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Sequent Calculi for SCI

Abstract. In this paper we are applying certain strategy described by Negri and Von Plato (Bull Symb Log 4(04):418–435, 1998), allowing construction of sequent calculi for axiomatic theories, to Suszko's *Sentential calculus with identity*. We describe two calculi obtained in this way, prove that the cut rule, as well as the other structural rules, are admissible in one of them, and we also present an example which suggests that the cut rule is not admissible in the other.

Keywords: Non-Fregean logics, Sequent calculi, Admissibility of cut, Propositional identity.

1. Introduction

The aim of this paper is to give a proof-search oriented account of *Sentential calculus with identity* (SCI) of Suszko [1,8,10]. Our construction is based on a multisuccedent, context-sharing sequent calculus for *Classical propositional logic* (CPL) and is inspired by a strategy of obtaining sequent calculi for extensions of CPL (see [4,7] for examples).

This strategy provides cut-free systems with admissible structural rules and subformula property, provided the construction of non-logical rules follows certain restrictions.

Other proof-theoretic approaches to SCI have been made before. Michaels [3] describes a sequent calculus with sequences of formulas instead of multisets. There are two SCI-specific rule schemas, differing in order of formula occurrence. These rules make use of substitution of identical subformulas and deriving classical equivalence from identity. A simplified account of this approach is given in [12]. In this instance the system is purely right-sided, with all sequents having empty antecedents. Wasilewska [13] provides a decidability algorithm for SCI founded on this calculus. Another system, based on the Rasiowa–Sikorski dual tableaux method, is presented in [2,9]. As in the earlier cases, its rule for the identity connective involves use of substitution. Following standard rules for identity in first-order logic, it allows replacing some occurrences of a subformula with formulas identical to it.

Calculi described here, namely $\mathbf{rG3}_{\mathsf{SCI}}$ and $\mathbf{lG3}_{\mathsf{SCI}}$, are both sound and complete. System $\mathbf{lG3}_{\mathsf{SCI}}$ has admissible structural rules, the cut rule in

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particular (these properties make this system well-suited for efficient proof-search procedures), but it is not the case with system $\mathbf{rG3}_{\mathsf{SCI}}$. What differs calculi introduced here from the existing systems mentioned above is that our calculi do not make use of a substitution rule.

One of the most intriguing observations of this work is that the basic, non-Fregean logic SCI fails to be fully compatible with the mentioned strategy, due to the non-atomic character of SCI-axioms. We chose to apply it, as it provides a proof-search oriented framework for constructing rules from axioms and structural analysis, in particular cut elimination. However, in this case the final calculi lack some properties normally obtained this way such as the subformula property.

The main points of interest of this paper are as follows:

- After introducing the strategy from [4–6] in Section 2 and the logic SCI in Section 3, we provide two sequent calculi for SCI in Section 4. First, we describe $\mathbf{lG3}_{\mathsf{SCI}}$, with left SCI-specific rules only. Next we present a description of the right system $\mathbf{rG3}_{\mathsf{SCI}}$.
- In Sections 4.1 and 4.2 we discuss the problem of admissibility of structural rules in $\mathbf{lG3}_{\mathsf{SCI}}$ and $\mathbf{rG3}_{\mathsf{SCI}}$.
- In Section 5 we study some derivable rules, in particular we provide derivations of the left and right substitution rules. This result brings a tool for comparison with other formalizations of SCI.

2. From Axioms to Rules

There is a number of strategies of building sequent calculi or natural deduction systems for axiomatic theories based on a certain logic (see for example [5–7,11]). The strategy we are interested in enables one to turn each axiom of a given axiomatic system into a rule of a corresponding sequent calculus in such a way that all structural rules—the cut rule in particular—are admissible in the generated calculus. The rules obtained correspond to the initial axiom (where P_i and Q_j are atomic):

$$P_1 \wedge \cdots \wedge P_m \to Q_1 \vee \cdots \vee Q_n \tag{1}$$

in either of the following manners:

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}\, L$$

$$\frac{\Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n, P_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n, P_m}{\Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n}\, R$$

Formulas specified in a conclusion of a left rule scheme or right rule scheme are called *principal formulas* of a given rule scheme, while formulas specified in premisses of a rule scheme are called *active formulas* of that rule scheme. For future reference let us observe that if $m = 0$ in the general form of an axiom 1, then 1 has the form $Q_1 \vee \cdots \vee Q_n$, and the corresponding left rule

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L$$

has no principal formulas and has $Q_1, \ldots, Q_n$ as active formulas. In the same situation, the corresponding right rule scheme of the following form:

$$\frac{}{\Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n}\ R$$

has no active formulas and has $Q_1, \ldots, Q_n$ as principal formulas.

If each axiom is transformed into a left (right) rule we obtain a *left* (*right*) *system*. Moreover, each left system constructed with this method needs to satisfy the following condition:

Definition 2.1. (*Closure condition*, [4]) If a system with nonlogical rules has a rule, where a substitution instance in the atoms produces a rule of the form:

$$\frac{Q_1, P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}\ R$$

then it also has to contain the rule:

$$\frac{Q_1, P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}\ R^*$$

Systems of rules for the theory of partial order are a perfect example of applying the Negri strategy. They are based on the following axioms:

(refl) $\forall x(x \leq x)$

(trans) $\forall x, y, z(x \leq y \wedge y \leq z \rightarrow x \leq z)$

These two rules are obtained by application of the left strategy:

$$\frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L_{refl} \qquad \frac{x \leq z, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}\ L_{trans}$$

and the following two by the right strategy:

$$\frac{}{\Gamma \Rightarrow \Delta, x \leq x}\ R_{refl} \qquad \frac{\Gamma \Rightarrow \Delta, x \leq z, x \leq y \quad \Gamma \Rightarrow \Delta, x \leq z, y \leq z}{\Gamma \Rightarrow \Delta, x \leq z}\ R_{trans}$$

Note that the rule L_{refl} has no principal formulas and R_{refl} has no active formulas.

