of atomic actions whose consecutive

[1] In discussions in ethical theory, the above law is called *Kant's Principle* or *Kant's law*. It is usually formulated in the agential form:

Anything morally obligatory for an agent must be within the agent's ability.

See [7], p. 62.

implementations lead from initial states to some expected or planned final states.[2]

The conceptual framework is semantically modeled on action systems defined as in [2]. This approach is closer to finite automata, where, commencing with some initial state, the automaton reaches a final state by means of performing strings of atomic actions labelled by the symbols of the alphabet Σ inherent to the automaton. The set of words accepted by the automaton (i.e., the regular language accepted by the automaton) is treated as the compound action that is compulsory.

This approach is also close to propositional dynamic logic, because action systems are in fact special multimodal Kripke frames. As is known, dynamic logic stems from the theory of computer science, where it is used to prove correctness properties of computer programs. A computer program is a sequence of actions of a certain kind. The paper by Meyer [10] presents a deontic logic of actions that is motivated by dynamic logic. The approach presented in this paper differs substantially from that of Meyer both with regard to syntax and semantics of the pertinent logical systems. The problem of finding tangent points between the algebraic structure of compound actions presented here and the algebra of actions in the sense of Meyer needs careful scrutiny.

This paper is a kind of a companion piece to the monograph [2]. It is also a sequel to the paper [3]. Some deontological issues, especially those concerning compound actions, that were merely outlined in the monograph, are presented here in a more systematic way in the form of a coherent and strict logical system $\models$. This system is semantically defined by providing its intended models in which the role of actions of various types (atomic, sequential and compound ones) is accentuated. Since the consequence relation $\models$ is not finitary, other semantically defined variants of $\models$ are defined. The focus is on the finitary system $\models_f$ in which only finite compound actions are admissible. An adequate axiom system for $\models_f$ is defined. The strong completeness theorem is the central result. The role of the canonical model is emphasized.

[2] Cf. the following short propaganda passage from the newspaper Pereslavl Week from the Stalinist times. The text reads: “The plan is the law, fulfilment is duty, overfulfillment is honor!”.

1. Task-Oriented Obligations and Their Models

This section is motivated by the fact that usually actions undertaken by people are purposeful and subordinated to reaching a definite goal.

The goal is often difficult to define. A good example is tax collection. The tax system is subjected to numerous conditions: economic, political, social, legal, etc. The tax system in a modern state is to make state institutions be able to act; on the other hand, it is supposed to guarantee satisfaction of the principles of the so-called social justice (whatever it means), secure covering of old age pensions, health insurance, benefits for the unemployed, and the like, at last—is to cause all or selected sectors of economy to develop in such a way that they should put only a minimum burden on the state central budget. These tasks are hard to reconcile. The tax system is usually very nuanced and can be easily spoiled with hasty decisions, especially ones of the populist character. Elaboration of it is typically a product of top class specialists in this domain.

While elaborating the tax system, one can discern—in the background—certain values which can be very different. In democratic countries, beside the above-mentioned principles of social justice, which are usually articulated in the fundamental legal acts and/or norms of social co-existence that have been worked out through centuries, values resulting in equality in the eyes of law, economic freedom, proper distribution of state's incomes, etc., are truly significant ones. These are obvious and well-known questions and there is no need to dwell on them here any longer.

A sequence of actions is a principal unit that is relevant from the viewpoint of purposeful action. While the conception of action performability presented in [3] abstracts from its teleological aspects, they are explicitly articulated in this paper as inherent components of the notion of obligation.

Inevitably, assuming performability of a sequence of actions, one must guarantee performability of each of its links. However, performability of each link is subjected to a certain purposeful intention. The basis of the conception presented in this paper is that purposeful actions derive from a state of things called the initial state and lead to certain intended states of things called final states. Agents have at their disposal an established repertoire of available simple (or atomic) actions. Combining them in the right manner in finite strings of atomic actions and performing them consecutively, they attain, if possible, a final state. The final states can be attained in many ways through executing different sequences of simple actions. All such sequences, when collected together, form a compound action. Following automata theory, we shall call them accepted words. The compound action consisting of all accepted words of atomic actions is therefore an obligatory action. This compound action, say A, has to be performed so that starting from the initial state, one reaches a final state. But not all sequential actions belonging to A are compulsory. To achieve the goal one selects a string of atomic actions belonging to A which he/she adheres to and then performs them. This selected sequence is obligatory. For example, studying at a university is not compulsory. It is merely a permitted action if some conditions are met as e.g. having the high-school exit exam. But in the initial state in which one is enrolled in a university, studying becomes an obligatory, compound action. Some strings of simpler actions such as enrolling in various courses, passing the final exams, paying a tuition etc. are then obligatory until earning the diploma, which is the final state.

Each *task* is defined by specifying two sets of states: the set of initial states, and the set of final states. Actions subordinated to a definite task are defined as sets of finite sequences of atomic actions whose consecutive performances lead from initial states to final states. Reaching a finite state terminates a sequence of atomic actions and marks implementing the task.

The semantic apparatus we introduce here is similar, though more complicated, to the one defined in [3].

Let $\langle \Sigma^*, \bullet, e\rangle$ be the free semigroup freely generated by a nonempty set of generators Σ. Thus, formally, the elements of Σ^* are finite sequences of members of Σ. $\bullet$ is the operation of concatenation of sequences. e stands for the empty sequence.

The elements of Σ are called *symbols of atomic actions* while the elements of Σ^* are referred to as *sequences* of symbols of atomic actions or simply *symbols* of *sequential actions*. We shall simply refer to the elements of Σ^* as to *words*. Each symbol $a \in \Sigma$ is treated as a word of length 1. Therefore we may assume that $\Sigma \subset \Sigma^*$.

From the linguistic viewpoint, the set Σ is an alphabet and the members of Σ^* are (finite) words over Σ. (Here the set Σ is allowed to be infinite.) $\wp(\Sigma^*)$ is the power set of Σ^*. Thus, the elements of $\wp(\Sigma^*)$ are subsets of Σ^*. From the linguistic perspective, the elements of $\wp(\Sigma^*)$ are formal languages over the alphabet Σ. But here we adhere to the terminology of action theory and call the elements of $\wp(\Sigma^*)$ *symbols of compound actions* over Σ. Accordingly, the symbols of compound actions are the same objects as languages over Σ.

Some notation. The elements of Σ will be marked as a, b, c, d with indices if necessary. Sequential actions (words) will be denoted by x, y, z, w with indices if necessary. In turn, compound actions will be marked by capital letters A, B, C etc.

Models

DEFINITION 1.1. A *model* is defined as a triple

$$\langle W, V, V_R \rangle,$$

where

- W is a non-empty set, called the set of *states*,
- V is a mapping assigning to each symbol $a \in \Sigma$ a binary relation $V(a)$ defined on the set W.
- V_R is a mapping assigning to each symbol $a \in \Sigma$ a binary relation $V_R(a)$ such that $V_R(a) \subseteq V(a)$. ■

$V(a)$ is called the *atomic action* of a on the states of W and any pair $\langle u, w \rangle \in V(a)$ is called a *possible performance* of $V(a)$.

Although for each a, $V_R(a)$ is also a relation on W, it is not qualified as an atomic action; its role is different. Speaking metaphorically, the mapping V_R defines the limits of freedom in the model M; this remark can be made precise—see [2]. V_R imposes limitations of a definite type on the possibility of direct transitions from some states to others. A great variety of possible interpretations of V_R is obtained, choosing—in a proper way—interesting classes of action systems. To mention only the most important of these interpretations: the mapping V_R can be interpreted as a physical possibility of a transition from some states into others, or as psychological admissibility for a given man, or as compatibility with a social role, or as compatibility with labour regulations of a given institution. Apart from physical limitations, it is often necessary in some action systems to take into account restrictions that are imposed by law and its regulations. These are *deontic* action systems—some actions in such systems are legally forbidden, e.g., on the strength of traffic regulations, though they may be physically feasible (actions in *fraudem legis*). V_R may also reflect religious commitmens. Each of the mentioned interpretations is bound with a selection of certain mapping V_R defined on an appropriate set of states W.

In the context of deontology, the elements of $V_R(a)$ are called permitted performances of the action $V(a)$. Thus, if $\langle u, w \rangle \in V_R(a)$, this pair is called a *permitted performance* of $V(a)$ (in the sense of V_R). The binary relation $V_R(a)$ is also called the *resultant relation* of the action $V(a)$ in the model.

A model $\langle W, V, V_R\rangle$ is *deterministic* if for every $a \in \Sigma$, the relation $V(a)$ is a unary total function, i.e., it is a function whose domain is W. In any deterministic model, every relation $V_R(a)$ is a partial function being the restriction of $V(a)$ to a non-empty subset of W.

Deterministic models play a significant role in the presented approach, because their logical power is the same as the class of all relational models. More specifically, the logical system $\models$ we shall introduce is semantically defined by means of *all* relational models. But in view of Adequacy Theorem (Theorem 3.12), $\models$ is characterized by a single model M_c, the canonical model of $\models$. The model M_c is deterministic. Consequently, $\models$ is also complete with respect to the class of determininistic models.

V is inductively extended on the set Σ^* of words by means of the composition of binary relations. It is assumed that for the empty word e, $V(e)$ is the diagonal of W,

$$V(e) := \mathbf{0}_W.$$

Then, for any word $x \in \Sigma^*$ and any $a \in \Sigma$,

$$V(xa) := V(x) \circ V(a),$$

where $\circ$ is the composition operations of relations. Thus, if $x = a_1 \dots a_m$, then

$$V(x) = V(a_1) \circ \cdots \circ V(a_m),$$

that is, $u\, V(x)\, w$ if and only if there exists a sequence of states $u_1 \dots u_m$ with $u_m = w$ such that $u\, V(a_1)\, u_1 \dots u_{m-1}\, V(a_m)\, u_m$.

$V(x)$ is called the action of the sequence x on the states of W.

According to the above definition, two symbols a and b of atomic actions, when combined into the word ab, determine the binary relation $V(a) \circ V(b)$, the composition of the relations $V(a)$ and $V(b)$.

V is extended onto any arbitrary subsets $A \subseteq \Sigma^*$:

$$V(A) := \bigcup\{V(x) : x \in A\}.$$

$V(A)$ is also a binary relation on W.

If the set A is empty, then $V(\emptyset)$ is the empty relation on W. If $A = \Sigma^*$, then $V(\Sigma^*)$, the set-theoretic union of the relations $V(x)$, $x \in \Sigma^*$, may be a proper binary relation on W.

We also define:

$$V_R(e) \text{ is a subrelation of the diagonal } V(e),$$

and for any non-empty word $x \in \Sigma^*$ and any $a \in \Sigma$,

$$V_R(xa) := V_R(x) \circ V_R(a).$$

Thus if $x = a_1 \dots a_m$, then $u\, V_R(x)\, w$ holds if and only if there is a sequence of states $u_1 \dots u_m$ with $u_m = w$ such that $u\, V_R(a_1)\, u_1 \dots u_{m-1}\, V_R(a_m)\, u_m$.

The mapping V_R is extended onto arbitrary subsets of Σ^*. For any set $A \subseteq \Sigma^*$ we define:

$$V_R(A) := \bigcup \{V_R(x) : x \in A\}.$$

$V_R(x)$ is called the *resultant relation of the sequence* x and $V_R(A)$ is the *resultant relation of the compound action* A. ■

DEFINITION 1.2. A *task-oriented model* is any quintuple of the form $M = \langle W, V, V_R, I, F\rangle$, where $\langle W, V, V_R\rangle$ is a model and I, F are subsets of W.

I is the set of *initial* states and F is the set of *final* states. The pair $\langle I, F\rangle$ is called a *task* assigned to the model $\langle W, V, V_R\rangle$. ■

NOTES. 1. The definition of a task is borrowed from automata theory. In the description of an automaton one distinguishes the initial state and the set of final states. Here it is assumed that there may be more than one initial state.

