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A Study in Grzegorczyk Point-Free Topology Part II: Spaces of Points

Abstract. In the second installment to Gruszczyński and Pietruszczak (Stud Log, 2018. https://doi.org/10.1007/s11225-018-9786-8) we carry out an analysis of spaces of points of Grzegorczyk structures. At the outset we introduce notions of a *concentric* and ω-*concentric* topological space and we recollect some facts proven in the first part which are important for the sequel. Theorem 2.9 is a strengthening of Theorem 5.13, as we obtain stronger conclusion weakening Tychonoff separation axiom to mere regularity. This leads to a stronger version of Theorem 6.10 (in form of Corollary 2.10). Further, we show that Grzegorczyk points are maximal contracting filters in the sense of De Vries (Compact spaces and compactifications, Van Gorcum and Comp. N.V., 1962), but the converse inclusion is not necessarily true. We also compare the notions of a *Grzegorczyk point* and an *ultrafilter*, and establish several properties of topological spaces based on Grzegorczyk structures. The main results of the paper are representation and completion theorems for G-structures. We prove both set-theoretical and topological representation theorems for various classes of G-structures. We also present topological object duality theorem for the class of complete G-structures and the class of concentric spaces, both restricted to structures which satisfy countable chain condition. We conclude the paper with proving equivalence of the original Grzegorczyk axiom with the one accepted by us as axiom (G).

Keywords: Grzegorczyk structures, Point-free topology, Region-based topology, Mereology, Mereological fields, Mereological structures, Representation theorems, Regular spaces, Concentric spaces.

7. First-Countable G-Structures

The first axiom of countability for topological spaces says that at every point of a given space there exists a countable base. We have the following counterpart of this property for quasi-separation structures: a quasi-separation structure $\mathfrak{R}$ is *first-countable* iff each pre-point Q of $\mathfrak{R}$ contains a countable subset which is coinitial with Q. This subset is also a pre-point of $\mathfrak{R}$, and both pre-points generate the same point from $\mathbf{Pt}_{\mathfrak{R}}$:

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Corollary 7.1. *Let* $\mathfrak{R} \in \mathsf{qSep}$ *and* $Q \in \mathbf{Q}_{\mathfrak{R}}$. *If* $X \subseteq Q$ *and* X *is coinitial with* Q, *then* $X \in \mathbf{Q}_{\mathfrak{R}}$ *and* $\mathrm{F}_X = \mathrm{F}_Q$.

Proof. Firstly, Lemma 5.6 says that all subsets of Q which are coinitial with Q also belong to $\mathbf{Q}_{\mathfrak{R}}$. Secondly, by Corollary 5.4, $\mathrm{F}_X = \mathrm{F}_Q$, since $X \subseteq \mathrm{F}_Q$. ■

Let $\mathbf{Q}^{\omega}_{\mathfrak{R}}$ be the family of countable pre-points of $\mathfrak{R}$. Then directly from Proposition 6.1, Corollary 7.1 and definition of first-countable quasi-separation structures we obtain that for these kind of structures in definition of G-structures instead of the family $\mathbf{Q}_{\mathfrak{R}}$ we can use the family $\mathbf{Q}^{\omega}_{\mathfrak{R}}$:

Theorem 7.2. *A first-countable quasi-separation structure* $\mathfrak{R}$ *is a Grzegorczyk structure iff* $\mathfrak{R}$ *satisfies the following conditions*:

$$\forall_{x,y\in R}\big(x \bigcirc y \Longrightarrow \exists_{Q\in\mathbf{Q}^{\omega}_{\mathfrak{R}}}\exists_{z\in Q}\, z \sqsubseteq x \sqcap y\big), \tag{$\omega\mathrm{G}_{\bigcirc}$}$$

$$\forall_{x,y\in R}\big(x \mathrel{\mathsf{C}} y \,\wedge\, x \wr y \Longrightarrow \exists_{Q\in\mathbf{Q}^{\omega}_{\mathfrak{R}}}\forall_{z\in Q}(z \bigcirc x \,\wedge\, z \bigcirc y)\big)\,. \tag{$\omega\mathrm{G}_{\wr}$}$$

We say that a G-structure is *first-countable*, if it is a first-countable quasi-separation structure.

Let us remind that a subset of R is an *antichain* iff its any two distinct elements are exterior to each other. We say that $\mathfrak{R}$ satisfies the *countable chain condition* (abbrv.: c.c.c.) iff every antichain of its regions is countable.

Lemma 7.3. *All quasi-separation structures with c.c.c. are first-countable.*

Proof. Let $\mathfrak{R}$ be a quasi-separation structure satisfying c.c.c. and $Q \in \mathbf{Q}_{\mathfrak{R}}$. Then, by Lemma 5.8, there is $Q' \in \mathbf{Q}_{\mathfrak{R}}$ such that $Q' \subseteq Q$, Q' is coinitial with Q, and Q' is well-ordered chain in $\mathfrak{R}$. Therefore Q' is countable, because $\mathfrak{R}$ satisfies c.c.c. (the proof of this fact is similar to the proof of Lemma 2 from [8, p. 62] for Boolean algebras). ■

For completeness of presentation notice that the above lemma is not reversible even in the class of atomic G-structures with unity.

Example 7.1. Let S be an arbitrary uncountable set and $\mathrm{FC}(S)$ be the family of all finite and co-finite subsets of S. As is well-known, $\langle \mathrm{FC}(S), \subseteq, \emptyset, S\rangle$ is an atomic incomplete Boolean lattice, where $\{\{x\} \mid x \in S\}$ is the set of atoms. $\mathrm{FC}(S)$ does not satisfy c.c.c., but all its chains are countable (see, e.g., Exercise 2 in [8, p. 64] or Exercise 2 in [9, p. 46]).

We put $\mathrm{FC}^+(S) := \mathrm{FC}(S) \setminus \{\emptyset\}$. Then $\langle \mathrm{FC}^+(S), \subseteq\rangle$ is an atomic incomplete mereological field with the unity S.[1] Therefore, by propositions 4.2

[1] See Theorems 2.4 and 2.8; and also [10, Sections III.7.6 and III.9.5] and [11, Theorems III.1.2 and III.1.3].

and 6.8, $\langle \mathrm{FC}^{+}(S), \subseteq, \wr \rangle$ (where $)(:= \wr$, i.e., $X \mathrel{)(} Y \;:\Longleftrightarrow\; X \cap Y = \emptyset$, and so $\ll = \subseteq$) is an atomic incomplete G-structure with the unity S. The G-structure does not satisfy c.c.c., but all its chains are countable and so all its pre-points are countable. ■

8. G-Structures Associated with Concentric Topological Spaces

As we noted at the end of Section 4, with an arbitrary topological space $\mathcal{T} = \langle S, \mathscr{O} \rangle$ is associated the quasi-separation structure $\mathtt{qsep}\mathcal{T} := \langle \mathrm{r}\mathscr{O}^{+}, \subseteq, \mathtt{][} \rangle$, where $\mathtt{][}$ is a separation relation in $\mathrm{r}\mathscr{O}^{+}$ defined as:

$$U \mathrel{\mathtt{][}} V \;:\Longleftrightarrow\; \mathrm{Cl}\, U \cap \mathrm{Cl}\, V = \emptyset. \tag{df$\,\mathtt{][}$}$$

If $\mathcal{T}$ is weakly regular, then $\mathtt{qsep}\mathcal{T}$ is a complete separation structure and therefore we refer to this structure as $\mathtt{sep}\mathcal{T}$. It is known that every regular space is also weakly regular.

Let us also remind that, by Proposition 4.3(2), the relation $\ll$ in $\mathrm{r}\mathscr{O}^{+}$, defined by (df$\,\ll$), meets:

$$U \ll V \iff \mathrm{Cl}\, U \subseteq V.$$

In Part I we analyzed T_1-spaces such that all points of these spaces have bases that meet condition (R1) (cf. Lemma 5.11, Proposition 5.12, Theorem 6.9 and Lemmas A.2 and A.3). Thus—due to geometric appeal—we will say that a topological space $\mathcal{T} = \langle S, \mathscr{O} \rangle$ is *concentric* iff it is a T_1-space and for any $p \in S$ there is a base $\mathscr{B}^p$ at p such that:

$$\forall_{U,V \in \mathscr{B}^p} \big(U = V \vee \mathrm{Cl}\, U \subseteq V \vee \mathrm{Cl}\, V \subseteq U \big)\,. \tag{R1}$$

Firstly, with the term 'concentric space', we can rewrite Lemmas A.2 and 5.11 as follows:

Lemma 8.1. *For any concentric topological space $\mathcal{T} = \langle S, \mathscr{O} \rangle$:*

1. *$\mathcal{T}$ is regular (and so it is also weakly regular).*
2. *For any $p \in S$ let $\mathscr{B}^p$ be a base at p satisfying* (R1)*. Then:*
   (i) *The family $\mathrm{r}\mathscr{B}^p := \{\mathrm{Int}\,\mathrm{Cl}\, B \mid B \in \mathscr{B}^p\}$ is a base at p satisfying* (R1) *and included in $\mathrm{r}\mathscr{O}^{+}$.*
   (ii) *By Lemma 5.11, $\mathscr{B}^p$ and $\mathrm{r}\mathscr{B}^p$ meet the following conditions:*

$$\forall_{U \in \mathscr{X}} \exists_{V \in \mathscr{X}}\; \mathrm{Cl}\, V \subseteq U, \tag{R2}$$

$$\forall_{A,B \in \mathcal{P}(S)} (\forall_{U \in \mathscr{X}}\; U \cap A \neq \emptyset \neq U \cap B \Rightarrow \mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset)\,, \tag{R3}$$

$$\bigcap \mathscr{X} = \{p\}\,. \tag{$\star$}$$

Let us also notice that fulfillment of (R1) by a family $\mathscr{R}$ contained in $\mathrm{r}\mathscr{O}^+$ is equivalent to the fact that the following relation in $\mathrm{r}\mathscr{O}^+$:

$$\begin{aligned} U \mathrel{\underline{\ll}} V \ &:\Longleftrightarrow\ U \ll V \ \vee\ U = V, \\ &\Longleftrightarrow\ \mathrm{Cl}\, U \subseteq V \ \vee\ U = V \end{aligned} \tag{df $\underline{\ll}$}$$

linearly orders $\mathscr{R}$ (i.e., $\underline{\ll}$ is antisymmetric, transitive and total on $\mathscr{R}$).

Secondly, recall from Appendix that a topological space satisfies the *countable chain condition* (again, c.c.c. for short) iff each pairwise disjoint family of its open sets is countable. Using the term 'concentric space', we can rewrite together Lemmas A.3 and A.5 as follows:

LEMMA 8.2. *All second-countable concentric topological spaces are perfectly normal and satisfy c.c.c.*

Moreover, directly from definitions and Lemma 7.3, we have:

LEMMA 8.3. 1. *If a topological space $\mathcal{T}$ satisfies c.c.c., then* $\mathtt{qsep}\mathcal{T}$ *satisfies c.c.c. and so it is first-countable.*

2. *For any semiregular topological space $\mathcal{T}$ we have: $\mathcal{T}$ satisfies c.c.c. iff* $\mathtt{qsep}\mathcal{T}$ *satisfies c.c.c.*

Thirdly, using the term 'concentric space', we can rewrite Theorem 6.9 as follows:

THEOREM 8.4. *For each concentric topological space $\mathcal{T}$, the complete separation structure* $\mathtt{sep}\mathcal{T}$ *is a G-structure.*

Thus, for any concentric space $\mathcal{T}$, $\mathtt{sep}\mathcal{T}$ will be called the *G-structure associated with* $\mathcal{T}$ and denoted by: $\mathfrak{G}\mathcal{T}$. Directly from Theorem 8.4 and Lemmas 8.1 and 8.3 we obtain:

COROLLARY 8.5. *For any concentric topological space $\mathcal{T}$: $\mathcal{T}$ satisfies c.c.c. iff $\mathfrak{G}\mathcal{T}$ satisfies c.c.c.*

Moreover, directly from Lemma 8.1 we obtain:

COROLLARY 8.6. *Let $\mathcal{T} = \langle S, \mathscr{O}\rangle$ be a concentric space and for any point p let $\mathscr{B}^p$ be a base at p satisfying* (R1). *Then the regular base $\mathrm{r}\mathscr{B}^p$ from Lemma 8.1 is a pre-point of the G-structure $\mathfrak{G}\mathcal{T}$.*

We say that a topological space $\mathcal{T} = \langle S, \mathscr{O}\rangle$ is *ω-concentric* iff it is a T_1-space and for any $p \in S$ there is a *countable* base at p for which the condition (R1) holds. Obviously, all ω-concentric spaces are first-countable and concentric. We will show, see (8.2), that also all first-countable concentric spaces are ω-concentric. Now, directly from Theorem 8.4 and Corollary 8.6, we obtain:

Corollary 8.7. *For any concentric topological space $\mathcal{T}$: $\mathcal{T}$ is ω-concentric iff the G-structure $\mathfrak{G}\mathcal{T}$ is first-countable.*

Furthermore, in virtue of Lemma 5.15, we have:

Theorem 8.8. *All concentric topological spaces with c.c.c. are ω-concentric.*

Proof. Let $\mathcal{T} = \langle S, \mathscr{O} \rangle$ be a concentric space satisfying c.c.c. Then it is a T_1-space and for any $p \in S$ there is a base $\mathscr{B}^p$ at p which is a chain of open sets satisfying (R1). It suffices to consider only the case when $\mathscr{B}^p$ is infinite. In this case, in the light of Lemma 5.15, there is a sequence $(U^p_n)_{n\in\omega}$ of sets from $\mathscr{B}^p$ which is coinitial with $\mathscr{B}^p$ and such that $\mathrm{Cl}\, U^p_{n+1} \subsetneq U^p_n$, for any $n \in \omega$. Finally, $(U^p_n)_{n\in\omega}$ is a base at p, since $\mathscr{B}^p$ is a base at p and $(U^p_n)_{n\in\omega}$ is coinitial with $\mathscr{B}^p$. ∎

In the light of the above theorem, for topological spaces we get:

$$\begin{aligned}&\text{the class of concentric spaces satisfying c.c.c.}\\ &\quad = \text{the class of } \omega\text{-concentric spaces satisfying c.c.c.}\end{aligned} \tag{8.1}$$

Now we prove that also:

Theorem 8.9. *All first-countable regular topological spaces are ω-concentric.*

Proof. Let $\mathcal{T} = \langle S, \mathscr{O} \rangle$ be a first-countable regular space and $p \in S$. Enumerate all elements of $\mathscr{B}^p$: $B_0, B_1, \ldots, B_n, \ldots$, and notice that regularity entails that $\{B \in \mathscr{B}^p \mid \mathrm{Cl}\, B \subseteq B_n \cap B_m\} \neq \emptyset$, for all $n, m \in \omega$.

Thus, by induction, we can define the following function $F\colon \omega \to \mathscr{B}^p$: $F(0) \in \{B \in \mathscr{B}^p \mid \mathrm{Cl}\, B \subseteq B_0\}$ and $F(n+1) \in \{B \in \mathscr{B}^p \mid \mathrm{Cl}\, B \subseteq B_n \cap F(n)\}$. We put $\mathscr{Q}^p := \{F(n) \mid n \in \omega\}$. $\mathscr{Q}^p$ is coinitial with $\mathscr{B}^p$, therefore $\mathscr{Q}^p$ is a (countable) base at p. It fulfills (R1), since $\mathrm{Cl}\, F(n+1) \subseteq F(n)$. ∎

In the light of Theorem 8.9 and Lemma 8.1, for the above-mentioned classes of topological spaces we get:

$$\begin{aligned}&\text{the class of first-countable regular spaces } \subsetneq\\ &\text{the class of } \omega\text{-concentric spaces } \subseteq\\ &\text{the class of first-countable concentric spaces } \subsetneq\\ &\text{the class of concentric spaces } \subsetneq\\ &\text{the class of regular spaces.}\end{aligned}$$

Therefore for topological spaces we obtain:

$$\begin{aligned}&\text{the class of first-countable regular spaces } =\\ &\text{the class of } \omega\text{-concentric spaces } =\\ &\text{the class of first-countable concentric spaces.}\end{aligned} \tag{8.2}$$

Notice that, by Theorems 8.4 and 8.9, we can strengthen Theorem 6.10:

COROLLARY 8.10. *If $\mathcal{T}$ is a first-countable regular topological space, then the complete first-countable separation structure* $\mathtt{sep}\mathcal{T}$ *is a G-structure.*

Moreover, from Theorem 8.9 and Lemma 8.2, we obtain:

COROLLARY 8.11. *All second-countable regular spaces are ω-concentric and perfectly normal satisfying c.c.c.*

The following theorem demonstrates that Grzegorczyk construction of points is "correct", in the sense that from point of view of motivations of region-based topology formulated in the first part, for each concentric topological space $\mathcal{T}$ satisfying c.c.c., the points of $\mathcal{T}$ are in bijective correspondence with the points of $\mathfrak{G}\mathcal{T}$.

