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Pure Modal Logic of Names and Tableau Systems

Abstract. By a pure modal logic of names (PMLN) we mean a quantifier-free formulation of such a logic which includes not only traditional categorical, but also modal categorical sentences with modalities *de re* and which is an extension of Propositional Logic. For categorical sentences we use two interpretations: a “natural” one; and Johnson and Thomason’s interpretation, which is suitable for some reconstructions of Aristotelian modal syllogistic (Johnson in *Notre Dame J Form Logic* 30(2):271–284, 1989; Thomason in *J Philos Logic* 22(2):111–128, 1993 and *J Philos Logic* 26:129–141, 1997. In both cases we use Johnson-like models (1989). We also analyze different kinds of versions of PMLN, for both general and singular names. We present complete tableau systems for the different versions of PMLN. These systems enable us to present some decidability methods. It yields “strong decidability” in the following sense: for every inference starting with a finite set of premises (resp. every syllogism, every formula) we can specify a finite number of steps to check whether it is logically valid. This method gives the upper bound of the cardinality of models needed for the examination of the validity of a given inference (resp. syllogism, formula).

Keywords: Pure modal logic of names, Semantics, Tableaus, Decidability, Modal syllogistic.

Introduction

By a pure modal logic of names (PMLN) we mean a quantifier-free formulation of such a logic, in which to traditional categorical sentences we add also modal categorical sentences with modalities *de re* and which is an extension of Propositional Logic. In this paper we present complete tableau systems for this logic as well as for modal syllogisms in Johnson-like models for categorical sentences. Johnson [4] and Thomason [14, 15] used a natural interpretation for general sentences of the form ‘Every *a* is necessarily *b*’ and ‘No *a* is possibly *b*’. In Section 8 we describe pure modal logics for both general and singular names and some tableau systems for the logics.

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We will in this paper take a natural interpretation of these four kinds of sentence, since we do not aim to reconstruct Aristotelian syllogistic. However, to reconstruct Aristotelian modal syllogistic, in [4, 14, 15] the authors have assumed unnatural interpretations of particular sentences of the form: ‘Some a is necessarily b ’ and ‘Some a is not possibly b ’, and as a consequence also of general sentences of the form: ‘Every a is possibly b ’ and ‘No a is necessarily b ’ (see Section 9). In the last part of the paper the unnatural interpretation is discussed and may be compared with the former results. Summing up, the general aim of the paper is to examine natural as well as unnatural interpretations of modal syllogism in the light of modern tableau tools.

In this paper we present a special kind of tableau, which we call a “mini-max” tableau. Mini-max tableaux involve a very low number of proof steps. They give us a decidability method for PMLN (as well as for modal syllogistic) which yields what we call “strong decidability”. It means that for every inference starting with a finite set of premises (resp. every syllogism, every formula) we can specify a finite number of steps to check whether it is logically valid. This method gives also the upper bound of the cardinality of models needed for the examination of the validity of a given inference (resp. syllogism, formula).

For syllogisms, as in [3], we can also provide another decidability method which estimates a limit of the length of a tableau for a given syllogism. This paper is both a significant extension of [3] and a modification of it.

1. Pure Modal Logic of Names and Modal Syllogisms

1.1. Syntax

Formulas of modal syllogisms. We assume a countably infinite set \mathbf{GL} which contains schematic general name letters: $a_0, a_1, a_2, a_3, \dots$ (for the first three of these letters we will use abbreviations: ‘ a ’, ‘ b ’, and ‘ c ’, respectively). We assume that the letters from \mathbf{GL} represent various general names.

Following Thomason [14, 15] a modal syllogistic formula is an expression of any of the following forms: $A\alpha\beta$, $I\alpha\beta$, $E\alpha\beta$, $O\alpha\beta$, $A^\square\alpha\beta$, $A^\diamond\alpha\beta$, $I^\square\alpha\beta$, $I^\diamond\alpha\beta$, $E^\square\alpha\beta$, $E^\diamond\alpha\beta$, $O^\square\alpha\beta$, $O^\diamond\alpha\beta$, for $\alpha, \beta \in \mathbf{GL}$. The notation reflects a way to read non-modal and apodeictic propositions, respectively: Aab : b belongs to every a , Eab : b belongs to no a , Iab : b belongs to some a , Oab : b does not belong to some a ; $A^\square ab$: b of necessity belongs to every a ; $E^\square ab$: b of necessity belongs to no a ; $I^\square ab$: b belongs of necessity to some a ; $O^\square ab$: b of necessity does not belong to some a (see, e.g., [1] and [7, p. 10]). Similar

readings hold for formulas with ‘ \diamond ’, where ‘of necessity’ is replaced with ‘of possibility’.

Let $\mathbf{For}_{\text{syll}}$ be the set of all syllogistic formulas. All formulas with symbols A, I, E, or O we will call *non-modal*. For any $\bullet \in \{\text{blank}, \square, \diamond\}$ all formulas with I^\bullet or O^\bullet we will call *particular* (for short: *p-formulas*), and all formulas with A^\bullet or E^\bullet we will call *general* (for short: *g-formulas*).

We present English glosses for modal syllogistic formulas under a formal interpretation which reflects a natural way of understanding them [see, e.g., 7, pp. 10–35]. Moreover, the interpretations are expressed by translations into formulas of monadic first-order modal logic.¹

- | | |
|---|---|
| 1. Aab: Every a is b | $\forall_x(ax \supset bx)$ |
| $A^\square ab$: Every a must be b , Every a is necessarily b | $\forall_x(ax \supset \square bx)$ |
| $A^\diamond ab$: Every a may be b , Every a is possibly b | $\forall_x(ax \supset \diamond bx)$ |
| 2. Eab: No a is b | $\neg \exists_x(ax \wedge bx)$ |
| $E^\square ab$: No a may be b , No a is possibly b | $\neg \exists_x(ax \wedge \diamond bx)$ |
| $E^\diamond ab$: No a must be b , No a is necessarily b | $\neg \exists_x(ax \wedge \square bx)$ |
| 3. Iab: Some a is b | $\exists_x(ax \wedge bx)$ |
| $I^\square ab$: Some a must be b , Some a is necessarily b | $\exists_x(ax \wedge \square bx)$ |
| $I^\diamond ab$: Some a may be b , Some a is possibly b | $\exists_x(ax \wedge \diamond bx)$ |
| 4. Oab: Some a is not b | $\exists_x(ax \wedge \neg bx)$ |
| $O^\square ab$: Some a must not be b | $\exists_x(ax \wedge \square \neg bx)$ |
| Some a is not possibly b | $\exists_x(ax \wedge \neg \diamond bx)$ |
| $O^\diamond ab$: Some a may not be b | $\exists_x(ax \wedge \diamond \neg bx)$ |
| Some a is not necessarily b | $\exists_x(ax \wedge \neg \square bx)$ |

Syllogisms. By *syllogism* we mean any pair $\langle \Psi, \varphi \rangle$, where Ψ is a non-empty finite subset² of $\mathbf{For}_{\text{syll}}$ (for short: $\Psi \in \mathcal{P}_{\text{fin}}(\mathbf{For}_{\text{syll}}) \setminus \{\emptyset\}$) and $\varphi \in \mathbf{For}_{\text{syll}}$.³

Formulas of PMLN. All formulas of PMLN are built with members of $\mathbf{For}_{\text{syll}}$ by the use of classical propositional connectives and parentheses. These formulas are members of \mathbf{For} which is the smallest set such that:

¹Formulas of monadic first-order modal logic are defined in a standard way: ‘ ax ’ and ‘ bx ’ stands for ‘ x is a ’ and ‘ x is b ’, respectively; ‘ \square ’ and ‘ \diamond ’ stand for the *de dicto* operators ‘it is necessary that’ and ‘it is possible that’, respectively.

²For any set S let $\mathcal{P}(S)$ (resp. $\mathcal{P}_{\text{fin}}(S)$) be the family of all (resp. finite) subsets of S .

³The denotation of the term ‘syllogism’ is in our paper wider than in [1].

- $\mathbf{For}_{\text{syll}} \subseteq \mathbf{For}$;
- if $\varphi \in \mathbf{For}$, then $\neg \varphi \in \mathbf{For}$;
- if $\varphi, \psi \in \mathbf{For}$, then $\neg(\varphi \circ \psi) \in \mathbf{For}$, where $\circ \in \{\wedge, \vee, \supset, \equiv\}$.

For any $\Phi \in \mathcal{P}(\mathbf{For})$ let $\mathbf{GL}(\Phi)$ be the set of letters from \mathbf{GL} which occur in at least one formula of Φ .

1.2. Semantics

Models and the notion of truth for For. A *model*⁴ for **For** is any quadruple $\langle D, d, d^\square, d^\diamond \rangle$, where D is a set (*domain*; possibly empty) and d, d^\square, d^\diamond are functions from \mathbf{GL} into $\mathcal{P}(D)$ satisfying the following condition:

$$(\star) \quad d^\square(\alpha) \subseteq d(\alpha) \subseteq d^\diamond(\alpha), \text{ for any } \alpha \in \mathbf{GL}.$$

Intuitively, $d(a)$ is the denotation of a in \mathfrak{M} , i.e., it is the set of all objects which are a . Moreover, $d^\square(a)$ (resp. $d^\diamond(a)$) is the set of all objects which are necessarily (resp. possibly) a .⁵

Let \mathbf{Mod} be the class of all models for **For**. The *empty model* \mathfrak{M}_\emptyset is the only model with the empty universe. This model has also *empty denotation* functions, i.e., $d(\alpha) = d^\square(\alpha) = d^\diamond(\alpha) = \emptyset$, for any $\alpha \in \mathbf{GL}$.

We use the following interpretation of formulas from **For** in model $\mathfrak{M} = \langle D, d, d^\square, d^\diamond \rangle$, for all $\alpha, \beta \in \mathbf{GL}$, $\varphi, \psi \in \mathbf{For}$, and $\bullet \in \{\text{blank}, \square, \diamond\}$:

$$\mathfrak{M} \models A^\bullet \alpha \beta \text{ iff } d(\alpha) \subseteq d^\bullet(\beta),$$

$$\mathfrak{M} \models E^\bullet \alpha \beta \text{ iff } d(\alpha) \cap d^*(\beta) = \emptyset,$$

where $*$ = blank, \diamond , \square , for \bullet = blank, \square , \diamond , respectively,

$$\mathfrak{M} \models I^\bullet \alpha \beta \text{ iff } d(\alpha) \cap d^\bullet(\beta) \neq \emptyset,$$

$$\mathfrak{M} \models O^\bullet \alpha \beta \text{ iff } d(\alpha) \not\subseteq d^*(\beta),$$

where $*$ = blank, \diamond , \square , for \bullet = blank, \square , \diamond , respectively,

$$\mathfrak{M} \models \neg \varphi \text{ iff } \mathfrak{M} \not\models \varphi,$$

$$\mathfrak{M} \models (\varphi \wedge \psi) \text{ iff } \mathfrak{M} \models \varphi \text{ and } \mathfrak{M} \models \psi,$$

$$\mathfrak{M} \models (\varphi \vee \psi) \text{ iff } \mathfrak{M} \models \varphi \text{ or } \mathfrak{M} \models \psi,$$

$$\mathfrak{M} \models (\varphi \supset \psi) \text{ iff } \mathfrak{M} \not\models \varphi \text{ or } \mathfrak{M} \models \psi,$$

$$\mathfrak{M} \models (\varphi \equiv \psi) \text{ iff either } \mathfrak{M} \models \varphi \text{ and } \mathfrak{M} \models \psi, \text{ or } \mathfrak{M} \not\models \varphi \text{ and } \mathfrak{M} \not\models \psi.$$

⁴The notion of model is based on [15], where Thomason adopted Johnson's notion from [4] and his own notion from [14].

⁵All pairs of the form $\langle D, d \rangle$ are models for non-modal formulas [see, e.g., 8–11].

If $\mathfrak{M} \models \varphi$ then we say that the formula φ is *true* in \mathfrak{M} . Otherwise, i.e. $\mathfrak{M} \not\models \varphi$, we say that it is *false* in \mathfrak{M} . If for any ψ from $\Psi \subseteq \mathbf{For}$ we have $\mathfrak{M} \models \psi$, then we write $\mathfrak{M} \models \Psi$.

Note that the empty model \mathfrak{M}_\emptyset plays the same role as for any set D the model $\mathfrak{M}_{D,\emptyset} := \langle D, \alpha \mapsto \emptyset, \alpha \mapsto \emptyset, \alpha \mapsto \emptyset \rangle$ which has empty denotation functions.

LEMMA 1.1. *For any set D and any $\varphi \in \mathbf{For}$: $\mathfrak{M}_{D,\emptyset} \models \varphi$ iff $\mathfrak{M}_\emptyset \models \varphi$.*

Logical entailment and equivalence. Logically valid formulas and syllogisms.

For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$, we say that Ψ *logically entails* φ (we write: $\Psi \models \varphi$) iff there is no model \mathfrak{M} in \mathbf{Mod} such that $\mathfrak{M} \models \Psi$ and $\mathfrak{M} \not\models \varphi$. If $\Psi = \{\psi_1, \dots, \psi_n\}$ ($n > 0$), then we write: $\psi_1, \dots, \psi_n \models \varphi$. Formulas $\varphi, \psi \in \mathbf{For}$ are *logically equivalent* (we write: $\varphi \models \psi$) iff $\varphi \models \psi$ and $\psi \models \varphi$. Moreover, a formula φ is *logically valid* (we write: $\models \varphi$) iff $\emptyset \models \varphi$.