If one considers such a compound action as baking a bread, then I specifies initial conditions and ingredients that are relevant to this action such as all purpose flour, seasonings and various components as well as the type of oven in the bakery and its adjustment etc. The set F specifies types of bread to be baked like rye bread, brown bread, baguettes etc. Other factors are irrelevant here.

2. Real-life situations are more involved than the ones abstractly modelled by means of the above set theoretic constructs. In multi-agent systems usually one specifies a finite family of sets of initial states, each set selected for each agent who initializes actions. The agents perform sequences of concerted actions, which collectively form a complicated graph of mutual dependencies among the agents and their actions. In simple cases these graphs are finite trees, where each leaf is labelled with an initial set, and the root of the tree is labelled by the set of final sets, cf. the notes on agency in *Final remarks.* ■

Deontology

The list of the definitions of deontic operators we shall give is parallel to the way formal languages are introduced in linguistics: one first defines words over an alphabet, and then languages as sets of words. Accordingly, we first define the deontic operators on sequential actions and then on compound actions, the latter viewed as sets of sequential actions. This distinction is motivated by the different notions of obligation and permission when applied to sequential and compound actions.

Throughout this section $M = \langle W, V, V_R, I, F\rangle$ is a task-oriented model. The notion of a permitted sequential action $V(x)$ is defined as follows.

DEFINITION 1.3. Let x be in Σ^*. The sequential action $V(x)$ is *permitted in a state* $u \in W$ if and only if there exists a state w such that $u\, V_R(x)\, w$.

In particular, $V(e)$ is *permitted in* u if and only if $u\, V_R(e)\, u$.

Thus, in the developed form, if $x = a_1 \dots a_m$, then $V(x)$ is permitted in u if and only if there exists a finite path of transitions $u_1\, V_R(a_1)\, u_2\, V_R(a_2) \dots u_m\, V_R(a_m)\, u_{m+1}$ between states such that $u = u_1$. Since $V_R(a_i) \subseteq V(a_i)$, *each* transition $u_i\, V_R(a_i)\, u_{i+1}$ is accomplished by means of the atomic action $V(a_i)$ for $i = 1, \dots, m$. (The task $\langle I, F\rangle$ does not intervene in the definition of permission.) Accordingly, the empty word e is permitted in u if and only if $u\, V_R(e)\, u$. ■

DEFINITION 1.4. A compound action $V(A)$, where $A \subseteq \Sigma^*$, is *permitted in a state* u if and only if for *some* sequence $x \in A$, the action $V(x)$ is permitted in u. ■

The above definition represents "minimalistic" attitude towards the permission of compound actions—a compound action is permitted in a state in virtue of the fact that merely *some* string of atomic actions belonging to $V(A)$ is permitted in this state. A stronger standpoint is possible, viz, one may require that *all* sequences of atomic actions belonging to $V(A)$ is permitted at this state. This option, though legitimate, is not discussed in this work.

DEFINITION 1.5. A sequential action $V(x)$ is *obligatory in a state* u if $u \in I$ and there exists a state $w \in F$ such that $u\, V_R(x)\, w$.

Thus if $x = a_1 \dots a_m$, this means that there exists a finite path of transitions $u_1\, V_R(a_1)\, u_2\, V_R(a_2) \dots u_m\, V_R(a_m)\, u_{m+1}$ between states such that $u = u_1 \in I$ and $w = u_{m+1} \in F$. (The transition $u_i\, V_R(a_i)\, u_{i+1}$ is thus accomplished by consecutively performing the atomic actions $V(a_i)$ for $i = 1, \dots, m$.)

It follows that the empty word e is obligatory in u if and only if $u \in I \cap F$ and $u\, V_R(e)\, u$. ■

The above definition involves the task $\langle I, F \rangle$ associated with the model M. Thus sequential actions are obligatory only at initial states and, when performed, they lead to a final state. There may be several sequential actions x that are obligatory at some state $u \in I$ and that lead to states in F.

Since the notion of an agent as well as spatio-temporal situational components of actions are not incorporated into the above formalism, it does not make sense to say here that an agent is obliged to perform *several* obligatory sequential actions simultaneously. This issue of agency of compound actions is not analysed in this paper at length, see *Final Remarks* below.

As each atomic action $a \in \Sigma$ qualifies as a sequential action of length 1, it follows that $V(a)$ is obligatory at u if and only if $u \in I$ and for some final state $w \in F$ it is the case that $u\, V_R(a)\, w$. ■

NOTE. There is an analogy between the above notion of obligation of $V(x)$ for finite words and the notion of acceptance of a word by a finite automaton. Definition 1.5 does not imply that if $V(x)$ is obligatory in some state u, then for any non-empty prefix y of x, the action $V(y)$ is obligatory in u as well. Thus obligation is not inherited by non-empty prefixes of an obligatory word. The same phenomenon occurs in finite automata—if a word is accepted, then not all prefixes of this word are accepted.

One may also consider a stronger notion of obligation that takes into account the above inheritance of obligation. This would result in introducing a hierarchy of (sub)tasks of the task (I, F) for the obligatory action $V(x)$. Thus if y is a prefix of x and $x = yz$, then the final set of subtask for $V(y)$ would be included in the initial set for the subtask for $V(z)$. The semantics for such hierarchical obligations could be defined by means of a suitable modification of the models discussed in this paper. This option, though interesting, is not discussed here. ■

DEFINITION 1.6. A compound action A over Σ is *obligatory in a state* u if and only if some sequential action $x \in A$ is obligatory in u. ■

Note that *every* sequential or compound action which is obligatory in u is permitted in u. Therefore the above semantics of deontology of actions validates Kant's Principle. According to Definition 1.6, $V(A)$ is obligatory in a state u if and only if for *some* word $x \in A$, the sequential action $V(x)$ is obligatory in u. The following example illustrates this definition. Suppose we are given a compound action A as e.g. *Learning calculus 166 to pass the exam.* If this action is obligatory in some state prior to the exam date, each agent of this action (a student) has, according to the syllabus, a variety of logically legitimate paths to follow which would lead him from the current state of his mathematical knowledge to the state in which he gets a positive grade in the exam. A thus consists of many sequential actions. He selects one of many alternative plans of learning calculus—he may first learn volumes done in washers and shells, arc length and areas of surfaces, work and centres of mass, integration by parts, then learn limits of sequence and functions, infinite series, indefinite integrals etc. There are various options available here. It is up to him which path he chooses in accordance with his preferences. Each such a path represents a word from the language forming the above compound action. Thus, though this action is obligatory to him, only *one* path (word) is obligatory, viz. the one he selects. It would be an absurdity to claim that all conceivable paths (words) are obligatory for him in a given state.

If B is a compound action being a singleton, $B = \{x\}$ for some word x, that is obligatory, then the agent of B cannot of course choose a sequence in B, but proceeds according to the string of actions x. We shall later return to the issues the above definitions evoke.

The issues of agency and obligation are correlated but these links are not expressible in the present formalism. (Agents of an action belong to the situational envelope on action systems; but in this work the latter is reducible to the set of states of the system. States do not involve agents.)

Suppose that I have a strict tutor who selects exactly one path $x \in A$ I should follow to learn calculus 166. The decision of the tutor is the source of my obligation to the sequential action x—this action is obligatory and I am the agent of x. Does it imply that I am also the agent of A? Intuitively yes, because I also perform A. On the other hand, suppose I have a less demanding tutor who advises me to select only one path y in A at my discretion to learn calculus 166 (because any sequence in A leads from the set of initial states to the set of final states.) In this case I am the agent of A. I am also the agent of the sequential action $y \in A$ chosen by me. This action is also obligatory. There is however a certain semantic difference between the first and the second situation. We thus see that mutual relationships holding between agency and obligations are not straightforward; they are open to further scrutiny.

For each initial state $u \in I$ there exists the largest compound obligatory action, viz., the action $V(A_u)$, where A_u consists of all words x such that $V(x)$ is obligatory in u:

$$A_u := \{x \in \Sigma^* : (\exists w \in F)\, u\, V_R(u)\, w\}.$$

It is not difficult to show that if W and Σ are finite sets, A_u is a regular language in the sense of formal linguistics. (To define languages from other levels of the Chomsky hierarchy, one must incorporate situational components into the above picture of action and work not with elementary action systems of the form (W, V, V_R) but with more involved situational action systems in the sense of [2].)

One may argue that the above picture of obligation could be modified by introducing a preference relation on the set $A_{I,F}$ of all sequential actions x starting in an initial state and ending in the set of final states:

$$A_{I,F} := \bigcup_{u \in I} A_u.$$

The preference relation need not be a linear order. The most preferred *sequential* actions, i.e., maximal elements of $A_{I,F}$, would become obligatory. It would be then plausible to say that a *compound* action A is obligatory in a state u if $u \in I$ and A contains a best preferred sequential action $x \in A_u$. Such a solution is justified by the fact that when working with various performable strings of actions involved in action plans, one usually prefers sequences which are are less laborious and more economical. This pragmatical standpoint would require a suitable modification of Definition 1.6. The above option is not analysed in this paper.

Definition 1.7. A sequential action $V(x)$ is *forbidden in* u if and only if it is not permitted in u.

A compound action $V(A)$, $A \subseteq \Sigma^*$, is forbidden in u if and only if all sequential actions $V(x)$, $x \in A$, are forbidden in u. ■

We assign to each sequential action $V(x)$, where $x \in \Sigma^*$, the proposition $\boldsymbol{P}V(x)$ consisting of all states $u \in I$ such that $V(x)$ is permitted in u. Analogously, for each compound action $V(A)$ with $A \subseteq \Sigma^*$, we define the proposition $\boldsymbol{P}V(A)$ consisting of states $u \in I$ in which $V(A)$ is permitted.

Definition 1.8.

(p1) $u \in \boldsymbol{P}V(x) \Leftrightarrow_{df} V(x)$ is permitted in u

($\Leftrightarrow$ there exists a state w such that $u\, V_R(x)\, w$).

(p2) $u \in \boldsymbol{P}V(A) \Leftrightarrow_{df} V(A)$ is permitted in u

($\Leftrightarrow$ there exists a word $x \in A$ such that $u \in \boldsymbol{P}V(x)$).

■

It follows that

(1) $$\boldsymbol{P}V(A) = \bigcup \{\boldsymbol{P}V(x) : x \in A\}.$$

According to (1), to permit the compound action $V(A)$ it suffices to permit only *one* of its instances $V(x)$, where $x \in A$.

If $A = \emptyset$, then $\boldsymbol{PV}(\emptyset) := \emptyset$. If $A = \{e\}$, then $\boldsymbol{PV}(e) := \{u \in W : u\, V_R(e)\, u\}$.

For each symbol $a \in \Sigma$, the proposition $\boldsymbol{PV}(a)$ coincides with the domain of the relation $V_R(a)$, and $\boldsymbol{PV}(x)$ is equal to the domain of $V_R(x)$.

Analogously, we also assign to each sequential action $V(x)$, where $x \in \Sigma^*$, the proposition $\boldsymbol{OV}(x)$ consisting of all states $u \in I$ such that $V(x)$ is obligatory in u. Moreover, for each compound action $V(A)$ with $A \subseteq \Sigma^*$, we define the proposition $\boldsymbol{OV}(A)$ consisting of states $u \in I$ in which $V(A)$ is obligatory.

Definition 1.9.

(o1) $u \in \boldsymbol{OV}(x) \Leftrightarrow_{df} V(x)$ is obligatory in u

($\Leftrightarrow u \in I$ and there exists a state $w \in F$ such that $u\, V_R(x)\, w$),

(o2) $u \in \boldsymbol{OV}(A) \Leftrightarrow_{df} V(A)$ is obligatory in u

($\Leftrightarrow$ there exists a word $x \in A$ such that $u \in \boldsymbol{OV}(x)$.

■

It follows that

(2) $\boldsymbol{OV}(A) = \bigcup\{\boldsymbol{OV}(x) : x \in A\}$.

As to prohibited actions, we define:

Definition 1.10.

(f1) $u \in \boldsymbol{FV}(x) \Leftrightarrow_{df} V(x)$ is forbidden in u

($\Leftrightarrow V(x)$ is not permitted in u).