THEOREM 8.12. *Let $\mathcal{T} = \langle S, \mathscr{O}\rangle$ be a concentric space satisfying c.c.c. For any $p \in S$ let $\mathrm{r}\mathscr{B}^p$ be a pre-point of $\mathfrak{G}\mathcal{T}$ from Corollary 8.6. Then:*

1. *For any $p \in S$ we have $\mathrm{F}_{\mathrm{r}\mathscr{B}^p} = \{U \in \mathrm{r}\mathscr{O} \mid p \in U\}$.*
2. $\mathbf{Pt}_{\mathfrak{G}\mathcal{T}} = \{\mathrm{F}_{\mathrm{r}\mathscr{B}^p} : p \in S\}$.
3. *The mapping* $\mathrm{b}\colon S \ni p \mapsto \mathrm{F}_{\mathrm{r}\mathscr{B}^p} \in \mathbf{Pt}_{\mathfrak{G}\mathcal{T}}$ *is a bijection.*

PROOF. *Ad* 1. For any $p \in S$, since $\mathrm{r}\mathscr{B}^p$ is a base at p, for any $U \in \mathrm{r}\mathscr{O}$ we have: $U \in \mathrm{F}_{\mathrm{r}\mathscr{B}^p}$ iff $\exists_{B\in\mathrm{r}\mathscr{B}^p} B \subseteq U$ iff $p \in U$.

Ad 2. "$\subseteq$" Let $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{G}\mathcal{T}}$, i.e., for some pre-point $\mathscr{Q}$ of $\mathfrak{G}\mathcal{T}$ we have $\mathfrak{p} = \mathrm{F}_{\mathscr{Q}} := \{U \in \mathrm{r}\mathscr{O} \mid \exists_{X\in\mathscr{Q}} X \subseteq U\}$. Then $|\bigcap\mathscr{Q}| = 1$, by Lemma 8.1 and Theorem 5.17. So for some $p \in S$ we have $\{p\} = \bigcap\mathscr{Q}$. We show that $\mathfrak{p} \subseteq \mathrm{F}_{\mathrm{r}\mathscr{B}^p}$; and so $\mathfrak{p} = \mathrm{F}_{\mathscr{B}^p}$, by Corollary 5.3. Indeed, if $U \in \mathfrak{p}$ then for some $Q \in \mathscr{Q} \subseteq \mathrm{r}\mathscr{O}^+$ we have $Q \subseteq U$. But, since $\mathrm{r}\mathscr{B}^p$ is a base at p, there is a $B \in \mathrm{r}\mathscr{B}^p$ such that $p \in B \subseteq Q$. Hence $p \in U$; and so $U \in \mathrm{F}_{\mathrm{r}\mathscr{B}^p}$, by 8.13.

"$\supseteq$" By Corollary 8.6, for any $p \in S$, the filter $\mathrm{F}_{\mathrm{r}\mathscr{B}^p}$ is a point of $\mathfrak{G}\mathcal{T}$.

Ad 3. Obviously, b is one-to-one and it is surjective, by 2. ■

Let us recall that for any $n \in \omega^+$, $\mathfrak{G}\mathcal{E}^n := \langle \mathrm{r}\mathscr{E}_+(\mathbb{R}^n), \subseteq, \mathtt{][}\rangle$ is the complete G-structure associated with the topological Euclidean space $\mathcal{E}^n$. For any $p \in \mathbb{R}^n$ the family $\mathcal{B}^p$ of all open balls with center at p is a base at p and, in virtue of Corollary 5.18, $\mathcal{B}^p$ is a pre-point of $\mathfrak{G}\mathcal{E}^n$. So the filter $\mathrm{F}_{\mathcal{B}^p} := \{U \in \mathrm{r}\mathscr{E}_+(\mathbb{R}^n) \mid \exists_{B\in\mathcal{B}^p} B \subseteq U\}$ is a point of $\mathfrak{G}\mathcal{E}^n$. Thus, in Theorem 8.12, we can substitute pre-points $\mathcal{B}^p$ by $\mathscr{Q}^p$.

THEOREM 8.13. 1. *For any $p \in \mathbb{R}^n$ we have $\mathrm{F}_{\mathcal{B}^p} = \{U \in \mathrm{r}\mathscr{E}_+(\mathbb{R}^n) \mid p \in U\}$.*

2. $\mathbf{Pt}_{\mathfrak{G}\mathcal{E}^n} = \{\mathrm{F}_{\mathcal{B}^p} : p \in \mathbb{R}^n\}$.
3. *The mapping* $\mathrm{b}\colon \mathbb{R}^n \ni p \mapsto \mathrm{F}_{\mathcal{B}^p} \in \mathbf{Pt}_{\mathfrak{G}\mathcal{E}^n}$ *is a bijection.*

PROOF. Analogously to the proof of Theorem 8.12; we only change: $\mathcal{T}$ to $\mathcal{E}^n$; $\mathfrak{G}\mathcal{T}$ to $\mathfrak{G}\mathcal{E}^n$; and $\mathscr{Q}^p$ to $\mathcal{B}^p$. ■

9. Atoms in G-Structures. Finite G-Structures

Let $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle$ be any G-structure. First of all, we show that each atom is non-tangentially included in itself and creates a pre-point of $\mathfrak{R}$.

PROPOSITION 9.1. *For any atom* $a \in \mathrm{At}_{\mathfrak{R}}$: $a \ll a$ *and* $\{a\} \in \mathbf{Q}_{\mathfrak{R}}$.

PROOF. Let $a \in \mathrm{At}_{\mathfrak{R}}$. Firstly, Proposition 6.2 says that every region has at least one non-tangential part. But, by $(\mathrm{I}^{\sqsubseteq}_{\ll})$, if $a \in \mathrm{At}_{\mathfrak{R}}$ and $x \ll a$, then $x = a$. So $a \ll a$. Secondly, for $\{a\}$ condition (r1) is trivial and (r2) follows from $a \ll a$. For (r3), suppose that $x, y \in R$ and $a \bigcirc x$ and $a \bigcirc y$. Then $a \sqsubseteq x$ and $a \sqsubseteq y$. Hence $x \bigcirc y$; and so $x \mathrel{\mathsf{C}} y$, by $(\mathrm{I}^{\mathsf{C}}_{\bigcirc})$. ■

By the above proposition, for atoms we obtain:

COROLLARY 9.2. 1. *For all* $a \in \mathrm{At}_{\mathfrak{R}}$ *and* $x \in R$:

- $a \mathrel{)(} x$ *iff* $a \wr x$;
- $a \mathrel{\mathsf{C}} x$ *iff* $a \bigcirc x$ *iff* $a \sqsubseteq x$ *iff* $a \ll x$;
- *if* $x \neq a$*, then* $a \mathrel{)(} x - a$.

2. *For all* $a, b \in \mathrm{At}_{\mathfrak{R}}$: *if* $a \neq b$ *then* $a \mathrel{)(} b$.

PROOF. *Ad* 9.2. Let $a \in \mathrm{At}_{\mathfrak{R}}$ and $x \in R$. By Proposition 9.1, we have $a \ll a$. Hence, firstly, by (df$\ll$), if $x \wr a$, then $x \mathrel{)(} a$. The other implication holds by $(\mathrm{I}^{\wr}_{)(})$. Secondly, by (3.7), if $a \sqsubseteq x$, then $a \ll x$. The other implication holds by $(\mathrm{I}^{\sqsubseteq}_{\ll})$. Thirdly, by (df$\ll$), if $x \neq a$, then $a \mathrel{)(} x - a$, since $a \wr x - a$.

Ad 2. Let $a, b \in \mathrm{At}_{\mathfrak{R}}$ and $a \neq b$. Then $b - a = b$. So $a \mathrel{)(} b$, by 9.2. ■

All finite G-structures (as finite mereological fields) are atomic. By the above corollary, for such structures we have:

COROLLARY 9.3. *If* $\mathfrak{R}$ *is finite, then the relation* $\ll$ *is reflexive.*

PROOF. Fix a region x. From Lemma 2.11, $x = a_1 \sqcup \cdots \sqcup a_n$, where $\{a_1, \ldots, a_n\} = \{a \in \mathrm{At}_{\mathfrak{R}} \mid a \sqsubseteq x\}$. Suppose that $z \wr x$. Then, by (2.3), for any $i \in \{1, \ldots, n\}$ we have $z \wr a_i$. By Corollary 9.2, also $z \mathrel{)(} a_i$. Hence, by (S4), $z \mathrel{)(} a_1 \sqcup \cdots \sqcup a_n = x$. Therefore $x \ll x$, by (df$\ll$). ■

The following proposition characterizes finite G-structures.

PROPOSITION 9.4. *For any structure* $\langle R, \sqsubseteq,)(\rangle$ *with two binary relations* $\sqsubseteq$ *and* $)($ *in* R*, the following conditions are eqivalent*:

(a) $\langle R, \sqsubseteq, \mathrel{)(} \rangle$ *is a finite G-structure,*

(b) $\langle R, \sqsubseteq \rangle$ *is a finite mereological field and* $\mathrel{)(} = \wr$ *,*

(c) $\langle R, \sqsubseteq \rangle$ *is a finite mereological structure and* $\mathrel{)(} = \wr$ *.*

Proof. "(a) ⇒ (b)" We have $\mathrel{)(} \subseteq \wr$, by $(\mathrm{I}^{\wr}_{)(})$. Moreover, from Corollary 9.3 and $(\mathrm{df} \ll)$ we have the converse inclusion. "(b) ⇒ (a)" From Propositions 4.2 and 6.8, since all finite mereological fields are atomic. "(b) ⇔ (c)" From Theorem 2.7 and Proposition 2.10. ■

10. Points vs. Contracting Filters of Quasi-Separation Structures

Let $\mathfrak{R}$ be an arbitrary quasi-separation structure. A filter F of $\mathfrak{R}$ is called *contracting*[2] iff for each $x \in F$ there is a $y \in F$ such that $y \ll x$. Let us remind that points of $\mathfrak{R}$ are filters generated by pre-points of $\mathfrak{R}$ (cf. Section 5). We will draw a comparison between points and contracting filters, the latter in its maximal form (i.e. maximal among contracting filters) being points in the sense of [2].

Directly from $(\mathrm{I}^{\sqsubseteq}_{\ll})$ it follows that:

Lemma 10.1. *For any contracting filter F of $\mathfrak{R}$ and any $x \in R$ we have*:

$$x \in F \iff \exists_{y \in F}\, y \sqsubseteq x \iff \exists_{y \in F}\, y \ll x.$$

Moreover, by means of the standard application of the Kuratowski-Zorn lemma (in its "strong" form), it is provable that:

Lemma 10.2. *Every contracting filter of $\mathfrak{R}$ can be extended to a maximal filter in the family of all contracting filters.*

For brevity, we introduce after [12] the following notation: for any subset X of R and any $y \in R$, the expression '$y \infty X$' means that y is connected with every member of X. Formally, for all $X \in \mathcal{P}(R)$ and $y \in R$ we put:

$$y \infty X \;:\!\iff \forall_{x \in X}\, x \mathrel{\mathsf{C}} y. \tag{df$\,\infty$}$$

We will also use notation '$X \infty y$' with the same meaning. Similarly, for all subsets X, Y of R, the expression '$X \infty Y$' means that each region from X is connected with each region from Y. Formally, for all $X, Y \in \mathcal{P}(R)$ we put:

$$X \infty Y \;:\!\iff \forall_{x \in X} \forall_{y \in Y}\, x \mathrel{\mathsf{C}} y \iff \forall_{x \in X}\, x \infty Y. \tag{df$'\,\infty$}$$

[2] In [2] and [3] these filters are called *concordant* and *round*, respectively. The name used by us comes from [12].

Thus, in both cases the relation ∞ is symmetric. The complement of ∞ will be denoted by '$\not\infty$'.

Let us begin with the following observations.

LEMMA 10.3. *For arbitrary filters F_1 and F_2:*

$$F_1 \subseteq F_2 \implies \forall_{x\in F_1}\forall_{y\in F_2}\, x \bigcirc y,$$
$$\forall_{x\in F_1}\forall_{y\in F_2}\, x \bigcirc y \implies F_1 \infty F_2.$$

PROOF. First, assume that $F_1 \subseteq F_2$, $x \in F_1$ and $y \in F_2$. Then also $x \in F_2$. Hence $x \bigcirc y$, by the definition of filters. Second, we use (df$'$ ∞) and ($\mathrm{I}^{\mathsf{C}}_{\circ}$). ■

LEMMA 10.4. *For any contracting filter F and any $x \in R$:*

$$x \infty F \iff \forall_{y\in F}\, y \bigcirc x.$$

PROOF. "⇒" Assume that (a) $x \infty F$ and (b) $y \in F$. Then, by (b), for some $y_0 \in F$ we have (c) $y_0 \ll y$. Moreover, (a) and (df ∞) entail (d) $y_0 \mathrel{\mathsf{C}} x$. Therefore $y \bigcirc x$, by (c), (d) and (df$'$$\ll$).

"⇐" Directly from (df ∞) and ($\mathrm{I}^{\mathsf{C}}_{\circ}$). ■

COROLLARY 10.5. 1. *For all contracting filters F_1 and F_2:*

$$F_1 \infty F_2 \iff \forall_{x\in F_1}\forall_{y\in F_2}\, x \bigcirc y.$$

2. *If $\mathfrak{R}$ has the unity $\mathsf{1}$, then for any contracting filter F and any $x \in R$:*

$$x \in F \iff x = \mathsf{1} \vee (x \neq \mathsf{1} \wedge -x \not\infty F).$$

PROOF. *Ad* 10.5. Directly from Lemma 10.4 and (df$'$ ∞).

Ad 2. "⇒" Assume that $x \in F$, $x \neq \mathsf{1}$ and $-x \infty F$. Then, by Lemma 10.4, we obtain a contradiction: $x \bigcirc -x$. "⇐". If $x = 1$, then $x \in F$. So suppose that $x \neq \mathsf{1}$ and $-x \not\infty F$. From this and Lemma 10.4, for some $x_0 \in F$ we have $x_0 \mathrel{)(} -x$. Hence $x_0 \sqsubseteq x$ and $x \in F$. ■

LEMMA 10.6. *If a contracting filter F satisfies the following condition:*

$$\forall_{x,y\in R}(x \infty F \wedge x \ll y \implies y \in F)\,, \tag{$*$}$$

then F is a maximal in the family of all contracting filters.

PROOF. Suppose that a contracting filter F satisfies ($*$). Let F' be any contracting filter such that $F \subseteq F'$. Then $F \infty F'$, in the light of Lemma 10.3. Now we show that $F' \subseteq F$. Indeed, assume that $x \in F'$. Then for some $x_0 \in F'$ both $x_0 \ll x$ and $x_0 \infty F$. Hence $x \in F$, by ($*$). ■

Notice that if $\mathfrak{R}$ has the unity then instead of condition ($*$) we can take two other conditions equivalents with it.

Proposition 10.7 [4]. *If $\mathfrak{R}$ has the unity $\mathsf{1}$, then for any contracting filter F the following conditions are equivalent*:

(a) *F satisfies* (∗).

(b) *F satisfies*:

$$\forall_{x,y\in R}\big(x \ll y \implies y \in F \vee (x \neq \mathsf{1} \wedge -x \in F)\big). \tag{$**$}$$

(c) *F satisfies*:

$$\forall_{x,y\in R}\,(x \infty F \infty y \Longrightarrow x \mathrel{\mathsf{C}} y). \tag{$***$}$$

Proof. "(a) ⇒ (b)" Suppose that $x \ll y$ and $y \notin F$. Then $x \not\infty F$, by (∗). Hence $x \neq \mathsf{1}$ and also $-x \neq \mathsf{1}$. Thus $-x \in F$, by Corollary 10.5(2).

"(b) ⇒ (c)" Suppose that $x \mathrel{)(} y$. Then $x \neq \mathsf{1} \neq y$ and so $-x \neq \mathsf{1} \neq -y$. Moreover, $x \ll -y$, by (3.9). Hence either $-x \in F$ or $-y \in F$, by (∗∗). Thus by Corollary 10.5(2), either $x \not\infty F$ or $y \not\infty F$.

"(c) ⇒ (a)" Suppose that $x \infty F$ and $x \ll y$. If $y = \mathsf{1}$ then $y \in F$; so let $y \neq \mathsf{1}$. Then $x \mathrel{)(} -y$, by (3.9). Hence $-y \not\infty F$, by (∗∗∗). Thus $y \in F$, by Corollary 10.5(2). ■

Let $\mathbf{Pt}_{\mathfrak{R}}$ be the family of all points in $\mathfrak{R}$.