Our main aim is to study sequent calculi for sentential calculus with identity (SCI), which was constructed according to the strategy just described. The idea is that specific axioms of SCI cannot be transformed into the form 1, due to the requirement that all formulas in 1 have to be atomic. We have decided to apply this strategy nevertheless and it has turned out that the left system has all desired properties (the admissibility of the cut rule in particular) whereas the cut rule is essential in the right system.

3. Hilbert-Style System for SCI

The language $\mathcal{L}_{\mathsf{SCI}}$ of the logic SCI is defined by the following BNF grammar, where $V = \{p_1, \ldots, p_n, \ldots\}$ is a countable set of propositional variables:

$$A, B \ ::= \ V \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid A \approx B$$

The axiom system for SCI, $\mathbf{H}_{\mathsf{SCI}}$, is obtained from any such system for CPL expressed in a language with the connectives $\bot$, $\wedge$, $\vee$ and $\rightarrow$ by the addition of the following axioms ($\otimes \in \{\wedge, \vee, \rightarrow, \approx\}$):

($\approx_1$) $A \approx A$

($\approx_2$) $(A \approx B) \rightarrow ((A \rightarrow \bot) \approx (B \rightarrow \bot))$

($\approx_3$) $(A \approx B) \rightarrow (B \rightarrow A)$

($\approx_4$) $((A \approx B) \wedge (C \approx D)) \rightarrow ((A \otimes C) \approx (B \otimes D))$

If a formula A is provable in $\mathbf{H}_{\mathsf{SCI}}$, we say it is a *thesis* of SCI.

In order to fit axiom $\approx_3$ to the general axiom form 1, the axiom $\approx_3$ can be restated by $(A \approx B) \rightarrow (\neg B \vee A)$. But the problem of atomic character of formulas involved in SCI-axioms still remains. Nevertheless, we find it interesting to study calculi which emerged from an application of this strategy to axiom systems which do not satisfy Negri's conditions on general axiom form.

Typically, soundness and completeness theorems for SCI are proven with regard to algebraic semantics, as in [1,8]. The only difference in our treatment of an algebraic semantics for SCI is that instead of a negation we use $\bot$ constant, which forces a small change in the notion of an SCI-model.[1]

[1] If a negation symbol is present, then $\mathcal{M} = \langle U, \sim, \sqcup, \sqcap, \rhd, \circ, D \rangle$ and we stipulate:

$$\sim a \in D \text{ iff } a \notin D$$

DEFINITION 3.1. (SCI-*model*) An SCI model is a structure

$$\mathcal{M} = \langle U, \bullet, \sqcup, \sqcap, \rhd, \circ, D \rangle$$

where U is non-empty set, $\bullet \in U$, D is a non-empty subset of U and the following conditions are satisfied for arbitrary $a, b \in U$:

1. $\bullet \notin D$
2. $a \sqcup b \in D$ iff $a \in D$ or $b \in D$
3. $a \sqcap b \in D$ iff $a \in D$ and $b \in D$
4. $a \rhd b \in D$ iff $a \notin D$ or $b \in D$
5. $a \circ b \in D$ iff $a = b$

DEFINITION 3.2. (SCI-*valuation*) Let v be a mapping from V to U. Then, v is extended to a mapping from the set $\mathbf{For}^{\mathsf{SCI}}$ to U, which fulfils the following conditions:

1. $v(\bot) = \bullet$
2. $v(A \vee B) = v(A) \sqcup v(B)$
3. $v(A \wedge B) = v(A) \sqcap v(B)$
4. $v(A \to B) = v(A) \rhd v(B)$
5. $v(A \approx B) = v(A) \circ v(B)$

Let v be an SCI-valuation in a model $\mathcal{M}$. A formula A is *satisfied* by v in $\mathcal{M}$ iff $v(A) \in D$. A formula is *true* in $\mathcal{M}$ if it is satisfied by all valuations in $\mathcal{M}$; a formula is SCI-*tautology* iff it is true in all SCI models.

4. Sequent Calculi for SCI

Our SCI-calculi are based on a slight modification of **G3cp**, a system for classical logic consisting of rules presented in Table 1. Calculus used here is not identical with **G3cp**, as its axiom scheme allows A to be arbitrary and in **G3cp** A must be atomic.[2] Γ, Δ are finite multisets of formulas. We will refer to sequents by $\phi, \psi, \ldots$.

Footnote 1 continued
It can be easily seen that these two conditions are equivalent. We can define $\sim$ as follows $\sim a ::= a \rhd \bullet$. Now assume $\sim a \in D$. Then $a \rhd \bullet \in D$ and $a \notin D$ or $\bullet \in D$. Since we have $\bullet \notin D$ we conclude that $a \notin D$.

[2]However, it is easily showable that these calculi are equivalent through proving by induction that any proof in our system can be transformed into a **G3cp** proof.

Table 1. Rules of **G3cp**

$A, \Gamma \Rightarrow \Delta, A$	$\dfrac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot$
$\dfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L_\wedge$	$\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R_\wedge$
$\dfrac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L_\vee$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R_\vee$
$\dfrac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L_\to$	$\dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ R_\to$

Table 2. Structural rules

$\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ L_w$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ R_w$	$\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ L_{ctr}$	$\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ R_{ctr}$

$$\frac{\Gamma' \Rightarrow \Delta', A \quad A, \Gamma'' \Rightarrow \Delta''}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

We refer to formulas specified in the premisses of a rule schema as *active formulas* of that rule and to these specified in the conclusion as *principal formulas* of this rule.

We use the notions of derivation height, cut-height and formula weight in a manner described in [7]. By height of a derivation we mean the maximal number of successive applications of the logical rules of **G3cp**. The structural rules from Table 2. are admissible and height-preserving.