(f2) $u \in \boldsymbol{FV}(A) \Leftrightarrow_{df} V(A)$ is forbidden in u

($\Leftrightarrow$ for every word $x \in A$ the action $V(x)$ is forbidden in u).

■

Thus

(3) $\boldsymbol{FV}(x) = W \backslash \boldsymbol{PV}(x)$, for all $x \in \Sigma^*$,

and

(4) $\boldsymbol{FV}(A) = \bigcap\{\boldsymbol{FV}(x) : x \in A\}$.

It follows that

$$\boldsymbol{FV}(A) = \bigcap\{W \backslash \boldsymbol{PV}(x) : x \in A\} = W \backslash \bigcup\{\boldsymbol{PV}(x) : x \in A\} = W \backslash \boldsymbol{PV}(A).$$

Thus the closure principle in the semantic form

(5) $\boldsymbol{F}V(A) = W \backslash \boldsymbol{P}V(A)$,

holds for all $A \subseteq \Sigma^*$.

NOTE. We shall refer to the obligation of compound actions in the sense of Definition 1.6 to as ∃-*obligation* due to the occurrence of the existential quantifier ∃ in the *definiens*. Analogously the notion of permission in the sense of Definition 1.4 is referred to as ∃-*permission*, and the prohibition in the sense of Definitions 1.7 is marked as ∀-*prohibition*.

The above remarks point out other options according to which one may define deontic operators on compound actions. E.g., a compound action $V(A)$, $A \subseteq \Sigma^*$, is *obligatory in a state* u if all sequential actions $V(x)$, $x \in A$, are obligatory in u. This form of obligation is referred to as the ∀-*obligation* due to the occurrence of the universal quantifier ∀ in the *definiens*. If obligation is considered as the ∀-*obligation*, formula (2) turns into

$$\boldsymbol{O}_{\forall} V(A) = \bigcap \{\boldsymbol{O}V(x) : x \in A\}.$$

By way of analogy, we may also define permission as the ∀-*permission*: a compound action $V(A)$, $A \subseteq \Sigma^*$, is ∀-*permitted in a state* u if *all* sequential actions $V(x)$, $x \in A$, are permitted at u. Then formula (1) is replaced by

$$\boldsymbol{P}_{\forall} V(A) = \bigcap \{\boldsymbol{P}V(x) : x \in A\}.$$

In turn, the formula

$$\boldsymbol{F}_{\exists} V(A) = \bigcup \{\boldsymbol{F}V(x) : x \in A\}$$

also represents a form of prohibition: $V(A)$ is prohibited on account of the fact that $\boldsymbol{F}V(x)$ is prohibited merely for *some* word $x \in A$.

It is clear that $\boldsymbol{O}_{\forall} V(A) \subseteq \boldsymbol{O}V(A)$, $\boldsymbol{P}_{\forall} V(A) \subseteq \boldsymbol{P}V(A)$, and $\boldsymbol{F}V(A) \subseteq \boldsymbol{F}_{\exists} V(A)$, for all $A \subseteq \Sigma^*$.

If obligations, permissions and prohibitions are understood as above, the closure principle also holds for compound actions. Indeed, in virtue of the above definitions we have that

$$\boldsymbol{P}_{\forall} V(A) = W \backslash \boldsymbol{F}_{\exists} V(A),$$

because $W \backslash \boldsymbol{F}_{\exists} V(A) = W \backslash \bigcup \{\boldsymbol{F}V(x) : x \in A\} = \bigcap \{W \backslash \boldsymbol{F}V(x) : x \in A\} = \bigcap \{\boldsymbol{P}V(x) : x \in A\} = \boldsymbol{P}_{\forall} V(A)$.

But we also may have mixed options: some deontological operators are strong and some are weak.

We have therefore an abundance of options. Which triple of deontological operators to choose for compound actions? There are altogether 8 options available here. We mention some:

1. ∃-obligation, ∃-permission, ∀-prohibition
2. ∀-obligation, ∀-permission, ∃-prohibition
3. ∃-obligation, ∀-permission, ∀-prohibition
4. ∃-obligation, ∀-permission, ∃-prohibition etc.

Option 1 is adopted in this paper. Option 4 is a good alternative. As mentioned above, each of the options 1 and 2 entails both the closure principle for compound actions and Kant's Principle. In the other options, the closure principle for compound actions may be invalidated. In some options even Kant's Principle is rejected, e.g., for ∀-obligation and ∃-permission. ■

Here is yet another example shedding some light on the problem of choosing right options in some situations. We consider the compound action termed *The morning routine of an adult man*. It consists of finite sequences of simpler actions. (The actions involved into the definition of a routine are treated as types.) This compound action is performed in the initial proposition which is conventionally named "in the morning, after getting up" (A definite morning hour is not specified here.) The routine encompasses finite sequences of simpler actions such as: shaving, taking shower, putting on cosmetics etc. Yet another sequence encompasses other actions which are rather seldom performed as e.g. cutting nails, or trimming hair. Some simple actions may be performed in different orders, e.g. first shaving and then taking shower, or conversely, but they are not altogether permutable, due to physical limitations. In other words, not all sequences of atomic actions are meaningful as e.g. taking shower first and then using the lavatory. The task is clear—from the initial proposition one wants to achieve the final proposition in which the morning routine is finished. Generally, both the initial and final propositions are not single states. The action *The morning routine of an adult man* is obligatory in the initial proposition (unless the agent is a slovenly person). But this obligation is of the weak form here: the man is obliged to perform only *one* deliberately chosen by him sequence of simple actions that leads from the initial to the final proposition. It would be an absurdity to claim that he is committed to perform in the morning all possible sequences of simple actions of the morning routine. On the other hand, not all sequences of simple actions included into the morning routine are permitted but only those that are are physically feasible or meaningful.

Summing up, we may say that in this example the obligation, permission and prohibition of the morning routine (treated as a compound action) are all taken in the sense of Definitions 1.4, 1.6, and 1.7. Eg.—the morning routine is forbidden if none of the sequences belonging to it can be performed. This example favors the first of the listed options. Consequently, the closure principle in this case is preserved.

2. The Language of Action Deontology and Its Semantics

We first define the language $\boldsymbol{L}$ of action deontology. It is assumed that Σ is a countable set of symbols. It follows that Σ^* is countably infinite.

Atomic formulas are expressions of the form

(i) $O(x), P(x), F(x)$,

where $x \in \Sigma^*$,

(ii) $O(A), P(A), F(A)$,

for any set $A \subseteq \Sigma^*$.

Note that there are uncountably many atomic formulas of the form (ii), because the set Σ^* is countably infinite.

As each letter $a \in \Sigma$ is qualified as a word of length 1, the group (i) encompasses all atomic formulas $O(a), P(a), F(a)$, $a \in \Sigma$, and the following three formulas $O(e), P(e), F(e)$, where e is the empty word. In turn, (ii) encompasses formulas in which A is the empty set.

Compound formulas are built from the above atomic formulas by means of applying the Boolean connectives $\rightarrow$ and $\neg$. The connectives such as $\vee, \wedge$ and $\leftrightarrow$ are defined in the standard way as appropriate abbreviations.

$\boldsymbol{L}$ is the set of all formulas.

There are no extra propositional variables. Thus the above language $\boldsymbol{L}$ defines Boolean interrelations holding merely between deontologically "loaded" formulas only. (But one may expand the vocabulary of $\boldsymbol{L}$ by enriching the set of atomic formulas by a countably infinite list of propositional variables $p_0, p_1, \ldots$ and then form compound formulas by applying the connectives $\rightarrow$ and $\neg$ as above.)

The grammatical resources of $\boldsymbol{L}$ are very limited; the grammar of $\boldsymbol{L}$ is too poor to recursively express the definition of $P(x)$ for $x \in \Sigma^*$ in terms of the constituents $P(a)$, where a occurs in x, in the form of a plausible logical axiom. E.g. it is not possible to define $P(xa)$ in terms of $P(x)$ and

$P(a)$ without resorting to a linguistic counterpart of the operation of composition of relations. Such a connective is absent in the vocabulary of $\boldsymbol{L}$. A similar remark applies to the definition of $O(x)$ for $x \in \Sigma^*$. As a result, the words of Σ^* are taken as smallest grammatical units in $\boldsymbol{L}$ formulas, and not the symbols of Σ. Accordingly, the formulas of shape (i) are defined as atomic.

Interpretations and Truth

We define the notion of truth of formulas of $\boldsymbol{L}$ in models. The notation

$$M, u \models \sigma$$

means that σ is true in a task-oriented model $M = \langle W, V, V_R, I, F\rangle$ in a state $u \in W$.

Definition 2.1.
(1). Let x be a word in Σ^*.

$M, u \models P(x) \Leftrightarrow_{df} u \in \boldsymbol{P}V(x)$, i.e, the action $V(x)$ is permitted in u in M.
$M, u \models O(x) \Leftrightarrow_{df} u \in \boldsymbol{O}V(x)$, i.e, the action $V(x)$ is obligatory in u in M.
$M, u \models F(x) \Leftrightarrow_{df} u \in \boldsymbol{F}V(x)$, i.e, the action $V(x)$ is forbidden in u in M.

(2). Suppose $A \subseteq \Sigma^*$.

$M, u \models P(A) \Leftrightarrow_{df} u \in \boldsymbol{P}V(A)$, i.e, the action $V(A)$ is permitted in u in M.
$M, u \models O(A) \Leftrightarrow_{df} u \in \boldsymbol{O}V(A)$, i.e, the action $V(A)$ is obligatory in u in M.
$M, u \models F(A) \Leftrightarrow_{df} u \in \boldsymbol{F}V(A)$, i.e, the action $V(A)$ is forbidden in u in M.

The definition of $M, u \models$ is extended onto compound formulas as in classical logic. Thus

$$M, u \models \phi \to \psi \Leftrightarrow_{df} \text{ it is not the case that } M, u \models \phi \text{ or } M, u \models \psi;$$

and

$$M, u \models \neg\phi \Leftrightarrow_{df} \text{ if it is not the case that } M, u \models \phi.$$

A formula σ is *true in* the model $M = \langle W, V, V_R, I, F\rangle$, in symbols:

$$M \models \sigma$$

if and only if $M, u \models \sigma$ for all $u \in W$; we then also say that σ is *valid* in the model M.

σ is *logically valid* if it is valid in every task-oriented model. ■

The Logical Consequence $\models$

The logic is semantically defined as a consequence relation $\models$ operating on the set of all formulas of $\boldsymbol{L}$ in the following way. Let X be a set of formulas of $\boldsymbol{L}$ and σ a formula. We say that σ *logically follows* from X, in symbols:

$$X \models \sigma$$

if for every task-oriented model $M = \langle W, V, V_R, I, F\rangle$ and every state $u \in W$, if $M, u \vDash \phi$ for all $\phi \in X$, then $M, u \vDash \sigma$. (The "big" symbol $\models$ should not be confused with the "small" symbol $\vDash$, because they bear different meanings.)

$\models$ satisfies the standard conditions imposed on consequence relations (see e.g. [17]). Moreover $\models$ validates the tautologies of classical logic expressed in $\boldsymbol{L}$.

$\models$ satisfies the Deduction Theorem (DT): for any set X of formulas and any formulas ϕ, ψ:

(DT) $\qquad X \models \phi \rightarrow \psi$ if and only if $X \cup \{\phi\} \models \psi$.

A set X of formulas of $\boldsymbol{L}$ is *inconsistent* in the sense of $\models$ if and only if $X \models \phi$ for all formulas ϕ, equivalently, $X \models \phi \wedge \neg\phi$ for some (equivalently, for all) ϕ; otherwise X is called *consistent*. A formula σ is *inconsistent* (resp. *consistent*) if the set $\{\sigma\}$ is inconsistent (consistent).

It is easy to see that σ is $\models$-inconsistent if and only if $M, u \vDash \sigma$ for no model M and no state u of M.