Note that, by Lemma 1.1, we obtain:

CLAIM 1.2. *For any $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$: $\Psi \models \varphi$ iff there is no non-empty model \mathfrak{M} in \mathbf{Mod} such that $\mathfrak{M} \models \Psi$ and $\mathfrak{M} \not\models \varphi$.*

We say that a syllogism $\langle \Psi, \varphi \rangle$ is *logically valid* iff $\Psi \models \varphi$. Of course, a syllogism $\langle \{\psi_1, \dots, \psi_n\}, \varphi \rangle$ is logically valid iff the formula $\ulcorner \psi_1 \supset (\dots \supset (\psi_n \supset \varphi)) \urcorner$ (resp. $\ulcorner (\psi_1 \wedge \dots \wedge \psi_n) \supset \varphi \urcorner$) is logically valid.

Note that for categorical sentences we obtain:

CLAIM 1.3. *For all $\alpha, \beta \in \mathbf{GL}$:*

1. $A^\bullet \alpha \beta \models \neg O^\bullet \alpha \beta$ and $E^\bullet \alpha \beta \models \neg I^\bullet \alpha \beta$, where $\bullet = \text{blank}, \diamond, \square$, for $\bullet = \text{blank}, \square, \diamond$, respectively.
2. For any $X \in \{A, I, E, O\}$: $X^\square \alpha \beta \models X \alpha \beta \models X^\diamond \alpha \beta$.
3. $I \alpha \beta \models I \beta \alpha$ and $E \alpha \beta \models E \beta \alpha$.
4. $I \alpha \alpha \models I^\diamond \alpha \alpha$.
5. $I^\bullet \alpha \beta \models I \alpha \alpha$ and $O^\bullet \alpha \beta \models I \alpha \alpha$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
6. $\models A \alpha \alpha$ and $\models A^\diamond \alpha \alpha$. So also $\models \neg O \alpha \alpha$ and $\models \neg O^\square \alpha \alpha$.
7. $\not\models A^\square \alpha \alpha$ and $\not\models I^\bullet \alpha \alpha$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
8. $A^\bullet \alpha \beta \not\models I^\bullet \alpha \beta$, $E^\bullet \alpha \beta \not\models O^\bullet \alpha \beta$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
9. $I^\bullet \alpha \beta \not\models I^\bullet \beta \alpha$ and $E^\bullet \alpha \beta \not\models E^\bullet \beta \alpha$, for any $\bullet \in \{\square, \diamond\}$.

Logical contradiction. We say that $\varphi, \psi \in \mathbf{For}$ are *logically contradictory* iff $\varphi \models \neg \psi$. Of course, φ and $\neg \varphi$ are logically contradictory. Moreover,

by Claim 1.3, for all $\alpha, \beta \in \mathbf{GL}$, if $\bullet = \text{blank}, \square, \diamond$ and $*$ = blank, \diamond, \square , respectively, then formulas in the following pairs are logically contradictory: $\{A^\bullet \alpha \beta, O^* \alpha \beta\}$ and $\{E^\bullet \alpha \beta, I^* \alpha \beta\}$. These are the only pairs of the syllogistic formulas which are logically contradictory.

1.3. Compactness of Entailment in Pure Modal Logic of Names

We can prove that the relation \models is compact just by embedding it into the consequence relation of Classical Quantified Logic (CQL), which is compact. To do this, we use a transformation of formulas from **For** into a set of formulas of a monadic first order language and then we will use the compactness property of CQL.

A transformation For into some monadic first-order language. For **For** we use a monadic first-order language \mathcal{L} that has the set of variables $\mathbf{Var} := \{x_0, x_1, x_2, \dots\}$ (for ' x_0 ' we will use ' x ') and the set of monadic predicates $\mathbf{P}_{\mathcal{L}} := \{\alpha^\bullet : \alpha \in \mathbf{GL} \text{ and } \bullet \text{ is either blank, or } \square, \text{ or } \diamond\}$. Moreover, \mathcal{L} does not have any other non-logical constants. Finally, let $\mathbf{For}_{\mathcal{L}}$ be the set of all formulas of \mathcal{L} which are obtained in the standard way.

We use the following transformation \mathbf{t} from **For** into $\mathbf{For}_{\mathcal{L}}$, where for all $\alpha, \beta \in \mathbf{GL}$, $\varphi, \psi \in \mathbf{For}$, and $\bullet \in \{\text{blank}, \square, \diamond\}$ we put:

$$\begin{aligned} \mathbf{t}(A^\bullet \alpha \beta) &:= \ulcorner \forall_x (\alpha x \supset \beta^\bullet x) \urcorner, \\ \mathbf{t}(E^\bullet \alpha \beta) &:= \ulcorner \neg \exists_x (\alpha x \wedge \beta^* x) \urcorner, \\ &\quad \text{where } * = \text{blank}, \diamond, \square, \text{ for } \bullet = \text{blank}, \square, \diamond, \text{ respectively,} \\ \mathbf{t}(I^\bullet \alpha \beta) &:= \ulcorner \exists_x (\alpha x \wedge \beta^\bullet x) \urcorner, \\ \mathbf{t}(O^\bullet \alpha \beta) &:= \ulcorner \exists_x (\alpha x \wedge \neg \beta^* x) \urcorner, \\ &\quad \text{where } * = \text{blank}, \diamond, \square, \text{ for } \bullet = \text{blank}, \square, \diamond, \text{ respectively,} \\ \mathbf{t}(\neg \varphi) &= \ulcorner \neg \mathbf{t}(\varphi) \urcorner, \\ \mathbf{t}(\varphi \circ \psi) &= \ulcorner \mathbf{t}(\varphi) \circ \mathbf{t}(\psi) \urcorner, \quad \text{for } \circ \in \{\wedge, \vee, \supset, \equiv\}. \end{aligned}$$

Moreover, for any $\Psi \subseteq \mathbf{For}$ we put $\mathbf{t}(\Psi) := \{\mathbf{t}(\psi) : \psi \in \Psi\}$. As we can see the transformation is an injection.

Models for \mathcal{L} . A *model for \mathcal{L}* is any pair $\mathfrak{M}_{\mathcal{L}} = \langle D, \mathfrak{I} \rangle$, where D is a non-empty set and \mathfrak{I} is a function from $\mathbf{P}_{\mathcal{L}}$ into $\mathcal{P}(D)$. Let $\mathbf{Mod}_{\mathcal{L}}$ be the class of all models of \mathcal{L} . For any model $\mathfrak{M}_{\mathcal{L}} = \langle D, \mathfrak{I} \rangle$, any function $v: \mathbf{Var} \rightarrow D$ is a *valuation of variables*. We introduce, in the standard way, the notion of *truth of formulas of $\mathbf{For}_{\mathcal{L}}$ in $\mathfrak{M}_{\mathcal{L}}$* (we write: $\mathfrak{M}_{\mathcal{L}} \models \varphi$). Of course, for all $\pi \in \mathbf{P}_{\mathcal{L}}$ and $x_i \in \mathbf{Var}$: $\mathfrak{M}_{\mathcal{L}} \models \pi x_i$ iff $v(x_i) \in \mathfrak{I}(\pi)$.

For all $\Psi \in \mathcal{P}(\mathbf{For}_{\mathcal{L}})$ and $\varphi \in \mathbf{For}_{\mathcal{L}}$, also as usual, we say that in CQL Ψ *logically entails* φ (we write: $\Psi \models_{\text{CQL}} \varphi$) iff there is no model $\mathfrak{M}_{\mathcal{L}} \in \text{Mod}_{\mathcal{L}}$ such that $\mathfrak{M}_{\mathcal{L}} \models \Psi$ and $\mathfrak{M}_{\mathcal{L}} \not\models \varphi$.

For any $\alpha \in \mathbf{GL}$ we consider the following sentences from $\mathbf{For}_{\mathcal{L}}$:

$$\begin{aligned} T_{\alpha}^{\square} &:= \forall_x (\alpha^{\square} x \supset \alpha x) \\ T_{\alpha}^{\diamond} &:= \forall_x (\alpha x \supset \alpha^{\diamond} x). \end{aligned}$$

Moreover, for any $\Phi \in \mathcal{P}(\mathbf{For})$ we put:

$$T_{\Phi} := \{T_{\alpha}^{\square} \in \mathbf{For}_{\mathcal{L}} : \alpha \in \mathbf{GL}(\Phi)\} \cup \{T_{\alpha}^{\diamond} \in \mathbf{For}_{\mathcal{L}} : \alpha \in \mathbf{GL}(\Phi)\}.$$

That is, $T_{\Phi} \in \mathcal{P}(\mathbf{For}_{\mathcal{L}})$ and it is the set of all formulas of the form T_{α}^{\square} or T_{α}^{\diamond} in which $\alpha \in \mathbf{GL}(\Phi)$. We obtain:

LEMMA 1.4. *For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$:*

$$\Psi \models \varphi \quad \text{iff} \quad T_{\Psi \cup \{\varphi\}} \cup \mathbf{t}(\Psi) \models_{\text{CQL}} \mathbf{t}(\varphi).$$

PROOF. “ \Rightarrow ” Let $\Psi \models \varphi$, $\mathfrak{M}_{\mathcal{L}} = \langle D, \mathcal{J} \rangle \in \text{Mod}_{\mathcal{L}}$, and $\mathfrak{M}_{\mathcal{L}} \models T_{\Psi \cup \{\varphi\}} \cup \mathbf{t}(\Psi)$. Then, firstly, we have:

$$(\star\star) \quad \mathcal{J}(\alpha^{\square}) \subseteq \mathcal{J}(\alpha) \subseteq \mathcal{J}(\alpha^{\diamond}), \text{ for any } \alpha \in \mathbf{GL}(\Psi \cup \{\varphi\}).$$

Second, we consider the following model $\mathfrak{M} = \langle D, d, d^{\square}, d^{\diamond} \rangle$ from $\text{Mod} \setminus \{\mathfrak{M}_{\emptyset}\}$, where for all $\alpha \in \mathbf{GL}$ and $\bullet \in \{\text{blank}, \square, \diamond\}$ we put:

$$d^{\bullet}(\alpha) := \begin{cases} \mathcal{J}(\alpha^{\bullet}) & \text{if } \alpha \in \mathbf{GL}(\Psi \cup \{\varphi\}) \\ \emptyset & \text{otherwise} \end{cases}$$

Of course, for any $\chi \in \Psi \cup \{\varphi\}$ we have $(*)$: $\mathfrak{M} \models \chi$ iff $\mathfrak{M}_{\mathcal{L}} \models \mathbf{t}(\chi)$. Hence $\mathfrak{M} \models \Psi$, since $\mathfrak{M}_{\mathcal{L}} \models \mathbf{t}(\Psi)$. Therefore $\mathfrak{M} \models \varphi$, by the first assumption. So $\mathfrak{M}_{\mathcal{L}} \models \mathbf{t}(\varphi)$, by $(*)$.

“ \Leftarrow ” Let $T_{\Psi \cup \{\varphi\}} \cup \mathbf{t}(\Psi) \models_{\text{CQL}} \mathbf{t}(\varphi)$, $\mathfrak{M} = \langle D, d, d^{\square}, d^{\diamond} \rangle \in \text{Mod} \setminus \{\mathfrak{M}_{\emptyset}\}$, and $\mathfrak{M} \models \Psi$. For all $\alpha \in \mathbf{GL}$ and $\bullet \in \{\text{blank}, \square, \diamond\}$ we put $\mathcal{J}(\alpha^{\bullet}) := d^{\bullet}(\alpha)$. Then $\mathfrak{M}_{\mathcal{L}} := \langle D, \mathcal{J} \rangle$ belongs to $\text{Mod}_{\mathcal{L}}$ and satisfies the condition $(\star\star)$ for any $\alpha \in \mathbf{GL}$. Moreover, for any $\psi \in \mathbf{For}$ we have $(*)$: $\mathfrak{M} \models \psi$ iff $\mathfrak{M}_{\mathcal{L}} \models \mathbf{t}(\psi)$. So, by $(\star\star)$, $\mathfrak{M}_{\mathcal{L}} \models T_{\Psi \cup \{\varphi\}} \cup \mathbf{t}(\Psi)$, since $\mathfrak{M} \models \Psi$. Hence $\mathfrak{M}_{\mathcal{L}} \models \mathbf{t}(\varphi)$, by the first assumption. Therefore $\mathfrak{M} \models \varphi$, by $(*)$. ■

THEOREM 1.5 (Compactness). *For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$:*

$$\Psi \models \varphi \quad \text{iff} \quad \text{there is } \Psi_{\text{fin}} \in \mathcal{P}_{\text{fin}}(\Psi) \text{ such that } \Psi_{\text{fin}} \models \varphi.$$

PROOF. “ \Rightarrow ” Let $\Psi \models \varphi$. Then $T_{\Psi \cup \{\varphi\}} \cup \mathbf{t}(\Psi) \models_{\text{CQL}} \mathbf{t}(\varphi)$, by Lemma 1.4. Hence, by the compactness of CQL, we obtain two finite sets $T_{\Psi \cup \{\varphi\}}^{\text{fin}}$ and

$\mathbf{t}(\Psi)_{\text{fin}}$ such that $\mathbf{T}_{\Psi \cup \{\varphi\}}^{\text{fin}} \cup \mathbf{t}(\Psi)_{\text{fin}} \models_{\text{CQL}} \mathbf{t}(\varphi)$. Moreover, since \mathbf{t} is an injection, so for some finite subset Ψ_{fin} of Ψ we have: $\mathbf{t}(\Psi)_{\text{fin}} = \mathbf{t}(\Psi_{\text{fin}})$. Hence $\Psi_{\text{fin}} \models \varphi$, by Lemma 1.4. ■

2. Auxiliary Singular Formulas for Tableaus

Auxiliary singular names. Let \mathbf{AI} be the infinite, countable set of the following *auxiliary indexes*: i_0, i_1, i_2, \dots (for the first two of these indexes we will use ‘ i ’ and ‘ j ’, respectively). These indexes are *auxiliary singular names* that are needed for the construction of tableaus.