Definition 4.1. (*Height of derivation*) A sequent of one of the following forms: (i) $A, \Gamma \Rightarrow \Delta, A$ (ii) $\bot, \Gamma \Rightarrow \Delta$ has a derivation of height 0. If a sequent ϕ has a derivation D of height n, then the following derivation:

$$\dfrac{\begin{matrix}\mathcal{D}\\ \vdots \\ \phi\end{matrix}}{\psi}\ R$$

has height $n+1$, where R is a one-premiss rule. If sequents ϕ' and ϕ'' have derivations $\mathcal{D}', \mathcal{D}''$ of heights n and m respectively, then the following derivation:

$$\dfrac{\begin{matrix}\mathcal{D}'\\ \vdots \\ \phi'\end{matrix} \quad \begin{matrix}\mathcal{D}''\\ \vdots \\ \phi''\end{matrix}}{\psi}\ R'$$

Table 3. Left rules for identity

$$\frac{A \approx A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ L^1_{\approx} \qquad \frac{A \approx B, \Gamma \Rightarrow \Delta, B \quad A \approx B, A, \Gamma \Rightarrow \Delta}{A \approx B, \Gamma \Rightarrow \Delta}\ L^2_{\approx}$$

$$\frac{(A \otimes B) \approx (C \otimes D), A \approx C, B \approx D, \Gamma \Rightarrow \Delta}{A \approx C, B \approx D, \Gamma \Rightarrow \Delta}\ L^3_{\approx}$$

$$\frac{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma \Rightarrow \Delta}{A \approx B, \Gamma \Rightarrow \Delta}\ L^{3*}_{\approx}$$

has height $max(m, n) + 1$, where R' is a two-premiss rule.

By $\vdash_n \phi$ we mean that the sequent ϕ is derivable with height no greater than n. By $\frac{\vdash_n \phi}{\vdash_m \psi} R$ we mean that the sequent ϕ is derivable with height no greater than n and an application of the rule R gives us ψ, which is derivable with height no greater than m.

The notions of cut-height and formula weight defined below are used in proving admissibility of structural rules and they follow the definitions from [7].

Definition 4.2. (*Cut-height*) The cut-height of an application of the cut rule in a derivation $\mathcal{D}$ is the sum of heights of derivations of two premisses of cut.

Definition 4.3. (*Formula weight*) Weight is a function from the set of all formulas to the set of natural numbers, which fulfils the following conditions:

1. $w(\bot) = 0$,
2. $w(p_i) = 1$, for each propositional variable p_i,
3. $w(A \otimes B) = w(A) + w(B) + 1$, where $\otimes \in \{\approx, \vee, \wedge, \rightarrow\}$.

4.1. Left Strategy

Following the presented strategy, we obtain a set of left rules listed below. It is worth noticing that none of these rules are directly obtained from axiom ($\approx_2$). In the absence of negation as a primitive connective, axiom ($\approx_2$) is provable with the use of rules $L^3_{\approx}$ and $L^1_{\approx}$ (in a sense it is a particular case of congruence). The rule $L^{3*}_{\approx}$ is required to fulfil the closure condition with regard to the rule $L^3_{\approx}$. They differ from the other rules, as they are in fact schemas of four different rules each, depending on the connective represented by $\otimes$. We will call these rules variants of $L^3_{\approx}$ and $L^{3*}_{\approx}$ respectively.

Soundness and completeness Let v be an SCI-valuation in a model $\mathcal{M}$. A sequent $\Gamma \Rightarrow \Delta$ is *sound* under a v iff a formula $\bigwedge \Gamma \to \bigvee \Delta$ is satisfied under v, where $\bigwedge$ and $\bigvee$ denotes conjunction of all formulas from the antecedent and disjunction of all formulas from the consequent respectively. A sequent is *valid*, when it is sound under every SCI-valuation in each model $\mathcal{M}$. A rule is *sound* iff soundness of the premisses of that rule entails soundness of the conclusion of that rule.

THEOREM 4.4. (Soundness) *If a sequent $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{lG3}_{\mathsf{SCI}}$, then it is SCI-valid.*

PROOF. A sequent $\Gamma \Rightarrow \Delta$ is sound under a SCI-valuation v in a given model $\mathcal{M}$ iff a corresponding formula of the form $\bigwedge \Gamma \to \bigvee \Delta$ is true under v in this model. This *denotational* interpretation makes it easy to check that each rule of $\mathbf{lG3}_{\mathsf{SCI}}$ preserves validity of sequents. Then by induction on the length of proof of the sequent $\Gamma \Rightarrow \Delta$ it is easy to show that it is SCI-valid. ■

THEOREM 4.5. (Simulation for $\mathbf{lG3}_{\mathsf{SCI}}$) *If A is provable in $\mathbf{H}_{\mathsf{SCI}}$ then the sequent $\Rightarrow A$ is provable in $\mathbf{lG3}_{\mathsf{SCI}}$.*

PROOF. We want to show that $\mathbf{lG3}_{\mathsf{SCI}}$ is able to simulate a Hilbert-style system for SCI, i.e. all axioms are provable in a sequent system and the *modus ponens* rule is derivable in it. Its derivation uses cut in an essential way:

$$\dfrac{\Rightarrow A \qquad \dfrac{\Rightarrow A \to B \qquad \dfrac{A \Rightarrow B, A \quad B, A \Rightarrow B}{A \to B, A \Rightarrow B}\, L_{\to}}{A \Rightarrow B}\, cut}{\Rightarrow B}\, cut$$

All SCI-specific axioms are provable. Let us consider axiom ($\approx_1$):

$$\dfrac{A \approx A \Rightarrow A \approx A}{\Rightarrow A \approx A}\, L^1_{\approx}$$

In the case of axiom ($\approx_2$) a proof has the following form:

$$\dfrac{\dfrac{\dfrac{\bot \approx \bot, A \approx B, (A \to \bot) \approx (B \to \bot) \Rightarrow (A \to \bot) \approx (B \to \bot)}{\bot \approx \bot, A \approx B \Rightarrow (A \to \bot) \approx (B \to \bot)}\, L^3_{\approx}}{\bot \approx \bot \Rightarrow (A \approx B) \to ((A \to \bot) \approx (B \to \bot))}\, R_{\to}}{\Rightarrow (A \approx B) \to ((A \to \bot) \approx (B \to \bot))}\, L^1_{\approx}$$