A set of formulas X is *closed* in the sense of $\models$, shortly: X is $\models$-*closed*, if $X \models \sigma$ implies $\sigma \in X$, for every formula σ. $\models$-closed sets are also called *theories of* $\models$. They collectively form a closure system on $\boldsymbol{L}$, denoted by $Th(\models)$.

Some Tautologies of $\models$

If x and y are words in Σ^*, then the notation

$$x \preccurlyeq y$$

means that x is a prefix of y, i.e., there exists a word z such that $xz = y$.

It is easy to see that if $e \neq x \preccurlyeq y$, i.e., x is a non-empty prefix of y, then

$$P(y) \rightarrow P(x)$$

is a tautology of $\models$. In particular, for every non-empty word $x \in \Sigma^*$ and any symbol $a \in \Sigma$, the formula

$$P(xa) \rightarrow P(x)$$

is a tautology. But $O(y) \to O(x)$ is not a tautology whenever $e \neq x \preccurlyeq y$.

For any subset $A \subseteq \Sigma^*$ and for any word $x \in A$ the formulas

$$P(x) \to P(A) \text{ and } O(x) \to O(A)$$

are validated by $\models$, i.e., they are tautologies of $\models$. Kant's Principle

$$O(A) \to P(A),$$

is also a tautology. In particular, for any word $x \in \Sigma$,

$$O(x) \to P(x),$$

is a tautology. The formula $\neg P(\emptyset)$ is also a tautology.[3]

If A is a non-empty finite set of words, $A = \{x_1, \ldots, x_n\}$, then the formulas

$$P(A) \leftrightarrow P(x_1) \vee \cdots \vee P(x_n),$$
$$O(A) \leftrightarrow O(x_1) \vee \cdots \vee O(x_n),$$

are validated by the semantic consequence $\models$. Moreover for any (possibly infinite) subsets A, B of Σ^*, $\models$ validates the formulas

$$P(A \cup B) \leftrightarrow P(A) \vee P(B), O(A \cup B) \leftrightarrow O(A) \vee O(B).$$

In the framework of some deontic action logics, where one says about parallel executions of actions by an agent, the last formula is sometimes referred to as a version of *Ross Paradox*. Here it is not a paradox, because the notion of an agent is not involved in the adopted semantics and the formula, according to its meaning, does not refer to parallel performances of the actions A and B.

3. Ultrasets and the Canonical Model

Ultrasets

A *Lindenbaum set* of $\models$ is a maximal consistent set Δ in the sense of $\models$.

Every Lindenbaum set Δ contains all instances of logical axioms of CPC as well as specific action tautologies.

By maximality, $\sigma \vee \tau \in \Delta$ if and only if $\sigma \in \Delta$ or $\tau \in \Delta$, for all formulas σ, τ; equivalently, $\neg\sigma \in \Delta$ if and only if $\sigma \notin \Delta$, for any formula σ. Moreover, also by maximality, each Lindenbaum set Δ is $\models$-closed.

[3] The formula $P(e)$ is not a tautology. It is valid in all models in which the relation $V_R(e)$ is the diagonal.

It follows from maximality of Δ that for every word $x \in \Sigma^*$, either $P(x) \in \Delta$ or $\neg P(x) \in \Delta$. We also have that for every word $x \in \Sigma^*$, either $O(x) \in \Delta$ or $\neg O(x) \in \Delta$. In particular, $O(e) \in \Delta$ or $\neg O(e) \in \Delta$ and each option is possible here.

Definition 3.1. A set Δ of formulas of $\boldsymbol{L}$ is called an *ultraset* of $\models$ if and only if Δ is a Lindenbaum set of $\models$ with two additional properties holding for all sets $A \subseteq \Sigma^*$:

(1) $O(A) \in \Delta \Leftrightarrow O(x) \in \Delta$ for some word $x \in A$,

(2) $P(A) \in \Delta \Leftrightarrow P(x) \in \Delta$ for some word $x \in A$. ■

Since the implications $O(x) \to O(A)$ and $P(x) \to P(A)$ are $\models$-valid, for all non-empty sets $A \subseteq \Sigma^*$ and all $x \in A$, we see that in (1) and (2) only the implication ($\Rightarrow$) matters.

Every ultraset, being maximal consistent, is $\models$-closed.

Although in the above formulas the variable A ranges over subsets of Σ^* and therefore it may be regarded as a second order variable, it is not subject to quantification on the level of the language $\boldsymbol{L}$—there are no second order quantifiers in $\boldsymbol{L}$ bounding subsets of Σ^*. In fact, in the notation "$a \in \Sigma$" the role of the symbol a is twofold: a may be treated as a definite element of Σ; but it can also be treated as a variable ranging over the elements of Σ. An analogous remark applies to the notation "$A \subseteq \Sigma^*$". This notational duality is characteristic to formal linguistics (see e.g. [8]).

The following fact immediately follows from the above definition:

Corollary 3.2. *A subset $\Delta \subseteq \boldsymbol{L}$ is an ultraset of $\models$ if and only if it is consistent and satisfies the following conditions:*

(1) *For any $\sigma, \tau \in \boldsymbol{L}$,*

$$\sigma \wedge \tau \Leftrightarrow \sigma \in \Delta \text{ and } \tau \in \Delta,$$

(2) *For any $\sigma \in \boldsymbol{L}$,*

$$\neg\sigma \in \Delta \Leftrightarrow \sigma \notin \Delta,$$

(3) *For any non-empty set $A \subseteq \Sigma^*$,*

$$O(A) \in \Delta \Rightarrow O(x) \in \Delta \text{ for some word } x \in A,$$

(4) *For any non-empty set $A \subseteq \Sigma^*$,*

$$P(A) \in \Delta \Rightarrow P(x) \in \Delta \text{ for some word } x \in A.$$

■

Ultrasets exist. We have:

Proposition 3.3. *Let $\langle W, V, V_R, I, F\rangle$ be a task-oriented model and u a state in W. Define*

$$\Delta_u := \{\sigma \in \boldsymbol{L} : M, u \models \sigma\}.$$

Then Δ_u is an ultraset.

Proof. Straightforward. ■

In fact, *every* Lindenbaum set of $\models$ is an ultraset. This fact follows from the following observation:

Theorem 3.4. *For every set of formulas Δ, the following conditions are equivalent:*

(i) *Δ is a Lindenbaum set of $\models$;*

(ii) *Δ is an ultraset set of $\models$;*

(iii) *There is a model $\langle W, V, V_R, I, F\rangle$ and a state u in W such that $\Delta = \Delta_u$.*

Proof. The implications (iii) ⇒ (ii) and (ii) ⇒ (i) are obvious.

(i) ⇒ (iii). We need some facts from the theory of deductive systems.

By a *base* for $Th(\models)$ we shall understand any family $\boldsymbol{B} \subseteq Th(\models)$ such that every theory of $\models$ is the intersection of some subfamily of $\boldsymbol{B}$. It follows from the definition of $\models$ that the sets Δ_u defined as in Proposition 3.3 (with u ranging over all states of arbitrary models M) form a base for $Th(\models)$.

Now let Δ be a Lindenbaum set of $\models$. Since Δ is $\models$-consistent and closed, it is the intersection of a non-empty family of sets of the form Δ_u. But inasmuch as Δ is maximal, Δ is equal to exactly one set of the form Δ_u. This shows that (iii) holds. ■

It follows from the above theorem that the family of ultrasets forms a basis for the closure system $Th(\models)$ of all closed theories of $\models$, i.e., for every theory $X \in Th(\models)$ there exists a family $\{\Delta_i : i \in I\}$ of ultrasets such that $X = \bigcap_{i\in I} \Delta_i$. Another corollary is that ultrasets are the only maximal consistent sets of the consequence relation $\models$.

Every ultraset is fully determined by the set of atomic formulas that are contained in it. This follows from the following fact:

Proposition 3.5. *For any ultrasets Δ and Δ', the following conditions are equivalent:*

(1) $\Delta = \Delta'$.

(2) $(\forall x \in \Sigma^*)(P(x) \in \Delta \Leftrightarrow P(x) \in \Delta')$ *and*
$(\forall x \in \Sigma^*)(O(x) \in \Delta \Leftrightarrow O(x) \in \Delta')$.

PROOF. The implication (1) $\Rightarrow$ (2) is immediate. To prove the reverse implication, assume (2) and then prove by induction on complexity of formulas that for any formula $\sigma \in \boldsymbol{L}$, $\sigma \in \Delta \Leftrightarrow \sigma \in \Delta'$. (This proof requires Corollary 3.2.) ■

A function $f : \Sigma^* \to \{0, 1\}$ is said to be *antitone* on Σ^* if $f(y) = 1$ implies $f(x) = 1$ whenever $x \preccurlyeq y$. In other words, f is antitone if $f(x) = 0$ implies that $f(y) = 0$ for all words y prefixed by x.

Let f and g be antitone functions defined on the set Σ^* with values in $\{0, 1\}$ such that $g \leq f$. ($g \leq f$ means that $g(x) \leq f(x)$ for every word x.)

For each word x the formulas $f(x)P(x)$ and $g(x)O(x)$ are defined as follows:

$$f(x)P(x) := \begin{cases} P(x) & \text{if } f(x) = 1 \\ \neg P(x) & \text{if } f(x) = 0. \end{cases}$$

$$g(x)O(x) := \begin{cases} O(x) & \text{if } g(x) = 1 \\ \neg O(x) & \text{if } g(x) = 0. \end{cases}$$

Let H be the set of all pairs $\langle f, g\rangle$ of such functions. (Thus $g \leq f$ for all $\langle f, g\rangle \in H$.) For each pair $\langle f, g\rangle \in H$, we define:

$$\Phi(f, g) := \{f(x)P(x) : x \in \Sigma^*\} \cup \{g(x)O(x) : x \in \Sigma^*\}.$$

According to the above definition, for every atomic formula of the form $P(x)$, either $P(x) \in \Phi(f, g)$ or $\neg P(x) \in \Phi(f, g)$. Similarly, for every atomic formula of the form $O(x)$, either $O(x) \in \Phi(f, g)$ or $\neg O(x) \in \Phi(f, g)$. In virtue of the fact that $g \leq f$ we also have that if $O(x) \in \Phi(f, g)$, then $P(x) \in \Phi(f, g)$ and $\neg P(x) \in \Phi(f, g)$ implies $\neg O(x) \in \Phi(f, g)$, for all words x. $\Phi(f, g)$ is called a *complete* set of atomic or negated atomic formulas.

THEOREM 3.6. *Each set $\Phi(f, g)$ and therefore each pair $\langle f, g\rangle \in H$, determines a unique ultraset that contains $\Phi(f, g)$.*

PROOF. The uniqueness of the ultraset generated by $\Phi(f, g)$ follows from the above proposition. This ultraset is denoted by $\Delta(f, g)$.

The critical property is the consistency of each set $\Phi(f, g)$ with respect to the logic $\models$. This fact implies the existence of $\Delta(f, g)$.

To handle this problem one may suitably modify the construction of the model presented in the subsection *Consistency of complete sets* of Section 4 in [3]. We shall omit the details. We shall directly pass to the construction of the canonical model for $\models$. This construction gives an insight into the problem of consistency of complete sets. ■

For every ultraset Δ, there is a unique pair $\langle f, g\rangle \in H$ such that $\Delta = \Delta(f, g)$. As the set Σ^* is countably infinite, it follows that H is of cardinality of the continuum. Consequently, there exists a continuum of ultrasets.

Let *Ultrasets* be the family of all ultrasets of $\models$.

For each word $x \in \Sigma^*$, we define the unary function $A(x)$ on the set *Ultrasets* as follows. For every ultraset Δ,

$$A(x)(\Delta) := \text{ the unique ultraset } \Delta' \text{ that includes the complete set } \{P(z) : z \in \Sigma^* \text{ and } P(xz) \in \Delta\} \cup \{O(z) : z \in \Sigma^* \text{ and } O(xz) \in \Delta\}.$$

(cf. [3], Section 4).