Auxiliary singular formulas and their negations. An *auxiliary singular formula* is any expression which has one of the following forms: $\ulcorner i\varepsilon\alpha \urcorner$, $\ulcorner i\varepsilon^\square\alpha \urcorner$, and $\ulcorner i\varepsilon^\diamond\alpha \urcorner$, for some $i \in \mathbf{AI}$ and $\alpha \in \mathbf{GL}$.⁶ We read, respectively, the auxiliary singular formulas ‘ $i\varepsilon a$ ’, ‘ $i\varepsilon^\square a$ ’, and ‘ $i\varepsilon^\diamond a$ ’ as follows: “ i is a ”, “ i is necessarily a ”, and “ i is possibly a ”.

If σ is an auxiliary singular formula, then $\ulcorner \neg \sigma \urcorner$ is its negation. Let Σ be the set of all auxiliary singular formulas and their negations. Moreover, let Σ^\neg be the set of all negations of singular formulas; so $\Sigma^\neg \subsetneq \Sigma$.

Notion of truth for the set $\mathbf{For} \cup \Sigma$. By Claim 1.2, the empty model \mathfrak{M}_\emptyset does not have an influence on the relation \models . Thus, we will use only non-empty models from \mathbf{Mod} . Moreover, for any non-empty domain D we introduce a function $\nu: \mathbf{AI} \rightarrow D$ which is a *valuation of indexes*. A valuation of indexes in a model is a valuation into its domain.

For any non-empty model $\mathfrak{M} = \langle D, d, d^\square, d^\diamond \rangle$ and any valuation of indexes $\nu: \mathbf{AI} \rightarrow D$ we can extend the notion of *truth* to the set $\mathbf{For} \cup \Sigma$. For all $i \in \mathbf{AI}$, $\alpha \in \mathbf{GL}$, $\sigma \in \Sigma$, and $\varphi \in \mathbf{For}$ we put:

$$\begin{aligned} \mathfrak{M}, \nu \models i\varepsilon^\bullet\alpha &\text{ iff } \nu(i) \in d^\bullet(\alpha), \text{ for any } \bullet \in \{\text{blank}, \square, \diamond\}, \\ \mathfrak{M}, \nu \models \neg\sigma &\text{ iff } \mathfrak{M}, \nu \not\models \sigma, \\ \mathfrak{M}, \nu \models \varphi &\text{ iff } \mathfrak{M} \models \varphi. \end{aligned}$$

Logical entailment for $\mathbf{For} \cup \Sigma$. For all $\Psi \in \mathcal{P}(\mathbf{For} \cup \Sigma)$ and $\varphi \in \mathbf{For} \cup \Sigma$, we say that Ψ *logically entails* φ (we write: $\Psi \models \varphi$) iff for any non-empty model \mathfrak{M} and any valuation ν in its domain: if $\mathfrak{M}, \nu \models \Psi$, then $\mathfrak{M}, \nu \models \varphi$.

⁶Although we use symbol ε , of course it is not Stanisław Leśniewski’s constant.

$\neg\neg\varphi$		$\varphi \wedge \psi$		$\neg(\varphi \vee \psi)$		$\neg(\varphi \supset \psi)$		$\varphi \equiv \psi$
\downarrow	(E $\neg\neg$)	\downarrow	(E \wedge)	\downarrow	(E $\neg\vee$)	\downarrow	(E $\neg\supset$)	\downarrow
φ		φ		$\neg\varphi$		φ		$\varphi \supset \psi$
		ψ		$\neg\psi$		$\neg\psi$		$\psi \supset \varphi$
								(E \equiv)

Table 1. The rule (E $\neg\neg$) and non-branching elimination rules (NBE)

CLAIM 2.1. For all $\alpha, \beta \in \mathbf{GL}$, $\imath \in \mathbf{AI}$, and $\bullet \in \{\text{blank}, \square, \diamond\}$:

- (i) $A^\bullet\alpha\beta$, $\imath\epsilon\alpha \models \imath\epsilon^\bullet\beta$ and $E^\bullet\alpha\beta$, $\imath\epsilon\alpha \models \neg\imath\epsilon^*\beta$, where $*$ = blank, \diamond , \square , for \bullet = blank, \square , \diamond , respectively.
- (ii) $\imath\epsilon^\square\alpha \models \imath\epsilon\alpha \models \imath\epsilon^\diamond\alpha$.

3. Tableaus

To examine $\Psi \models \varphi$ we will apply tableaus that are a kind of finite trees constructed from formulas of $\Psi \cup \{\neg\varphi\} \cup \Sigma$ by rules we introduce below.

3.1. Tableau Rules

Tableaus are build by so-called *tableau rules*. We will first outline the rules. Afterwards, we will provide a precise definition of a tableau that will formally specify their behaviour. We have seven groups of rules. Only the first three kinds decompose sentences with classical propositional connectives.

The first group consists of a single rule (E $\neg\neg$) of elimination of double negation. Formally, this rule is the set $\{\langle \neg\neg\varphi, \varphi \rangle : \varphi \in \mathbf{For}\}$ (cf. Table 1).

The second group has four members (E \wedge), (E $\neg\vee$), (E $\neg\supset$), (E \equiv) and they do not lead to branches in tableaus either. We will denote these rules by (NBE) for “non-branching elimination”. Formally, each rule in (NBE) is a set of triples of the form $\langle \varphi, \langle \psi, \chi \rangle \rangle$, where $\varphi, \psi, \chi \in \mathbf{For}$ have a particular form given by the relevant schemes from Table 1. The ordered pair $\langle \psi, \chi \rangle$ plays the role of a conjunction of conclusions.

The third group of rules, (E \vee), (E \supset), (E $\neg\wedge$), (E $\neg\equiv$), make two branches in tableaus (cf. Table 2). We will denote these rules by (BE) for “branching elimination”. Formally, each rule in (BE) is a set of ordered pairs $\langle \varphi, \{\psi_1, \psi_2\} \rangle$, where $\varphi, \psi_1, \psi_2 \in \mathbf{For}$ have a particular form given by the relevant schemes. In this case the set $\{\psi_1, \psi_2\}$ plays the role of alternative conclusions.

The rules of the other four types are divided into two categories: “rules of inference” and “rules of choice”.

$\begin{array}{c} \varphi \vee \psi \\ \swarrow \quad \searrow \\ \varphi \quad \psi \end{array} \quad (E\vee)$	$\begin{array}{c} \varphi \supset \psi \\ \swarrow \quad \searrow \\ \neg \varphi \quad \psi \end{array} \quad (E\supset)$	$\begin{array}{c} \neg(\varphi \wedge \psi) \\ \swarrow \quad \searrow \\ \neg \varphi \quad \neg \psi \end{array} \quad (E\neg\wedge)$	$\begin{array}{c} \neg(\varphi \equiv \psi) \\ \swarrow \quad \searrow \\ \varphi \wedge \neg \psi \quad \neg \varphi \wedge \psi \end{array} \quad (E\neg\equiv)$
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Table 2. Branching elimination rules (BE)

$\begin{array}{c} \neg A^\bullet \alpha \beta \\ \downarrow \\ O^* \alpha \beta \end{array}$	$\begin{array}{c} \neg O^\bullet \alpha \beta \\ \downarrow \\ A^* \alpha \beta \end{array}$	$\begin{array}{c} \neg E^\bullet \alpha \beta \\ \downarrow \\ I^* \alpha \beta \end{array}$	$\begin{array}{c} \neg I^\bullet \alpha \beta \\ \downarrow \\ E^* \alpha \beta \end{array}$
--	--	--	--

where $*$ = blank, \Diamond , \Box , for \bullet = blank, \Box , \Diamond , respectively.

Table 3. Rules inference (RI \neg)

For any $\bullet \in \{\text{blank}, \Box, \Diamond\}$:

$\begin{array}{c} A^\bullet \alpha \beta \\ \imath \varepsilon \alpha \\ \downarrow \\ \imath \varepsilon^\bullet \beta \end{array}$	$\begin{array}{c} E^\bullet \alpha \beta \\ \imath \varepsilon \alpha \\ \downarrow \\ \neg \imath \varepsilon^* \beta \end{array}$
--	---

where $*$ = blank, \Diamond , \Box , for \bullet = blank, \Box , \Diamond , respectively.

Table 4. Rules of inference for general formulas (RIg)

The fourth group of rules consists of twelve rules of inference; one rule for one type of categorical formulas. Each of these rules transforms the negation of a given categorical formula into a logically contradictory formula (cf. Table 3). So these twelve rules we denote by (RI \neg). Formally, each rule in (RI \neg) is a set of ordered pairs of the form $\langle \varphi, \psi \rangle$, where $\varphi, \psi \in \mathbf{For}$ have a particular form given by the relevant schemes.

The fifth group of rules consists of six rules of inference; one rule for one type of general formulas (cf. Table 4). So these six rules we denote by (RIg). Formally, each rule in (RIg) is a set of ordered pairs of the form $\langle \{\varphi, \sigma\}, \varsigma \rangle$, where $\varphi \in \mathbf{For}$ and $\sigma, \varsigma \in \Sigma$ have a particular form given by the relevant schemes. In this case $\{\varphi, \sigma\}$ plays the role of a set of premises.

The sixth group of rules consists of two rules of inference for auxiliary singular formulas (cf. Table 5). So these two rules we denote by (RIa). Formally, each rule in (RIa) is a set of ordered pairs of the form $\langle \sigma, \varsigma \rangle$, where $\sigma, \varsigma \in \Sigma \setminus \Sigma^\neg$ have a particular form given by the relevant schemes.

The seventh group of rules consists of six rules of choice for particular formulas; one rule for one each of particular formula (cf. Table 6). These six

$$\begin{array}{ccc}
\iota \varepsilon^{\Box} \alpha & & \iota \varepsilon \alpha \\
\downarrow & & \downarrow \\
\iota \varepsilon \alpha & & \iota \varepsilon^{\Diamond} \alpha
\end{array}$$

Table 5. Rules of inference for auxiliary singular formulas (RIa)

For any $\bullet \in \{\text{blank}, \Box, \Diamond\}$ and any new index ι :

$$\begin{array}{ccc}
\mathbf{I}^{\bullet} \alpha \beta & & \mathbf{O}^{\bullet} \alpha \beta \\
\downarrow & & \downarrow \\
\iota \varepsilon \alpha & & \iota \varepsilon \alpha \\
\iota \varepsilon^{\bullet} \beta & & \neg \iota \varepsilon^* \beta
\end{array}$$

where $*$ = blank, \Diamond , \Box , for \bullet = blank, \Box , \Diamond , respectively.

Table 6. Rules (RC) for particular formulas

rules we denote by (RC). Formally, each of rules of (RC) is a triple of the form $\langle \varphi, \langle \sigma, \varsigma \rangle \rangle$, where $\varphi \in \mathbf{For}$ and $\sigma, \varsigma \in \Sigma$ have a specific form given by the relevant schemes. In this case $\langle \sigma, \varsigma \rangle$ plays the role of a conjunction of conclusions.

3.2. Tableaus. Closed Tableaus. Open Tableaus

Let Φ be a non-empty (maybe infinite) subset of \mathbf{For} . A *tableau for Φ* is any NON-EMPTY FINITE TREE of formulas from $\mathbf{For} \cup \Sigma$ which meets all of the following four conditions:

1. The tree T has only one root (i.e., the node at the top) belonging to Φ .
2. From each of the nodes of T at most two edges (arrows) start.
3. If from a given node in T two edges start with formulas ψ_1 and ψ_2 , respectively, then in the initial path from the root to this node there is a node with the formula φ such that the pair $\langle \varphi, \{\psi_1, \psi_2\} \rangle$ belongs to some rule of (BE).
4. For any initial path P in T which has the form $(\varphi_1, \dots, \varphi_n)$ (i.e., φ_1 is the root of T), for any $l \leq n$ at least one of the following conditions is met:
 - (i) $\varphi_l \in \Phi$.
 - (ii) There is a $j < l$ such that $\langle \varphi_j, \varphi_l \rangle$ belongs to some rule of (E \neg), or of (RIa), or of (RN).
 - (iii) There is a $j < l$ such that for some $\psi \in \mathbf{For}$ the pair $\langle \varphi_j, \{\varphi_l, \psi\} \rangle$ belongs to some rule of (BE) and for the initial subpath P' of P which has the form $(\varphi_1, \dots, \varphi_j, \dots, \varphi_{l-1})$ the path (P', ψ) belongs to T .