The following derivation is a proof of axiom ($\approx_3$):

$$\dfrac{\dfrac{\dfrac{A \approx B, B \Rightarrow A, B \quad A, A \approx B, B \Rightarrow A}{A \approx B, B \Rightarrow A}\, L^2_{\approx}}{A \approx B \Rightarrow B \to A}\, R_\wedge}{\Rightarrow (A \approx B) \to (B \to A)}\, R_\to$$

The following derivation is a proof of axiom ($\approx_4$):

$$\dfrac{\dfrac{\dfrac{(A \otimes C) \approx (B \otimes D), A \approx B, C \approx D \Rightarrow (A \otimes C) \approx (B \otimes D)}{A \approx B, C \approx D \Rightarrow (A \otimes C) \approx (B \otimes D)}\, L^3_{\approx}}{(A \approx B) \wedge (C \approx D) \Rightarrow (A \otimes C) \approx (B \otimes D)}\, L_\wedge}{\Rightarrow ((A \approx B) \wedge (C \approx D)) \to ((A \otimes C) \approx (B \otimes D))}\, R_\to$$

■

Completeness of the left system is a corollary from the simulation theorem since $\mathbf{lG3}_{\mathsf{SCI}}$ is able to prove every formula, which is provable in $\mathbf{H}_{\mathsf{SCI}}$ and Hibert-style system for SCI is complete with respect to SCI-semantics.

Admissibility of cut in $\mathbf{lG3}_{\mathsf{SCI}}$ Let us first prove the admissibility of rules of weakening and contraction. Then we will prove the central result, namely the admissibility of the cut rule.

Theorem 4.6. (Admissibility of L_w and R_w) *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n A, \Gamma \Rightarrow \Delta$ *and* $\vdash_n \Gamma \Rightarrow \Delta, A$.

Proof. A very straightforward proof relies on the observation that one can always transform a given derivation of $\Gamma \Rightarrow \Delta$ into a derivation of $A, \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, A$ by adding a formula A to the antecedent or succedent of each sequent in the original derivation. ■

Theorem 4.7. (Admissibility of L_{ctr} and R_{ctr})

1. *If* $\vdash_n \Gamma \Rightarrow \Delta, A, A$, *then* $\vdash_n \Gamma \Rightarrow \Delta, A$.
2. *If* $\vdash_n A, A, \Gamma \Rightarrow \Delta$, *then* $\vdash_n A, \Gamma \Rightarrow \Delta$.

Proof. By induction on the height of derivation.

1. Assume $n = 0$. Then the sequent $\Gamma \Rightarrow \Delta, A, A$ is an (i) axiom or (ii) conclusion of $\bot$. It is easy to check that in these cases the sequent $\Gamma \Rightarrow \Delta, A$ is an axiom or conclusion of $\bot$ as well.
   Assume that admissibility of contraction holds up to n and let $\vdash_{n+1} \Gamma \Rightarrow \Delta, A, A$. Consider the last step of the proof. If A is not principal in the last rule, then we have:

$$\frac{\begin{array}{c}\mathcal{D}\\ \vdots\\ \vdash_n \Gamma' \Rightarrow \Delta', A, A\end{array}}{\vdash_{n+1} \Gamma \Rightarrow \Delta, A, A}\ R$$

By inductive hypothesis, $\vdash_n \Gamma' \Rightarrow \Delta', A$. Now we apply the last rule R to conclude $\Gamma \Rightarrow \Delta, A$ in at most $n+1$ steps.
If the contraction formula is principal in the conclusion of the last rule we have to consider only classical rules, which are described in [7].

2. Assume $n = 0$. Then the sequent $A, A, \Gamma \Rightarrow \Delta$ is an (i) axiom or (ii) conclusion of the rule for $\perp$. Naturally, in these cases sequent $A, \Gamma \Rightarrow \Delta$ is an axiom or conclusion of the rule for $\perp$.

Assume the theorem holds up to n, and let $\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta$. If the contraction formula A is not principal in the conclusion of the last applied rule R of a given derivation, then we have to consider the following derivation:

$$\frac{\vdash_n A, A, \Gamma' \Rightarrow \Delta'}{\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta}\ R$$

By inductive hypothesis we have that $\vdash_n A, \Gamma' \Rightarrow \Delta'$. By applying the rule R to this sequent we obtain $\vdash_{n+1} A, \Gamma \Rightarrow \Delta$.

Assume that the contraction formula is principal. We omit the classical cases and assume that a formula A has the form $B \approx C$ and the last rule used is either $L^2_{\approx}$, $L^3_{\approx}$ or $L^{3*}_{\approx}$. The active formulas of the premisses of these rules are not present in their conclusions. If only one occurrence of the contraction formula is principal, then we obtain the contracted premiss by inductive hypothesis and apply the appropriate rule to it. Let us consider the exception. Both contraction formulas are principal and the last rule applied is $L^3_{\approx}$:

$$\frac{\vdash_n (B \otimes B) \approx (C \otimes C), B \approx C, B \approx C, \Gamma \Rightarrow \Delta}{\vdash_{n+1} B \approx C, B \approx C, \Gamma \Rightarrow \Delta}\ L^3_{\approx}$$

By inductive hypothesis, $\vdash_n (B \otimes B) \approx (C \otimes C), B \approx C, \Gamma \Rightarrow \Delta$. We apply the rule $L^{3*}_{\approx}$, to conclude $B \approx C, \Gamma \Rightarrow \Delta$ in at most $n+1$ steps:

$$\frac{\vdash_n (B \otimes B) \approx (C \otimes C), B \approx C, \Gamma \Rightarrow \Delta}{\vdash_{n+1} B \approx C, \Gamma \Rightarrow \Delta}\ L^{3*}_{\approx}$$

In this instance, applying the rule obtained by the closure condidtion is necessary for proving admissibility of contraction. ■

THEOREM 4.8. (Admissibility of cut) *The cut rule of the form*:

$$\dfrac{\Gamma' \Rightarrow \Delta', C \quad C, \Gamma'' \Rightarrow \Delta''}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

is admissible in $\mathbf{lG3}_{\mathsf{SCI}}$.