In particular, for any $a \in \Sigma$,

$$A(a)(\Delta) := \text{ the unique ultraset } \Delta' \text{ that includes the complete set } \{P(z) : z \in \Sigma^* \text{ and } P(az) \in \Delta\} \cup \{O(z) : z \in \Sigma^* \text{ and } O(az) \in \Delta\}.$$

It follows from the above definition and Proposition 3.5 that for the empty word e:

(3) $A(e)$ *is the identity mapping in Ultrasets.*

PROPOSITION 3.7. *For any words $x, y \in \Sigma^*$, the function $A(xy)$ coincides with the composition of $A(x)$ and $A(y)$, i.e.,*

$$A(xy) = A(x) \circ A(y).$$

PROOF. ($\Rightarrow$). Assume $\Delta' = A(xy)(\Delta)$. We claim that $\Delta' = (A(x) \circ A(y))(\Delta)$. We have:

(1) $$(\forall u \in \Sigma^*)(P(xyu) \in \Delta \Leftrightarrow P(u) \in \Delta') \text{ and } (\forall u \in \Sigma^*)(O(xyu) \in \Delta \Leftrightarrow O(u) \in \Delta').$$

We claim that there exists an ultraset Γ such that $\Gamma = A(x)(\Delta)$ and $\Delta' = A(y)(\Gamma)$. We put:

$$\Phi := \{P(z) : P(xz) \in \Delta\} \cup \{\neg P(z) : \neg P(xz) \in \Delta\} \cup \{O(z) : O(xz) \in \Delta\} \cup \{\neg O(z) : \neg O(xz) \in \Delta\}.$$

Φ is a complete set. We then define Γ to be the unique ultraset that includes Φ. It follows from the definition of Γ that

(2) $$(\forall z \in \Sigma^*)(P(xz) \in \Delta \Leftrightarrow P(z) \in \Gamma) \text{ and } (\forall z \in \Sigma^*)(O(xz) \in \Delta \Leftrightarrow O(z) \in \Gamma).$$

So $\Gamma = A(x)(\Delta)$.

On the other hand, we also have that $\Delta' = A(y)(\Gamma)$. Indeed, suppose $u \in \Sigma^*$ and $P(yu) \in \Gamma$. Hence, by the definition of Γ, $P(xyu) \in \Delta$. Then (1) gives that $P(u) \in \Delta'$. Conversely, assume $P(u) \in \Delta'$. Then by (1), $P(xyu) \in \Delta$. As $\Delta A(x)\Gamma$, it follows by (2) that $P(yu) \in \Gamma$. Consequently,

(3) $$(\forall u \in \Sigma^*)(P(yu) \in \Gamma \Leftrightarrow P(u) \in \Delta').$$

Analogously one shows that

(4) $$(\forall u \in \Sigma^*)(O(yu) \in \Gamma \Leftrightarrow O(u) \in \Delta').$$

So $\Delta' = A(y)(\Gamma)$.

It follows that $\Delta' = (A(x) \circ A(y))(\Delta)$.

($\Leftarrow$). We assume that $\Delta' = (A(x) \circ A(y))(\Delta)$. We claim that $\Delta' = A(xy)(\Delta)$. There exists an ultraset Γ such that $\Gamma = A(x)(\Delta)$ and $\Delta' = A(y)(\Gamma)$. We show that $\Delta' = A(xy)(\Delta)$.

As $\Gamma = A(x)(\Delta)$ we have that:

(a) $$(\forall z \in \Sigma^*)(P(xz) \in \Delta \Leftrightarrow P(z) \in \Gamma) \text{ and } (\forall z \in \Sigma^*)(O(xz) \in \Delta \Leftrightarrow O(z) \in \Gamma)$$

As $\Delta' = A(y)(\Gamma)$, we also have that:

(b) $$(\forall u \in \Sigma^*)(P(yu) \in \Gamma \Leftrightarrow P(u) \in \Delta') \text{ and } (\forall u \in \Sigma^*)(O(yu) \in \Gamma \Leftrightarrow O(u) \in \Delta')$$

It follows from (a) and (b) that

(c) $$(\forall w \in \Sigma^*)(P(xyw) \in \Delta \Leftrightarrow P(w) \in \Delta') \text{ and } (\forall w \in \Sigma^*)(O(xyw) \in \Delta \Leftrightarrow O(w) \in \Delta'),$$

showing that $\Delta' = A(xy)(\Delta)$.

It follows from the above proposition that for any word $x = a_1 \dots a_m$ it is the case that

(4) $$A(a_1 \dots a_m) = A(a_1) \circ \dots \circ A(a_m).$$

∎

The Canonical Model

We set about constructing of a task-oriented deterministic model $M_c = \langle W, V, V_R, I, F\rangle$ for $\boldsymbol{L}$, which we shall call the *canonical model* of $\models$.

The set of states W of M_c is equal to *Ultrasets*.

The interpretation $V(a)$ of the symbols a of Σ is defined as expected, viz.,

$$V(a) := A(a),$$

for all $a \in \Sigma$.

The relations $V(x)$, $x \in \Sigma^*$, are recursively defined in terms of the relations $V(a)$ as in Section 1. It follows from the above equality and Proposition 3.7 that $V(x) = A(x)$, for all $x \in \Sigma^*$. Consequently, each sequential action $V(x)$ is a total function defined on *Ultrasets*. In particular, $V(e)$ is the identity map on *Ultrasets*.

The compound actions $V(A)$, $A \subseteq \Sigma^*$, are also defined as in Section 1. Thus, for any ultrasets Δ and Δ' we have that $\Delta V(A)\Delta'$ if and only if $\Delta V(x)\Delta'$ for some word $x \in \Sigma^*$. $V(A)$ is a binary relation on *Ultrasets*; it need not be a partial function. $V(\emptyset)$ is the empty set.

In the next step we define the mappings $V_R(a)$, $a \in \Sigma$. It is assumed that each $V_R(a)$ is the partial function being the restriction of $V(a)$ $(= A(a))$ to the set $\{\Delta \in W : P(a) \in \Delta\}$. The last set is the domain of $V_R(a)$.

$V_R(x)$ is then recursively defined for all words x in the standard way. For the empty word e, the function $V_R(e)$ is the restriction of the diagonal $V(e)$ to the set $\{\Delta \in W : P(e) \in \Delta\}$. Thus

$$\Delta' = V_R(e)(\Delta) \Leftrightarrow_{df} \Delta = \Delta' \wedge P(e) \in \Delta.$$

For any non-empty word $x \in \Sigma^*$ and any $a \in \Sigma$,

$$V_R(xa) := V_R(x) \circ V_R(a).$$

We define the sets I and F of initial and final states:

$$I := \{\Delta \in \textit{Ultrasets} : (\exists x \in \Sigma^*)\, O(x) \in \Delta\},$$
$$F := \{\Delta' \in \textit{Ultrasets} : (\exists \Delta \in \textit{Ultrasets})(\exists x \in \Sigma^*)\, \Delta\, V_R(x)\, \Delta'\}.$$

PROPOSITION 3.8. *For any ultraset Δ and any word $x \in \Sigma^*$ the following conditions are equivalent:*

(i) $P(x) \in \Delta$.

(ii) *The mapping $V_R(x)$ is defined at Δ, i.e., there is an ultraset Δ' such that $\Delta' = V_R(x)(\Delta)$.*

Proof. Fix an ultraset Δ. We shall prove the equivalence of (i) and (ii) by induction on the length of the words x of Σ^*.

Induction base. In view of the definitions of $V_R(e)$ and $V_R(a)$, the conditions (i) and (ii) are equivalent for the empty word e as well as for the symbols a of Σ.

Induction step. We assume that (i) and (ii) are equivalent for a word $x \in \Sigma^*$. We claim that this equivalence continues to hold for the word xa, for all $a \in \Sigma$.

We shall apply the graph-style notation for functions.

We first assume (ii) holds for xa. Hence that there exist ultrasets Γ, Δ' such that

(a) $$\Delta\, V_R(x)\, \Gamma\, V_R(a)\, \Delta'.$$

We want to show that $P(xa) \in \Delta$.

$\Gamma\, V_R(a)\, \Delta'$ in (a) means that

(b) $$\Gamma\, V(a)\, \Delta' \text{ and } P(a) \in \Gamma.$$

$\Delta\, V_R(x)\, \Gamma$ in (a) implies that $\Delta\, V(x)\, \Gamma$. Hence

$$(\forall z \in \Sigma^*)(P(xz) \in \Delta \Leftrightarrow P(z) \in \Gamma).$$

Putting $z = a$ and applying the second conjunct of (b), we obtain that $P(xa) \in \Delta$. So (i) holds.

Conversely, assume (i) holds, that is, $P(xa) \in \Delta$. We claim that there exists an ultraset Δ' such that $\Delta\, V_R(xa)\, \Delta'$. If $x = e$ we are done, by the definition of $V_R(a)$.

As x is non-empty, the assumption $P(xa) \in \Delta$ implies that $P(x) \in \Delta$. As $P(x) \in \Delta$, we have that there exists an ultraset Γ such that $\Delta\, V_R(x)\, \Gamma$. We claim that there exists an ultraset Δ' such that $\Gamma\, V_R(a)\, \Delta'$. We define:

$$\Phi' := \{P(z) : P(az) \in \Gamma\} \cup \{\neg P(z) : \neg P(az) \in \Gamma\} \cup \\ \{O(z) : O(az) \in \Gamma\} \cup \{\neg O(z) : \neg O(az) \in \Gamma\}.$$

Φ' is a complete set. Let Δ' be the unique ultraset that includes Φ'. It follows from the definition of Δ' that

$$(\forall z \in \Sigma^*)(P(az) \in \Gamma \Leftrightarrow P(z) \in \Delta')$$

and

$$(\forall z \in \Sigma^*)(O(az) \in \Gamma \Leftrightarrow O(z) \in \Delta').$$

So $\Gamma\, V(a)\, \Delta'$ holds, i.e., Δ' is the value of $V(a)$ at Γ. But we must also prove that $\Gamma\, V_R(a)\, \Delta'$, that is, we must show that $P(a) \in \Gamma$.

The proof that $P(a) \in \Gamma$ runs as follows. Since $\Delta\, V_R(x)\, \Gamma$, we also have that $\Delta\, V(x)\, \Gamma$. As $\Gamma\, V(a)\, \Delta'$, we therefore obtain that $\Delta\, V(xa)\, \Delta'$. The definition of $V(xa)$ implies that that

$$(\forall z \in \Sigma^*)(P(xaz) \in \Delta \Leftrightarrow P(z) \in \Delta').$$

In particular, for $z = e$,

$$P(xa) \in \Delta \Leftrightarrow P(e) \in \Delta'. \tag{c}$$

But, by the assumption, $P(xa) \in \Delta$. It folows from (c) that $P(e) \in \Delta'$.

As $\Gamma\, V(a)\, \Delta'$ holds, we have that

$$(\forall z \in \Sigma^*)(P(az) \in \Gamma \Leftrightarrow P(z) \in \Delta').$$

In particular, for $z = e$, we get that

$$P(a) \in \Gamma \Leftrightarrow P(e) \in \Delta'.$$

Since $P(e) \in \Delta'$, the above equivalence gives that $P(a) \in \Gamma$.

This concludes the proof that $\Gamma\, V_R(a)\, \Delta'$ holds, showing at the same time that $\Delta\, V_R(xa)\, \Delta'$.

The proof of the proposition is completed. ■

PROPOSITION 3.9. *In the canonical model $M_c = \langle W, V, V_R, I, F\rangle$, for every ultraset Δ and any word $x \in \Sigma^*$ the following conditions are equivalent:*

(i) $O(x) \in \Delta$,

(ii) $\Delta \in I$ *and there is an ultraset* $\Delta' \in F$ *such that* $\Delta' = V_R(x)(\Delta)$.

PROOF. Let x and Δ be arbitrary but fixed.

(i) ⇒ (ii). Assume $O(x) \in \Delta$. Then $P(x) \in \Delta$. In view of Proposition 3.8, the partial function $V_R(x)$ is defined in Δ, i.e., there exists an ultraset Δ' such that $\Delta' = V_R(x)(\Delta)$. As $\Delta \in I$, it follows that $\Delta' \in F$, by the definition of F. So (ii) holds.