- (iv) There are $j, k < l$ such that $\langle \{\varphi_j, \varphi_k\}, \varphi_l \rangle$ belongs to some rule of (RIg).
- (v) There are $j < l$ and $\psi \in \mathbf{For} \cup \Sigma$ such that for the initial subpath P' of P which has the form $(\varphi_1, \dots, \varphi_j, \dots, \varphi_l)$ the path (P', ψ) belongs to T and $\langle \varphi_j, \langle \varphi_l, \psi \rangle \rangle$ belongs to some rule either of (NBE) or of (RC), where in the second case the index i from φ_l and ψ does not occur in the initial subpath P'' of P which has the form $(\varphi_1, \dots, \varphi_{l-1})$;
- (vi) There is a $j < l - 1$ such that $\langle \varphi_j, \langle \varphi_{l-1}, \varphi_l \rangle \rangle$ belongs to some rule either of (NBE) or of (RC), where in the second case the index i from φ_{l-1} and φ_l does not occur in the initial subpath P' of P which has the form $(\varphi_1, \dots, \varphi_{l-2})$.

Clearly, in a tableau for some Φ only a finite subset of formulas in Φ that occur in the tableau forms a proof. So any tableau for Φ is also a tableau for each set of formulas which is a superset of Φ .

Any path from a root to a bottom is called a *branch*. A branch in a given tableau is *closed* iff there are formulas from $\mathbf{For} \cup \Sigma$ of the form φ and $\neg \varphi$ on two of its nodes; otherwise it is *open*. A tableau is *closed* iff every its branch is closed; otherwise is *open*.

4. Tableau Proofs. Soundness

For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$, we say that φ has a *tableau proof* from Ψ (we write: $\Psi \vdash \varphi$) iff there is a closed tableau for the set $\Psi \cup \{\neg \varphi\}$.

We prove that the relation \vdash is sound, i.e.:

THEOREM 4.1 (Soundness). *For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$:*

$$\text{if } \Psi \vdash \varphi \text{ then } \Psi \models \varphi.$$

PROOF. Assume that $\Psi \vdash \varphi$, i.e., there is a closed tableau T for $\Psi \cup \{\neg \varphi\}$. Suppose towards a contradiction that $\Psi \not\models \varphi$. Then there is a non-empty model $\mathfrak{M} = \langle D, d, d^\square, d^\diamond \rangle \in \mathbf{Mod}$ such that $\mathfrak{M} \models \Psi$, but $\mathfrak{M} \not\models \varphi$; so $\mathfrak{M} \models \Psi \cup \{\neg \varphi\}$.

Note that all tableau rules of the types (NBE) and (RN) preserve truth and moreover at least one conclusion derived from the application of a rule of type (BE) preserves truth. Hence for some closed branch B in T all formulas from \mathbf{For} which are in B are true in \mathfrak{M} , because $\mathfrak{M} \models \Psi \cup \{\neg \varphi\}$. Let \mathbf{For}_B be the set of all formulas from \mathbf{For} which are in B . So $\mathfrak{M} \models \mathbf{For}_B$.

If in branch B no rules of (RC) were used, then in B there would be only formulas from \mathbf{For}_B . But this contradicts the fact that B is closed. So we

suppose that on B some rule of (RC) is used at least once. Let Σ_B be the set of all auxiliary formulas on B which we obtain by rules of (RC) and let I_B be the set of all indexes which appear in the formulas from Σ_B . We have $I_B \neq \emptyset$. Since rules from the group (RC) were used only to formulas from \mathbf{For}_B and $\mathfrak{M} \models \mathbf{For}_B$, there is a valuation $\nu: I_B \rightarrow D$ such that $\mathfrak{M}, \nu \models \Sigma_B$. Since all other tableau rules preserve truth for \mathfrak{M} and ν , so by induction on the length of B , we show that all formulas on B are also true in \mathfrak{M} . But this contradicts the fact that B is closed. ■

5. Completeness

Since the logical consequence relation \models is compact, we can focus on tableaux for non-empty finite sets. Let Φ be any non-empty finite subset of \mathbf{For} .

In the light of Claim 1.3 we see that the rules of (RC) do not have to be applied to all p-formulas in a given branch of a given tableau. It is enough to apply the rules to the so-called *essential p-formulas in a given branch*. For any tableau T and any branch B of T a p-formula φ is *essential in B* iff there is no p-formula ψ on B such that $\psi \models \varphi$ and $\varphi \not\models \psi$.

Now we introduce a special kind of tableaux, called *mini-max*. They are maximal since all relevant expressions are decomposed, but minimal since only relevant expressions are decomposed.

We say that a tableau T for Φ is “*mini-max*” iff T satisfies the following conditions:

1. T begins with a single path in which there occur all members from Φ (we will call this path the *initial list*).
2. In each branch B of T , any rule that could be applied to a formula was applied exactly once, with the exception of the following cases:
 - rules of (RC) were applied only to essential p-formulas,
 - rules of (RC) were applied only to one of two equivalent formulas (see points 3 and 4 of Claim 1.3),
 - in B we have a pair of formulas χ and $\neg \chi$.

Of course, for any non-empty finite subset of \mathbf{For} there is a “mini-max” tableau. Note that formulas of the form $\neg \Box \alpha$ and $\neg \Box \alpha \alpha$ cannot appear in an open branch in “mini-max” tableaux (cf. Claim 1.3.6 and the rules of (RC) and (RIa)). All “mini-max” tableaux are *complete* in the standard sense [see, e.g., 13, p. 9].

In Sections 5–8 for a given branch B , let p_e^B be the number of all essential non-equivalent p-formulas in B . For the Completeness Theorem we use the following Completeness Lemma:

LEMMA 5.1. *For any non-empty finite subset Φ of **For** and any “mini-max” open tableau T for Φ , there is a model $\mathfrak{M}_T \in \mathbf{Mod}$ such that $\mathfrak{M}_T \models \Phi$ and the cardinality of \mathfrak{M}_T is equal to $\min\{p_e^B : B \text{ is an open branch in } T\}$.*

PROOF. Let B be an open branch in T such that B has the smallest number of essential formulas that are not equivalent to p-formulas in B among all open branches in T . By definition of T , all formulas from Φ are on B . We construct the following model $\mathfrak{M}_T = \langle D, d, d^\square, d^\diamond \rangle$ from \mathbf{Mod} . First, we put:

$$D = \{\iota \in \mathbf{AI} : \exists \alpha \in \mathbf{GL} \, \ulcorner \iota \varepsilon \alpha \urcorner \text{ is in } B \urcorner\}.$$

Because T is “mini-max”, we have: $D = \emptyset$ iff there is no p-formula in B iff $p_e^B = 0$. Moreover, $\text{Card}(D) = p_e^B$ and for any $\iota \in \mathbf{AI}$ we have:

$$\iota \in D \quad \text{iff} \quad \exists \alpha \in \mathbf{GL} \, \ulcorner \iota \varepsilon^\bullet \alpha \urcorner \text{ is in } B, \text{ for some } \bullet \in \{\text{blank}, \square, \diamond\}.$$

Second, for all $\alpha \in \mathbf{GL}$ and $\bullet \in \{\text{blank}, \square, \diamond\}$ we put:

$$d^\bullet(\alpha) := \{\iota \in D : \ulcorner \iota \varepsilon^\bullet \alpha \urcorner \text{ is in } B\}.$$

Now we show that \mathfrak{M}_T is a model from \mathbf{Mod} , i.e., for any α in \mathbf{GL} : $d^\square(\alpha) \subseteq d(\alpha) \subseteq d^\diamond(\alpha)$. Indeed, if $\iota \in d^\square(\alpha)$, then $\ulcorner \iota \varepsilon^\square \alpha \urcorner$ is in B . Hence, by some rule of (RIa), also $\ulcorner \iota \varepsilon \alpha \urcorner$ is in B , since T is “mini-max”. So $\iota \in d(\alpha)$. Similarly we show that $d(\alpha) \subseteq d^\diamond(\alpha)$.

Now we prove that for any $\varphi \in \mathbf{For}$ we obtain: if φ is in B , then $\mathfrak{M}_T \models \varphi$. The proof is by induction on the complexity of formulas.

(a) We consider the case of formulas of $\mathbf{For}_{\text{syll}}$.

Suppose that for some $\bullet \in \{\text{blank}, \square, \diamond\}$, $\ulcorner \mathbf{I}^\bullet \alpha \beta \urcorner$ is in B . Then, if $\ulcorner \mathbf{I}^\bullet \alpha \beta \urcorner$ is essential in B , then by some rule of (RC), for some $\iota \in \mathbf{AI}$ we have: $\ulcorner \iota \varepsilon \alpha \urcorner$ and $\ulcorner \iota \varepsilon^\bullet \beta \urcorner$ are in B .⁷ If $\ulcorner \mathbf{I}^\bullet \alpha \beta \urcorner$ is not essential in B then $\bullet \neq \square$. So if $\bullet = \text{blank}$ then $\ulcorner \mathbf{I}^\square \alpha \beta \urcorner$ is in B . Hence, by some rule of (RC), for some $\iota \in \mathbf{AI}$ we have: $\ulcorner \iota \varepsilon \alpha \urcorner$ and $\ulcorner \iota \varepsilon^\square \beta \urcorner$ are in B . So, by some rule of (RIa), $\ulcorner \iota \varepsilon \beta \urcorner$ is also in B . Moreover, if $\bullet = \diamond$ then $\ulcorner \mathbf{I}^* \alpha \beta \urcorner$ is in B , for some $*$ $\in \{\text{blank}, \square\}$. Hence, by some rule of (RC), for some $\iota \in \mathbf{AI}$ we have: $\ulcorner \iota \varepsilon \alpha \urcorner$ and $\ulcorner \iota \varepsilon^* \beta \urcorner$ are in B . So, by some rule of (RIa), $\ulcorner \iota \varepsilon^\diamond \beta \urcorner$ is also in B . Thus, in all cases, $\iota \in d(\alpha) \cap d^\bullet(\beta)$; so $\mathfrak{M}_T \models \mathbf{I}^\bullet \alpha \beta$.

Suppose that $\ulcorner \mathbf{O}^\bullet \alpha \beta \urcorner$ is in B , for some $\bullet \in \{\text{blank}, \square, \diamond\}$. Then, if $\ulcorner \mathbf{O}^\bullet \alpha \beta \urcorner$ is essential in B , then by some rule of (RC), for some $\iota \in \mathbf{AI}$ we have: $\ulcorner \iota \varepsilon \alpha \urcorner$ and $\ulcorner \neg \iota \varepsilon^* \beta \urcorner$ are in B , where $*$ $= \text{blank}, \diamond, \square$, for $\bullet = \text{blank}, \square, \diamond$,

⁷Sometimes the effect is because some rule of (RC) was applied to a formula equivalent to $\ulcorner \mathbf{I}^\bullet \alpha \beta \urcorner$, since B is a branch of a “mini-max”. For example, $\bullet = \text{blank}$ and rule of (RC) was applied to $\ulcorner \mathbf{I} \beta \alpha \urcorner$, or $\alpha = \beta$ and $\bullet \in \{\text{blank}, \diamond\}$, but rule of (RC) was applied either to $\ulcorner \mathbf{I} \alpha \alpha \urcorner$ or to $\ulcorner \mathbf{I}^\diamond \alpha \alpha \urcorner$.

respectively. Hence $\iota \in d(\alpha)$ and $\lceil \iota \varepsilon^* \beta \rceil$ is not in B , since B is open. If $\lceil O^\bullet \alpha \beta \rceil$ is not essential in B then $\bullet \neq \square$. So if $\bullet = \text{blank}$ then $\lceil O^\square \alpha \beta \rceil$ is in B . Therefore, by some rule of (RC), for some $\iota \in \mathbf{AI}$ we have: $\lceil \iota \varepsilon \alpha \rceil$ and $\lceil \neg \iota \varepsilon^\diamond \beta \rceil$ are in B . Hence $\iota \in d(\alpha)$ and $\lceil \iota \varepsilon^\diamond \beta \rceil$ is not in B , since B is open. So, by some rule of (RIa), also $\lceil \iota \varepsilon \beta \rceil$ is not in B . Moreover, if $\bullet = \diamond$ then $\lceil O^* \alpha \beta \rceil$ is in B , for some $* \in \{\text{blank}, \square\}$. Therefore, by some rule of (RC), for some $\iota \in \mathbf{AI}$ we have: $\lceil \iota \varepsilon \alpha \rceil$ and $\lceil \neg \iota \varepsilon^* \beta \rceil$ are in B , where $\star = \text{blank}, \diamond$, for $\star = \text{blank}, \square$, respectively. Hence $\iota \in d(\alpha)$ and $\lceil \iota \varepsilon^* \beta \rceil$ is not in B , since B is open. So, by some rule of (RIa), also $\lceil \iota \varepsilon^\diamond \beta \rceil$ is not in B . Thus, in all cases, $\iota \notin d^*(\beta)$, where $\star = \text{blank}, \diamond, \square$, for $\bullet = \text{blank}, \square, \diamond$, respectively. Therefore $d(\alpha) \not\subseteq d^*(\beta)$, i.e., $\mathfrak{M}_T \models O^\bullet \alpha \beta$.