PROOF. The proof is organized as in [7]. We consider only such cases, where the new rules for identity are applied.

I The cut formula C is not principal in the left premiss.
(1) The last rule applied was $L^1_{\approx}$. Cut-height equals $m + 1 + m'$:

$$\dfrac{\dfrac{\overset{m}{A \approx A, \Gamma' \Rightarrow \Delta', C}}{\Gamma' \Rightarrow \Delta', C}\ L^1_{\approx} \quad \overset{m'}{C, \Gamma'' \Rightarrow \Delta''}}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

This derivation is transformed into a derivation of lower cut-height (equal to $m + m'$):

$$\dfrac{\dfrac{A \approx A, \Gamma' \Rightarrow \Delta', C \quad C, \Gamma'' \Rightarrow \Delta''}{A \approx A, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ L^1_{\approx}$$

(2) The last rule applied was $L^2_{\approx}$. The cut height equals $max(m, n) + 1 + m'$:

$$\dfrac{\dfrac{\overset{m}{A \approx B, \Gamma' \Rightarrow \Delta', C, B} \quad \overset{n}{A \approx B, A, \Gamma' \Rightarrow \Delta', C}}{A \approx B, \Gamma' \Rightarrow \Delta', C}\ L^2_{\approx} \quad \overset{m'}{C, \Gamma'' \Rightarrow \Delta''}}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

This derivation is transformed into derivation with two cuts, each of which has lesser cut-height ($m + m'$ and $n + m'$ respectively):

$$\dfrac{\dfrac{A \approx B, \Gamma' \Rightarrow \Delta', C, B \quad C, \Gamma'' \Rightarrow \Delta''}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta'', B}\ cut \quad \dfrac{A \approx B, A, \Gamma' \Rightarrow \Delta', C \quad C, \Gamma'' \Rightarrow \Delta''}{A \approx B, A, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ L^2_{\approx}$$

(3) The last rule applied was $L^3_{\approx}$. Cut-height equals $m + 1 + m'$:

$$\dfrac{\dfrac{\overset{m}{(D \otimes F) \approx (E \otimes G), D \approx E, F \approx G, \Gamma' \Rightarrow \Delta', C}}{D \approx E, F \approx G, \Gamma' \Rightarrow \Delta', C}\ L^3_{\approx} \qquad \overset{m'}{C, \Gamma'' \Rightarrow \Delta''}}{D \approx E, F \approx G, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

This derivation is transformed into a derivation with lesser cut-height ($m + m'$):

$$\dfrac{\dfrac{(D \otimes F) \approx (E \otimes G), D \approx E, F \approx G, \Gamma' \Rightarrow \Delta', C \quad C, \Gamma'' \Rightarrow \Delta''}{(D \otimes F) \approx (E \otimes G), D \approx E, F \approx G, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut}{D \approx E, F \approx G, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ L^3_{\approx}$$

(4) The last rule applied was $L^{3*}_{\approx}$. Cut-height equals $m + 1 + m'$:

$$\dfrac{\dfrac{\overset{m}{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma' \Rightarrow \Delta', C}}{A \approx B, \Gamma' \Rightarrow \Delta', C}\ L^{3*}_{\approx} \qquad \overset{m'}{C, \Gamma'' \Rightarrow \Delta''}}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

This derivation is transformed into a derivation with lesser cut-height ($m + m'$):

$$\dfrac{\dfrac{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma' \Rightarrow \Delta', C \quad C, \Gamma'' \Rightarrow \Delta''}{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ L^{3*}_{\approx}$$

II When the cut-formula is principal in the left premiss only, we consider the last rule applied to the right premiss of cut. Some tranformations are analogous to the previous ones.

(1) The last rule applied was $L^1_{\approx}$. Cut-height equals $m + 1 + m'$:

$$\dfrac{\overset{m}{\Gamma' \Rightarrow \Delta', C} \qquad \dfrac{\overset{m'}{C, A \approx A, \Gamma'' \Rightarrow \Delta''}}{C, \Gamma'' \Rightarrow \Delta''}\ L^1_{\approx}}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

This derivation is transformed into a derivation of lesser cut-height (equal to $m + m'$):

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta', C \quad C, A \approx A, \Gamma'' \Rightarrow \Delta''}{A \approx A, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ L^1_{\approx}$$

(2) The last rule applied was $L^2_{\approx}$. The cut height equals $max(m,n)+1+m'$:

$$\dfrac{\overset{m'}{\Gamma'' \Rightarrow \Delta'', C} \quad \dfrac{\overset{m}{C, A \approx B, \Gamma' \Rightarrow \Delta', B} \quad \overset{n}{C, A \approx B, A, \Gamma' \Rightarrow \Delta'}}{C, A \approx B, \Gamma' \Rightarrow \Delta'}\, L^2_{\approx}}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, cut$$

This derivation is transformed into derivation with two cuts, each of which has lesser cut-height ($m + m'$ and $n + m'$ respectively):

$$\dfrac{\dfrac{\Gamma'' \Rightarrow \Delta'', C \quad C, A \approx B, \Gamma' \Rightarrow \Delta', B}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta'', B}\, cut \quad \dfrac{\Gamma'' \Rightarrow \Delta'', C \quad C, A \approx B, A, \Gamma' \Rightarrow \Delta'}{A \approx B, A, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, cut}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, L^2_{\approx}$$

(3) The last rule applied was $L^3_{\approx}$. The cut height equals $m + 1 + n$:

$$\dfrac{\overset{m}{\Gamma' \Rightarrow \Delta', E} \quad \dfrac{\overset{n}{(A \otimes C) \approx (B \otimes D), E, A \approx B, C \approx D, \Gamma'' \Rightarrow \Delta''}}{E, A \approx B, C \approx D, \Gamma'' \Rightarrow \Delta''}\, L^3_{\approx}}{A \approx B, C \approx D, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, cut$$

this derivation can be transformed into a derivation with cut height $m+n$:

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta', E \quad (A \otimes B) \approx (A \otimes B), E, A \approx A, B \approx B, \Gamma'' \Rightarrow \Delta''}{(A \otimes C) \approx (B \otimes D), A \approx B, C \approx D, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, cut}{A \approx B, C \approx D, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, L^3_{\approx}$$

(4) The last rule applied was $L^{3*}_{\approx}$. Cut-height equals $m + 1 + m'$:

$$\dfrac{\overset{m'}{\Gamma'' \Rightarrow \Delta'', C} \quad \dfrac{\overset{m}{(A \otimes A) \approx (B \otimes B), C, A \approx B, \Gamma' \Rightarrow \Delta'}}{C, A \approx B, \Gamma' \Rightarrow \Delta'}\, L^{3*}_{\approx}}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, cut$$

This derivation is transformed into a derivation with lesser cut-height ($m + m'$):

$$\dfrac{\dfrac{\Gamma'' \Rightarrow \Delta'', C \quad (A \otimes A) \approx (B \otimes B), C, A \approx B, \Gamma' \Rightarrow \Delta'}{(A \otimes A) \approx (B \otimes B), A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, cut}{A \approx B, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\, L^{3*}_{\approx}$$

III If the cut formula C is principal in both premises of the cut rule, only classical rules can be applied to C. ■

Table 4. Right rules for identity

$$\frac{}{\Gamma \Rightarrow \Delta, A \approx A}\ R^1_\approx \qquad \frac{A, \Gamma \Rightarrow \Delta, B, A \approx B}{A, \Gamma \Rightarrow \Delta, B}\ R^2_\approx$$

$$\frac{\Gamma \Rightarrow \Delta, (A \otimes B) \approx (C \otimes D), A \approx C \quad \Gamma \Rightarrow \Delta, (A \otimes B) \approx (C \otimes D), B \approx D}{\Gamma \Rightarrow \Delta, (A \otimes B) \approx (C \otimes D)}\ R^3_\approx$$

Note that in the cut rule:

$$\frac{\Gamma' \Rightarrow \Delta', C \quad C, \Gamma'' \Rightarrow \Delta''}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

the cut-formula C is not determined by anything that occurs in a conclusion. Thus the admissibility of the cut rule results in proof system for SCI, which is more useful in automatic proof search than the system with the cut rule.

4.2. Right Strategy

We now proceed to the right variant of sequent calculus for SCI, named $\mathbf{rG3}_{\mathsf{SCI}}$. As in the previous case, it is based on **G3cp** and constructed with the use of the strategy presented in Section 2.

As an example we prove the law of transitivity for equality in $\mathbf{rG3}_{\mathsf{SCI}}$ (where X stands for $(A \approx B) \approx (A \approx C)$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \approx B, B \approx C \Rightarrow A \approx C, X, A \approx A}\ R^1_\approx \quad A \approx B, B \approx C \Rightarrow A \approx C, X, B \approx C}{A \approx B, B \approx C \Rightarrow A \approx C, (A \approx B) \approx (A \approx C)}\ R^3_\approx}{A \approx B, B \approx C \Rightarrow A \approx C}\ R^2_\approx}{A \approx B \Rightarrow (B \approx C) \to (A \approx C)}\ R_\to}{\Rightarrow (A \approx B) \to ((B \approx C) \to (A \approx C))}\ R_\to$$

Soundness and completeness The following theorems have analogous proofs to those from Section 4.1; therefore we omit them to some extent.

THEOREM 4.9. (Soundness) *If a sequent is provable in* $\mathbf{rG3}_{\mathsf{SCI}}$*, then it is* SCI-*valid.*

THEOREM 4.10. (Simulation for $\mathbf{rG3}_{\mathsf{SCI}}$) *If A is provable in* $\mathbf{H}_{\mathsf{SCI}}$ *then the sequent* $\Rightarrow A$ *is provable in* $\mathbf{rG3}_{\mathsf{SCI}}$.

PROOF. As in $\mathbf{lG3}_{\mathsf{SCI}}$, we simulate the axiomatic system of SCI. The derivation of *modus ponens* is identical as in $\mathbf{lG3}_{\mathsf{SCI}}$. All SCI-specific axioms are provable. For example, the following derivation is a proof of axiom $\approx_4$, where

X stands for $(A \otimes C) \approx (B \otimes D)$.

$$\cfrac{\cfrac{\cfrac{A \approx B, C \approx D \Rightarrow X, A \approx B \quad A \approx B, C \approx D \Rightarrow X, C \approx D}{A \approx B, C \approx D \Rightarrow (A \otimes C) \approx (B \otimes D)}\, R^3_{\approx}}{(A \approx B) \wedge (C \approx D) \Rightarrow (A \otimes C) \approx (B \otimes D)}\, L_{\wedge}}{\Rightarrow ((A \approx B) \wedge (C \approx D)) \rightarrow ((A \otimes C) \approx (B \otimes D))}\, R_{\rightarrow}$$

■

Admissibility of weakening and contraction in rG3$_{\mathsf{SCI}}$ As before, proofs of admissibility of weakening and contraction follow the procedure described in [7]. We consider only non-classical cases.

THEOREM 4.11. (Admissibility of L_w and R_w) *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n A, \Gamma \Rightarrow \Delta$. *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n \Gamma \Rightarrow \Delta, A$.

PROOF. Adding the formula A in the antecedent and consequent in every sequent of the derivation of $\Gamma \Rightarrow \Delta$ will produce derivations of $A, \Gamma \Rightarrow \Delta$ in the former and $\Gamma \Rightarrow \Delta, A$ in the latter case. ■

THEOREM 4.12. (Admissibility of L_{ct} and R_{ct}) *If* $\vdash_n A, A, \Gamma \Rightarrow \Delta$, *then* $\vdash_n A, \Gamma \Rightarrow \Delta$. *If* $\vdash_n \Gamma \Rightarrow \Delta, A, A$, *then* $\vdash_n \Gamma \Rightarrow \Delta, A$.