Proposition 10.8. *Every point from $\mathbf{Pt}_{\mathfrak{R}}$ is a contracting filter.*

Proof. Let $x \in \mathfrak{p}$, i.e., there is $Q \in \mathbf{Q}_{\mathfrak{R}}$ such that $\mathfrak{p} = \{y \in R \mid \exists_{u\in Q}\, u \sqsubseteq y\}$. Then for some $u_0 \in Q$ we have $u_0 \sqsubseteq x$. Moreover, by the property (r2), for some $v_0 \in Q$, $v_0 \ll u_0$. From this and (3.7) we obtain that $v_0 \ll x$. ■

Example 10.1. For any $n > 0$, the structure $\mathfrak{G}\mathcal{E}^n$ is complete G-structure (cf. Section 8). There are G-structures in which there are contracting filters which are not Grzegorczyk points. For example, the set from Figure 4 is contracting but it is neither a pre-point nor a point. ■

From the Propositions 10.8 and 5.2(2), Lemmas 10.1 and 10.4 and Corollary 10.5 we have:

Corollary 10.9. 1. *For any $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ and any $x \in R$*:

$$x \in \mathfrak{p} \iff \exists_{y\in\mathfrak{p}}\, y \sqsubseteq x \iff \exists_{y\in\mathfrak{p}}\, y \ll x,$$
$$x \infty \mathfrak{p} \iff \forall_{y\in\mathfrak{p}}\, y \bigcirc x.$$

2. *For all $\mathfrak{p}, \mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}$*:

$$\mathfrak{p} \infty \mathfrak{q} \iff \forall_{x\in\mathfrak{p}}\forall_{y\in\mathfrak{p}}\, x \bigcirc y,$$
$$\mathfrak{p} \infty \mathfrak{q} \implies \mathfrak{p} = \mathfrak{q}.$$

3. *If* $\mathfrak{R}$ *has the unity* 1, *then for any* $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ *and any* $x \in R$:

$$x \in \mathfrak{p} \iff x = 1 \lor (x \neq 1 \land -x \not\circ \mathfrak{p}).$$

Finally observe that some maximal filters in the family of all contracting filters in $\mathfrak{R}$ do not have to be ultrafilters in $\mathfrak{R}$. Directly from Lemma 10.4 and Proposition 2.15 we obtain:

PROPOSITION 10.10. *For any contracting filter* F *the following conditions are equivalent*:

(a) $F \in \mathrm{Ult}(\mathfrak{R})$;

(b) $\forall_{x \in R}(x \infty F \Rightarrow x \in F)$.

11. Some Properties of Points of G-Structures

Now let $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle$ be any G-structure. We prove that the following conditions hold:

$$\forall_{x \in R} \exists_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}}\, x \in \mathfrak{p}, \tag{11.1}$$

$$\forall_{x,y \in R} \forall_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}} \big(x \in \mathfrak{p} \land y \in \mathfrak{p} \iff x \bigcirc y \land x \sqcap y \in \mathfrak{p}\big), \tag{11.2}$$

$$\forall_{x,y \in R} \big(x \bigcirc y \implies \exists_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}} (x \sqcap y \in \mathfrak{p} \land x \in \mathfrak{p} \land y \in \mathfrak{p}\big), \tag{11.3}$$

$$\forall_{x,y \in R} \big(x \bigcirc y \iff \exists_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}} (x \in \mathfrak{p} \land y \in \mathfrak{p})\big), \tag{11.4}$$

$$\forall_{x,y \in R} \big(x \mathrel{\mathsf{C}} y \iff \exists_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}} \forall_{z \in \mathfrak{p}} (z \bigcirc x \land z \bigcirc y)\big), \tag{11.5}$$

$$\forall_{x,y \in R} \big(x \ll y \iff \forall_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}} (y \in \mathfrak{p} \lor \exists_{z \in \mathfrak{p}}\, z \wr x)\big). \tag{11.6}$$

PROOF. *Ad* (11.1): By (6.2), for any $x \in R$ there is $Q \in \mathbf{Q}_{\mathfrak{R}}$ such that for some $z \in Q$ we have $z \sqsubseteq x$. Therefore $x \in \mathfrak{p} := \{y \in R \mid \exists_{u \in Q}\, u \sqsubseteq y\}$.

Ad (11.2): From the definition of filters and (2.4).

Ad (11.3): From (11.1) and (11.2).

Ad (11.4): From (11.2) and (11.3).

Ad (11.5): "$\Rightarrow$" Let $x \mathrel{\mathsf{C}} y$. Then, by (G), there is $Q \in \mathbf{Q}_{\mathfrak{R}}$ such that for any $v \in Q$: $v \bigcirc x$ and $v \bigcirc y$. We put $\mathfrak{p} := \{z \in R \mid \exists_{u \in Q}\, u \sqsubseteq z\}$ and let $z \in \mathfrak{p}$ be arbitrary. Then for some $w \in Q$ we have: $w \sqsubseteq z$, $w \bigcirc x$, and $w \bigcirc y$. Hence $z \bigcirc x$ and $z \bigcirc y$, by (MF).

"$\Leftarrow$" Let $\mathfrak{p}$ be a point such that for any $z \in \mathfrak{p}$: $z \bigcirc x$ and $z \bigcirc y$. For some $Q_{\mathfrak{p}} \in \mathbf{Q}_{\mathfrak{R}}$ we have that $\mathfrak{p}$ is generated by $Q_{\mathfrak{p}}$. So $Q_{\mathfrak{p}} \subseteq \mathfrak{p}$. Therefore also for any $z \in Q_{\mathfrak{p}}$: $z \bigcirc x$ and $z \bigcirc y$. Thus, by (r3), we have $x \mathrel{\mathsf{C}} y$.

Ad (11.6) "$\Rightarrow$" Let $x \ll y$. Assume for a contradiction that for some point $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ we have (a) $y \notin \mathfrak{p}$ and $\forall_{z \in \mathfrak{p}}\, z \bigcirc x$. Hence (b) $\forall_{z \in \mathfrak{p}}\, z \not\sqsubseteq y$, and therefore we obtain (c) $\forall_{z \in \mathfrak{p}}\, z - y \mathrel{)(} x$ (since $z - y \wr y$ and $x \ll y$). The point $\mathfrak{p}$ is generated by some $\mathbf{Q}_{\mathfrak{R}} \ni Q_{\mathfrak{p}} \subseteq \mathfrak{p}$. Thanks to (b) we have (d): $\forall_{u,v \in Q_{\mathfrak{p}}}\, u \bigcirc v - y$. Indeed, by (r1) and ($\mathrm{I}^{\sqsubseteq}_{\ll}$), either $v \sqsubseteq u$ or $u \sqsubseteq v$. In the first case: $v - y \sqsubseteq v \sqsubseteq u$. In the second case: $u - y \sqsubseteq v - y$ and $u - y \sqsubseteq u$; so $v - y \bigcirc u$. Since $Q_{\mathfrak{p}} \neq \emptyset$, we pick a member v_0 thereof. Thus, by (r3), (a) and (d), we have $v_0 - y \mathrel{\mathsf{C}} x$, which contradicts (c).

"$\Leftarrow$" Suppose that $x \not\ll y$, i.e., there is $u_0 \in R$ such that (a) $u_0 \wr y$ and (b) $u_0 \mathrel{\mathsf{C}} x$. Then, by (b) and (11.5), there is $\mathfrak{p}_0 \in \mathbf{Pt}_{\mathfrak{R}}$ such that (c): $\forall_{z \in \mathfrak{p}_0}(z \bigcirc u_0 \wedge z \bigcirc x)$. Thus $y \notin \mathfrak{p}$, by (a) and (c). ■

Remark 11.1. Condition (11.1) is a counterpart of the first one of two axioms accepted in [1] instead of (G). The second one of these axioms is implication "from left to right" in condition (11.5).

In [1] points are defined in a different way than in [6], since by *a representative of a point* the authors mean an arbitrary subset S of R satisfying two conditions (these are A and B at the beginning of Section 3 in [1, p. 434]): the first one says that X is totally ordered by $\ll$ and has no minimal element with respect to $\ll$ (that is: $\forall_{x \in S} \exists_{y \in S}(y \ll x \wedge y \neq x)$), the second one is (r3). This entails that—unlike in this paper—all representatives of points are infinite. Another important consequence is also the fact that all structures from [1] are non-atomic (cf. Proposition 4.2). ■

Thanks to (11.1) we may prove the following:

Theorem 11.1. *$\mathfrak{R}$ is finite iff the set $\mathbf{Pt}_{\mathfrak{R}}$ is finite.*

Proof. "$\Rightarrow$" Obvious. "$\Leftarrow$" Suppose that $\mathfrak{R}$ is infinite. Then $\mathfrak{R}$ has an infinite antichain A, in virtue of Lemma 2.12(2). By (11.1), for any $y \in A$ there is $\mathfrak{p}_y \in \mathbf{Pt}_{\mathfrak{R}}$ such that $y \in \mathfrak{p}_y$. Since A is an antichain and every point is a filter, elements of $\{\mathfrak{p}_y \mid y \in A\} \subseteq \mathbf{Pt}_{\mathfrak{R}}$ must be pairwise distinct, and therefore the family $\mathbf{Pt}_{\mathfrak{R}}$ is also infinite. ■

Obviously, in the light of Corollary 10.9, conditions (11.5) and (11.6) can be written as:

$$\forall_{x,y \in R}\big(x \mathrel{\mathsf{C}} y \iff \exists_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}}\, x \infty \mathfrak{p} \infty y\big), \tag{11.5$'$}$$

$$\forall_{x,y \in R}\big(x \ll y \iff \forall_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}}(x \infty \mathfrak{p} \Rightarrow y \in \mathfrak{p})\big). \tag{11.6$'$}$$

From (3.9), (11.6) and the definition of filters follows that if $\mathfrak{R}$ has the unity $\mathbf{1}$, then:

$$\forall_{x \in R \setminus \{\mathbf{1}\}}\big(x \mathrel{)(} -x \iff \forall_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}}(x \in \mathfrak{p} \;\vee\; -x \in \mathfrak{p})\big). \tag{11.7}$$

Indeed, for any $x \neq \mathbf{1}$: $x \mathrel{)(} -x$ iff $x \ll x$ iff $\forall_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}}(x \in \mathfrak{p} \vee \exists_{z \in \mathfrak{p}}\, z \wr x)$ iff $\forall_{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}}(x \in \mathfrak{p} \vee -x \in \mathfrak{p})$; since for any $z \in R$: $z \wr x$ iff $z \sqsubseteq -x$.

Corollary 5.3 says that in an arbitrary quasi-separation structure any given point is a maximal filter in the family of all points. However, by (11.6′), Lemma 10.6 and Proposition 10.8, this result can be strengthened.

PROPOSITION 11.2. *In an arbitrary G-structure every point is a maximal filter in the family of all contracting filters. Formally, for any* $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ *and any contracting filter* F*: if* $\mathfrak{p} \subseteq F$*, then* $\mathfrak{p} = F$*.*

PROOF. By (11.6′), any point $\mathfrak{p}$ satisfies condition ($*$) from Lemma 10.6, i.e., we have: $\forall_{x,y \in R}(x \infty \mathfrak{p} \ \wedge \ x \ll y \Longrightarrow y \in \mathfrak{p})$. Thus, we use Proposition 10.8 and Lemma 10.6. ∎

We will strengthen Example 10.1 by showing that there is a G-structure in which in the family of all contracting filters there is a maximal element which is not a point. This may even happen in complete G-structures satisfying the following condition:

$$\forall_{x,y \in R}\,(x \ll y \Longrightarrow \exists_{z \in R}\, x \ll z \ll y)\,, \tag{IA$^\bullet$}$$

which is the point-free counterpart of the interpolation condition (IA) from Section 4 (see also page 835).

EXAMPLE 11.1. The complete G-structure $\mathfrak{G}\mathcal{E}^1$ (cf. Section 8) satisfies (IA$^\bullet$). Let $\mathcal{S}$ be the family $\{(n, +\infty) \mid n \in \omega\}$ of open infinite segments in $\mathbb{R}$. We consider the contracting filter $\mathcal{F}_{\mathcal{S}}$ generated by $\mathcal{S}$ and its contracting maximal extension $\mathcal{F}^*_{\mathcal{S}}$ in the family of all contracting filters. Notice that $\mathcal{F}^*_{\mathcal{S}}$ does not satisfy (11.5′), so it is not a member of $\mathbf{Pt}_{\mathfrak{G}\mathcal{E}^1}$. To see that, we define two open subset of $\mathbb{R}$:

$$U := \operatorname{Int}\operatorname{Cl}\bigcup\nolimits_{n\in\omega}(4n, +\infty) \quad \text{and} \quad V := \operatorname{Int}\operatorname{Cl}\bigcup\nolimits_{n\in\omega}(4n+2, +\infty)\,.$$

We have $V \infty \mathcal{F}^*_{\mathcal{S}} \infty U$, yet $\operatorname{Cl} V \cap \operatorname{Cl} U = \emptyset$, i.e., $V \mathrel{][} U$. ∎

12. Points vs. Ultrafilters of G-Structures

Let $\mathfrak{M} = \langle R, \sqsubseteq \rangle$ be an arbitrary mereological field (with or without unity). Then Proposition 2.15 says that F is an ultrafilter of $\mathfrak{M}$ iff for any $x \in R$ either $x \in F$ or there is a $y \in F$ such that $y \wr x$. We obtain:

LEMMA 12.1. *For any* $x \in R$*:* $x \in \mathrm{At}_{\mathfrak{M}}$ *iff* $\mathrm{F}_x \in \mathrm{Ult}(\mathfrak{M})$*.*

PROOF. "⇒" Assume for a contradiction that (a) $x \in \mathrm{At}_{\mathfrak{M}}$ and (b) $\mathrm{F}_x \notin \mathrm{Ult}(\mathfrak{M})$. Then, by (a), for any $z \in R$ either $x \sqsubseteq z$ or $x \wr z$. Moreover, by (b) and Proposition 2.15, for some $z_0 \in R$ both $x \not\sqsubseteq z_0$ and for any $y \in \mathrm{F}_x$ we have $y \bigcirc z_0$. But $x \in \mathrm{F}_x$. So we obtain a contradiction: $x \wr z_0$ and $x \bigcirc z_0$.

"$\Leftarrow$" In the light of Proposition 2.15, if $\mathrm{F}_x \in \mathrm{Ult}(\mathfrak{M})$ then for any $z \in R$ either $x \sqsubseteq z$ or both $x \sqsubseteq y_0$ and $y_0 \wr z$. Hence for any $z \in R$ either $x \sqsubseteq z$ or $x \wr z$. So $x \in \mathrm{At}_{\mathfrak{M}}$. ■

A filter F of $\mathfrak{M}$ is said to be *free* iff there is no $x \in R$ such that for any $y \in F$ we have $x \sqsubseteq y$. We obtain:

LEMMA 12.2. *Each ultrafilter of $\mathfrak{M}$ either is free or principal generated by an atom.*

PROOF. Let F be an ultrafilter of $\mathfrak{M}$ that is not free. Then for some $x_0 \in R$ for any $y \in F$ we have $x_0 \sqsubseteq y$. Moreover, by Proposition 2.15, if $x_0 \notin F$ then there is a $y \in F$ such that $y \wr x_0$; and so we obtain a contradiction. Hence $x_0 \in F$. Therefore $F = \mathrm{F}_{x_0}$. Moreover, $x_0 \in \mathrm{At}_{\mathfrak{M}}$, by Lemma 12.1. ■

Let us remind:

LEMMA 12.3. 1. *Every infinite Boolean lattice has a free ultrafilter.*

2. [cf. 7, Lemma 43] *In every complete Boolean lattice no free ultrafilter is generated by a chain.*

Hence, by Theorems 2.6 and 2.8 we obtain, respectively:

4. *Every infinite mereological field with unity has a free ultrafilter.*
5. *In every mereological structure no free ultrafilter is generated by a chain.*

Thanks to existence of free ultrafilters we have a general way of constructing G-structures for which all filters are contracting and the family of points is properly included in the family of all free ultrafilters.

PROPOSITION 12.4. *Let $\langle R, \sqsubseteq\rangle$ be an atomic infinite mereological structure. Then in the complete G-structure $\mathfrak{R} := \langle R, \sqsubseteq, \wr\,\rangle$ all filters are contracting and there is a free ultrafilter which is not a point.*

PROOF. Let $\langle R, \sqsubseteq\rangle$ be any atomic infinite mereological structure with unity 1. By Proposition 6.8, $\mathfrak{R} := \langle R, \sqsubseteq, \wr\,\rangle$ is a complete G-structure, where $)(:= \wr$ and $\ll = \sqsubseteq$. Therefore all filters of $\mathfrak{R}$ are contracting.

In the atomic infinite complete Boolean lattice $\mathfrak{B} := \langle R^{\mathit{0}}, \sqsubseteq^{\mathit{0}}, \mathit{0}, 1\rangle$ (cf. Theorem 2.8) the set $\{-a \mid a \in \mathrm{At}_{\mathfrak{B}}\}$ of co-atoms has finite intersection property and generates the co-finite free filter F_{c} of $\mathfrak{B}$. By the Kuratowski-Zorn lemma, the filter F_{c} can be extended to a free ultrafilter U of $\mathfrak{B}$. But, by Lemma 12.3(2), no free ultrafilter of $\mathfrak{B}$ is generated by a chain. Hence U is not generated by any chain. Therefore $U \notin \mathbf{Pt}_{\mathfrak{R}}$, since all members of $\mathbf{Q}_{\mathfrak{R}}$ are chains. ■

Now let $\mathfrak{R}$ be an arbitrary G-structure (with or without unity). First, directly from (11.6) and Proposition 2.15 we obtain:

Proposition 12.5. $\mathbf{Pt}_{\mathfrak{R}} \subseteq \mathrm{Ult}(\mathfrak{R})$ *iff the relation* $\ll$ *is reflexive* (*and also* $)(= \wr$ *and* $\ll = \sqsubseteq$, *by Proposition 3.3*).