Suppose that $\lceil A^\bullet \alpha \beta \rceil$ is in B , for some $\bullet \in \{\text{blank}, \square, \diamond\}$. If $\iota \in d(\alpha)$ then $\lceil \iota \varepsilon \alpha \rceil$ is in B . So also $\lceil \iota \varepsilon^\bullet \beta \rceil$ is in B , by some rule of (RIg). Hence $\iota \in d^*(\beta)$. Thus, $d(\alpha) \subseteq d^*(\beta)$, i.e., $\mathfrak{M}_T \models A^\bullet \alpha \beta$.

Suppose that $\lceil E^\bullet \alpha \beta \rceil$ is in B , for some $\bullet \in \{\text{blank}, \square, \diamond\}$. If $\iota \in d(\alpha)$ then $\lceil \iota \varepsilon \alpha \rceil$ is in B . Hence, by some rule of (RIg), $\lceil \neg \iota \varepsilon^* \beta \rceil$ is in B , where $\star = \text{blank}, \diamond, \square$, for $\bullet = \text{blank}, \square, \diamond$, respectively. So $\lceil \iota \varepsilon^* \beta \rceil$ is not in B , since B is open. Hence $\iota \notin d^*(\beta)$. Thus, $d(\alpha) \cap d^*(\beta) = \emptyset$, i.e., $\mathfrak{M}_T \models E^\bullet \alpha \beta$.

(b) Now we consider the case of negation of formulas from **For**_{syll}. Suppose that $\lceil \neg I^\bullet \alpha \beta \rceil$ (resp. $\lceil \neg O^\bullet \alpha \beta \rceil$, $\lceil \neg A^\bullet \alpha \beta \rceil$, $\lceil \neg E^\bullet \alpha \beta \rceil$) is in B , for some $\bullet \in \{\text{blank}, \square, \diamond\}$. Then, by some rule of (RN), $\lceil E^* \alpha \beta \rceil$ (resp. $\lceil A^* \alpha \beta \rceil$, $\lceil O^* \alpha \beta \rceil$, $\lceil I^* \alpha \beta \rceil$) is in B , where $\star = \text{blank}, \diamond, \square$, for $\bullet = \text{blank}, \square, \diamond$, respectively. So, by (a), $\mathfrak{M}_T \models E^* \alpha \beta$ (resp. $\mathfrak{M}_T \models A^* \alpha \beta$, $\mathfrak{M}_T \models O^* \alpha \beta$, $\mathfrak{M}_T \models I^* \alpha \beta$). Hence $\mathfrak{M}_T \models \neg I^\bullet \alpha \beta$ (resp. $\mathfrak{M}_T \models \neg O^\bullet \alpha \beta$, $\mathfrak{M}_T \models \neg A^\bullet \alpha \beta$, $\mathfrak{M}_T \models \neg E^\bullet \alpha \beta$).

(c) For other complex formulas we naturally consider inductive steps. We use the elimination rules and the fact that T is “mini-max”.

Since all formulas from Φ are in B , so $\mathfrak{M}_T \models \Phi$. ■

By Lemma 5.1 and Theorem 1.5 (compactness of the relation \models) we obtain that the tableau approach is complete.

THEOREM 5.2 (Completeness). *For all $\Psi \subseteq \mathbf{For}$ and $\varphi \in \mathbf{For}$:*

$$\text{if } \Psi \models \varphi, \text{ then } \Psi \vdash \varphi.$$

PROOF. Let $\Psi \models \varphi$. Then, by the compactness of \models , there is a finite subset Ψ_{fin} of Ψ such that $\Psi_{\text{fin}} \models \varphi$. Now we suppose towards contradiction that $\Psi_{\text{fin}} \not\models \varphi$. Then there is a “mini-max” open tableau T for $\Psi_{\text{fin}} \cup \{\neg \varphi\}$. Hence, by Lemma 5.1, there is a model $\mathfrak{M}_T \in \mathbf{Mod}$ such that $\mathfrak{M}_T \models \Psi_{\text{fin}} \cup \{\neg \varphi\}$. So we obtain a contradiction: $\Psi_{\text{fin}} \not\models \varphi$. Thus, $\Psi_{\text{fin}} \vdash \varphi$. So also $\Psi \vdash \varphi$. ■

6. Decidability and Cardinalities of Models

The decidability of $\Psi \models \varphi$ for the pure modal logic of names is considered in the case when Ψ is finite. Thus, let $\Psi \in \mathcal{P}_{\text{fin}}(\mathbf{For})$. The decidability of $\Psi \models \varphi$ can be tested in the following two ways.

First, we have already established decidability on the basis of Lemma 1.4, since the relation \models_{CQL} is decidable. However, we dare for more! We want to estimate the cardinalities of models for the logic under research that are less than the cardinalities of models for CQL. It is known that, for $\Psi \in \mathcal{P}(\mathbf{For}_{\mathcal{L}})$ and $\varphi \in \mathbf{For}_{\mathcal{L}}$, in order to check whether $\Psi \models_{\text{CQL}} \varphi$ we have to examine models with a cardinality not bigger than 2^k , where k is the number of monadic predicates of $\mathbf{P}_{\mathcal{L}}$ that occur in the formulas from the set $\Psi \cup \{\varphi\}$. Let us note that we mean not only letters from \mathbf{GL} , but also all instances of the form α^\bullet , where $\alpha \in \mathbf{GL}$ and $\bullet \in \{\text{blank}, \square, \diamond\}$.

Either by Lemma 1.4 and the decidability of monadic first-order logic, or by Lemma 5.1 and Theorem 4.1 we obtain:

THEOREM 6.1 (Decidability). *For all $\Psi \in \mathcal{P}_{\text{fin}}(\mathbf{For})$ and $\varphi \in \mathbf{For}$, the problem whether $\Psi \models \varphi$ is decidable.*

PROOF. For $\Psi \cup \{\neg\varphi\}$ we build some “mini-max” tableau. If the tableau is closed, then $\Psi \vdash \varphi$. So $\Psi \models \varphi$, by soundness (Theorem 4.1). If the tableau is open, then $\Psi \not\models \varphi$, by Lemma 5.1. ■

As we declared, we would like to estimate the cardinality of a given model to check if $\Psi \models \varphi$.⁸ First, we obtain the following theorem for formulas that have a disjunctive normal form. Note that each formula χ from \mathbf{For} can be written in a disjunctive normal form χ^d . Moreover, let $\Psi^d := \{\psi^d : \psi \in \Psi\}$.

THEOREM 6.2. *For all $\Psi \in \mathcal{P}_{\text{fin}}(\mathbf{For})$ and $\varphi \in \mathbf{For}$ the following conditions are equivalent:*

- (a) $\Psi \not\models \varphi$.
- (b) $\Psi^d \not\models \varphi^d$.
- (c) *For some mini-max open tableau T for $\Psi^d \cup \{(\neg\varphi)^d\}$ there is a model $\mathfrak{M}_T \in \mathbf{Mod}$ such that $\mathfrak{M}_T \models \Psi^d \cup \{(\neg\varphi)^d\}$ and the cardinality of \mathfrak{M}_T is equal to $\min\{p_e^B : B \text{ is an open branch in } T\}$.*
- (d) $\Psi^d \not\models \varphi^d$.

⁸In [3], we carried out a similar examination for modal syllogisms. There, we also gave the length of tableaux for modal syllogisms. But those tableaux were not branching. Here, we are not concerned with the length of tableaux.

(e) $\Psi \not\models \varphi$.

PROOF. “(a) \Leftrightarrow (e)” and “(b) \Leftrightarrow (d)” By Theorems 4.1 and 5.2.

“(b) \Rightarrow (c)” Let $\Psi^d \not\models \varphi^d$. Then all tableaux for $\Psi^d \cup \{(\neg\varphi)^d\}$ are open and there exists a “mini-max” open tableau for $\Psi^d \cup \{(\neg\varphi)^d\}$. So we use Lemma 5.1. “(c) \Rightarrow (d)” By definition. “(d) \Leftrightarrow (e)” This case is obvious. ■

Now we assume the following notations. For any formula χ , let k_χ be the number of conjunctions occurring in χ^d . For any i from $\{1, \dots, k_\chi\}$, let C_χ^i be the set of all non-negated p-formulas and all negated g-formulas that occur in the i -th conjunction in χ^d .⁹

We put the members of Ψ as a sequence (ψ_1, \dots, ψ_n) (for $\Psi = \emptyset$ we put $n := 0$). Let Λ be the set of all sequences $(l_1, \dots, l_n, l_{n+1})$ of numbers such that $1 \leq l_i \leq k_{\psi_i}$, for $i = 1, \dots, n$, and $1 \leq l_{n+1} \leq k_{\neg\varphi}$. Then for any $\lambda = (l_1, \dots, l_n, l_{n+1}) \in \Lambda$ we define the following set:

$$B^\lambda := C_{\psi_1}^{l_1} \cup \dots \cup C_{\psi_n}^{l_n} \cup C_{\neg\varphi}^{l_{n+1}}.$$

Clearly, any set B^λ , for $\lambda \in \Lambda$, corresponds to a branch in a tableau build from $\Psi^d \cup \{(\neg\varphi)^d\}$. For any $\lambda \in \Lambda$ we define the essential p-formulas in the set B^λ , as for a given branch of a given tableau (see p. 13). Let $p_e^{B^\lambda}$ be the number of all essential, non-equivalent p-formulas in B^λ .

Once again let us note that formulas having one of the forms $\lceil O^\square \alpha \alpha \rceil$, $\lceil O \alpha \alpha \rceil$, $\lceil \neg A^\diamond \alpha \alpha \rceil$, and $\lceil \neg A \alpha \alpha \rceil$ cannot appear in any open branch in “mini-max” tableaux. So let Λ^* be the set of all $\lambda \in \Lambda$ such that no such formula is a member of set B^λ . We put $\text{Max}(\Psi, \varphi) := \max\{p_e^{B^\lambda} : \lambda \in \Lambda^*\}$.¹⁰ From Theorem 6.2 we obtain:

THEOREM 6.3. *For all $\Psi \in \mathcal{P}_{\text{fin}}(\mathbf{For})$ and $\varphi \in \mathbf{For}$:*

$\Psi \models \varphi$ iff there is no $\mathfrak{M} \in \mathbf{Mod}$ such that $\mathfrak{M} \models \Psi \cup \{\neg\varphi\}$ and the power of \mathfrak{M} is not greater than $\text{Max}(\Psi, \varphi)$.

PROOF. “ \Rightarrow ” Obvious. “ \Leftarrow ” Suppose that $\Psi \not\models \varphi$. Then, by Theorem 6.2, for some mini-max open tableau T for $\Psi^d \cup \{(\neg\varphi)^d\}$ there is a model \mathfrak{M}_T from \mathbf{Mod} such that $\mathfrak{M}_T \models \Psi^d \cup \{(\neg\varphi)^d\}$ and the cardinality of \mathfrak{M}_T is equal to $\min\{p_e^B : B \text{ is an open branch in } T\}$. Hence the cardinality of \mathfrak{M}_T is not greater than $\text{Max}(\Psi, \varphi)$. Of course, $\mathfrak{M}_T \models \Psi \cup \{\neg\varphi\}$, too. ■

⁹We apply of course rules from (RN) to negated g-formulas and receive corresponding non-negated p-formulas.

¹⁰Note that $\text{Max}(\Psi, \varphi)$ does not depend on the enumerated members of set Ψ and moreover $\text{Max}(\Psi, \varphi) = \text{Max}(\emptyset, \bigwedge \Psi \rightarrow \varphi)$.

Let $\mathbf{S} = \langle \Psi, \varphi \rangle$ be such a syllogism, where $\Psi \in \mathcal{P}_{\text{fin}}(\mathbf{For}_{\text{syll}}) \setminus \{\emptyset\}$ and $\varphi \in \mathbf{For}_{\text{syll}}$ (see p. 3). Then the set Λ has one element and any tableau for $\Psi \cup \{\neg\varphi\}$ has got only one branch. Of course, if either some formula of the form $\ulcorner O^\square \alpha \alpha \urcorner$ or $\ulcorner O \alpha \alpha \urcorner$ belongs to Ψ , or φ has the form $\ulcorner A^\diamond \alpha \alpha \urcorner$ or $\ulcorner A \alpha \alpha \urcorner$, then $\Psi \models \varphi$. In the other case we obtain that an upper estimation of the cardinality of models is the number $p_e^{\mathbf{S}}$ of essential, but non-equivalent p-formulas in $\Psi \cup \{\bar{\varphi}\}$, where $\bar{\varphi}$ is the formula of $\mathbf{For}_{\text{syll}}$ which we obtain from $\ulcorner \neg\varphi \urcorner$, by the suitable rule of (RN).

Of course, if $\mathbf{S} = \langle \Psi, \varphi \rangle$ is non-modal then, in the second case, $p_e^{\mathbf{S}}$ is the number of non-modal particular non-equivalent formulas in the set $\Psi \cup \{\bar{\varphi}\}$. In [2, 3] some larger estimations were made. An upper estimation of the cardinality of models was the number of non-modal particular formulas in $\Psi \cup \{\bar{\varphi}\}$. However, in those papers two various approaches to the presented problem were outlined.

In [9, 11] the cardinalities of models for non-modal pure logic of names were also examined. For any non-modal formula φ from \mathbf{For} the upper estimation of cardinalities of its models is $\frac{1}{2}n(n+1)$, where n is the number of letters from \mathbf{GL} occurring in φ . It is difficult to compare that number with number $\text{Max}(\emptyset, \varphi)$ as it does not make sense to try to count the number of non-modal particular non-equivalent formulas built with letters occurring in φ .