(ii) ⇒ (i). Assume (ii). As $\Delta \in I$, we obtain that $O(x) \in \Delta$, by the definition of I. Hence (i) holds. ■

LEMMA 3.10. (The Truth Lemma). *Let Δ be an arbitrary ultraset in the canonical model M_c of $\models$. Then for any formula ϕ,*

(1) $M_c, \Delta \models \phi$ *if and only if* $\phi \in \Delta$.

THE PROOF is by induction on complexity of formulas.

We first prove:

CLAIM 1. *For every word $x \in \Sigma^*$,*

$$M_c, \Delta \models P(x) \Leftrightarrow P(x) \in \Delta.$$

PROOF OF THE CLAIM. Let x be an arbitrary word. We have:

$M_c, \Delta \models P(x) \Leftrightarrow$ (by the definition of $\models$)
$\Delta \in \boldsymbol{PV}(x)$ in $M_c \Leftrightarrow$ (by Definition 1.7.(p1))
There exists an ultraset Δ' such that $\Delta' = V_R(x)(\Delta) \Leftrightarrow$
$P(x) \in \Delta$.

The last equivalence follows from Proposition 3.8. ■

CLAIM 2. *For every word $x \in \Sigma^*$,*

$$M_c, \Delta \models O(x) \Leftrightarrow O(x) \in \Delta.$$

PROOF OF THE CLAIM. Let x be an arbitrary word. We have:

$M_c, \Delta \models O(x) \Leftrightarrow$ (by the definition of $\models$)
$\Delta \in \boldsymbol{OV}(x)$ in $M_c \Leftrightarrow$ (by Definition 1.9.(o1))
$\Delta \in I$ and there exists an ultraset $\Delta' \in F$ such that $\Delta' = V_R(x)(\Delta) \Leftrightarrow$
$O(x) \in \Delta$.

The last equivalence follows from Proposition 3.9. ■

CLAIM 3. *For every non-empty set $A \subseteq \Sigma^*$,*

$$M_c, \Delta \models P(A) \Leftrightarrow P(A) \in \Delta.$$

PROOF OF THE CLAIM. Suppose A is a non-empty subset of Σ^*. Then:

$M_c, \Delta \models P(A) \Leftrightarrow$ (by the definition of $\models$)
$\Delta \in \boldsymbol{PV}(A) \Leftrightarrow$
$\Delta \in \boldsymbol{PV}(x)$ for some word $x \in A \Leftrightarrow$
$M_c, \Delta \models P(x)$ for some word $x \in A \Leftrightarrow$ (by Claim 1)
$P(x) \in \Delta$ for some word $x \in A \Leftrightarrow$
$P(A) \in \Delta$.

The last equivalence is due to the fact that Δ is an ultraset. ■

CLAIM 4. *For every non-empty set $A \subseteq \Sigma^*$,*

$$M_c, \Delta \models O(A) \Leftrightarrow O(A) \in \Delta.$$

Proof of the claim. Suppose A is a non-empty subset of Σ^*. Then:

$$\begin{aligned}
&M_c, \Delta \models O(A) \Leftrightarrow \text{ (by the definition of satisfaction)}\\
&\Delta \in \boldsymbol{OV}(A) \Leftrightarrow\\
&\Delta \in \boldsymbol{OV}(x) \text{ for some word } x \in A \Leftrightarrow\\
&M_c, \Delta \models O(x) \text{ for some word } x \in A \Leftrightarrow \text{ (by Claim 2)}\\
&O(x) \in \Delta \text{ for some word } x \in A \Leftrightarrow\\
&O(A) \in \Delta.
\end{aligned}$$

The last equivalence is due to the fact that Δ is an ultraset. ■

It follows from the above claims and Corollary 3.2 that the the equivalence (1) continues to hold for arbitrary Boolean combinations of atomic formulas. This concludes the proof of the lemma. ■

Corollary 3.11. *For every word* $x \in \Sigma^*$,

$$M_c, \Delta \models F(x) \Leftrightarrow F(x) \in \Delta.$$

For every set $A \subseteq \Sigma^*$,

$$M_c, \Delta \models F(A) \Leftrightarrow F(A) \in \Delta.$$

■

Some Other Properties of the Consequence $\models$

The consequence relation determined by the canonical model $M_c = \langle W, V, V_R, I, F\rangle$ on $\boldsymbol{L}$ agrees with $\models$. More specifically, we define the consequence relation $\models_c$ on $\boldsymbol{L}$ as follows. For any set $X \subseteq \boldsymbol{L}$ and any formula $\sigma \in \boldsymbol{L}$ we put:

$$X \models_c \sigma \Leftrightarrow_{df} (\forall \Delta \in W)(M_c, \Delta \models X \Rightarrow M, \Delta \models \sigma).$$

(The symbol "$M_c, \Delta \models X$" means that $M_c, \Delta \models \phi$ holds for all $\phi \in X$.) $\models_c$ is the consequence relation defined by M_c.

Theorem 3.12. (The Adequacy Theorem). $\models \; = \; \models_c$.

Proof. The inequality $\models \; \leq \; \models_c$ is immediate, because $\models$ is semantically defined by the class of all models that includes the canonical model.

To prove the opposite inequality, suppose that for some set $X \subseteq \boldsymbol{L}$ and a formula $\sigma \in \boldsymbol{L}$ it is *not* the case that $X \models \sigma$. We show that $X \models_c \sigma$ does not hold. According to the definition of $\models$, there exists a model $N = \langle W, V, V_R, I, F\rangle$ and a state $u \in W$ such that $N, u \models X$ and $N, u \not\models \sigma$. (N need not be the canonical model.) We then define: $\Delta_u := \{\phi \in \boldsymbol{L} : N, u \models \phi\}$.

Δ_u is an ultraset of $\models$, $X \subseteq \Delta_u$ and $\sigma \notin \Delta_u$. Passing to the canonical model M_c we obtain, by the Truth Lemma, that $M_c, \Delta_u \models X$ and $M_c, \Delta_u \not\models \sigma$. Consequently, $X \models_c \sigma$ does not hold. ■

Theorem 3.12 implies that the above semantics of deontologically-loaded actions can be based on models in which actions are total unary functions defined on the set of states. This resembles the situation in the theory of finite automata—from the linguistic perspective finite deterministic automata suffice to establish the reach of this theory.

We shall establish some other facts concerning $\models$.

Since $\boldsymbol{L}$ contains formulas of infinite length as e.g. $P(A)$ and $O(A)$, where $A \subseteq \Sigma^*$ is an infinite set, one cannot expect that the system $\models$ is finitary. Indeed, we have:

THEOREM 3.13. *If Σ has at least two elements, then the consequence $\models$ is not finitary.*

PROOF. We shall argue as in the proof of Theorem 5.13 in [3], suitably accommodating the proof of Proposition 5.12. Let a and b be different symbols in Σ. We define:

$$A := \{ab^n a : n \geq 1\}.$$

Note that A is a regular set. (A is the set-theoretic difference of the regular set $\{ab^n a : n \geq 0\}$ and $\{aa\}$.)

Let N be the set of positive integers.

CLAIM.

(1) $\{\neg P(ab^n a) : n \in N\} \models \neg P(A)$;

(2) *For every finite subset $N_f \subset N$, it is not the case that*

$$\{\neg P(ab^n a) : n \in N_f\} \models \neg P(A).$$

PROOF OF THE CLAIM. As to the first statement, suppose $M = \langle W, V, V_R, I, F\rangle$ is a model for $\boldsymbol{L}$ in which $V = V_R$, and $u \in W$ is a state such that $M, u \models \neg P(ab^n a)$ for all $n \in N$. This means that $u \notin \boldsymbol{PV}(ab^n a)$ for all $n \in N$ in M. As $\boldsymbol{PV}(A) = \bigcup_{n \in N} \boldsymbol{PV}(ab^n a)$, it follows that $u \notin \boldsymbol{PV}(A)$. Hence $M, u \models \neg P(A)$.

To prove the other statement, it suffices to show that for every positive integer m it is *not* the case that $\{\neg P(ab^n a) : n \leq m\} \models \neg P(A)$. To this end we fix m and take a relational model $M = (W, V, V_R)$ in which $V = V_R$, a state $u \in W$ and a unique sequence of different states $u_0, u_1, \ldots, u_m, u_{m+1}$ and w_{m+1} such that

$$u\,V(a)\,u_0\,V(b)\,u_1\,V(b)\,u_2 \dots u_m\,V(b)\,u_{m+1}\,V(a)\,w_{m+1}$$

and for each n, $0 \leq n \leq m$, there is no state w such that $u_n\,V(a)\,w$.

Such a model M can be easily defined. It follows that the action $V(a)$ is not permitted in each state u_n, $0 \leq n \leq m$, but it is permitted in u_{m+1}. Consequently, by the uniqueness of the above sequence of states, we have that $M, u \models \neg P(ab^n a)$ for all $n \leq m$. On the other hand, $M, u \models P(A)$, because $M, u \models P(ab^{m+1}a)$. Thus it is not the case that $\{\neg P(ab^n a) : n \leq m\} \models \neg P(A)$.

This proves the claim and concludes the proof of the theorem. ■

Speaking figuratively, unbounded "pumping" the symbol b in each word of the set A accounts for the fact that $\models$ is not finitary. The logic $\models$ allows for such unbounded iterations of permitted actions.

CONJECTURE. *If Σ has one element only, then the consequence $\models$ is not finitary.* ■

Note however that if Σ is a singleton, then for every non-empty set $A \subseteq \Sigma^*$ there is a word $x \in A$ such that $\models P(A) \leftrightarrow P(x)$. We claim that this equivalence is not true for the connective O.

Yet another interesting problem is a charaterization of $\models$ in terms of (possibly infinitary) rules of inference. $\{\neg P(ab^n a) : n \in N\}/\neg P(A)$ is an example of such an infinite rule.

4. Deontology of Finite Actions

The fact that the semantically defined consequence $\models$ is infinitary, nullifies the possibility of presenting it in the form of an axiom system based on finitary axioms and rules. Our plan is to replace $\models$ by a finitary consequence relation defined on a fragment of the language $\boldsymbol{L}$. We restrict here the semantic discourse on deontology of compound actions to the special case when all compound formal actions in question are *finite*. They form a countably infinite subfamily of the power set of Σ^*. The advantage of such limitation consists in the fact that the resulting semantic consequence relation, being an analogue of $\models$, is finitary.

We define the sublanguage $\boldsymbol{L}_f$ of $\boldsymbol{L}$ as follows. Atomic formulas of $\boldsymbol{L}_f$ are expressions of the form:

(i) $O(x),\ P(x),\ F(x)$,

where $x \in \Sigma^*$,

(ii) $O(A),\ P(A),\ F(A)$,

for any finite set $A \subset \Sigma^*$.

Since the set Σ^* is countably infinite, the above set of atomic formulas is countably infinite as well.

As each letter $a \in \Sigma$ is qualified as word of length 1, the group (i) encompasses all atomic formulas of the form $O(a), P(a), F(a)$, $a \in \Sigma$. (i) also encompasses the following three formulas $O(e), P(e), F(e)$, where e is the empty word. In turn, (ii) encompasses formulas in which A is the empty set.

Compound formulas are built from the above atomic formulas by means of applying the Boolean connectives $\rightarrow$ and $\neg$.

$\boldsymbol{L}_f$ marks the set of all so defined formulas. Since classical logic is assumed in $\boldsymbol{L}_f$, the other Boolean connectives such as $\vee$, $\wedge$ and $\leftrightarrow$ are defined in the standard way as appropriate abbreviations.

Models for $\boldsymbol{L}_f$ are the same as for the language $\boldsymbol{L}$. Satisfaction in models is also defined as for $\boldsymbol{L}$ with the only exception that the extended valuations $V(A)$ are defined only for finite sets $A \subset \Sigma^*$.

$\models_f$ is the semantic consequence relation in $\boldsymbol{L}_f$ defined in an analogous way as $\models$ in $\boldsymbol{L}$. It follows that $\models_f$ is the restriction of $\models$ to $\boldsymbol{L}_f$.