Remark 12.1. If $\mathbf{Pt}_{\mathfrak{R}} \subseteq \mathrm{Ult}(\mathfrak{R})$, then $\ll$ is reflexive and so, by (11.6′), we have: $\forall_{x\in R}\forall_{\mathfrak{p}\in\mathbf{Pt}_{\mathfrak{R}}}(x \infty \mathfrak{p} \Rightarrow x \in \mathfrak{p})$. And this is in line with propositions 10.8 and 10.10 . ■

Moreover, we obtain:

Proposition 12.6. *If* $\mathfrak{R}$ *has the unity* 1, *then* $\mathbf{Pt}_{\mathfrak{R}} \subseteq \mathrm{Ult}(\mathfrak{R})$ *iff for any* $x \in R\setminus\{1\}$ *we have* $x \,)(\, -x$.

Proof. Let $\mathfrak{R}$ have the unity 1. Then, by Proposition 2.13, for any $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$: $\mathfrak{p} \in \mathrm{Ult}(\mathfrak{R})$ iff for any $x \in R\setminus\{1\}$ either $x \in \mathfrak{p}$ or $-x \in \mathfrak{p}$ iff for any $x \in R\setminus\{1\}$ we have $x \,)(\, -x$, by (11.7). ■

From (11.1) and Corollary 5.3 we obtain that points may be principal filters only if they are generated by atoms, i.e.:

Proposition 12.7. *For all* $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ *and* $x \in R$: *if* $\mathfrak{p} = \mathrm{F}_x$, *then* $x \in \mathrm{At}_{\mathfrak{R}}$.

What's more, for any $x \in R$ *the following conditions are equivalent*:

(a) $\mathrm{F}_x \in \mathbf{Pt}_{\mathfrak{R}}$,

(b) $x \in \mathrm{At}_{\mathfrak{R}}$,

(c) $\mathrm{F}_x \in \mathrm{Ult}(\mathfrak{R})$.

Proof. "(a) $\Rightarrow$ (b)" If $x \notin \mathrm{At}_{\mathfrak{R}}$ then for some $x_0 \in R$ we have $x_0 \sqsubset x$. Moreover, by (11.1), for some $\mathfrak{p}_0 \in \mathbf{Pt}_{\mathfrak{R}}$ we have $x_0 \in \mathfrak{p}_0$. Thus, $\mathrm{F}_x \subsetneq \mathfrak{p}_0$. Hence, $\mathrm{F}_x \notin \mathbf{Pt}_{\mathfrak{R}}$, in virtue of Corollary 5.3.[3]

"(b) $\Rightarrow$ (a)" If $x \in \mathrm{At}_{\mathfrak{R}}$, then $\{x\} \in \mathbf{Q}_{\mathfrak{R}}$, by Proposition 9.1. Thus, $\mathrm{F}_x \in \mathbf{Pt}_{\mathfrak{R}}$.

"(b) $\Leftrightarrow$ (c)" By Lemma 12.1. ■

This, on the other hand, for complete G-structures entails:

Proposition 12.8 [4]. *If* $\mathfrak{R}$ *is complete, then for each* $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}} \cap \mathrm{Ult}(\mathfrak{R})$ *there is an* $a \in \mathrm{At}_{\mathfrak{R}}$ *such that* $\mathfrak{p} = \mathrm{F}_a$.

Proof. Let $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}} \cap \mathrm{Ult}(\mathfrak{R})$. Then, by Lemma 12.3(4), $\mathfrak{p}$ is not free, since $\mathfrak{p}$ is generated by a chain. So, by Lemma 12.2, $\mathfrak{p} = \mathrm{F}_a$ for some $a \in \mathrm{At}_{\mathfrak{R}}$. ■

[3] Of course, also $\mathrm{F}_x \notin \mathrm{Ult}(\mathfrak{R})$. We get therefore another justification for "(c) $\Rightarrow$ (b)".

For completeness of presentation, we provide an incomplete G-structure with unity for which the above proposition fails.

EXAMPLE 12.1. Let $\mathrm{FC}(\omega)$ be the family of all finite and co-finite subsets of ω. It is well-known that $\langle \mathrm{FC}(\omega), \subseteq, \emptyset, \omega\rangle$ is an atomic incomplete Boolean lattice, where $\{\{n\} \mid n \in \omega\}$ is the set of atoms. We put $\mathrm{FC}^+(\omega) := \mathrm{FC}(\omega) \setminus \{\emptyset\}$. Then $\langle \mathrm{FC}^+(\omega), \subseteq\rangle$ is an atomic incomplete mereological field with the unity ω. Thus, by propositions 4.2 and 6.8, $\mathfrak{FC} := \langle \mathrm{FC}^+(\omega), \subseteq, \wr\rangle$ (where $)(\, := \wr$, i.e., $X \mathrel{)(} Y \;:\!\iff\; X \cap Y = \emptyset$ and so $\ll\, = \,\subseteq$) is an atomic incomplete G-structure with the unity ω. The structure has only one free ultrafilter, the family $\mathrm{F_c}$ of all co-finite subsets of ω. The filter $\mathrm{F_c}$ is generated by the chain $Q := \{\omega \setminus \{0, \dots, n\} \mid n \in \omega\}$. Indeed, let K be a co-finite subset of ω with the complement $\{k_1, \dots, k_n\}$ in which k_* is the largest number. Then $\omega \setminus \{0, \dots, k_*\} \subseteq K$.

Now notice that the family Q is a pre-point. Indeed, conditions (r1) and (r2) are obvious. For (r3) suppose that sets X and Y from $\mathrm{FC}^+(\omega)$ have non-empty intersections with each set from Q. Then X and Y are co-finite, and so $X \cap Y \neq \emptyset$.

Thus, $\mathrm{F_c}$ is both a point and an ultrafilter, but there is no $n \in \omega$ such that $\mathrm{F_c} = \mathrm{F}_{\{n\}}$, i.e., $\mathrm{F_c}$ in not generated by any atom. ∎

Now notice that:

PROPOSITION 12.9. *If $\mathfrak{R}$ is finite, then $\mathfrak{R}$ is complete and* $\mathbf{Pt}_{\mathfrak{R}} = \mathrm{Ult}(\mathfrak{R}) = \{\mathrm{F}_a \mid a \in \mathrm{At}_{\mathfrak{R}}\}$.

PROOF. Let $\mathfrak{R}$ be finite. Then, in virtue of Proposition 2.10, $\mathfrak{R}$ is based on a complete mereological structure. Thus, $\mathbf{Pt}_{\mathfrak{R}} \subseteq \mathrm{Ult}(\mathfrak{R})$, by Corollary 9.3 and Proposition 12.5. For the converse inclusion we apply Lemma 2.14 and suppose that $F \in \mathrm{Ult}(\mathfrak{R}) = \mathrm{Ult}(\mathfrak{R}) = \{\mathrm{F}_a \mid a \in \mathrm{At}_{\mathfrak{R}}\}$. Hence for some $a \in \mathrm{At}_{\mathfrak{R}}$ we have $F = \mathrm{F}_a := \{x \in R \mid a \sqsubseteq x\}$. But $\{a\} \in \mathbf{Q}_{\mathfrak{R}}$, by Proposition 9.1. So $\mathrm{F}_a \in \mathbf{Pt}_{\mathfrak{R}}$. ∎

We also get the converse implication:

PROPOSITION 12.10. *If $\mathfrak{R}$ is complete and* $\mathrm{Ult}(\mathfrak{R}) \subseteq \mathbf{Pt}_{\mathfrak{R}}$, *then $\mathfrak{R}$ is finite.*

PROOF. Suppose that $\mathfrak{R}$ is complete, 1 is its unity, and $\mathrm{Ult}(\mathfrak{R}) \subseteq \mathbf{Pt}_{\mathfrak{R}}$. By $(\mathrm{I}^{\sqsubseteq}_{\ll})$ and (r1), all elements of $\mathbf{Q}_{\mathfrak{R}}$ are chains in the non-trivial complete Boolean lattice $\langle R^{\mathit{0}}, \sqsubseteq^{\mathit{0}}, \mathit{0}, 1\rangle$ (cf. Theorem 2.8). Hence, by the assumption, every ultrafilter of $\mathfrak{R}$ is generated by a chain. So, by Lemma 12.3(4), every ultrafilter is principal, and by Lemma 12.3(4), R^{0} and R are finite.

∎

Example 12.1 shows that also Proposition 12.10 fails for some incomplete G-structures.

Example 12.2. Let us continue with Example 12.1. Since F_{c} is the only free ultrafilter of $\mathfrak{FC}$, the remaining ultrafilters of $\mathfrak{FC}$ are principal and generated by atoms. By Proposition 12.7, all principal filters are points. Thus, $\mathrm{Ult}(\mathfrak{FC}) \subseteq \mathbf{Pt}_{\mathfrak{FC}}$, however, the G-structure $\mathfrak{FC}$ is infinite. ■

From the Kuratowski-Zorn lemma and Corollary 5.3 we have:

Proposition 12.11. $\mathrm{Ult}(\mathfrak{R}) \subseteq \mathbf{Pt}_{\mathfrak{R}}$ *iff* $\mathbf{Pt}_{\mathfrak{R}} = \mathrm{Ult}(\mathfrak{R})$, *i.e., if all ultrafilters are points, then also all points are ultrafilters.*

Proof. Let $\mathrm{Ult}(\mathfrak{R}) \subseteq \mathbf{Pt}_{\mathfrak{R}}$ and $\mathfrak{p}$ be an arbitrary point. Then $\mathfrak{p}$ is a filter in $\mathfrak{R}$. So, by the Kuratowski-Zorn lemma, for some $F \in \mathrm{Ult}(\mathfrak{R})$ we have $\mathfrak{p} \subseteq F$. However, $\mathrm{Ult}(\mathfrak{R}) \subseteq \mathbf{Pt}_{\mathfrak{R}}$. So $F \in \mathbf{Pt}_{\mathfrak{R}}$ and $\mathfrak{p} = F$, in light of Corollary 5.3. Thus, $\mathbf{Pt}_{\mathfrak{R}} \subseteq \mathrm{Ult}(\mathfrak{R})$. ■

13. Internal Points and Adherent Points of Regions

Let $\mathfrak{R} = \langle R, \sqsubseteq, \,)(\, \rangle$ be any G-structure. We introduce an operation that assigns to an arbitrary region $x \in R$ the set of all points of whose x is an element. Formally we have the operation $\mathbf{Irl}\colon R \to \mathcal{P}^+(\mathbf{Pt}_{\mathfrak{R}})$ such that:

$$\mathbf{Irl}(x) := \{\, \mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}} \mid x \in \mathfrak{p} \,\}. \tag{df Irl}$$

Elements of $\mathbf{Irl}(x)$ will be called *internal points of* the region x. In light of (11.1), every region has an internal point, i.e., for any $x \in R$ we have:

$$\mathbf{Irl}(x) \neq \emptyset. \tag{13.1}$$

In light of (df **Irl**) and Corollary 5.4 we obtain:

Proposition 13.1. *For any* $Q \in \mathbf{Q}_{\mathfrak{R}}$:

$$\{\mathrm{F}_Q\} = \bigcap\{\mathbf{Irl}(x) \mid x \in \mathrm{F}_Q\} = \bigcap\{\mathbf{Irl}(x) \mid x \in Q\}.$$

Thus, for any $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ *we have* $\{\mathfrak{p}\} = \bigcap\{\mathbf{Irl}(x) \mid x \in Q\}$, *where* $\mathfrak{p} = \mathrm{F}_Q$.[4]

Proof. Let $Q \in \mathbf{Q}_{\mathfrak{R}}$ and $\mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}$. If $\mathfrak{q} \in \bigcap\{\mathbf{Irl}(x) \mid x \in Q\}$, then $Q \subseteq \mathfrak{q}$. So $\mathrm{F}_Q = \mathfrak{q}$, by Corollary 5.4. Thus, $\bigcap\{\mathbf{Irl}(x) \mid x \in Q\} \subseteq \{\mathrm{F}_Q\} \subseteq \bigcap\{\mathbf{Irl}(x) \mid x \in \mathrm{F}_Q\} \subseteq \bigcap\{\mathbf{Irl}(x) \mid x \in Q\}$, since $Q \subseteq \mathrm{F}_Q \in \bigcap\{\mathbf{Irl}(x) \mid x \in \mathrm{F}_Q\}$. ■

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[4] The last statement of the proposition is formulated without a proof in [6, p. 234].

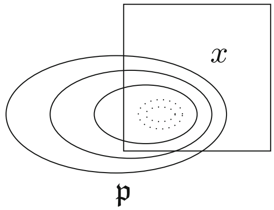

Figure 1. The point $\mathfrak{p}$ is internal of the region x

The spatial intuitions behind the concept of *internal point* are depicted in Figure 1 and are very well expressed by the following binary relation in $\mathbf{Pt}_{\mathfrak{R}} \times R$:

$$\mathfrak{p} \lessdot x \ :\Longleftrightarrow\ \mathfrak{p} \in \mathbf{Irl}(x) \ :\Longleftrightarrow\ x \in \mathfrak{p}. \tag{df $\lessdot$}$$

Of course, if $\mathfrak{p} \lessdot x$, then we also say that $\mathfrak{p}$ is an *internal point of the region* x.

Due to $\mathbf{Pt}_{\mathfrak{R}} \neq \emptyset$, the definition of points, (df **Irl**), (11.4), and (11.2), respectively, the following conditions hold:

$$\forall_{x,y \in R}\big(x \bigcirc y \iff \mathbf{Irl}(x) \cap \mathbf{Irl}(y) \neq \emptyset\big)\,, \tag{13.2}$$

$$\forall_{x,y \in R}\big(x \bigcirc y \Longrightarrow \mathbf{Irl}(x \sqcap y) = \mathbf{Irl}(x) \cap \mathbf{Irl}(y)\big)\,. \tag{13.3}$$

Remark 13.1. Condition (13.3) is analogous to the standard property of the ordinary interior operation $\operatorname{Int}\colon \mathcal{P}(S) \to \mathcal{P}(S)$, for a topological space $\langle S, \mathscr{O}\rangle$: $\operatorname{Int}(X \cap Y) = \operatorname{Int} X \cap \operatorname{Int} Y$, for all $X, Y \in \mathcal{P}(S)$ (see Appendix in the first part of the paper). So we can see that the operation $\mathbf{Irl}\colon R \to \mathcal{P}^{+}(\mathbf{Pt}_{\mathfrak{R}})$ indeed deserves the name of *interior of a region*. Other properties of Int are not expressible by means of **Irl**, since formulas '$\mathbf{Irl}(x) \sqsubseteq x$' and '$\mathbf{Irl}(x) = x$' are meaningless. In the presented theory the notion of *open region* does not make any sense. ■

Further, by (13.3), ($\mathrm{sep}_{\sqsubseteq}$), and (11.1), we have:

$$\mathbf{Pt}_{\mathfrak{R}} = \textstyle\bigcup_{x \in R} \mathbf{Irl}(x), \tag{13.4}$$

$$\forall_{x,y \in R}\big(x \sqsubseteq y \iff \mathbf{Irl}(x) \subseteq \mathbf{Irl}(y)\big), \tag{13.5}$$

$$\forall_{x,y \in R}\big(x = y \iff \mathbf{Irl}(x) = \mathbf{Irl}(y)\big), \tag{13.6}$$

$$\forall_{x \in R}\big(\mathbf{Irl}(x) = \mathbf{Pt}_{\mathfrak{R}} \iff x \text{ is the unity of } \mathfrak{R}\big). \tag{13.7}$$

Proof. *Ad* (13.4): Let $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$, i.e., for some $\emptyset \neq Q_{\mathfrak{p}} \in \mathbf{Q}_{\mathfrak{R}}$ we have $\mathfrak{p} = \{y \in R \mid \exists_{x \in Q_{\mathfrak{p}}}\, x \sqsubseteq y\}$. Then there is an $x \in Q_{\mathfrak{p}}$ such that $\mathfrak{p} \in \mathbf{Irl}(x)$, since $Q_{\mathfrak{p}} \subseteq \mathfrak{p}$.