Moreover, even for the non-modal pure logic of names, it would be a mistake to compare our results with the results presented by Kulicki [5, 6]. First, Kulicki used a so-called *strong interpretation* for formulas of the form $\ulcorner A \alpha \beta \urcorner$: $\mathfrak{M} \models A \alpha \beta$ iff $\emptyset \neq d(\alpha) \subseteq d(\beta)$. So we have: $A \alpha \beta \models I \alpha \beta$. Second, Kulicki examined the cardinalities of models only for Horn formulas which have the form $\ulcorner (\chi_1 \wedge \dots \wedge \chi_n) \rightarrow \chi_{n+1} \urcorner$, where χ_i is $\ulcorner A \alpha \beta \urcorner$ or $\ulcorner I \alpha' \beta' \urcorner$, for some $\alpha, \beta, \alpha', \beta' \in \mathbf{GL}$. These Horn formulas correspond only to syllogisms that are limited to those two kind of categorical sentences.

7. Tableau Proofs with Other Version of Tableau Contradiction

To make tableau proofs shorter we may use an extended version of the notion of *tableau contradiction* and apply it to our earlier notions of the closed tableau and the tableau proof. Hence, to the pairs of tableau contradictory formulas we defined earlier we add all pairs of logically contradictory syllogistic formulas from $\mathbf{For}_{\text{syll}}$ (see p. 5). So, a branch in a given tableau is closed, if at some of its node there is a formula either of the form $\ulcorner O \alpha \alpha \urcorner$

$Iab, Aab \vee Eab \vdash Aab$ (in both versions):

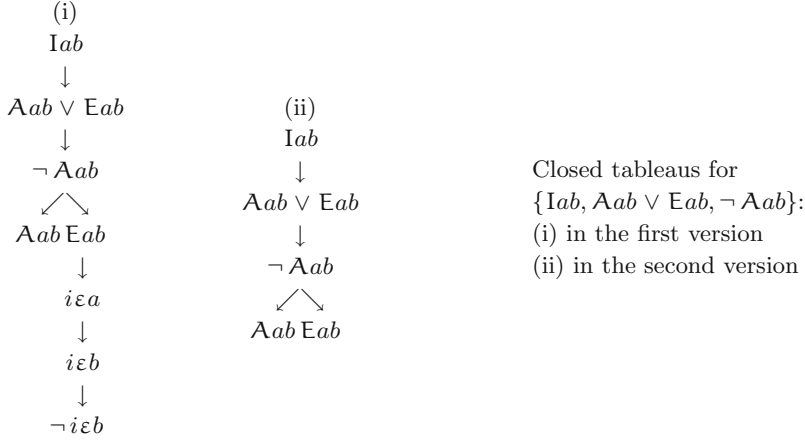


Table 7. Example of proofs with regular and extended notion of tableau contradiction.

or of the from $\lceil O^\square \alpha \alpha \rceil$ (cf. Claim 1.3.6). Due to that we have the following fact:

PROPOSITION 7.1. $\Psi \vdash \varphi$ in the first version iff $\Psi \vdash \varphi$ in the second version.

PROOF. “ \Rightarrow ” This case is obvious.

“ \Leftarrow ” Suppose that $\Psi \vdash \varphi$ in the second version and for $\Psi \cup \{\neg\varphi\}$ there is a closed tableau T such that T has a branch B which is not closed in the first version of tableau contradiction. Then on B we have either $\lceil O\alpha\alpha \rceil$, or $\lceil O^\square \alpha \alpha \rceil$, or both $\lceil A^\bullet \alpha \beta \rceil$ and $\lceil O^* \alpha \beta \rceil$, or both $\lceil E^\bullet \alpha \beta \rceil$ and $\lceil I^* \alpha \beta \rceil$, where $*$ = blank, \diamond , \square for \bullet = blank, \square , \diamond , respectively. Hence, by use of appropriate rules of the types (RC), (RIa), and (RIg) we obtain, in the first case: $\lceil i\epsilon\alpha \rceil$ and $\lceil \neg i\epsilon\alpha \rceil$; in the second case: $\lceil i\epsilon\alpha \rceil$, $\lceil \neg i\epsilon^\diamond\alpha \rceil$, and $\lceil i\epsilon^\diamond\alpha \rceil$; in the third case: $\lceil i\epsilon\alpha \rceil$, $\lceil \neg i\epsilon^\bullet\beta \rceil$, and $\lceil i\epsilon^\bullet\beta \rceil$; in the fourth case: $\lceil i\epsilon\alpha \rceil$, $\lceil i\epsilon^*\beta \rceil$, and $\lceil \neg i\epsilon^*\beta \rceil$. So we obtain a branch B' which is closed in the first version (see the example in Table 7). ■

8. Pure Modal Logic for Both General and Singular Names

So far we have considered the pure modal logic of names only for general names. It includes indexes for auxiliary singular names only in tableau proofs. Now—beside the letters from **GL** representing general names—we would like to use also schematic letters representing «real» singular names.

We shall show that our examination of the logic of general names from the former sections can be successfully repeated for the logic of both kinds of names.

8.1. Formulas

We fix a countably infinite set **SL** which contains schematic letters: n_0, n_1, n_2, \dots (for the first two schematic letters we will use ‘ n ’ and ‘ m ’, respectively). We assume that these letters represent various singular names.

A *singular formula* is any expression which has one of the following forms: $\ulcorner \nu \varepsilon \alpha \urcorner$, $\ulcorner \nu \varepsilon^\square \alpha \urcorner$, and $\ulcorner \nu \varepsilon^\diamond \alpha \urcorner$, where $\nu \in \mathbf{SL}$ and $\alpha \in \mathbf{GL}$. Let **For_{sin}** be the set of all singular formulas.

Now, formulas are made of formulas **For_{syll}** and **For_{sin}** by the use of Boolean connectives and brackets. Let **For⁺** be the smallest set such that

- **For_{syll}** \cup **For_{sin}** \subseteq **For⁺**;
- if $\varphi \in \mathbf{For}^+$, then $\ulcorner \neg \varphi \urcorner \in \mathbf{For}^+$;
- if $\varphi, \psi \in \mathbf{For}^+$, then $\ulcorner (\varphi \circ \psi) \urcorner \in \mathbf{For}^+$, where $\circ \in \{\wedge, \vee, \supset, \equiv\}$.

8.2. Semantics

Models and the notion of truth for For⁺. A *model for For⁺* we call any quintuple $\langle D, d^s, d, d^\square, d^\diamond \rangle$, where D is a non-empty set, d^s is a function from **SL** into D , and the quadruple $\langle D, d, d^\square, d^\diamond \rangle$ is a non-empty model from **Mod** for **For** (see p. 4). Let **Mod⁺** be the class of all models for **For⁺**.

The notion of *truth* for **For_{sin}** is similar to that which we used for **Σ** (here denotation function d^s replaces valuation v). We use the following interpretation of formulas from **For⁺** in all models $\mathfrak{M} = \langle D, d^s, d, d^\square, d^\diamond \rangle \in \mathbf{Mod}^+$, $\nu \in \mathbf{SL}$, $\alpha \in \mathbf{GL}$, and $\bullet \in \{\text{blank}, \square, \diamond\}$:

$$\mathfrak{M} \models \nu \varepsilon^\bullet \alpha \text{ iff } d^s(\nu) \in d^\bullet(\alpha).$$

The remaining conditions have been left unchanged (the are on p. 4).

Logical entailment, equivalence, and contradiction. These three notions are introduced for **For⁺** in the same way as for **For** on p. 5. We obtain:

CLAIM 8.1. *For all $\alpha, \beta \in \mathbf{GL}$ and $\nu \in \mathbf{SL}$:*

1. $A^\bullet \alpha \beta, \nu \varepsilon \alpha \models \nu \varepsilon^\bullet \beta$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
2. $E^\bullet \alpha \beta, \nu \varepsilon \alpha \models \neg \nu \varepsilon^\bullet \beta$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
3. $\nu \varepsilon^\square \alpha \models \nu \varepsilon \alpha \models \nu \varepsilon^\diamond \alpha$.
4. $\nu \varepsilon \alpha, \nu \varepsilon^\bullet \beta \models I^\bullet \alpha \beta$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.

5. $\nu\epsilon\alpha, \neg\nu\epsilon^\bullet\beta \models \mathbf{O}^*\alpha\beta$, where $*$ = blank, \Diamond , \Box , for \bullet = blank, \Box , \Diamond , respectively.

Compactness of logical entailment. We obtain this result in a way similar to how obtained it for **For** in Section 1.3. However, we use a monadic first-order logic with individual constants [see, e.g., 12].

Now we use the monadic first-order language \mathcal{L}^+ with monadic predicates from $\mathbf{P}_{\mathcal{L}}$ and individual constants from \mathbf{SL} . So we have also atomic formulas of the form $\ulcorner \pi\nu \urcorner$, where $\pi \in \mathbf{P}_{\mathcal{L}}$ and $\nu \in \mathbf{SL}$. Let $\mathbf{For}_{\mathcal{L}^+}$ be the set of all formulas of \mathcal{L}^+ which we obtain in a standard way.

The transformation \mathbf{t} from p. 6 is enriched with the following condition for all $\nu \in \mathbf{SL}$ and $\alpha \in \mathbf{GL}$:

$$\mathbf{t}(\nu\epsilon^\bullet\alpha) := \ulcorner \alpha^\bullet\nu \urcorner, \text{ for any } \bullet \in \{\text{blank}, \Box, \Diamond\}.$$

Ordinarily, a *model* for \mathcal{L}^+ is any pair $\mathfrak{M}_{\mathcal{L}^+} = \langle D, \mathfrak{I} \rangle$, where D is a non-empty set and \mathfrak{I} is a function from \mathbf{SL} into D and from $\mathbf{P}_{\mathcal{L}}$ into $\mathcal{P}(D)$.

We obtain the relevant counterpart of Lemma 1.4. Hence, by the compactness of Classical Quantified Logic, we also obtain the relevant version of Theorem 1.5, which is that the relation \models for \mathbf{For}^+ is also compact.

Soundness, completeness and decidability. We use the same notion of tableau and the same tableau rules as in Section 3. The only difference is that now in the description of the rules a variable ' i ' takes values from the union of sets **AI** and **SL**, whereas in the former case it only took values from the set **AI**. Using the above rules we obtain the following facts that correspond to those in Claim 8.1:

CLAIM 8.2. *For all $\alpha, \beta \in \mathbf{GL}$ and $\nu \in \mathbf{SL}$:*

1. $\mathbf{A}^\bullet\alpha\beta, \nu\epsilon\alpha \vdash \nu\epsilon^\bullet\beta$, for any $\bullet \in \{\text{blank}, \Box, \Diamond\}$.
2. $\mathbf{E}^\bullet\alpha\beta, \nu\epsilon\alpha \vdash \neg\nu\epsilon^\bullet\beta$, for any $\bullet \in \{\text{blank}, \Box, \Diamond\}$.
3. $\nu\epsilon^\Box\alpha \vdash \nu\epsilon\alpha \vdash \nu\epsilon^\Diamond\alpha$.
4. $\nu\epsilon\alpha, \nu\epsilon^\bullet\beta \vdash \mathbf{I}^\bullet\alpha\beta$, for any $\bullet \in \{\text{blank}, \Box, \Diamond\}$.
5. $\nu\epsilon\alpha, \neg\nu\epsilon^\bullet\beta \vdash \mathbf{O}^*\alpha\beta$, where $*$ = blank, \Diamond , \Box , for \bullet = blank, \Box , \Diamond , respectively.

PROOF. 4. $\nu\epsilon\alpha, \nu\epsilon^\bullet\beta, \neg\mathbf{I}^\bullet\alpha\beta \xrightarrow{\text{RN}} \mathbf{E}^*\alpha\beta \xrightarrow{\text{RI}_1} \neg\nu\epsilon^\bullet\beta \times$

5. $\nu\epsilon\alpha, \neg\nu\epsilon^\bullet\beta, \neg\mathbf{O}^*\alpha\beta \xrightarrow{\text{RN}} \mathbf{A}^\bullet\alpha\beta \xrightarrow{\text{RI}_1} \nu\epsilon^\bullet\beta \times$ ■

Similarly, we obtain a suitable counterpart of Lemma 5.1, where for any branch B , l^B is the number of letters from **SL** occurring in singular formulas on B :

LEMMA 8.3. *For any non-empty finite subset Φ of **For**⁺ and any “mini-max” open tableau T for Φ , there is a model $\mathfrak{M}_T \in \text{Mod}^+$ such that $\mathfrak{M}_T \models \Phi$ and its cardinality is equal to $\min\{p_e^B + l^B + 1 : B \text{ is an open branch in } T\}$.*

PROOF. Let B be an open branch in T such that B has the smallest number p_e^B among all open branches in T . By definition of T , all formulas of Φ are on B . We construct the following model $\mathfrak{M}_T = \langle D, d^s, d, d^\square, d^\diamond \rangle$ from Mod^+ .

If $p_e^B + l^B = 0$, then $D := \{0\}$ and for all $\bullet \in \{\text{blank}, \square, \diamond\}$, $\alpha \in \mathbf{GL}$, and $\nu \in \mathbf{SL}$ we put: $d^\bullet(\alpha) := \emptyset$ and $d^s(\nu) := 0$.