PROOF. For derivation height $n = 0$, the sequents $A, A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, A, A$ are axioms or conclusions of either $\bot$ or $R^1_{\approx}$. If so, sequents $A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, A$ are axioms or conclusions of either $\bot$ or $R^1_{\approx}$ as well. For the inductive step, let us assume height-preserving left and right contraction for derivations of height up to n. Assume $\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta$ and $\vdash_{n+1} \Gamma \Rightarrow \Delta, A, A$. In both cases, if the formula A is not principal in the last used rule, we apply the inductive hypothesis to its premisses and then we apply that very rule (or the inversion lemma in the case of classical connectives), obtaining the contracted formula with derivation height of $n + 1$.

If the formula A is principal in the last step of the derivation, we have two cases in which the last rule used is non-classical.

1. The last rule applied was $R^2_{\approx}$. Then we have two subcases, depending on the side of the sequent on which the contracted formula occurs. For left contraction, $\Delta = \Delta', B$ and the derivation has the following form:

$$\cfrac{\vdash_n A, A, \Gamma \Rightarrow \Delta', B, A \approx B}{\vdash_{n+1} A, A, \Gamma \Rightarrow \Delta', B}\, R^2_{\approx}$$

   By applying the inductive hypothesis to the premiss, we have $\vdash_n A, \Gamma \Rightarrow \Delta', B, A \approx B$. Using the rule $R^2_{\approx}$, we obtain $\vdash_{n+1} A, \Gamma \Rightarrow \Delta', B$.

For right contraction, $\Gamma = B, \Gamma'$ and the derivation to consider is:

$$\frac{\vdash_n B, \Gamma' \Rightarrow \Delta, A, A, B \approx A}{\vdash_{n+1} B, \Gamma' \Rightarrow \Delta, A, A}\ R^2_\approx$$

An application of the inductive hypothesis to the premiss gives us $\vdash_n B, \Gamma' \Rightarrow \Delta, A, B \approx A$. With the use of rule $R^2_\approx$, we conclude $B, \Gamma' \Rightarrow \Delta, A$ in at most $n + 1$ steps.

2. The last rule applied is $R^3_\approx$. Then $A = (B \otimes C) \approx (D \otimes E)$. We need to consider the following derivation:

$$\frac{\vdash_n \Gamma \Rightarrow \Delta, A, A, B \approx D \quad \vdash_n \Gamma \Rightarrow \Delta, A, A, C \approx E}{\vdash_{n+1} \Gamma \Rightarrow \Delta, A, A}\ R^3_\approx$$

Applying the inductive hypothesis to the premisses gives us $\vdash_n \Gamma \Rightarrow \Delta, A, B \approx D$ and $\vdash_n \Gamma \Rightarrow \Delta, A, C \approx E$. Using $R^3_\approx$ we infer $\Gamma \Rightarrow \Delta, A$ in at most $n + 1$ steps. ■

The problem of cut-admissibility in $\mathbf{rG3}_{\mathsf{SCI}}$ Although weakening and contraction rules are admissible in $\mathbf{rG3}_{\mathsf{SCI}}$, the rule of cut seems to be essential in proving certain SCI-theses. Let us consider the following example[3]:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p, q \Rightarrow p \quad p, q \Rightarrow q}{p, q \Rightarrow p \wedge q}\ R_\wedge \qquad \dfrac{p \wedge q, (p \wedge q) \approx r \Rightarrow r, (p \wedge q) \approx r}{p \wedge q, (p \wedge q) \approx r \Rightarrow r}\ R^2_\approx}{p, q, (p \wedge q) \approx r \Rightarrow r}\ cut}{p, q, \Rightarrow ((p \wedge q) \approx r) \to r}\ R_\to}{p \Rightarrow q \to (((p \wedge q) \approx r) \to r)}\ R_\to}{\Rightarrow p \to (q \to (((p \wedge q) \approx r) \to r))}\ R_\to$$

The cut formula $p \wedge q$ is principal in both premises of the cut rule. The rule $R_\wedge$ has been applied to the left premiss of cut and the rule $R^2_\approx$ has been applied to the right premiss of cut. All leafs of the derivation above are axioms, therefore cut-height equals 2. If the rule applied to the right premiss of cut was $L_\wedge$, we would transform the following derivation:

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta', p \quad \Gamma' \Rightarrow \Delta', q}{\Gamma' \Rightarrow \Delta', p \wedge q}\ R_\wedge \qquad \dfrac{p, q, \Gamma'' \Rightarrow \Delta''}{p \wedge q, \Gamma'' \Rightarrow \Delta''}\ L_\wedge}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut$$

[3] I would like to thank Dorota Leszczyńska-Jasion for pointing out to me this particular example.

into:

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta', q \qquad \dfrac{\Gamma' \Rightarrow \Delta', p \quad p, q, \Gamma'', \Rightarrow \Delta''}{q, \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ cut}{\Gamma', \Gamma', \Gamma'' \Rightarrow \Delta', \Delta', \Delta''}\ cut}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}\ L_{ctr}, R_{ctr}$$

and the complexity of cut-formula would be reduced. Application of a similar strategy to the original example (we omit three consecutive applications of $R_{\to}$) gives us the following derivation:

$$\dfrac{\dfrac{q \Rightarrow q \qquad \dfrac{p \Rightarrow p \quad p, q, (p \wedge q) \approx r \Rightarrow r}{q, p, (p \wedge q) \approx r \Rightarrow r}\ cut}{p, p, q, (p \wedge q) \approx r \Rightarrow r}\ cut}{p, q, (p \wedge q) \approx r \Rightarrow r}\ L_{ctr}$$

in which we have reduced the complexity of cut formula but we have also obtained a problematic sequent labelling the rightmost leaf. Thus this derivation is not a proof of the sequent $p, q, (p \wedge q) \approx r \Rightarrow r$. There is a way to recover the provability of this sequent by applying contraction rules and the cut rule:

$$\dfrac{\dfrac{q \Rightarrow q \qquad \dfrac{p \Rightarrow p \qquad \dfrac{\dfrac{p, q \Rightarrow p \quad p, q \Rightarrow q}{p, q \Rightarrow p \wedge q}\ R_{\wedge} \quad \dfrac{p \wedge q, (p \wedge q) \approx r \Rightarrow r, (p \wedge q) \approx r}{p \wedge q, (p \wedge q) \approx r \Rightarrow r}\ R^2_{\approx}}{p, q, (p \wedge q) \approx r \Rightarrow r}\ cut}{q, p, (p \wedge q) \approx r \Rightarrow r}\ cut}{p, p, q, (p \wedge q) \approx r \Rightarrow r}\ cut}{p, q, (p \wedge q) \approx r \Rightarrow r}\ L_{ctr}$$

but this derivation leads us nowhere due to the fact that the cut height of the last application of the cut rule is not reduced compared to the original derivation (and is equal to 2).