Ad (13.5): If $x \sqsubseteq y$, then $x \sqcap y = x$. Hence, by (13.3), we have $\mathbf{Irl}(x) = \mathbf{Irl}(x \sqcap y) = \mathbf{Irl}(x) \cap \mathbf{Irl}(y)$; so $\mathbf{Irl}(x) \subseteq \mathbf{Irl}(y)$. Conversely, let $x \not\sqsubseteq y$. Then,

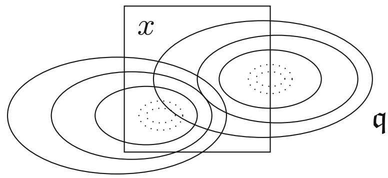

Figure 2. Both points $\mathfrak{p}$ and $\mathfrak{q}$ are adherent to the region x, but $\mathfrak{q}$ is not internal of x

by (11.1), there exists $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ such that $x - y \in \mathfrak{p}$. So $x \in \mathfrak{p}$ and $y \notin \mathfrak{p}$ (i.e., $\mathbf{Irl}(x) \nsubseteq \mathbf{Irl}(y)$), since $\mathfrak{p}$ is a filter, $x - y \sqsubseteq x$, and $x - y \wr y$.

Ad (13.6): By (13.5) and reflexivity and antisymmetry of $\sqsubseteq$.

Ad (13.7): "$\Rightarrow$" By (13.5). "$\Leftarrow$" Since the unity belongs to all points. ∎

After [1] we introduce another binary relation in $\mathbf{Pt}_{\mathfrak{R}} \times R$:

$$\mathfrak{p}\,\mathbf{A}\,x \;:\Longleftrightarrow\; \forall_{y\in\mathfrak{p}}\, y \bigcirc x \;\Longleftrightarrow\; x \infty \mathfrak{p}\,. \tag{df A}$$

If $\mathfrak{p}\,\mathbf{A}\,x$, we say that the point $\mathfrak{p}$ *is adherent to the region* x. The intuitions associated with the operation are presented in Figure 2.

By the definition of filters, (df **Irl**), (df $\lessdot$) and (df **A**), the relation $\lessdot$ is included in the relation **A**, i.e., all internal points of a region are also its adherent points, i.e., for any $x \in R$ we have:

$$\mathbf{Irl}(x) \subseteq \mathbf{A}(x) := \{\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}} \mid \mathfrak{p}\,\mathbf{A}\,x\}. \tag{13.8}$$

Directly from (df **A**) and (2.3) the operation $\mathbf{A}\colon R \to \mathcal{P}^{+}(\mathbf{Pt}_{\mathfrak{R}})$ has the following property:

$$\forall_{x,y\in R}\, \mathbf{A}(x \sqcup y) = \mathbf{A}(x) \cup \mathbf{A}(y)\,, \tag{13.9}$$

Remark 13.2. Referring to Remark 13.1, note that condition (13.9) is analogous to the property of the ordinary closure operation $\mathrm{Cl}\colon \mathcal{P}(S) \to \mathcal{P}(S)$, for a topological space $\langle S, \mathscr{O}\rangle$: $\mathrm{Cl}(X \cup Y) = \mathrm{Cl}\,X \cup \mathrm{Cl}\,Y$, for all $X, Y \in \mathcal{P}(S)$. ∎

The relation $\sqsubseteq$ can also be expressed by means of **A** and **Irl**:

$$\mathbf{A}(x) \subseteq \mathbf{A}(y) \iff x \sqsubseteq y \iff \mathbf{Irl}(x) \subseteq \mathbf{A}(y), \tag{13.10}$$

$$x = y \iff \mathbf{A}(x) = \mathbf{A}(y). \tag{13.11}$$

Ad (13.10): Let $x \sqsubseteq y$. Then $x \sqcup y = y$ and by (13.9), $\mathbf{A}(y) = \mathbf{A}(x \sqcup y) = \mathbf{A}(x) \cup \mathbf{A}(y)$; so $\mathbf{A}(x) \subseteq \mathbf{A}(y)$. Thus $\mathbf{Irl}(x) \subseteq \mathbf{A}(y)$, by (13.8). Now let $x \not\sqsubseteq y$. Then, by (11.1), there exists $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ such that $x - y \in \mathfrak{p}$. Hence $\mathfrak{p} \in$

$\mathbf{Irl}(x-y) \subseteq \mathbf{Irl}(x)$, by (13.5). Yet $\mathfrak{p} \notin \mathbf{A}(y)$, since $x - y \wr y$. In consequence $\mathbf{Irl}(x) \nsubseteq \mathbf{A}(y)$ and $\mathbf{A}(x) \nsubseteq \mathbf{A}(y)$, by (13.8).

Ad (13.11): By (13.10) and and reflexivity and antisymmetry of $\sqsubseteq$.

Using (11.5), (11.6), and (13.8), respectively, for all $x, y \in R$ we obtain:

$$\begin{aligned} x \mathrel{\mathsf{C}} y &\iff \mathbf{A}(x) \cap \mathbf{A}(y) \neq \emptyset, \\ x \mathrel{)(} y &\iff \mathbf{A}(x) \cap \mathbf{A}(y) = \emptyset, \end{aligned} \tag{13.12}$$

$$x \ll y \iff \mathbf{A}(x) \subseteq \mathbf{Irl}(y). \tag{13.13}$$

$$x \ll x \iff \mathbf{A}(x) = \mathbf{Irl}(x). \tag{13.14}$$

If $\mathfrak{R}$ has the unity $\mathsf{1}$ then for any $x \in R \setminus \{\mathsf{1}\}$:

$$\begin{aligned} \mathbf{Irl}(-x) &= \mathbf{Pt}_{\mathfrak{R}} \setminus \mathbf{A}(x), \\ \mathbf{A}(-x) &= \mathbf{Pt}_{\mathfrak{R}} \setminus \mathbf{Irl}(x). \end{aligned} \tag{13.15}$$

Indeed, $\mathfrak{p} \in \mathbf{Irl}(-x)$ iff $-x \in \mathfrak{p}$ iff (by Corollary 10.9) $\exists_{y \in \mathfrak{p}}\ y \sqsubseteq -x$ iff $\exists_{y \in \mathfrak{p}}\ y \wr x$ iff $\neg \forall_{y \in \mathfrak{p}}\ y \bigcirc x$. For the second equality apply $x = - - x$.

Remark 13.3. In reference to remarks 13.1 and 13.2 , observe that condition (13.15) is analogous to the properties of the operations Int and Cl, for a topological space $\langle S, \mathscr{O} \rangle$: $\operatorname{Int} X = S \setminus \operatorname{Cl}(S \setminus X)$ and $\operatorname{Cl} X = S \setminus \operatorname{Int}(S \setminus X)$, for any $X \in \mathcal{P}(S)$. ■

14. Set-Theoretical Version of Representation Theorem

Let $\mathfrak{R}_1 = \langle R_1, \sqsubseteq_1,)(_1 \rangle$ and $\mathfrak{R}_2 = \langle R_2, \sqsubseteq_2,)(_2 \rangle$ be relational structures with binary relations. A *strong homomorphism* from $\mathfrak{R}_1$ into $\mathfrak{R}_2$ is a map $h \colon R_1 \to R_2$ such that for all $x, y \in R_1$:

$$\begin{aligned} x \sqsubseteq_1 y &\iff h(x) \sqsubseteq_2 h(y), \\ x \mathrel{)(_1} y &\iff h(x) \mathrel{)(_2} h(y). \end{aligned}$$

A one-to-one (resp. onto) strong homomorphism is called an *embedding* or a *monomorphism* (resp. an *epimorphism*). A map is an *isomorphism* from $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ iff it is both an embedding and an epimorphism iff it is a bijective strong homomorphism. We say that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are *isomorphic* ($\mathfrak{R}_1 \cong \mathfrak{R}_2$) iff there exists an isomorphism from $\mathfrak{R}_1$ onto $\mathfrak{R}_2$. If e is an embedding, then $\mathfrak{R}_1$ is isomorphic to $\langle e[R_1], \sqsubseteq_2|_{e[R_1]},)(_2|_{e[R_1]} \rangle$ *via* e.

Lemma 14.1. *If $\mathfrak{R}_1$ is a G-structure and e is an embedding from $\mathfrak{R}_1$ into $\mathfrak{R}_2$, then $\langle e[R_1], \sqsubseteq_2|_{e[R_1]},)(_2|_{e[R_1]} \rangle$ is also a G-structure.*

Let $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle$ be any G-structure. A *representation* of $\mathfrak{R}$ is an isomorphism from $\mathfrak{R}$ into a G-structure whose universe is included in $\mathcal{P}^{+}(\mathbf{Pt}_{\mathfrak{R}})$.

A representation $\imath$ is *reduced* iff the image $\imath[R]$ separates the points of $\mathbf{Pt}_{\mathfrak{R}}$, i.e., iff for any two distinct $\mathfrak{p}, \mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}$, there is an $x \in R$ such that $\mathfrak{p} \in \imath(x)$ and $\mathfrak{q} \notin \imath(x)$. A representation $\imath$ is *perfect* iff for all $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ and $x \in R$: $x \in \mathfrak{p}$ iff $\mathfrak{p} \in \imath(x)$. We can construct a G-structure for which the operation **Irl** will be a reduced and perfect representation of $\mathfrak{R}$.

Firstly, notice that the image $\mathbf{Irl}[R]$ is included in $\mathcal{P}^{+}(\mathbf{Pt}_{\mathfrak{R}})$. Secondly, since by (13.6) the operation **Irl** is one-to-one, in the family $\mathbf{Irl}[R]$ we can introduce the following binary relation:

$$X \;)(\; Y \;:\!\iff\; \mathbf{A} \circ \mathbf{Irl}^{-1}(X) \cap \mathbf{A} \circ \mathbf{Irl}^{-1}(Y) = \emptyset. \qquad (\mathrm{df}\,)(\,)$$

It means that for any $x, y \in R$ we have:

$$\mathbf{Irl}(x) \;)(\; \mathbf{Irl}(y) \iff \mathbf{A}(x) \cap \mathbf{A}(y) = \emptyset. \qquad (14.1)$$

Thus, we can put $\mathbf{Irl}[\mathfrak{R}] := \langle \mathbf{Irl}[R], \subseteq,)(\, \rangle$.

THEOREM 14.2. 1. *The operation* **Irl** *is an isomorphism of* $\mathfrak{R}$ *onto* $\mathbf{Irl}[\mathfrak{R}]$.

2. $\mathbf{Irl}[\mathfrak{R}]$ *is a G-structure.*
3. *The operation* **Irl** *is a reduced and perfect representation of* $\mathfrak{R}$.
4. *If* $\mathfrak{R}$ *has the unity* 1, *then* $\mathbf{Irl}[\mathfrak{R}]$ *has the unity* $\mathbf{Pt}_{\mathfrak{R}}$ *and* $\mathbf{Irl}(1) = \mathbf{Pt}_{\mathfrak{R}}$.
5. $\mathfrak{R}$ *is complete iff* $\mathbf{Irl}[\mathfrak{R}]$ *is complete.*

PROOF. *Ad* 1. By (13.5), (13.6), (13.12) and (14.1).

Ad 2. From Lemma 14.1 and 14.2.

Ad 3. Directly from definitions, for any two distinct $\mathfrak{p}, \mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}$, there is an $x \in R$ such that $x \in \mathfrak{p}$ and $x \notin \mathfrak{q}$, so $\mathfrak{p} \in \mathbf{Irl}(x)$ and $\mathfrak{q} \notin \mathbf{Irl}(x)$. Moreover, for all $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ and $x \in R$: $x \in \mathfrak{p}$ iff $\mathfrak{p} \in \mathbf{Irl}(x)$.

Ad 4. By (13.7). *Ad* 5. Obvious. ∎

15. Topological Spaces for G-Structures

In this section we move on to the construction of a topological space in the set of points (as sets of regions) of a given G-structure. To this end a standard method of generating of topology via basis will be applied.

15.1. From the Base of Topology to the Family of Open Sets

Let $\mathfrak{R} = \langle R, \sqsubseteq,)(\, \rangle$ be an arbitrary G-structure. As basis for the set $\mathbf{Pt}_{\mathfrak{R}}$ may be taken the image of R with respect to the operation **Irl**:

$$\mathbf{B}_{\mathfrak{R}} := \mathbf{Irl}[R] := \{\, \mathbf{Irl}(x) \mid x \in R \,\}. \qquad (\mathrm{df}\,\mathbf{B}_{\mathfrak{R}})$$

Indeed, for $x, y \in R$, if $\mathfrak{p} \in \mathbf{Irl}(x) \cap \mathbf{Irl}(y)$, then $x \circ y$ and so $\mathfrak{p} \in \mathbf{Irl}(x \sqcap y) = \mathbf{Irl}(x) \cap \mathbf{Irl}(y)$, by (13.2) and (13.3). Moreover, directly from the defintions of the families $\mathbf{Q}_{\mathfrak{R}}$ and $\mathbf{Pt}_{\mathfrak{R}}$, for any $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ there is an $x \in R$ such that $\mathfrak{p} \in \mathbf{Irl}(x)$, i.e., $x \in \mathfrak{p}$.

Thus—in the standard way—we can introduce the family $\mathsf{O}_{\mathfrak{R}}$ of open sets on $\mathbf{Pt}_{\mathfrak{R}}$ which is generated by $\mathsf{B}_{\mathfrak{R}}$. That is, for any $\Gamma \in \mathcal{P}(\mathbf{Pt}_{\mathfrak{R}})$ we put:

$$\Gamma \in \mathsf{O}_{\mathfrak{R}} \;:\Longleftrightarrow\; \Gamma = \bigcup \mathsf{B},\ \text{for some subfamily } \mathsf{B} \text{ of } \mathsf{B}_{\mathfrak{R}}. \tag{df $\mathsf{O}_{\mathfrak{R}}$}$$

The family $\mathsf{O}_{\mathfrak{R}}$ will be called the *Grzegorczyk topology* (shortly: *G-topology*) of $\mathfrak{R}$, while the topological space $\langle \mathbf{Pt}_{\mathfrak{R}}, \mathsf{O}_{\mathfrak{R}} \rangle$ takes the name of *Grzegorczyk topological space of* $\mathfrak{R}$ (shortly: *G-topological space of* $\mathfrak{R}$) and we put $\mathtt{Gts}\,\mathfrak{R} := \langle \mathbf{Pt}_{\mathfrak{R}}, \mathsf{O}_{\mathfrak{R}} \rangle$. Let $\mathrm{r}\mathsf{O}_{\mathfrak{R}}$ be the family of all regular open sets in $\mathtt{Gts}\,\mathfrak{R}$; $\mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}} := \mathrm{r}\mathsf{O}_{\mathfrak{R}} \setminus \{\emptyset\}$. Further, $\mathsf{Cl}_{\mathfrak{R}}$ and $\mathsf{Clop}_{\mathfrak{R}}$ are the families of all closed and all clopen sets in $\mathtt{Gts}\,\mathfrak{R}$, respectively.

In [6] the author did not use the base $\mathsf{B}_{\mathfrak{R}}$ for the introduction of the topology $\mathsf{O}_{\mathfrak{R}}$. He introduced the family of open sets by the definition D_2 [cf. 6, p. 232] which corresponds to the following property of the family $\mathsf{O}_{\mathfrak{R}}$:

$$\begin{aligned} \Omega \in \mathsf{O}_{\mathfrak{R}} &\iff \forall_{\mathfrak{p} \in \Omega} \exists_{x \in \mathfrak{p}}\ \mathbf{Irl}(x) \subseteq \Omega \\ &\iff \forall_{\mathfrak{p} \in \Omega} \exists_{x \in \mathfrak{p}} \forall_{\mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}} (x \in \mathfrak{q} \Rightarrow \mathfrak{q} \in \Omega). \end{aligned} \tag{15.1}$$

PROOF. "$\Rightarrow$": Let $\Gamma \in \mathsf{O}_{\mathfrak{R}}$, i.e., for some subfamily B of $\mathsf{B}_{\mathfrak{R}}$ we have $\Gamma = \bigcup \mathsf{B}$. Suppose that $\mathfrak{p} \in \Gamma$, i.e., there is a $B_0 \in \mathsf{B}$ such that $\mathfrak{p} \in B_0$. Then for some $x_0 \in R$ we have $\mathfrak{p} \in \mathbf{Irl}(x_0) = B_0 \subseteq \Gamma$.