If $p_e^B + l^B > 0$, then we put:

$$D := \{0\} \cup \{\iota \in \mathbf{AI} : \exists \alpha \in \mathbf{GL} \text{ } \ulcorner \iota \varepsilon \alpha \urcorner \text{ is on } B\} \cup \\ \{\nu \in \mathbf{SL} : \exists \alpha \in \mathbf{GL} \text{ } \ulcorner \nu \varepsilon^\bullet \alpha \urcorner \text{ is on } B, \text{ for some } \bullet \in \{\text{blank}, \square, \diamond\}\}.$$

Because T is “mini-max”, $\text{Card}(D) = p_e^B + l^B + 1$ and for any $\iota \in \mathbf{SL} \cup \mathbf{AI}$ we have:

$$\iota \in D \quad \text{iff} \quad \exists \alpha \in \mathbf{GL} \text{ } \ulcorner \iota \varepsilon^\bullet \alpha \urcorner \text{ is on } B, \text{ for some } \bullet \in \{\text{blank}, \square, \diamond\}.$$

Moreover, for all $\alpha \in \mathbf{GL}$ and $\bullet \in \{\text{blank}, \square, \diamond\}$ we put:

$$d^\bullet(\alpha) := \{\iota \in \mathbf{SL} \cup \mathbf{AI} : \ulcorner \iota \varepsilon^\bullet \alpha \urcorner \text{ is on } B\}.$$

So $0 \notin d^\bullet(\alpha)$. Moreover, for any $\nu \in \mathbf{SL}$ we put:

$$d^s(\nu) := \begin{cases} \nu & \text{if } \nu \in D \\ 0 & \text{if } \nu \notin D \end{cases}$$

$\mathfrak{M}_T := \{D, d^s, d, d^\square, d^\diamond\}$ belongs to Mod^+ , because it satisfies condition (\star) , as shown in the proof of Lemma 5.1.

Now we show that for any $\varphi \in \mathbf{For}^+$ we obtain: if φ is on B , then $\mathfrak{M}_T \models \varphi$. The proof is by induction on the complexity of formulas.

Suppose that for some $\bullet \in \{\text{blank}, \square, \diamond\}$, $\ulcorner \nu \varepsilon^\bullet \alpha \urcorner$ is on B . Then $\nu \in D$, $d^s(\nu) = \nu$, and $\nu \in d^\bullet(\alpha)$. So $\mathfrak{M}_T \models \nu \varepsilon^\bullet \alpha$.

Suppose that $\ulcorner \neg \nu \varepsilon^\bullet \alpha \urcorner$ is on B , for some $\bullet \in \{\text{blank}, \square, \diamond\}$. Then $\ulcorner \nu \varepsilon^\bullet \alpha \urcorner$ is not on B , since B is open. So $\nu \notin d^\bullet(\alpha)$. Hence in both cases $d^s(\nu) = \nu$ and $d^s(\nu) = 0$ we have $d^s(\nu) \notin d^\bullet(\alpha)$. So $\mathfrak{M}_T \not\models \nu \varepsilon^\bullet \alpha$ and $\mathfrak{M}_T \models \neg \nu \varepsilon^\bullet \alpha$.

The rest is as in the proof of Lemma 5.1. ■

We also have soundness theorem (a counterpart of Theorem 4.1), completeness theorem (a counterpart of Theorem 5.2), and decidability (a counterpart of theorems 6.2 and 6.3). In the last case we replace merely a number p_e^B with a number $p_e^B + l^B + 1$; and also in the calculation of the number $\text{Max}(\Psi, \varphi)$.

Its decidability is a result of the fact that relation \models_{CQL} is decidable. In addition, we would like to estimate cardinalities of models for \mathbf{For}^+ that are less than the cardinalities of models for CQL. It is known that for $\Psi \in \mathcal{P}(\mathbf{For}_{\mathcal{L}^+})$ and $\varphi \in \mathbf{For}_{\mathcal{L}^+}$ to check whether $\Psi \models_{\text{CQL}} \varphi$ it is enough to examine all models whose cardinalities are not greater than 2^k , where k is a number of all monadic predicates from $\mathbf{P}_{\mathcal{L}}$ occurring in the set $\Psi \cup \{\varphi\}$. So the number of individual constants occurring in this set is not essential [see 12, Corollary 2].¹¹

9. Aristotelian Modal Syllogisms

9.1. Semantics

As we mentioned in the introduction, in [4, 14, 15] the aim was to reconstruct Aristotelian modal syllogistic [1], where the pairs of formulas $\text{I}^\square \alpha \beta$ and $\text{I}^\square \beta \alpha$ (and so also $\text{E}^\diamond \alpha \beta$ and $\text{E}^\diamond \beta \alpha$) are equivalent. So in [4, 14, 15] for models of the form $\langle D, d, d^\square, d^\diamond \rangle$, satisfying among others condition (\star) from p. 4, a different interpretation of I^\square and E^\diamond we use:

$$\begin{aligned} \mathfrak{M} \models \text{I}^\square \alpha \beta & \text{ iff } d^\square(\alpha) \cap d^\square(\beta) \neq \emptyset, \\ \mathfrak{M} \models \text{E}^\diamond \alpha \beta & \text{ iff } \mathfrak{M} \not\models \text{I}^\square \alpha \beta \text{ iff } d^\square(\alpha) \cap d^\square(\beta) = \emptyset. \end{aligned}$$

Moreover, for some reasons given in [4, 14, 15] another interpretation of O^\square and so of A^\diamond were used:

$$\begin{aligned} \mathfrak{M} \models \text{O}^\square \alpha \beta & \text{ iff } d^\square(\alpha) \not\subseteq d^\diamond(\beta), \\ \mathfrak{M} \models \text{A}^\diamond \alpha \beta & \text{ iff } \mathfrak{M} \not\models \text{O}^\square \alpha \beta \text{ iff } d^\square(\alpha) \subseteq d^\diamond(\beta). \end{aligned}$$

¹¹The latter becomes important if we consider a monadic predicate logic with equality. Then an examination of whether $\Psi \models_{\text{CQL}} \varphi$ requires only the examination of all models with a cardinality not greater than $l + 2^k \cdot \max\{n, 1\}$, where k is a number of all monadic predicates occurring in the set $\Psi \cup \{\varphi\}$, l (resp. n) is a number of all individual constants (resp. variables) occurring in the non-tautological equations from this set, and the tautological equations have the form $\ulcorner A = A \urcorner$. If in the set $\Psi \cup \{\varphi\}$ there are no variables then the above estimation can be reduced to $\max\{l, 1\} + \min\{2^k - 1, q - \max\{l, 1\}\}$, where q is the number of individual constants occurring in this set [see 12, p. 52].

In [4, 14, 15] various conditions on a denotation for models are imposed. For example, in [14, 15] (\star) is assumed plus for all $\alpha, \beta \in \mathbf{GL}$:

$$d^\square(\alpha) \neq \emptyset, \quad (1^{\text{Ar}})$$

$$\text{if } d(\alpha) \cap d^\diamond(\beta) \neq \emptyset \text{ then } d^\diamond(\alpha) \cap d(\beta) \neq \emptyset, \quad (2^{\text{Ar}})$$

$$\text{if } d(\alpha) \cap d(\beta) \neq \emptyset \text{ then } d(\alpha) \cap d^\square(\beta) \neq \emptyset, \quad (3^{\text{Ar}})$$

$$\text{if } d(\alpha) \subseteq d^\square(\beta) \text{ then } d^\diamond(\alpha) \subseteq d^\diamond(\beta). \quad (4^{\text{Ar}})$$

Let Mod_{Ar} be the class of all models satisfying conditions (\star) and (1^{Ar}) – (4^{Ar}) . For this class we construct in the standard way the relation \models_{Ar} . Using the revised interpretation and the conditions (\star) , (1^{Ar}) , and (2^{Ar}) we obtain:

CLAIM 9.1. *For all $\alpha, \beta \in \mathbf{GL}$:*

1. $A^\bullet \alpha \beta \models_{\text{Ar}} \neg O^\bullet \alpha \beta$ and $E^\bullet \alpha \beta \models_{\text{Ar}} \neg I^\bullet \alpha \beta$, where $\bullet = \text{blank}, \diamond, \square$, for $\bullet = \text{blank}, \square, \diamond$, respectively.
2. For any $X \in \{A, I, E, O\}$: $X^\square \alpha \beta \models_{\text{Ar}} X \alpha \beta \models_{\text{Ar}} X^\diamond \alpha \beta$.
3. $I^\bullet \alpha \beta \models_{\text{Ar}} I^\bullet \beta \alpha$ and $E^\bullet \alpha \beta \models_{\text{Ar}} E^\bullet \beta \alpha$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
4. $\models_{\text{Ar}} I^\bullet \alpha \alpha$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.¹²
5. $\models_{\text{Ar}} A \alpha \alpha$ and $\models_{\text{Ar}} A^\diamond \alpha \alpha$.
6. $A^\bullet \alpha \beta \models_{\text{Ar}} I^\bullet \alpha \beta$ and $E^\bullet \alpha \beta \models_{\text{Ar}} O^\bullet \alpha \beta$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
7. $\not\models_{\text{Ar}} A^\square \alpha \alpha$.

9.2. Tableau Rules

We use the rules (RIg) only for $\bullet \in \{\text{blank}, \square\}$ and the rules (RC) only for $\bullet \in \{\text{blank}, \diamond\}$. For the revised interpretation of I^\square , E^\diamond , O^\square , A^\diamond we describe tableau rules in Table 8.

However, the rules (RIg), for $\bullet \in \{\text{blank}, \square\}$, $(\text{RI}_1^{\text{Ar}})$, (RIa) , (RN) , (RC) , for $\bullet \in \{\text{blank}, \square\}$, and (RC^{Ar}) are not sufficient for a reconstruction of the theory from [14], since they do not reflect conditions (1^{Ar}) – (4^{Ar}) imposed on models. These conditions are reflected by the rules in Table 9. The rule (1_r^{Ar}) says that $\ulcorner \varepsilon^\square \alpha \urcorner$ can be put in any place in a tableau, but must be a new index in the tableau.

¹²We find condition (1^{Ar}) non-intuitive. Why should it be that ‘Some thief must be a thief’ is logically valid in both interpretations of I^\square (in the natural interpretation for $\models I^\square aa$ we use both (\star) and (1^{Ar}))?

$$\begin{array}{ccc}
\begin{array}{c} A^\diamond \alpha \beta \\ \iota \varepsilon^\square \alpha \\ \downarrow \\ \iota \varepsilon^\diamond \beta \end{array} & \begin{array}{c} E^\diamond \alpha \beta \\ \iota \varepsilon^\square \alpha \\ \downarrow \\ \neg \iota \varepsilon^\square \beta \end{array} & (RI_1^{Ar}) \\
\begin{array}{c} I^\square \alpha \beta \\ \downarrow \\ j \varepsilon^\square \alpha \\ j \varepsilon^\square \beta \end{array} & \begin{array}{c} O^\square \alpha \beta \\ \downarrow \\ j \varepsilon^\square \alpha \\ \neg j \varepsilon^\diamond \beta \end{array} & (RC^{Ar})
\end{array}$$

where ι is any index

where j is a new index in a tableau

Table 8. Rules (RI_1^a) and (RC^a)

$$\begin{array}{ccc}
\begin{array}{c} \downarrow \\ j \varepsilon^\square \alpha \end{array} & \begin{array}{c} \iota \varepsilon \alpha \\ \iota \varepsilon^\diamond \beta \\ \downarrow \\ j \varepsilon^\diamond \alpha \\ j \varepsilon \beta \end{array} & \begin{array}{c} \iota \varepsilon \alpha \\ \iota \varepsilon \beta \\ \downarrow \\ j \varepsilon \alpha \\ j \varepsilon^\square \beta \end{array} \\
(1_r^{Ar}) & (2_r^{Ar}) & (3_r^{Ar})
\end{array}$$

$$\begin{array}{c} A^\square \alpha \beta \\ \iota \varepsilon^\diamond \alpha \\ \downarrow \\ \iota \varepsilon^\diamond \beta \end{array} \quad (4_r^{Ar})$$

where ι is any index, but j is a new index in a tableau

Table 9. Additional rules

Using the above rules we obtain the following facts corresponding to those in Claim 9.1:

CLAIM 9.2. For all $\alpha, \beta \in \mathbf{GL}$:

1. $A^\bullet \alpha \beta \vdash_{Ar} \neg O^\bullet \alpha \beta$ and $E^\bullet \alpha \beta \vdash_{Ar} \neg I^\bullet \alpha \beta$, where $\bullet = \text{blank}, \diamond, \square$, for $\bullet = \text{blank}, \square, \diamond$, respectively.
2. For any $X \in \{A, I, E, O\}$: $X^\square \alpha \beta \vdash_{Ar} X \alpha \beta \vdash_{Ar} X^\diamond \alpha \beta$.
3. $I^\bullet \alpha \beta \vdash_{Ar} I^\bullet \beta \alpha$ and $E^\bullet \alpha \beta \vdash_{Ar} E^\bullet \beta \alpha$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
4. $\vdash_{Ar} I^\bullet \alpha \alpha$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.
5. $\vdash_{Ar} A \alpha \alpha$ and $\vdash_{Ar} A^\diamond \alpha \alpha$.
6. $A^\bullet \alpha \beta \vdash_{Ar} I^\bullet \alpha \beta$ and $E^\bullet \alpha \beta \vdash_{Ar} O^\bullet \alpha \beta$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.