The Logic $\vdash$

We shall syntactically characterize the above semantically defined consequence relation $\models_f$ in terms of a system of logical axioms and rules of inference. (In fact, only one primitive rule of inference is needed here—the rule of detachment.) To this end we first define an inferential consequence relation in $\boldsymbol{L}_f$, denoted by $\vdash$. The consequence $\vdash$ is an extension of classical propositional logic (CPC).

Every formula of $\boldsymbol{L}_f$ which is an instance of a tautology of CPC is logically valid. But there are also logically valid formulas specific to the deontology of actions.

As an axiom system of classical logic we adopt the following laws:

(a_1) $\phi \rightarrow (\psi \rightarrow \phi)$

(a_2) $\phi \rightarrow (\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

(a_3) $\neg\phi \rightarrow (\phi \rightarrow \psi)$

(a_4) $(\neg\phi \rightarrow \phi) \rightarrow \phi$,

where ϕ, ψ, χ are arbitrary formulas.

We adopt the following specific deontological axioms:

(d_1) $P(xa) \rightarrow P(x)$,

where x is any non-empty word and $a \in \Sigma$.

(d$_2$) $O(x) \to P(x)$,

(d$_3$) $F(x) \leftrightarrow \neg P(x)$,

where $x \in \Sigma^*$ and $a \in \Sigma$;

(d$_4$) $\neg P(\emptyset)$.

Moreover, for every non-empty *finite* set $A = \{x_1, \ldots, x_n\} \subset \Sigma^*$ we adopt the axioms:

(d$_5$) $P(A) \leftrightarrow P(x_1) \vee \cdots \vee P(x_n)$,

(d$_6$) $O(A) \leftrightarrow O(x_1) \vee \cdots \vee O(x_n)$,

(d$_7$) $F(A) \leftrightarrow F(x_1) \wedge \cdots \wedge F(x_n)$.

(d$_1$)–(d$_3$) and (d$_5$)–(d$_7$) are schemes of axioms. Each word $x \in \Sigma^*$ and each finite set $A \subseteq \Sigma^*$ define a separate formula of the above form.

The formula $P(e)$ is *not* assumed as an axiom.

The detachment rule given by the scheme $\phi, \phi \to \psi/\psi$ is the only primitive rule of inference.

We define

$$\vdash$$

to be the consequence relation in $\boldsymbol{L}$ determined by the above specific deontological axioms, the above axiom system for CPC and the detachment rule. Thus $X \vdash \sigma$ means that there is proof of σ from X carried out by means of the above logical axioms and the detachment rule. $\vdash$ is called the *inferential consequence* in $\boldsymbol{L}$.

$\vdash$ is finitary. Since $\vdash$ is based on classical logic and the detachment as the only primitive rule, $\vdash$ obeys the Deduction Theorem which means that for any set X of formulas and any formulas ϕ, ψ:

$$X \vdash \phi \to \psi \quad \text{if and only if} \quad X \cup \{\phi\} \vdash \psi.$$

It is easy to see that for any finite set $A \subset \Sigma^*$ the formulas

$$O(A) \to P(A) \quad \text{and} \quad F(A) \leftrightarrow \neg P(A)$$

are theses of $\vdash$. Moreover, for any two finite sets $A, B \subset \Sigma^*$:

$$O(A \cup B) \leftrightarrow O(A) \vee O(B),$$
$$P(A \cup B) \leftrightarrow P(A) \vee P(B)$$

are theses of $\vdash$ as well. For any words x, y with $x \neq e$, the formula

$$P(xy) \to P(x)$$

is a thesis too. This can be shown by applying axioms (d_1).

We also see that for each finite A, the formulas

$$P(x) \to P(A) \quad \text{and} \quad O(x) \to O(A)$$

are theses of $\vdash$, for all $x \in A$.

The axioms (d_5)–(d_7) *de facto* eliminate atomic formulas of the form $P(A), O(A)$ and $F(A)$ from $\boldsymbol{L}_f$, because each such formula for $A = \{x_1, \ldots, x_n\}$ can be replaced by the deductively equivalent formula $P(x_1) \vee \cdots \vee P(x_n)$, $O(x_1) \vee \cdots \vee O(x_n)$ and $F(x_1) \wedge \cdots \wedge F(x_n)$, respectively.

A set X is *inconsistent* if *all* formulas are $\vdash$-consequences of X, equivalently, if a formula of the form $\phi \wedge \neg\phi$ is derivable from X by means of the above logical axioms and *Modus Ponens*; otherwise X is *consistent.* A formula σ is *inconsistent* if the set $\{\sigma\}$ is inconsistent. Analogously one defines consistency of a formula.

Since the above axioms are validated in all models, we see that $\models_f$ is stronger than $\vdash$.

$\vdash$ is a variant of CPC. Therefore the (algebraic) closure system $Th(\vdash)$ has a base consisting of *Lindenbaum* sets of $\vdash$, i.e., maximal consistent subsets of $\boldsymbol{L}_f$.

Due to the axiom (d_5) and (d_6), each Lindenbaum set Δ is an ultraset in the sense that for any non-empty finite set $A \subset \Sigma^*$, $P(A) \in \Delta$ implies that $P(x) \in \Delta$ for some $x \in A$ and, analogously, $O(A) \in \Delta$ implies that $O(x) \in \Delta$ for some $x \in A$. Moreover, for any words x and y, $P(xy) \in \Delta$ implies $P(x) \in \Delta$.

Lindenbaum sets of $\vdash$ are therefore characterized in the same way as in Corollary 3.2 (with A restricted to finite sets).

The $\vdash$-counterparts of Proposition 3.3 and Theorem 3.4 also hold (but restricted to the language $\boldsymbol{L}_f$).

The canonical model model $M_c = \langle W, V, V_R, I, F\rangle \vdash$ is defined in a fully analogous way as for $\models$. (The only difference is that the system $\vdash$ is syntactically defined by means of logical axioms and the detachment rule, and not through models. But the axioms of $\vdash$ enable us to define all components of M_c in the same manner as in the case of the system $\models$.)

The set of states W is equal to the family of Lindenbaum sets of $\vdash$, and hence ultrasets in the above sense.

The unary functions $V(a) = A(a)$, $a \in \Sigma$, are defined on the set W similarly as in Section 3, i.e.,

$$\Delta' = A(a)(\Delta) \Leftrightarrow_{df} (\forall z \in \Sigma^*)(P(az) \in \Delta \Leftrightarrow P(z) \in \Delta') \text{ and } (\forall z \in \Sigma^*)(O(az) \in \Delta \Leftrightarrow O(z) \in \Delta').$$

In other words, the value of $A(a)$ at Δ is the unique ultraset Δ' that includes the set $\{P(z) : z \in \Sigma^* \text{ and } P(az) \in \Delta\} \cup \{O(z) : z \in \Sigma^* \text{ and } O(az) \in \Delta\}$.

The partial functions $V_R(a)$ are also defined as in Section 3, viz., each $V_R(a)$ is the partial function being the restriction of $V(a)$ $(= A(a))$ to the set $\{\Delta \in W : P(a) \in \Delta\}$. The last set is the domain of $V_R(a)$.

Proposition 3.5, Theorem 3.6, Propositions 3.7–3.9 from Section 3 continue to hold for the system $\vdash$.

Lemma 3.10 also holds for the above canonical model of $\vdash$:

LEMMA 4.1. (Truth Lemma). *Let Δ be an arbitrary Lindenbaum set in the above canonical model M_c. Then for any formula ϕ of $\boldsymbol{L}_f$:*

$$M_c, \Delta \models \phi \textit{ if and only if } \phi \in \Delta.$$

■

The Extended Completeness Theorem

The following fact is the main result of this part of the paper:

THEOREM 4.2. (The Extended Completeness Theorem). $\vdash \, = \, \models_f$.

PROOF. The inclusion $\vdash \, \subseteq \, \models_f$ is immediate, because the axioms of $\vdash$ are logically valid and *Modus Ponens* is a rule of $\models_f$.

To prove the reverse inclusion, let us assume that X is a set of formulas of L_f and σ is a formula such that it is *not* the case that $X \vdash \sigma$. We shall show that σ does not follows from X in the sense of the other consequence relation. There is a Lindenbaum set Δ_0 of $\vdash$ such that $X \subseteq \Delta_0$ and $\sigma \notin \Delta_0$.

Let $M_c = \langle W, V, V_R, I, F\rangle$ be the canonical model of $\vdash$ defined as above. Hence $\Delta_0 \in W$.

As $X \subseteq \Delta_0$ and $\sigma \notin \Delta_0$ we obtain, by Lemma 4.1, that every formula of X is true in M_c at Δ_0. Since $\sigma \notin \Delta_0$, we have that σ is not true in M_c at Δ_0. Consequently, it is not the case that $X \models_f \sigma$.

This shows the inclusion $\models_f \, \subseteq \, \vdash$. ■

NOTES 1. The above approach retains the closure principle for arbitrary actions. A more refined and nuanced framework that rejects this principle is available. It is based on two transition relations between states in models. It extends the semantics of the system $\boldsymbol{DL}^+$ presented in [2].

Regular Actions

Yet another option consists in restricting the semantic discourse on deontology of compound actions to *regular* actions. We recall that if A and B are compound actions, then $AB := \{xy : x \in A \text{ and } y \in B\}$ is the concatenation (or the composition) of A and B. We also define: $A^0 := \{e\}$, $A^{n+1} := A^n A$ for any natural number $n \in \omega$, and $A^* := \bigcup_{n\in\omega} A^n$. The action A^* is the *Kleene closure* of A.

The countably infinite family of REG(Σ) of regular sets over Σ is the least family of subsets of Σ^* that includes the sets $\emptyset$, $\{e\}$ and $\{a\}$ for all $a \in \Sigma$, and is closed with respect to the operations of set-theoretic union, concatenation and the Kleene closure: if A and B are regular sets, then $A \cup B$, AB and A^* are regular sets as well. Equivalently, REG(Σ) is recursively defined by applying regular expressions. It follows that every finite set of words is regular, see eg. [4].

We define the sublanguage $\boldsymbol{L}_{reg}$ of $\boldsymbol{L}$ as follows. Atomic formulas of $\boldsymbol{L}_{reg}$ are expressions of the form:

(i) $\qquad O(A), P(A), F(A),$

for any *regular* set $A \subset \Sigma^*$.

(i) encompasses the formulas

(ii) $\qquad O(x), P(x), F(x),$

where $x \in \Sigma^*$. ($O(x)$ is identified here with $O(\{x\})$. Similarly for the other formulas.)

Since the set Σ^* is countably infinite and the family of regular set over Σ is also countably infinite, the above set of atomic formulas is countably infinite as well.

Compound formulas are built from the above atomic formulas by means of applying the Boolean connectives $\to$ and $\neg$.

$\boldsymbol{L}_{reg}$ marks the countably infinite set of all so defined formulas. Since classical logic is assumed in $\boldsymbol{L}_{reg}$, the other Boolean connectives such as $\vee, \wedge$ and $\leftrightarrow$ are defined in the standard way as appropriate abbreviations.

The set $\boldsymbol{L}_{reg}$ is larger than the language $\boldsymbol{L}_f$ of finite actions. Thus $\boldsymbol{L}_f \subset \boldsymbol{L}_{reg} \subset \boldsymbol{L}$.

Models for $\boldsymbol{L}_{reg}$ are the same as for the language $\boldsymbol{L}$. Satisfaction in models is also defined as for $\boldsymbol{L}$ with the only exception that the extended valuations $V(A)$ are defined only for regular sets $A \subset \Sigma^*$.

$\models_{reg}$ is the semantic consequence relation in $\boldsymbol{L}_{reg}$ defined in an analogous way as $\models$ in $\boldsymbol{L}$. It follows that $\models_{reg}$ is the restriction of $\models$ to $\boldsymbol{L}_{reg}$.