"$\Leftarrow$": We have $\Gamma = \bigcup\{B \in \mathsf{B}_{\mathfrak{R}} \mid B \subseteq \Gamma\}$, i.e., $\Gamma \in \mathsf{O}_{\mathfrak{R}}$. Clearly, $\bigcup\{B \in \mathsf{B}_{\mathfrak{R}} \mid B \subseteq \Gamma\} \subseteq \Gamma$. For the converse inclusion we assume that $\mathfrak{p} \in \Gamma$. Then for some $x_0 \in \mathfrak{p}$ we have $\mathbf{Irl}(x_0) \subseteq \Gamma$. ∎

15.2. Interiors and Closures of Sets in G-Topological Spaces

Let $\mathfrak{R}$ be an G-structure. Using the fact that the family $\mathsf{B}_{\mathfrak{R}}$ is a basis of $\mathtt{Gts}\,\mathfrak{R}$ we see that the operation of interior, $\mathrm{Int}\colon \mathcal{P}(\mathbf{Pt}_{\mathfrak{R}}) \to \mathcal{P}(\mathbf{Pt}_{\mathfrak{R}})$, has the following property: for all $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ and $\Gamma \subseteq \mathbf{Pt}_{\mathfrak{R}}$

$$\begin{aligned} \mathfrak{p} \in \mathrm{Int}\,\Gamma &\iff \exists_{x \in \mathfrak{p}}\ \mathbf{Irl}(x) \subseteq \Gamma, \\ &\iff \exists_{x \in \mathfrak{p}} \forall_{\mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}} (x \in \mathfrak{q} \Rightarrow \mathfrak{q} \in \Gamma). \end{aligned} \tag{15.2}$$

Indeed, $\mathfrak{p} \in \mathrm{Int}\,\Gamma$ iff $\exists_{\Delta \in \mathsf{B}_{\mathfrak{R}}}\ \mathfrak{p} \in \Delta \subseteq \Gamma$ iff $\exists_{x \in R}\ \mathfrak{p} \in \mathbf{Irl}(x) \subseteq \Gamma$ iff $\exists_{x \in \mathfrak{p}}\ \mathbf{Irl}(x) \subseteq \Gamma$ iff $\exists_{x \in \mathfrak{p}} \forall_{\mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}} (x \in \mathfrak{q} \Rightarrow \mathfrak{q} \in \Gamma)$.

We also obtain that the operation of closure, $\mathrm{Cl}\colon \mathcal{P}(\mathbf{Pt}_{\mathfrak{R}}) \to \mathcal{P}(\mathbf{Pt}_{\mathfrak{R}})$, satisfies the following for all $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ and $\Gamma \subseteq \mathbf{Pt}_{\mathfrak{R}}$:

$$\begin{aligned}\mathfrak{p} \in \mathrm{Cl}\,\Gamma &\iff \forall_{x\in\mathfrak{p}}\ \mathbf{Irl}(x) \cap \Gamma \neq \emptyset,\\ &\iff \forall_{x\in\mathfrak{p}}\exists_{\mathfrak{r}\in\Gamma}\ x \in \mathfrak{r} \iff \mathfrak{p} \subseteq \bigcup \Gamma\,.\end{aligned} \tag{15.3}$$

Indeed, $\mathfrak{p} \in \mathrm{Cl}\,\Gamma$ iff $\forall_{\Delta\in\mathbf{B}_{\mathfrak{R}}}(\mathfrak{p} \in \Delta \Rightarrow \Delta \cap \Gamma \neq \emptyset)$ iff $\forall_{x\in R}(\mathfrak{p} \in \mathbf{Irl}(x) \Rightarrow \mathbf{Irl}(x) \cap \Gamma \neq \emptyset)$ iff $\forall_{x\in\mathfrak{p}}\ \mathbf{Irl}(x) \cap \Gamma \neq \emptyset$ iff $\forall_{x\in\mathfrak{p}}\exists_{\mathfrak{r}\in\Gamma}\ x \in \mathfrak{r}$ iff $\mathfrak{p} \subseteq \bigcup \Gamma$.

Moreover, by (15.3) and (13.2), for all $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ and $x \in R$ we have:

$$\mathfrak{p} \in \mathrm{Cl}\,\mathbf{Irl}(x) \iff \forall_{y\in\mathfrak{p}}\ y \bigcirc x. \tag{15.4}$$

Indeed, $\mathfrak{p} \in \mathrm{Cl}\,\mathbf{Irl}(x)$ iff $\forall_{y\in\mathfrak{p}}\ \mathbf{Irl}(y) \cap \mathbf{Irl}(x) \neq \emptyset$ iff $\forall_{y\in\mathfrak{p}}\ y \bigcirc x$.

Directly from (15.4) we obtain that for each region the set of its adherent points is the closure of the set of its internal points:

$$\mathbf{A}(x) = \mathrm{Cl}\,\mathbf{Irl}(x). \tag{15.5}$$

Now by (15.5) and, respectively, (13.12)–(13.14), we can express the relations C, $)($ and $\ll$ by means of topological notions. For all $x, y \in R$:

$$\begin{aligned}x \mathrel{\mathsf{C}} y &\iff \mathrm{Cl}\,\mathbf{Irl}(x) \cap \mathrm{Cl}\,\mathbf{Irl}(y) \neq \emptyset,\\ x \mathrel{)(} y &\iff \mathrm{Cl}\,\mathbf{Irl}(x) \cap \mathrm{Cl}\,\mathbf{Irl}(y) = \emptyset,\end{aligned} \tag{15.6}$$

$$x \ll y \iff \mathrm{Cl}\,\mathbf{Irl}(x) \subseteq \mathbf{Irl}(y), \tag{15.7}$$

$$x \ll x \iff \mathrm{Cl}\,\mathbf{Irl}(x) = \mathbf{Irl}(x). \tag{15.8}$$

After Grzegorczyk we demonstrate that:

Theorem 15.1. *For any $x \in R$, $\mathbf{Irl}(x) \in \mathrm{r}\mathbf{O}^{+}_{\mathfrak{R}}$. So $\mathbf{B}_{\mathfrak{R}} \subseteq \mathrm{r}\mathbf{O}^{+}_{\mathfrak{R}}$.*

In this way we will have shown that the space $\mathtt{Gts}\,\mathfrak{R}$ is semiregular (although, through Theorem 16.2, we will obtain that this space is also concentric; and so it is regular, by Lemma 8.1).

Proof of Theorem 15.1. Firstly, $\mathbf{Irl}(x) \subseteq \mathrm{Int}\,\mathrm{Cl}\,\mathbf{Irl}(x)$, since $\mathbf{Irl}(x) \in \mathbf{O}_{\mathfrak{R}}$. Secondly, let $\mathfrak{p} \in \mathrm{Int}\,\mathrm{Cl}\,\mathbf{Irl}(x)$. Then, by (15.2), for some $y_0 \in \mathfrak{p}$ we have $\mathbf{Irl}(y_0) \subseteq \mathrm{Cl}\,\mathbf{Irl}(x)$. We show $(*)$: $\forall_{z\in R}(z \bigcirc y_0 \Rightarrow z \bigcirc x)$. Indeed, let z be arbitrary region such that $z \bigcirc y_0$. Then, by (11.3), for some $\mathfrak{q}_0 \in \mathbf{Pt}_{\mathfrak{R}}$ we have $y_0 \in \mathfrak{q}_0$ and $z \in \mathfrak{q}_0$. So $\mathfrak{q}_0 \in \mathbf{Irl}(y_0) \subseteq \mathrm{Cl}\,\mathbf{Irl}(x)$. Thus, $z \bigcirc x$, by (15.4).

From $(*)$ we obtain $y_0 \sqsubseteq x$, because $\sqsubseteq$ is separative. Hence $x \in \mathfrak{p}$, and so $\mathfrak{p} \in \mathbf{Irl}(x)$. ■

Finally notice that if $\mathfrak{R}$ has the unity 1, then—using (15.5) and (13.15)—for any $x \in R \setminus \{1\}$ we obtain:

$$\mathbf{Irl}(-x) = \mathrm{Int}(\mathbf{Pt}_{\mathfrak{R}} \setminus \mathbf{Irl}(x)). \tag{15.9}$$

16. The Separation Axioms for G-Topological Spaces

Let $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle$ be any G-structure and $\mathtt{Gts}\,\mathfrak{R} := \langle \mathbf{Pt}_{\mathfrak{R}}, \mathbf{O}_{\mathfrak{R}} \rangle$ be its G-topological space generated by $\mathbf{B}_{\mathfrak{R}}$. First we get:

Lemma 16.1. *For any $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ there is a subfamily of $\mathbf{B}_{\mathfrak{R}}$ that is a base at $\mathfrak{p}$ and satisfies* (R1)*. Moreover, if $\mathfrak{R}$ is first-countable, then there is a countable base at $\mathfrak{p}$ which satisfies* (R1)*.*

Proof. Let $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$, i.e., for some $Q \in \mathbf{Q}_{\mathfrak{R}}$ we have $\mathfrak{p} = \mathrm{F}_Q = \{x \in R \mid \exists_{y \in Q}\, y \sqsubseteq x\}$. We put $\mathsf{B}_Q := \{\mathbf{Irl}(y) \mid y \in Q\}$.

Firstly, we show that the family B_Q is a base at $\mathfrak{p}$. Suppose that $\Omega \in \mathbf{O}_{\mathfrak{R}}$ and $\mathfrak{p} \in \Omega$. Then for some $B \in \mathbf{B}_{\mathfrak{R}}$ we have $\mathfrak{p} \in B \subseteq \Omega$. But for some $x_0 \in R$ we have $B = \mathbf{Irl}(x_0)$ and $x_0 \in \mathfrak{p}$. Hence for some $y_0 \in Q$ we have $y_0 \sqsubseteq x_0$. So $\mathbf{Irl}(y_0) \subseteq \mathbf{Irl}(x_0)$, by (13.5). Moreover, $\mathfrak{p} \in \mathbf{Irl}(y_0) \subseteq \Omega$ and $\mathbf{Irl}(y_0) \in \mathsf{B}_Q$, since $y_0 \in Q \subseteq \mathfrak{p}$.

Secondly, we show that the family B_Q satisfies (R1). Suppose that $B_1, B_2 \in \mathsf{B}_Q$ and $B_1 \neq B_2$. So for some $x_1, x_2 \in Q$ we have that $B_1 = \mathbf{Irl}(x_1)$ and $B_2 = \mathbf{Irl}(x_2)$. By (13.6) we have $x_1 \neq x_2$. Hence, by (r1), either $x_1 \ll x_2$ or $x_2 \ll x_1$. We now apply (15.7).

Moreover, in the light of Corollary 7.1, if $\mathfrak{R}$ is first-countable, then there is a countable $Q' \in \mathbf{Q}_{\mathfrak{R}}$ which is coinitial with Q and such that $\mathfrak{p} = \mathrm{F}_Q = \mathrm{F}_{Q'}$. Of course, the family $\mathsf{B}_{Q'} := \{\mathbf{Irl}(y) \mid y \in Q'\}$ is countable. As for B_Q, we show that $\mathsf{B}_{Q'}$ is a base at $\mathfrak{p}$ and satisfies (R1). ∎

From the above lemma we obtain:

Theorem 16.2. 1. $\mathtt{Gts}\,\mathfrak{R}$ *is concentric.*

2. *If $\mathfrak{R}$ is first-countable, then* $\mathtt{Gts}\,\mathfrak{R}$ *is ω-concentric.*

Proof. $\mathtt{Gts}\,\mathfrak{R}$ is a Urysohn space and so it is also a T_1-space. Indeed, suppose that $\mathfrak{p}, \mathfrak{q} \in \mathbf{Pt}_{\mathfrak{R}}$ are distinct points. Then, by Corollary 10.9(2), there are $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$ such that $x \mathrel{)(} y$. So $\mathfrak{p} \in \mathbf{Irl}(x)$, $\mathfrak{q} \in \mathbf{Irl}(y)$ and $\mathrm{Cl}\,\mathbf{Irl}(x) \cap \mathrm{Cl}\,\mathbf{Irl}(y) = \emptyset$, by (15.6). Now apply Lemma 16.1. ∎

Let us remember, see (8.2), that the class of ω-concentric topological spaces is equal to both the class of first-countable concentric topological spaces and the class of first-countable regular topological spaces.

It seems that nothing more can be said about separation axioms for G-topological spaces of G-structures without making additional assumptions about these structures. There is more to be said, however, about G-topological spaces built on G-structures created from regular open sets of some topological spaces (see Section 19).

17. G-Structures for G-Topological Spaces

Let $\mathfrak{R}$ be any G-structure and $\mathtt{Gts}\ \mathfrak{R} := \langle \mathbf{Pt}_{\mathfrak{R}}, \mathsf{O}_{\mathfrak{R}} \rangle$ be its G-topological space. Theorem 16.2 says that $\mathtt{Gts}\ \mathfrak{R}$ is concentric. With $\mathtt{Gts}\ \mathfrak{R}$—as with all concentric topological spaces (see Theorem 8.4)—is associated the complete G-structure $\langle \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}, \subseteq, \mathtt{][} \rangle$ (see (df $\mathtt{][}$) on p. 811). Thus, according to the convention from page 812, we will denote it by $\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})$. The following diagram illustrates adopted conventions:

$$\underset{}{\mathfrak{R}} \quad \mapsto \quad \underset{\text{concentric}}{\mathtt{Gts}\ \mathfrak{R}} \quad \mapsto \quad \underset{\text{complete G-structure}}{\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})}$$

Now notice that, respectively, from the fact that $\mathsf{B}_{\mathfrak{R}} := \mathbf{Irl}[R] \subseteq \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$ is a base of $\mathtt{Gts}\ \mathfrak{R}$, Theorems 16.2 and 8.8 and (13.2) we obtain (cf. Corollary 8.5 and (8.1)):

Proposition 17.1. *The following conditions are equivalent*:

(a) $\mathfrak{R}$ *satisfies c.c.c.*

(b) $\mathtt{Gts}\ \mathfrak{R}$ *satisfies c.c.c.* (*and so it is* ω*-concentric*).

(c) $\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})$ *satisfies c.c.c.*

Thus, by Proposition 17.1 and Theorem 8.9, for G-structures satisfying c.c.c. we obtain:

$$\underset{\text{satisfies c.c.c.}}{\mathfrak{R}} \quad \mapsto \quad \underset{\substack{\text{satisfies c.c.c.}\\ \omega\text{-concentric}}}{\mathtt{Gts}\ \mathfrak{R}} \quad \mapsto \quad \underset{\text{satisfies c.c.c.}}{\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})}$$

Furthermore, for first-countable G-structures (which form a wider class than the class of G-structures satisfying c.c.c.; see Lemma 7.3) from Theorem 16.2 and Corollary 8.7 we obtain:

$$\underset{\text{first-countable}}{\mathfrak{R}} \quad \mapsto \quad \underset{\omega\text{-concentric}}{\mathtt{Gts}\ \mathfrak{R}} \quad \mapsto \quad \underset{\text{first-countable}}{\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})}$$

18. Completions and the Topological Version of Representation Theorems

Let $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle$ be any G-structure. Theorem 15.1 says that $\mathbf{Irl}[R] \subseteq \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$. Hence, by (13.5), (13.6) and (15.6), we obtain:

Proposition 18.1. $\mathbf{Irl}$ *is an embedding of* $\mathfrak{R}$ *into* $\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})$.

By (14.1) and (15.5), the relation $)($ is equal to $][$ on $\mathbf{Irl}[R] =: \mathsf{B}_{\mathfrak{R}}$ (see p. 829). Furthermore, $\mathsf{B}_{\mathfrak{R}} \subseteq \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$ and $\mathsf{B}_{\mathfrak{R}}$ is dense in $\mathtt{Gts}\,\mathfrak{R}$, since $\mathsf{B}_{\mathfrak{R}}$ is a base of $\mathtt{Gts}\,\mathfrak{R}$. Therefore we obtain:

Proposition 18.2. 1. $\mathbf{Irl}[\mathfrak{R}] = \langle \mathsf{B}_{\mathfrak{R}}, \subseteq,][\rangle$.

2. *The G-structure* $\mathbf{Irl}[\mathfrak{R}]$ *is a dense substructure of* $\mathfrak{G}(\mathtt{Gts}\,\mathfrak{R})$.

So, by using Proposition 18.1, we obtain:

Corollary 18.3. *The pair* $\langle \mathfrak{G}(\mathtt{Gts}\,\mathfrak{R}), \mathbf{Irl}\rangle$ *is a completion of* $\mathfrak{R}$.[5]

We prove that the complete G-structure $\mathfrak{G}(\mathtt{Gts}\,\mathfrak{R})$ is equal to the G-structure $\mathbf{Irl}[\mathfrak{R}]$ if and only if $\mathfrak{R}$ is complete. For the proof of this fact we need the following lemma.

Lemma 18.4. *For any* $X \in \mathcal{P}(R)$ *that has a mereological sum,* $\mathbf{Irl}(\bigsqcup X) = \operatorname{Int}\operatorname{Cl}\bigcup_{x\in X}\mathbf{Irl}(x)$.

Proof. Suppose that $X \in \mathcal{P}(X)$ has a mereological sum. Then $X \neq \emptyset$ and $\bigsqcup X := (\iota\, z)\; z \;\mathsf{sum}\; X$.