PROOF. 3. $I^\square \alpha \beta, \neg I^\square \beta \alpha \xrightarrow{RC^{Ar}} j \varepsilon^\square \alpha, j \varepsilon^\square \beta \xrightarrow{RN} E^\diamond \beta \alpha \xrightarrow{RI_1^{Ar}} \neg j \varepsilon^\square \beta \times$.

$I^\diamond \alpha \beta, \neg I^\diamond \beta \alpha \xrightarrow{RC} i \varepsilon \alpha, i \varepsilon^\diamond \beta \xrightarrow{2_r^{Ar}} j \varepsilon^\diamond \alpha, j \varepsilon \beta \xrightarrow{RN} E^\square \beta \alpha \xrightarrow{RI_1^{Ar}} \neg j \varepsilon^\diamond \alpha \times$.

4. $\neg I^\square \alpha \alpha \xrightarrow{RN} E^\diamond \alpha \alpha \xrightarrow{1_r^{Ar}} j \varepsilon^\square \alpha \xrightarrow{RI_1^{Ar}} \neg j \varepsilon^\square \alpha \times$

$\neg I \alpha \alpha \xrightarrow{RN} E \alpha \alpha \xrightarrow{1_r^{Ar}} j \varepsilon^\square \alpha \xrightarrow{RI_2} j \varepsilon \alpha \xrightarrow{RI_1} \neg j \varepsilon \alpha \times$

$\neg I^\diamond \alpha \alpha \xrightarrow{RN} E^\square \alpha \alpha \xrightarrow{1_r^{Ar}} j \varepsilon^\square \alpha \xrightarrow{RI_2} j \varepsilon \alpha \xrightarrow{RI_2} j \varepsilon^\diamond \alpha \xrightarrow{RI_1} \neg j \varepsilon^\diamond \alpha \times$

6. $A^\square \alpha \beta, \neg I^\square \alpha \beta \xrightarrow{RN} E^\diamond \alpha \beta \xrightarrow{1_r^{Ar}} j \varepsilon^\square \alpha \xrightarrow{RI_1^{Ar}} \neg j \varepsilon^\square \beta \xrightarrow{RI_2} j \varepsilon \alpha \xrightarrow{RI_1} j \varepsilon^\square \beta \times$

$A^\diamond \alpha \beta, \neg I^\diamond \alpha \beta \xrightarrow{RN} E^\square \alpha \beta \xrightarrow{1_r^{Ar}} j \varepsilon^\square \alpha \xrightarrow{RI_1^{Ar}} j \varepsilon^\diamond \beta \xrightarrow{RI_2} j \varepsilon \alpha \xrightarrow{RI_1} \neg j \varepsilon^\diamond \beta \times \blacksquare$

Under the unnatural interpretations of I^\square , E^\diamond , O^\square and A^\diamond which we have assumed here, which satisfy conditions (\star) , (1^{Ar}) – (4^{Ar}) we can reformulate and prove counterparts of the facts given in Sections 3–6.

9.3. Soundness

We prove that the revised relation \vdash_{Ar} is sound, i.e.:

THEOREM 9.3 (Soundness). *For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$:*

$$\text{if } \Psi \vdash_{Ar} \varphi \text{ then } \Psi \models_{Ar} \varphi.$$

PROOF. Assume that $\Psi \vdash_{Ar} \varphi$, i.e., there is a closed tableau T for $\Psi \cup \{\neg\varphi\}$. Suppose towards contradiction that $\Psi \not\models_{Ar} \varphi$. Then there is a non-empty model $\mathfrak{M} = \langle D, d, d^\square, d^\diamond \rangle \in \text{Mod}_{Ar}$ such that $\mathfrak{M} \models_{Ar} \Psi$, but $\mathfrak{M} \not\models_{Ar} \varphi$; so $\mathfrak{M} \models_{Ar} \Psi \cup \{\neg\varphi\}$.

Note that all tableau rules of the types (NBE) and (RN) preserve truth and moreover at least one conclusion derived from the application of a rule of type (BE) preserves truth. Hence for some closed branch B in T all formulas from \mathbf{For} which are in B are true in \mathfrak{M} , because $\mathfrak{M} \models \Psi \cup \{\neg\varphi\}$. Let \mathbf{For}_B be the set of all formulas from \mathbf{For} which are in B . So $\mathfrak{M} \models \mathbf{For}_B$.

If in B no rule of type (RC) was used, then in B there would be only formulas of \mathbf{For}_B . But this contradicts the fact that B is closed. So we suppose that in B at least once some rule of type (RC) or (RC^{Ar}) was used. Let Σ_B^1 be the set of auxiliary formulas in B which we obtain by rules of (RC) and (RC^{Ar}) . Let I_B^1 be the set of all indexes which appear in formulas from Σ_B^1 . We have $I_B^1 \neq \emptyset$. Since we apply the rules of (RC) and (RC^{Ar}) only to formulas from \mathbf{For}_B and $\mathfrak{M} \models \mathbf{For}_B$, there is a valuation $\nu_1: I_B^1 \rightarrow D$ such that $\mathfrak{M}, \nu_1 \models \Sigma_B^1$. Then we suppose we could use also a rule of type (1_r^{Ar}) – (3_r^{Ar}) . Let Σ_B^2 be the set of auxiliary formulas in B which we obtain by rules of (1_r^{Ar}) – (3_r^{Ar}) . Let I_B^2 be the set of all indexes which appear in formulas from Σ_B^2 . We have $I_B^2 \neq \emptyset$. Since we apply the rules of (1_r^{Ar}) – (3_r^{Ar}) only to formulas from Σ_B^1 and $\mathfrak{M} \models \Sigma_B^1$, by interpretation and the conditions (1^{Ar}) – (3^{Ar}) , there is a valuation $\nu_2: I_B^2 \rightarrow D$ such that $\mathfrak{M}, \nu_2 \models \Sigma_B^2$. Moreover, by interpretation and the condition (4^{Ar}) , all other tableau rules preserve truth for \mathfrak{M} and $\nu_1 \cup \nu_2$. Hence, by induction on the length of B , we show that all formulas in B are true in \mathfrak{M} , too. But this contradicts the fact that B is closed. ■

9.4. Compactness, Completeness, and Decidability

By appropriate modification to the relevant proofs, we can obtain the counterparts of the facts from Sections 3–6.

For all $\alpha, \beta \in \mathbf{GL}$ besides the formulas T_α^\square and T_α^\diamond we consider the following sentences from $\mathbf{For}_\mathcal{L}$:

$$\begin{aligned} 1_\alpha^{\text{Ar}} &:= \exists_x \alpha^\square x \\ 2_{\alpha, \beta}^{\text{Ar}} &:= \exists_x (\alpha^\diamond x \wedge \beta x) \supset \exists_x (\alpha x \wedge \beta^\diamond x) \\ 3_{\alpha, \beta}^{\text{Ar}} &:= \exists_x (\alpha x \wedge \beta x) \supset \exists_x (\alpha x \wedge \beta^\square x) \\ 4_{\alpha, \beta}^{\text{Ar}} &:= \forall_x (\alpha x \supset \beta^\square x) \supset \forall_x (\alpha^\diamond x \supset \beta^\diamond x) \end{aligned}$$

Moreover, for any $\Phi \in \mathcal{P}(\mathbf{For})$ we put:

$$\begin{aligned} \text{Ar}_\Phi &:= T_\Phi \cup \{1_\alpha^{\text{Ar}} \in \mathbf{For}_\mathcal{L} : \alpha \in \mathbf{GL}(\Phi)\} \\ &\cup \{k_{\alpha, \beta}^{\text{Ar}} \in \mathbf{For}_\mathcal{L} : k = 2, 3, 4 \text{ \& } \alpha, \beta \in \mathbf{GL}(\Phi)\}. \end{aligned}$$

We obtain the results corresponding to Lemma 1.4 and Theorem 1.5:

LEMMA 9.4. *For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$:*

$$\Psi \models_{\text{Ar}} \varphi \quad \text{iff} \quad \text{Ar}_{\Psi \cup \{\varphi\}} \cup \mathbf{t}(\Psi) \models_{\text{CQL}} \mathbf{t}(\varphi).$$

THEOREM 9.5 (Compactness). *For all $\Psi \in \mathcal{P}(\mathbf{For})$ and $\varphi \in \mathbf{For}$:*

$$\Psi \models_{\text{Ar}} \varphi \quad \text{iff} \quad \text{there is } \Psi_{\text{fin}} \in \mathcal{P}_{\text{fin}}(\Psi) \text{ such that } \Psi_{\text{fin}} \models_{\text{Ar}} \varphi.$$

We can easily specify the notion of a *mini-max tableau* for any non-empty finite subset Φ of \mathbf{For} , where the rule (1_r^{Ar}) we use only for some α that occurs in some formula from Φ . For each such α we use this rule exactly once and the formula of the form $\ulcorner j\varepsilon^\square \alpha \urcorner$ we put in the initial list. Moreover, we do not apply rules (RC) to formulas $\ulcorner I^\bullet \alpha \urcorner$, for any $\bullet \in \{\text{blank}, \square, \diamond\}$.

For Completeness Theorem we use the following Completeness Lemma:¹³

LEMMA 9.6. *Let Φ be any non-empty finite subset of \mathbf{For} and T be any “mini-max” open tableau for Φ . Then there is a model $\mathfrak{M}_T \in \text{Mod}_{\text{Ar}}$ such that $\mathfrak{M}_T \models \Phi$.*

PROOF. Let B be an open branch in T . By definition, all formulas of Φ are in B . As in the proof of Lemma 5.1 we construct a model $\mathfrak{M}_T = \langle D, d, d^\square, d^\diamond \rangle$ from Mod_{Ar} .

Now we show that \mathfrak{M}_T is a model from Mod_{Ar} , i.e., for all $\alpha, \beta \in \mathbf{GL}$ we have: (\star) and (1_r^{Ar}) – (4_r^{Ar}) . Condition (\star) we can show as in the proof of Lemma 5.1.

For (1_r^{Ar}) : By rule (1_r^{Ar}) , for any α that occurs in some formula from Φ on the initial list we have $\ulcorner j\varepsilon^\square \alpha \urcorner$. Therefore $d^\square(\alpha) \neq \emptyset$.

¹³We will not be concerned here with the problem of minimal models, as we were in the case of Lemma 5.1.

For (2^{Ar}): Suppose that $d(\alpha) \cap d^\diamond(\beta) \neq \emptyset$. Then for some $i \in \mathbf{AI}$ formulas $\ulcorner i\epsilon\alpha \urcorner$ and $\ulcorner i\epsilon^\diamond\beta \urcorner$ are in B . In B we also used the rule (2^{Ar}). Hence for some new $j \in \mathbf{AI}$ formulas $\ulcorner j\epsilon^\diamond\alpha \urcorner$ and $\ulcorner j\epsilon\beta \urcorner$ are in B . Therefore $d^\diamond(\alpha) \cap d(\beta) \neq \emptyset$.

For (3^{Ar}): Suppose that $d(\alpha) \cap d(\beta) \neq \emptyset$. Then for some $i \in \mathbf{AI}$ formulas $\ulcorner i\epsilon\alpha \urcorner$ and $\ulcorner i\epsilon\beta \urcorner$ are in B . In B we also used the rule (3^{Ar}). Hence for some new $j \in \mathbf{AI}$ formulas $\ulcorner j\epsilon\alpha \urcorner$ and $\ulcorner j\epsilon^\square\beta \urcorner$ are in B . So $d(\alpha) \cap d^\square(\beta) \neq \emptyset$.

For (4^{Ar}): Suppose that we have $d(\alpha) \subseteq d^\square(\beta)$. Then, by (1^{Ar}) and (RI₂), a formula $\ulcorner A^\square\alpha\beta \urcorner$ is in B . So we can use the rule (4^{Ar}) to get $d^\diamond(\alpha) \subseteq d^\diamond(\beta)$, if $d^\diamond(\alpha) \neq \emptyset$, and in consequence some $\ulcorner i\epsilon^\diamond\alpha \urcorner$ is in the branch.

Now we show that for any $\varphi \in \mathbf{For}$ we obtain: if φ is in B , then $\mathfrak{M}_T \models \varphi$. The proof is by induction on the complexity of formulas as in the proof of Lemma 5.1. We use the rules of (RI₁^{Ar}) and of (RC^{Ar}) only for A^\diamond , E^\diamond , I^\square , and O^\square . ■

We have a counterpart of Theorem 5.2:

THEOREM 9.7 (Completeness). *For all $\Psi \subseteq \mathbf{For}$ and $\varphi \in \mathbf{For}$:*

$$\text{if } \Psi \models_{\text{Ar}} \varphi, \text{ then } \Psi \vdash_{\text{Ar}} \varphi.$$

Finally, we obtain a theorem corresponding to Theorem 6.1:

THEOREM 9.8 (Decidability). *For all $\Psi \in \mathcal{P}_{\text{fin}}(\mathbf{For})$ and $\varphi \in \mathbf{For}$, the problem whether $\Psi \models_{\text{Ar}} \varphi$ is decidable.*

Both theorems just give an alternative decision procedure for determining validity in the Thomason-Johnson semantics of Aristotelian modal syllogisms.

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