Theorem 4.3. $\models_{reg}$ *is not finitary.*

Proof. This follows from the proof of Theorem 3.13, because the action A defined there is regular. ■

Final Remarks

Agency. The approach presented in this work abstracts from situational aspects of action other than states of the system. A more sophisticated framework of action that takes into accout the situational envelope of action systems as well as an ordering of the set of states is developed in [2]. The notion of a situational action system plays the central role in this framework. The crucial issue concerns agency. Agents as well as states are part of situations. The problem consists in elaborating a consistent and adequate theory of agency for compound agents. Such an approach would be probably conceptually different from the well-known stit framework of action and agency—see [9].

The issue is how one can meaningfully and consistently speak of deontology of actions performed by agents. In other words, the focus is on the meanings attached to statements of the form "a definite agent is permitted (is obliged) to perform an atomic action $V(a)$ in a given situation". These statements are paraphrased in an equivalent form as "An action $V(a)$ is permitted (is obligatory) in a state w for a definite agent S". Agents of actions are treated as specialized constituents of situational envelopes of elementary action systems and deontological commitments are agential—these constituents do not directly refer to actions but to the agents of these actions. Some remarks on the relationship between actions and their agents can be given from the perspective of context-free grammars in Greibach normal form (GNF), because the situational interpretation of context-free grammars may offer a coherent, although simplified, picture of the deontology of concerted actions performed by a set of agents. We shall present here a couple of remarks devoted to this issue without entering into a detailed discussion of the subject.

We recall that every combinatorial grammar (over Σ) contains apart from the alphabet Σ, a finite set Γ of auxiliary symbols, a finite set of productions and the start symbol α which is always an element of Γ. More formally, a *combinatorial grammar* is a quadruple

$$G = (\Sigma, \Gamma, \mathtt{P}, \alpha),$$

where Σ is the given alphabet, also called the *terminal alphabet*, Γ is an *auxiliary alphabet* (the members of Γ are called *nonterminals* or *variables* or *syntactic categories*), $\mathtt{P}$ is a finite subset of $(\Sigma \cup \Gamma)^* \times (\Sigma \cup \Gamma)^*$ called the *list of productions* of the grammar; the fact that $(x, y) \in \mathtt{P}$ is written as $x \to y$. x is the *predecessor* and y is the *successor* of the production $x \to y$. α is a distinguished element of Γ called the *start symbol*.

$L(\mathbf{G}) \subseteq \Sigma^*$ is the language generated by $\mathbf{G}$. Thus, a string x is in $L(\mathbf{G})$ if x consists solely of terminals and the string can be derived from α. The symbols of the alphabet Γ appear only while deriving the words of $L(\mathbf{G})$; they do not occur in the words of $L(\mathbf{G})$. Thus, Γ plays an auxiliary role in the process of defining the language $L(\mathbf{G})$.

A grammar $\mathbf{G} = (\Sigma, \Gamma, \mathtt{P}, \alpha)$ is *context-free* if each production of $\mathtt{P}$ is of the form $X \to x$, where X is a variable (i.e., a non-terminal symbol) and x is a string of symbols from $(\Sigma \cup \Gamma)^*$.

Theorem 4.4. (Sheila Greibach [5]). *Every context-free language without e is generated by a grammar $\mathbf{G}$ for which every production is of the form $X \to a\delta$, where X is a variable, a is a terminal and δ is a (possibly empty) string of variables. Furthermore, every word of the language $L(\mathbf{G})$ can be derived by means of a leftmost derivation in $\mathbf{G}$.*

The proof of the above theorem is constructive—an algorithm is provided which, for every context free-grammar $\mathbf{G}$ in which no production of the form $v \to e$ occurs, converts it to Greibach normal form. We omit the details. ■

Since the theory of context-free languages is inherently linked with push-down automata, Σ is also called the *input alphabet* and Γ—*the stack alphabet*.

The formal apparatus of context-free grammars and pushdown automata can be accommodated to the study of the problem of agency in action theory. Such a move will require a certain terminological switch. Terminal symbols of a given grammar, viz, the elements of Σ, are, as yet, consistently called action symbols, the elements of Σ^*—sequential actions while the elements of Γ will be referred to as *agent symbols* (names of agents, or agents, for short).

Γ^* is the set consisting of all finite sequences of Γ. Each sequence $\gamma \in \Gamma^*$ of variables is called a *queue* of agents. (In automata theory the elements of Γ are called *stack symbols* and the elements of Γ^* are called states of the stack. We depart from this terminology here.)

Let $M = \langle W, V, V_R \rangle$ be a model. (We have not yet defined initial and final states.) The Cartesian product

$$S := W \times \Gamma^*$$

is called the set of *possible situations*. Accordingly, each situation is represented as an ordered pair

$$s = (w, \gamma),$$

where w is a state and γ is a queue of agents assigned to w. γ is also called the *label* of s.

$X\gamma$ denotes a queue of agents in which the agent X occupies the first place. (In the theory of pushdown automata, $X\gamma$ is called a stack with X on the top of it.)

Given a grammar $\mathbf{G} = (\Sigma, \Gamma, \mathtt{P}, \alpha)$ in the Greibach normal form, we define the transition relation Tr between situations of S. Transitions are determined by productions of $\mathtt{P}$ and states of W. To this end we shall read productions in a certain uniform way.

Let $X \to a\delta$ be a fixed production. We shall interprete this production as follows:

> *the agent X performs the action represented by a; after performing it, X is replaced by the queue of agents δ.*

Let s and s' be situations. We shall say that s *is transformed into* s' *in accordance with the production* $X \to a\delta$ if for some string γ and states w, w', it is the case that $s = (w, X\gamma)$, $s' = (w', \delta\gamma)$ and $wV_R(a)w'$.

Thus passing from s to s' is accomplished by means of the action $V(a)$ leading from the state w to the state w' and "replacing" the first agent X in the queue $X\gamma$ by the queue of agents δ. This results in the queue $\delta\gamma$.

We then write $s\,Tr\,s'$ for any such a situation s' obtained from s in accordance with some production of $\mathbf{G}$. Tr is the *transition relation* between situations.

Let s be a situation. Let $V(a)$ be an atomic action, where $a \in \Sigma$. We shall say that the action $V(a)$ is *permitted for the agent* X in the situation s if and only if s is of the form $(w, X\gamma)$ and there exists a production $X \to a\delta$ in $\mathbf{G}$ and a state w' such that $wV_R(a)w'$ holds.

In other words, permissibility of $V(a)$ for X in $s = (w, X\gamma)$ means that, for some production $X \to a\delta$, the action $V(a)$ of the agent X in s turns the situation $s = (w, X\gamma)$ into the situation $s' = (w', \delta\gamma)$. It follows that $s\,Tr\,s'$.

We may also say that X is the *agent of the action* $V(a)$ *in the situation* s.

We thus see that the functioning of such 'pushdown' situational action system is determined by simultaneous transititions between states of W accomplished by actions of $V(a)$, $a \in \Sigma$, and accompanying transitions between queues of agents that perform the consecutive actions. These moves are all determined by the productions of $\mathbf{G}$. The relation Tr thus organizes the rules of cooperation between agents while performing the actions.

In an analogous way one may define the notion "$V(a)$ is obligatory for the agent X in a situation s". This notion takes into account initial situations and final situations as well as the agent α. Let I be a non-empty set of states. We call it the set of initial states. *Initial situations* are of the form $s = (w, \alpha)$, where w is a state belonging to I and α is the start symbol. Final situations are of the form $s = (w, e)$, where w is a state and the stack e is empty. We shall omit the definitional details, because the relationship betewen agency and context free grammars will be discussed at length in another paper.

Inferential bases and weak adequacy. The fact that $\models$ and $\models_{reg}$ are infinitary systems gives rise to a number of questions. We mention three problems.

1. Give an axiom system for $\models$ endowed with a recursive list of possibly infinitary rules of inference that is adequate for $\models$.
2. An analogous question concerns $\models_{reg}$. If A and B are regular actions, then $P(AB) \to P(A)$ and $P(A_n) \to P(A^*)$, $n \in \omega$, are tautologies of $\models_{reg}$. Moreover $\models_{reg}$ validates the following ω-type infinitary rule of inference:
$$\{\neg P(A^n) : n \in \omega\}/\neg P(A^*).$$
$\models_{reg}$ also validates the formulas $O(A) \to P(A)$ and $O(A^n) \to O(A^*)$, $n \in \omega$, and the rule
$$\{\neg O(A^n) : n \in \omega\}/\neg O(A^*).$$
We ask how to characterize $\models_{reg}$ in terms of logical axioms and (possibly infinitary) rules of inference as the least consequence operation that validates these axioms and rules.
3. Although the above results show that the logical systems $\models$ and $\models_{reg}$ are inherently infinitary, the Weak Completeness Theorem appears to be an interesting option. In other words, we ask about axiomatizations of not the whole systems but only of the sets of their tautologies. The problem whether there exists a recursive set of logical axioms and finitary Hilbert-style rules that would axiomatize the sets of tautologies of $\models_{reg}$ needs special scrutiny.

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References

[1] Apostel, L., Game theory and the interpretation of deontic logic, *Logique & Analyse* 3:70–90, 1960.

[2] Czelakowski, J., *Freedom and Enforcement in Action. Elements of Formal Action Theory*, (in the series: Trends in Logic, Vol. 42), Springer, Berlin, 2015.

[3] Czelakowski, J., *Performability of Actions and Logic*, 2018, to appear.

[4] Fischer, M.J., and R.E. Ladner, Propositional Dynamic Logic of Regular Programs, *Journal of Computer and System Sciences* 18:194–211, 1979.

[5] Greibach, S.A., A New Normal Form Theorem for Context-Free Phrase Structure Grammars, *Journal of the Association for Computing Machinery* (*ACM*) 12:42–52, 1965.

[6] Harel, D., D. Kozen, and J. Tiuryn, *Dynamic Logic*, MIT Press. Mass, 2000.

[7] Hilpinen, R., and P. McNamara, Deontic Logic: A Historical Survey and Introduction, in D. Gabbay, J. Horty, X. Parent, R. van der Meyden, and L. van der Torre (eds.), *Handbook of Deontic Logic and Normative Systems*, College Publications, London, 2013.

[8] Hopcroft, J.E., and J.D. Ullman, *Introduction to Automata Theory, Languages, and Computation*, Addison–Wesley, Reading, Mass, 1979.

[9] Horthy, J.F., *Agency and Deontic Logic*, Oxford University Press, Oxford–New York, 2001.

[10] Meyer, J.-J.Ch., A Different Approach to Deontic Logic: Deontic Logic Viewed as a Variant of Dynamic Logic, *Notre Dame Journal of Formal Logic* 29(1):109–136, 1987.

[11] Nowakowska, M., *Language of Motivation and Language of Actions*, Mouton, The Hague, 1973.

[12] Nowakowska, M., Formal Theory of Actions, *Behavioral Science* 18:393–413, 1973.

[13] Nowakowska, M., *Teoria działania* (*Action Theory*), in Polish, PWN, Warszawa, 1979.

[14] Segerberg, K., A Deontic Logic of Action, *Studia Logica* 41:269–282, 1982.

[15] Segerberg, K., Models for Action, in B.K. Matilal and J.L. Shaw (eds.), *Analytical Philosophy in Comparative Perspective*, D. Reidel, Dordrecht, 1985, pp. 161–171.

[16] Tamminga, A., Deontic Logic for Strategic Games, *Erkenntnis* 78:183–200, 2013.

[17] Wójcicki, R., *Theory of Logical Calculi. Basic Theory of Consequence Operations*, Kluwer, Dordrecht, 1988.

[18] von Wright, G.H., *Norm and Action*, Routledge and Kegan Paul, London, 1963.

[19] von Wright, G.H., *An Essay in Deontic Logic and the General Theory of Action*, North Holland, Amsterdam, 1968.

[20] von Wright, G.H., On the Logic of Norms and Actions, in R. Hilpinen (ed.), *New Studies in Deontic Logic*, D. Reidel, Dordrecht, 1981, pp. 3–25.

J. Czelakowski
Institute of Mathematics
Opole University
Opole
Poland
jczel@math.uni.opole.pl