"$\subseteq$" Using (15.3), we prove that $\mathbf{Irl}(\bigsqcup X) \subseteq \operatorname{Cl}\bigcup_{x\in X}\mathbf{Irl}(x)$, which gives that $\mathbf{Irl}(\bigsqcup X) \subseteq \operatorname{Int}\operatorname{Cl}\bigcup_{x\in X}\mathbf{Irl}(x)$, since $\mathbf{Irl}(\bigsqcup X)$ is open. Thus, let $\mathfrak{p} \in \mathbf{Irl}(\bigsqcup X)$, i.e., $\bigsqcup X \in \mathfrak{p}$. If $y \in \mathfrak{p}$ then $y \bigcirc \bigsqcup X$. Hence for some $z_0 \in R$ we have $z_0 \sqsubseteq y$ and $z_0 \sqsubseteq \bigsqcup X$. By (13.5), we have $\mathbf{Irl}(z_0) \subseteq \mathbf{Irl}(y)$. By (df$\bigsqcup$) and (df$\,\mathsf{sum}$), for some $x_0 \in X$ we have $x_0 \bigcirc z_0$. Hence $\mathbf{Irl}(x_0) \cap \mathbf{Irl}(z_0) \neq \emptyset$, by (13.2). Therefore also $\mathbf{Irl}(y) \cap \mathbf{Irl}(x_0) \neq \emptyset$. So $\mathbf{Irl}(y) \cap \bigcup_{x\in X}\mathbf{Irl}(x) \neq \emptyset$. From this, by (15.3), we infer that $\mathfrak{p} \in \operatorname{Cl}\bigcup_{x\in X}\mathbf{Irl}(x)$.

"$\supseteq$" We have $\forall_{x\in X}\; x \sqsubseteq \bigsqcup X$. Hence $\forall_{x\in X}\; \mathbf{Irl}(x) \subseteq \mathbf{Irl}(\bigsqcup X)$, by (13.5); and so $\bigcup_{x\in X}\mathbf{Irl}(x) \subseteq \mathbf{Irl}(\bigsqcup X)$. But, by Theorem 15.1, the set $\mathbf{Irl}(\bigsqcup X)$ is regular open, so the conclusion follows. ■

Theorem 18.5. $\mathsf{B}_{\mathfrak{R}} = \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$ *iff* $\mathfrak{R}$ *is complete.*

Proof. "$\Rightarrow$" If $\mathsf{B}_{\mathfrak{R}} = \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$ then $\mathbf{Irl}[\mathfrak{R}] = \mathfrak{G}(\mathtt{Gts}\,\mathfrak{R})$ and so $\mathbf{Irl}[\mathfrak{R}]$ is complete. Hence $\mathfrak{R}$ is also complete, by Theorem 14.2(5).

"$\Leftarrow$" Theorem 15.1 says that $\mathsf{B}_{\mathfrak{R}} \subseteq \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$. For the converse inclusion suppose that $\mathfrak{R}$ is complete and $\Omega \in \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$. Then for some $X \in \mathcal{P}(R)$ we have $\Omega = \bigcup_{x\in X}\mathbf{Irl}(x)$, since $\Omega \in \mathsf{O}_{\mathfrak{R}}$ and $\mathsf{B}_{\mathfrak{R}}$ is a base of $\mathtt{Gts}\,\mathfrak{R}$. So, by Lemma 18.4, $\Omega = \operatorname{Int}\operatorname{Cl}\Omega = \mathbf{Irl}(\bigsqcup X)$, since Ω is regular open and X has a mereological sum. To conclude, $\Omega \in \mathsf{B}_{\mathfrak{R}}$. ■

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[5]For G-structures $\mathfrak{R}_1 = \langle R_1, \sqsubseteq_1,)(_1\rangle$ and $\mathfrak{R}_2 = \langle R_2, \sqsubseteq_2,)(_2\rangle$ and a mapping $e\colon R_1 \to R_2$, a pair $\langle \mathfrak{R}_2, e\rangle$ is said to be a *completion* of $\mathfrak{R}_1$ iff both $\mathfrak{R}_2$ is complete, e is an *embedding* from $\mathfrak{R}_1$ into $\mathfrak{R}_2$ and the image $e[R_1]$ is dense in $\mathfrak{R}_2$ (i.e., for any $y \in R_2$ there is an $x \in R_1$ such that $e(x) \sqsubseteq_2 y$).

Thus, by means of Theorems 14.2 and 18.5 , we obtain:

THEOREM 18.6. *The operation* **Irl** *is an isomorphism from* $\mathfrak{R}$ *onto* $\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})$ *iff* $\mathfrak{R}$ *is complete.*

REMARK 18.1. Proposition 4.3(4) says that if $\mathtt{Gts}\ \mathfrak{R}$ is a normal space then the relation $\ll$ on $\mathrm{r}\mathbf{O}^{+}_{\mathfrak{R}}$ is dense in $\mathfrak{G}(\mathtt{Gts}\ \mathfrak{R})$, i.e., satisfies:

$$\begin{aligned} &\forall_{U,V\in \mathrm{r}\mathbf{O}^{+}_{\mathfrak{R}}}\big(U \ll V \Longrightarrow \exists_{W\in \mathrm{r}\mathbf{O}^{+}_{\mathfrak{R}}}(U \ll W \ \wedge\ W \ll V)\big),\\ &\forall_{U,V\in \mathrm{r}\mathbf{O}^{+}_{\mathfrak{R}}}\big(\mathrm{Cl}\,U \subseteq V \Longrightarrow \exists_{W\in \mathrm{r}\mathbf{O}^{+}_{\mathfrak{R}}}(\mathrm{Cl}\,U \subseteq W \ \wedge\ \mathrm{Cl}\,W \subseteq V)\big). \end{aligned} \tag{IA}$$

Thus, by Theorem 18.6, if $\mathfrak{R}$ is complete and $\mathtt{Gts}\ \mathfrak{R}$ is a normal space, then the relation $\ll$ on R is dense in $\mathfrak{R}$, i.e., it satisfies condition ($\mathrm{IA}^{\bullet}$). ■

After all the work so far we are in a position to obtain representation theorems that characterize G-structures. Firstly, from Theorems 14.2, 16.2(1) and 18.6 and Proposition 18.2(1), respectively, we obtain:

THEOREM 18.7. 1. *Every G-structure is isomorphic to a dense substructure of a G-structure for a concentric topological space.*

2. *Every complete G-structure is isomorphic to a G-structure for a concentric topological space.*

Secondly, from Theorems 14.2, 16.2(1) and 18.6 and Propositions 17.1 and 18.2(2), respectively, we obtain:

THEOREM 18.8. 1. *Any G-structure satisfying c.c.c. is isomorphic to a dense substructure of a G-structure satisfying c.c.c. for an ω-concentric topological space satisfying c.c.c.*

2. *Any complete G-structure satisfying c.c.c. is isomorphic to a G-structure satisfying c.c.c. for an ω-concentric topological space satisfying c.c.c.*

Thirdly, from Theorems 14.2, 16.2(2) and 18.6 , respectively, we obtain:

THEOREM 18.9. 1. *Any first-countable G-structure is isomorphic to a dense substructure of a G-structure for an ω-concentric space.*

2. *Any complete first-countable G-structure is isomorphic to a G-structure for an ω-concentric space.*

Fourthly, from Theorems 14.2, 16.2 and 18.6 , respectively, we obtain (cf. Corollary 8.11):

THEOREM 18.10. 1. *Any countable G-structure is isomorphic to a dense substructure of a G-structure for a second-countable regular space.*

2. *Any complete countable G-structure is isomorphic to a G-structure for a second-countable regular space.*

Proof. For any countable G-structure $\mathfrak{R}$, the G-topological space $\mathtt{Gts}\ \mathfrak{R}$ is regular and second-countable, since $\mathbf{Irl}[R]$ is a countable base of $\mathtt{Gts}\ \mathfrak{R}$. ■

19. G-Topological Spaces of G-Structures for Concentric Topological Spaces

With a concentric topological space $\mathcal{T}$ we associate the complete G-structure $\mathfrak{G}\mathcal{T} := \langle \mathrm{r}\mathscr{O}^+, \subseteq, \mathtt{][} \rangle$. So we can consider the following G-topological space $\mathtt{Gts}(\mathfrak{G}\mathcal{T}) := \langle \mathbf{Pt}_{\mathfrak{G}\mathcal{T}}, \mathbf{O}_{\mathfrak{G}\mathcal{T}} \rangle$ with the base $\mathbf{B}_{\mathfrak{G}\mathcal{T}} = \{ \mathbf{Irl}(U) \mid U \in \mathrm{r}\mathscr{O}^+ \}$. By Theorem 8.12 we obtain:

Theorem 19.1. *If a concentric space $\mathcal{T}$ satisfies c.c.c., then the bijection* $\mathrm{b}\colon S \ni p \mapsto \mathrm{F}_{\mathrm{r}\mathscr{B}^p} \in \mathbf{Pt}_{\mathfrak{G}\mathcal{T}}$ *from Theorem 8.12 is a homeomorphism of $\mathcal{T}$ onto* $\mathtt{Gts}(\mathfrak{G}\mathcal{T})$.

Proof. We show that the bijection $\mathrm{b}\colon S \ni p \mapsto \mathrm{F}_{\mathrm{r}\mathscr{B}^p} \in \mathbf{Pt}_{\mathfrak{G}\mathcal{T}}$ from Theorem 8.12 is both continuous and open.

First, for any $\Omega \in \mathbf{B}_{\mathfrak{R}}$, where $\Omega = \mathbf{Irl}(U)$ for some $U \in \mathrm{r}\mathscr{O}^+$, we show that $\mathrm{b}^{-1}[\Omega] \in \mathscr{O}$. Indeed, by Theorem 8.12(1), for any $p \in S$: $p \in \mathrm{b}^{-1}[\mathbf{Irl}(U)]$ iff $\mathrm{b}(p) \in \mathbf{Irl}(U)$ iff $\mathrm{F}_{\mathrm{r}\mathscr{B}^p} \in \mathbf{Irl}(U)$ iff $U \in \mathrm{r}\mathscr{O}^p$ iff $p \in U$. Thus, $\mathrm{b}^{-1}[\mathbf{Irl}(U)] = U$.

Second, for any $U \in \mathrm{r}\mathscr{O}^+$, using (15.1), we show that $\mathrm{b}[U] \in \mathbf{O}_{\mathfrak{R}}$. Suppose that $\mathfrak{p} \in \mathrm{b}[U] = \{\mathrm{F}_{\mathrm{r}\mathscr{B}^p} \mid p \in U\}$. Then, by Theorem 8.12(1), for some $p \in U$ we have $\mathfrak{p} = \mathrm{F}_{\mathrm{r}\mathscr{B}^p} = \{V \in \mathrm{r}\mathscr{O}^+ \mid p \in V\}$. So $U \in \mathfrak{p}$. Now let $\mathfrak{q}$ be an arbitrary point of $\mathbf{Pt}_{\mathfrak{G}\mathcal{T}}$ such that $U \in \mathfrak{q}$. Then for some $q \in S$ we have $\mathfrak{q} = \mathrm{F}_{\mathrm{r}\mathscr{B}^q} = \{V \in \mathrm{r}\mathscr{O}^+ \mid q \in V\}$ and so $q \in U$. So $\mathfrak{q} \in \mathrm{b}[U]$. ■

Clearly, if a concentric space $\mathcal{T}$ satisfies c.c.c., then $\mathtt{Gts}(\mathfrak{G}\mathcal{T})$ must also satisfy c.c.c. Therefore Theorems 18.8(2) and 19.1 yield the following object duality for subclasses of Grzegorczyk structures and concentric spaces:

Theorem 19.2. *Every complete G-structure satisfying c.c.c. is isomorphic to a G-structure for a concentric space satisfying c.c.c.*; *and every concentric c.c.c. space is homeomorphic to a concentric space satisfying c.c.c. for some complete G-structure satisfying c.c.c.*

Corollary 8.11 says that all second-countable regular topological spaces are ω-concentric and perfectly normal satisfying c.c.c. Thus, by Theorem 19.1 and the fact that having countable basis is a topological property, i.e., it is preserved under homeomorphisms, we obtain:

Corollary 19.3. *If $\mathcal{T}$ is second-countable regular, then $\mathcal{T}$ and* $\mathtt{Gts}(\mathfrak{G}\mathcal{T})$ *are homeomorphic and* $\mathtt{Gts}(\mathfrak{G}\mathcal{T})$ *is second-countable, ω-concentric and perfectly normal satisfying c.c.c.*

In reference to Section 8, with the Euclidean topological space $\mathcal{E}^n$ we associate the complete G-structure $\mathfrak{G}\mathcal{E}^n := \langle \mathrm{r}\mathscr{E}_+(\mathbb{R}^n), \subseteq, \mathtt{]\![} \rangle$, for any $n \in \omega^+$. In the light of Theorem 19.1 we obtain:

Theorem 19.4. *The bijection* $\mathrm{b}\colon \mathbb{R}^n \ni p \mapsto \mathrm{F}_{\mathcal{B}^p} \in \mathbf{Pt}_{\mathfrak{G}\mathcal{E}^n}$ *from Theorem 8.13 is a homeomorphism of $\mathcal{E}^n$ onto* $\mathtt{Gts}(\mathfrak{G}\mathcal{E}^n) := \langle \mathbf{Pt}_{\mathfrak{G}\mathcal{E}^n}, \mathbf{O}_{\mathfrak{G}\mathcal{E}^n} \rangle$.

Remark 19.1. To conclude, we observe that in light of Corollary 19.3 and Theorem 18.10(2) it is tempting to draw a conclusion that we obtain another object duality, this time for countable *G*-structures and second-countable topological spaces. Unfortunately this is not the case, since given second-countable regular space $\mathcal{T}$, $\mathfrak{G}\mathcal{T}$ does not have to be countable. The standard Euclidean topology $\mathcal{E}^n$ and $\mathfrak{G}\mathcal{E}^n$ serve a suitable counterexample. ■

20. Some Properties of G-Topological Spaces

20.1. G-Topological Spaces for Finite G-Structures

If a G-structure $\mathfrak{R}$ is finite, then we not only have that $\mathbf{B}_{\mathfrak{R}} = \mathrm{r}\mathbf{O}^+_{\mathfrak{R}}$, but also:

Proposition 20.1. *If $\mathfrak{R}$ is finite, then* $\mathtt{Gts}\ \mathfrak{R}$ *is discrete and* $\mathbf{O}_{\mathfrak{R}} = \mathbf{B}_{\mathfrak{R}}$.

Proof. If $\mathfrak{R}$ is finite, then the set $\mathbf{Pt}_{\mathfrak{R}}$ is finite, by Theorem 11.1. $\mathtt{Gts}\ \mathfrak{R}$ is a T_1-space, therefore $\mathtt{Gts}\ \mathfrak{R}$ is discrete. By Proposition 12.9, $\mathbf{Pt}_{\mathfrak{R}} = \mathrm{Ult}(\mathfrak{R}) = \{\mathrm{F}_a \mid a \in \mathrm{At}_{\mathfrak{R}}\}$. Now we show that $\mathcal{P}(\mathbf{Pt}_{\mathfrak{R}}) \subseteq \mathbf{B}_{\mathfrak{R}}$. Suppose that $\Gamma \in \mathcal{P}(\mathbf{Pt}_{\mathfrak{R}})$. Then for some $a_1, \ldots, a_n \in \mathrm{At}_{\mathfrak{R}}$ we have $\Gamma = \{\mathrm{F}_{a_1}, \ldots, \mathrm{F}_{a_n}\}$. Since all points are ultrafilters, for any $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$: $\mathfrak{p} \in \mathbf{Irl}(a_1 \sqcup \cdots \sqcup a_n)$ iff $a_1 \sqcup \cdots \sqcup a_n \in \mathfrak{p}$ iff $\exists_{1\leqslant i\leqslant n}\ a_i \in \mathfrak{p}$ iff $\exists_{1\leqslant i\leqslant n}\ \mathrm{F}_{a_i} = \mathfrak{p}$ iff $\mathfrak{p} \in \Gamma$. Hence $\Gamma = \mathbf{Irl}(a_1 \sqcup \cdots \sqcup a_n)$, i.e., $\Gamma \in \mathbf{B}_{\mathfrak{R}}$. ■

20.2. G-Topological Spaces vs. Stone Spaces of G-Structures

Let $\mathfrak{R} \in \mathbf{G}$. As for Boolean lattices (algebras), the *Stone space of* $\mathfrak{R}$ (or the *space of ultrafilters* of $\mathfrak{R}$; shortly: $\mathtt{Ult}\mathfrak{R}$) is the topological space whose set of points is the set $\mathrm{Ult}(\mathfrak{R})$ of all ultrafilters of $\mathfrak{R}$ and the topology is generated by the base $\mathrm{s}[R] := \{\mathrm{s}(x) \mid x \in R\}$, where the Stone map $\mathrm{s}\colon R \to \mathcal{P}(\mathrm{Ult}(\mathfrak{R}))$ is defined by $\mathrm{s}(x) := \{F \in \mathrm{Ult}(\mathfrak{R}) \mid x \in F\}$ (cf. Section 2.9).

If $\mathfrak{R}$ has the unity $\mathbf{1}$ then the Stone space of $\mathfrak{R}$ is identical with the Stone space of the non-trivial Boolean algebra $\langle R^0, +, \cdot, -, 0, 1\rangle$ (cf. Theorems 2.5

and 2.6). As is well known: *A Boolean algebra is finite iff its Stone space is finite and discrete.*

In general, $\mathtt{Gts}\,\mathfrak{R}$ is different from $\mathtt{Ult}\mathfrak{R}$, since—usually—both $\mathbf{Pt}_{\mathfrak{R}} \nsubseteq \mathrm{Ult}(\mathfrak{R})$ and $\mathrm{Ult}(\mathfrak{R}) \nsubseteq \mathbf{Pt}_{\mathfrak{R}}$. Indeed, firstly, by Proposition 12.5, if $\mathbf{Pt}_{\mathfrak{R}} \subseteq \mathrm{Ult}(\mathfrak{R})$ then the relation $\ll$ is reflexive, $)(\,=\, \wr$ and $\ll \,=\, \sqsubseteq$; and in the case when $\mathfrak{R}$ has the unity 1, for any $x \in R \setminus \{1\}$ we have $x \;)(\; -x$. Secondly, by Proposition 12.10, if $\mathrm{Ult}(\mathfrak{R}) \subseteq \mathbf{Pt}_{\mathfrak{R}}$ and $\mathfrak{R}$ is complete, then $\mathfrak{R}$ is finite. Moreover, by Proposition 12.11 (for which we use the Kuratowski-Zorn lemma): $\mathrm{Ult}(\mathfrak{R}) \subseteq \mathbf{Pt}_{\mathfrak{R}}$ iff $\mathbf{Pt}_{\mathfrak{R}} = \mathrm{Ult}(\mathfrak{R})$.

Finally, by Proposition 12.9, if $\mathfrak{R}$ is finite, then $\mathfrak{R}$ is complete and $\mathbf{Pt}_{\mathfrak{R}} = \mathrm{Ult}(\mathfrak{R}) = \{\mathrm{F}_a \mid a \in \mathrm{At}_{\mathfrak{R}}\}$. Moreover, for any $x \in R$ we have $\mathrm{s}(x) = \mathbf{Irl}(x)$. So $\mathtt{Gts}\,\mathfrak{R} = \mathtt{Ult}\mathfrak{R}$. Notice that, by Proposition 20.1, in this case $\mathtt{Gts}\,\mathfrak{R}$ is discrete and $\mathsf{B}_{\mathfrak{R}} = \mathsf{O}_{\mathfrak{R}}$, as for the finite Boolean algebra $\langle R^{\mathit{0}}, +, \cdot, -, \mathit{0}, 1\rangle$.

20.3. Atoms and Isolated Points: Atomic G-Structures

We have the following characterization of the set $\mathrm{At}_{\mathfrak{R}}$ of atoms of $\mathfrak{R} \in \mathsf{G}$:

Proposition 20.2. *For any $x \in R$: $x \in \mathrm{At}_{\mathfrak{R}}$ iff the set $\mathbf{Irl}(x)$ is a singleton.*

Proof. "$\Rightarrow$" Suppose that $x \in \mathrm{At}_{\mathfrak{R}}$ and $\mathfrak{p}, \mathfrak{q} \in \mathbf{Irl}(x)$, i.e., $x \in \mathfrak{p}$ and $x \in \mathfrak{q}$. If $y \in \mathfrak{p}$, then $y \circ x$ and so $x \sqsubseteq y$. Therefore $y \in \mathfrak{q}$. Thus, $\mathfrak{p} = \mathfrak{q}$, by Corollary 5.3. So $\mathbf{Irl}(x)$ is a singleton.

"$\Leftarrow$" Suppose that $x \notin \mathrm{At}_{\mathfrak{R}}$. Then, by (MF), for some $y_1, y_2 \in R$ we have $y_1 \sqsubset x$, $y_2 \sqsubset x$, and $y_1 \wr y_2$. Hence $\mathbf{Irl}(y_1) \cap \mathbf{Irl}(y_2) = \emptyset$, by (13.2). So for some $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathbf{Pt}_{\mathfrak{R}}$ we have $\mathfrak{p}_1 \in \mathbf{Irl}(y_1)$, $\mathfrak{p}_2 \in \mathbf{Irl}(y_2)$ and $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Since $y_1 \in \mathfrak{p}_1$, $y_2 \in \mathfrak{p}_2$ and all points are filters, both $x \in \mathfrak{p}_1$ and $x \in \mathfrak{p}_2$, i.e., $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathbf{Irl}(x)$. Therefore the set $\mathbf{Irl}(x)$ is not a singleton. ∎

A point $\mathfrak{p}$ of $\mathbf{Pt}_{\mathfrak{R}}$ is *isolated* in the G-topological space $\mathtt{Gts}\,\mathfrak{R}$ iff $\{\mathfrak{p}\} \in \mathsf{O}_{\mathfrak{R}}$. As an important consequence of the above definition we get:

Proposition 20.3. *For any isolated point $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ there is $a \in \mathrm{At}_{\mathfrak{R}}$ such that $\{\mathfrak{p}\} = \mathbf{Irl}(a)$.*

Proof. Suppose that $\{\mathfrak{p}\} \in \mathsf{O}_{\mathfrak{R}}$. Then $\{\mathfrak{p}\} = \bigcup_{x \in X} \mathbf{Irl}(x)$, for some $X \in \mathcal{P}(\mathbf{Pt}_{\mathfrak{R}})$. We show that X is a singleton. Assume for a contradiction that for some $x_1, x_2 \in X$ we have $x_1 \neq x_2$. Then $\emptyset \neq \mathbf{Irl}(x_1) \neq \mathbf{Irl}(x_2) \neq \emptyset$, by (13.1) and (13.6). So $\mathbf{Irl}(x_1) \cup \mathbf{Irl}(x_2)$ is not a singleton.

Thus, for some $x \in R$ we have $\{\mathfrak{p}\} = \mathbf{Irl}(x)$. From this and Proposition 20.2 follows that $x \in \mathrm{At}_{\mathfrak{R}}$. ∎

Thus, the above two propositions entail:

Corollary 20.4. *A point* $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ *is isolated iff there is an* $a \in \mathrm{At}_{\mathfrak{R}}$ *such that* $\{\mathfrak{p}\} = \mathbf{Irl}(a)$*. So the isolated points of* $\mathtt{Gts}\,\mathfrak{R}$ *are in bijective correspondence with the atoms of* $\mathfrak{R}$*. More specifically, an isolated point* $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ *corresponds to an atom* $a \in \mathrm{At}_{\mathfrak{R}}$ *iff* $\mathbf{Irl}(a) = \{\mathfrak{p}\}$*.*

Corollary 20.5. $\mathrm{At}_{\mathfrak{R}} \neq \emptyset$ *iff there is at least one isolated point.*

Proof. "⇒" If $x \in \mathrm{At}_{\mathfrak{R}}$ then $\mathbf{Irl}(x)$ is a singleton, by Proposition 20.2. So the only element of the set $\mathbf{Irl}(x)$ is an isolated point.

"⇐" By Proposition 20.3. ■

Proposition 20.6. $\mathfrak{R}$ *is atomic iff the set of all isolated points is dense in* $\mathtt{Gts}\,\mathfrak{R}$*.*

Proof. "⇒" A set is dense in $\mathtt{Gts}\,\mathfrak{R}$ iff it has at least one common point with each set from $\mathbf{O}^+_{\mathfrak{R}}$. Assume that $\mathfrak{R}$ is atomic and $\Omega \in \mathbf{O}^+_{\mathfrak{R}}$. Then for some $x \in R$ we have $\mathbf{Irl}(x) \subseteq \Omega$. Moreover, for some $a \in \mathrm{At}_{\mathfrak{R}}$ we have $a \sqsubseteq x$. Hence $\mathbf{Irl}(a) \subseteq \mathbf{Irl}(x)$, by (13.5). But, by Proposition 20.2, for some $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ we have $\mathbf{Irl}(a) = \{\mathfrak{p}\}$. So $\mathfrak{p}$ is isolated and $\mathfrak{p} \in \Omega$.

"⇐" Suppose that the set of all isolated points is dense in $\mathtt{Gts}\,\mathfrak{R}$. Then for any $x \in R$ there is an isolated point $\mathfrak{p} \in \mathbf{Pt}_{\mathfrak{R}}$ such that $\mathfrak{p} \in \mathbf{Irl}(x)$. So, by Proposition 20.3, there is an $a \in \mathrm{At}_{\mathfrak{R}}$ such that $\{\mathfrak{p}\} = \mathbf{Irl}(a)$. Hence $a \in \mathfrak{p}$. So also $a \bigcirc x$, since $x \in \mathfrak{p}$. Therefore $a \sqsubseteq x$. ■

Remark 20.1. The facts from this subsection are counterparts of the well know property of Boolean algebras [see, e.g., 9, Proposition 7.18]: The isolated points of Stone space of a Boolean algebra $\mathfrak{B}$ are in bijective correspondence with its atoms. More specifically, an isolated point $U \in \mathrm{Ult}(\mathfrak{B})$ corresponds to an atom $a \in \mathrm{At}_{\mathfrak{B}}$ iff $\mathrm{s}(a) = \{U\}$. Moreover, a Boolean algebra $\mathfrak{B}$ is atomic iff the set of all isolated points is dense in $\mathtt{Ult}\mathfrak{B}$. ■

20.4. Connected G-Topological Spaces

By conditions (13.14) and (15.8), for any $x \in R$ we obtain:

$$x \ll x \iff \mathbf{Irl}(x) \in \mathbf{Clop}_{\mathfrak{R}}. \tag{20.1}$$

Proposition 20.7. *If* $\mathtt{Gts}\,\mathfrak{R}$ *is connected then*:

1. *For any* $x \in R$ *which is not the unity we have* $x \not\ll x$*.*
2. *If* $\mathfrak{R}$ *has the unity* $\mathtt{1}$*, then for any* $x \in R \setminus \{\mathtt{1}\}$ *we have* $x \mathrel{\mathsf{C}} -x$*.*
3. *If* $\mathfrak{R}$ *is non-trivial, then* $\mathrm{At}_{\mathfrak{R}} = \emptyset$*.*

Proof. *Ad* 1. Assume for a contradiction that for $x \in R$ which is not the unity we have $x \ll x$. Hence, by (20.1), the set $\mathbf{Irl}(x)$ is clopen in $\mathtt{Gts}\,\mathfrak{R}$. But by (13.1) and (13.7), we have $\emptyset \neq \mathbf{Irl}(x) \neq \mathbf{Pt}_{\mathfrak{R}}$. Hence $\mathsf{Clop}_{\mathfrak{R}} \neq \{\emptyset, \mathbf{Pt}_{\mathfrak{R}}\}$, i.e., $\mathtt{Gts}\,\mathfrak{R}$ is not connected.

Ad 2. By 1 and (3.9).

Ad 3. Assume that $\mathfrak{R}$ is non-trivial. Then, by Proposition 9.1, if $a \in \mathrm{At}_{\mathfrak{R}}$, then a is not the unity and $a \ll a$. Hence, by 1, we have $\mathrm{At}_{\mathfrak{R}} = \emptyset$. ■

Further, for complete G-structures we obtain:

Proposition 20.8. *Let $\mathfrak{R}$ be complete and satisfy the following condition*:

$$\forall_{x \in R}\,(x \neq 1 \Longrightarrow x \,\mathsf{C}\, {-x}). \tag{C6}$$

Then $\mathtt{Gts}\,\mathfrak{R}$ is connected.

Proof. Suppose that $\mathfrak{R}$ is complete and $\mathtt{Gts}\,\mathfrak{R}$ is not connected. By Theorem 18.5 we have $\mathsf{B}_{\mathfrak{R}} = \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}$. Moreover, there is $\Omega \in \mathsf{Clop}_{\mathfrak{R}}$ such that $\emptyset \neq \Omega \neq \mathbf{Pt}_{\mathfrak{R}}$. So $\Omega \in \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}} = \mathsf{B}_{\mathfrak{R}}$ and for some $x \in R$, $\Omega = \mathbf{Irl}(x)$. Since $\Omega \neq \mathbf{Pt}_{\mathfrak{R}}$, we have that $x \neq 1$, by (13.7). Therefore $x \ll x$ and $x \,)(\, {-x}$, by (20.1) and (3.9), respectively. ■

21. On Grzegorczyk's Original Formulation of G-Structures

Grzegorczyk's original formulation of his theory in [6] is slightly different from ours and in this section we would like to explain those differences.

21.1. Completeness of Mereological Fields

In [6] Grzegorczyk did not explicitly assume that mereological fields are complete (and so that G-structures are complete). As Theorem 18.6 shows, axiom ($\exists\mathsf{sum}$) is necessary to prove that two G-structures $\mathfrak{R}$ and $\mathtt{Gts}\,\mathfrak{R} := \langle \mathrm{r}\mathsf{O}^{+}_{\mathfrak{R}}, \subseteq, \mathtt{][} \rangle$ are isomorphic. However, such a result can be found in [6, p. 235] with its proof appealing to completeness of underlying structure. So it is reasonable to admit that axiom ($\exists\mathsf{sum}$) was implicitly assumed by Grzegorczyk.

21.2. The Original Version of Axiom (G)

Instead of (G), Grzegorczyk formulates the following axiom (or rather the one which is equivalent to the following): for all connected $x, y \in R$ there exists an $X \in \mathcal{P}^{+}(R)$ such that X, x, and y satisfy (r1)–(r3), the condition

(g2) for any $z \in X$ we have $z \bigcirc x$ and $z \bigcirc y$

and the following condition:

(g1$^{\mathrm{org}}$) $\exists_{z\in R}\big(z \in X \ \wedge\ (x \sqsubseteq y \Rightarrow z \sqsubseteq x)\big)$.

That is, the Grzegorczyk's axiom has the form which is equivalent to the following:

$$\forall_{x,y\in R}\big(x \mathrel{\mathsf{C}} y \Longrightarrow \exists_{X\in\mathcal{P}(R)}\ X,\ x,\ y \text{ satisfy (r1)–(r3), (g1}^{\mathrm{org}}\text{), (g2)}\big). \quad (\mathrm{G}^{\mathrm{org}})$$

Only after he defines the notion of *representative of a point* (in other words, the notion of *pre-point*). However, we will prove that both formulations with (G) and (G$^{\mathrm{org}}$), respectively, yield equivalent theories.

Notice that the role of the first conjunct of (g1$^{\mathrm{org}}$) is to ensure the non-emptiness of the set of regions satisfying all the conditions. In the definition of the family $\mathbf{Q}_{\mathfrak{R}}$ we require that only non-empty sets are representatives of points and we use this family in (G). So we do not have to say anything about non-emptiness of the set postulated by the axiom in (g1). Actually Grzegorczyk defines representatives of points[6] in a way analogous to us, but the order of axioms and definitions is different. As we mentioned above – first, he introduces axiom (G$^{\mathrm{org}}$) and then defines counterpart of our set $\mathbf{Q}_{\mathfrak{R}}$. Thus he must somehow ensure non-emptiness of sets that generate points and he chooses the way described.

Theorem 21.1. *For any quasi-separation structure* $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle$ *the conditions* (G) *and* (G$^{\mathrm{org}}$) *are equivalent.*

Proof. "G $\Rightarrow$ (G$^{\mathrm{org}}$)" Suppose that $x \mathrel{\mathsf{C}} y$. Then, by (G), for some $Q_0 \in \mathbf{Q}_{\mathfrak{R}}$ we have: Q_0, x, and y satisfy (g1) and (g2). Now suppose that $x \sqsubseteq y$. Then also $x \bigcirc y$. Hence, by (g1), for some $z_0 \in Q_0$ we have $z_0 \sqsubseteq x \sqcap y = x$. So we obtain: $x \sqsubseteq y \Rightarrow \exists_{z\in Q_0} z \sqsubseteq x$. But this is equivalent to: $\exists_{z\in Q_0}(x \sqsubseteq y \Rightarrow z \sqsubseteq x)$ and $\exists_{z\in R}(z \in Q_0 \wedge (x \sqsubseteq y \Rightarrow z \sqsubseteq x))$. Thus, Q_0, x, and y satisfy (g1$^{\mathrm{org}}$). So we obtain (G$^{\mathrm{org}}$).

"(G$^{\mathrm{org}}$) $\Rightarrow$ (G$_\bigcirc$) $\wedge$ (G$_\wr$)" (see Proposition 6.1) Firstly, we show that (G$_\bigcirc$) holds. Let $x \bigcirc y$. Then by reflexivity of C and (G$^{\mathrm{org}}$) applied to $x \sqcap y$, for some $X_0 \subseteq R$ satisfying (r1)–(r3) we have: X_0, $x/x \sqcap y$, and $y/x \sqcap y$ satisfy (g1$^{\mathrm{org}}$). But, by (g1$^{\mathrm{org}}$), we get $X_0 \neq \emptyset$, so $X_0 \in \mathbf{Q}_{\mathfrak{R}}$. Moreover, by (g1$^{\mathrm{org}}$) and reflexivity of $\sqsubseteq$, for some $z_0 \in X_0$ we have $z_0 \sqsubseteq x \sqcap y$. Secondly, also (G$_\wr$) holds. Suppose that $x \mathrel{\mathsf{C}} y$ and $x \wr y$. Then $X \in \mathbf{Q}_{\mathfrak{R}}$, since by (g1$^{\mathrm{org}}$) we have $X \neq \emptyset$. The rest we obtain by (g2). ■

[6]See the predicate $Q(x)$ in [6, p. 232].

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