We suppose that the cut rule is not admissible in $\mathbf{rG3}_{\mathsf{SCI}}$. Our diagnosis focuses on the role on axiom $\approx_2$, $(A \approx B) \to (B \to A)$, which corresponds to the right rule of the form:

$$\dfrac{A, \Gamma \Rightarrow \Delta, B, A \approx B}{A, \Gamma \Rightarrow \Delta, B}\ R^2_{\approx}$$

Note that the problematic rule is based on the so-called *bridge axiom* i.e. an axiom which governs the relation between non-classical connective, identity, and implication. Looking at this rule from the proof-search oriented perspective (i.e. bottom-up) we conclude that it allows us to synthesise identity on the right side of a premiss. The internal structure of A and B is not important in this synthesis which is the main reason why the strategy of reducing the complexity of cut-formulas does not work in this particular case of $R^2_{\approx}$. It is interesting that this problem does not arise in the left system, which causes deep asymmetry between left and right system for SCI.

5. Some Derivability Results

In the case of sequent calculi for SCI, proving some simple SCI-tautologies may be very complex due to the rule corresponding to the axiom $\approx_4$. Therefore it is interesting to study some derivable rules, which can be used in proof search in SCI.

5.1. Left System

Let us first prove that the rule:

$$\frac{B \approx A, \Gamma \Rightarrow \Delta}{A \approx B, \Gamma \Rightarrow \Delta}\ L_{sym}$$

is derivable in $\mathbf{lG3}_{\mathsf{SCI}}$ (Z stands for the formula $(B \approx A) \approx (B \approx B)$):

$$\dfrac{\dfrac{\dfrac{Z, B \approx B, A \approx B, \Gamma \Rightarrow \Delta, B \approx B \quad \dfrac{B \approx A, \Gamma \Rightarrow \Delta}{Z, B \approx A, B \approx B, A \approx B, \Gamma \Rightarrow \Delta}\ L_w}{Z, B \approx B, A \approx B, \Gamma \Rightarrow \Delta}\ L^2_{\approx}}{B \approx B, A \approx B, \Gamma \Rightarrow \Delta}\ L^3_{\approx}}{A \approx B, \Gamma \Rightarrow \Delta}\ L^1_{\approx}$$

Moreover, the transitivity rule:

$$\frac{A \approx C, A \approx B, B \approx C, \Gamma \Rightarrow \Delta}{A \approx B, B \approx C, \Gamma \Rightarrow \Delta}\ L_{trans}$$

is derivable in $\mathbf{lG3}_{\mathsf{SCI}}$ by (Z stands for the formula $(A \approx B) \approx (A \approx C)$):

$$\dfrac{\dfrac{\dfrac{Z, A \approx A, A \approx B, B \approx C, \Gamma \Rightarrow \Delta, A \approx B \quad \mathcal{D}}{Z, A \approx A, A \approx B, B \approx C, \Gamma \Rightarrow \Delta}\ L^2_{\approx}}{A \approx A, A \approx B, B \approx C, \Gamma \Rightarrow \Delta}\ L^3_{\approx}}{A \approx B, B \approx C, \Gamma \Rightarrow \Delta}\ L^1_{\approx}$$

where $\mathcal{D}$ is:

$$\frac{A \approx C, A \approx B, B \approx C, \Gamma \Rightarrow \Delta}{Z, A \approx C, A \approx A, A \approx B, B \approx C, \Gamma \Rightarrow \Delta}\ L_w$$

The following rule (*modus ponens* for $\approx$):

$$\frac{B, A, A \approx B, \Gamma \Rightarrow \Delta}{A, A \approx B, \Gamma \Rightarrow \Delta}\ L_{mp}$$

is derivable in $\mathbf{lG3}_{\mathsf{SCI}}$ by:

$$\dfrac{B \approx A, A, \Gamma \Rightarrow \Delta, A \quad \dfrac{B, A, A \approx B, \Gamma \Rightarrow \Delta}{B, A, B \approx A, \Gamma \Rightarrow \Delta}\,L_{sym}}{\dfrac{A, B \approx A, \Gamma \Rightarrow \Delta}{A, A \approx B, \Gamma \Rightarrow \Delta}\,L_{sym}}\,L^2_{\approx}$$

Let us consider the following substitution rule:

$$\frac{\lambda(B), \lambda(A), A \approx B, \Gamma \Rightarrow \Delta}{\lambda(A), A \approx B, \Gamma \Rightarrow \Delta}\,L_{sub}$$

where $\lambda(A)$ denotes a formula which has formula A as its subformula, and $\lambda(B)$ is a result of replacing some occurrences of A in $\lambda(A)$ by a formula B. Versions of the rule L_{sub} are used in [2,9] as well as in [3,12] to formalize SCI.

Proposition 1. *The rule L_{sub} is admissible in* $\mathbf{lG3}_{\mathsf{SCI}}$.

Proof. $\mathbf{lG3}_{\mathsf{SCI}}$ is proved to be complete with respect to SCI semantics and the rule L_{sub} is sound. ■

Moreover, we can prove the following:

Proposition 2. *The rule L_{sub} is derivable in* $\mathbf{lG3}_{\mathsf{SCI}}$ *with contraction rules and the cut rule.*

Proof. Let us consider the following application of *cut*:

$$\dfrac{\dfrac{\lambda(A), A \approx B \Rightarrow \lambda(B) \quad \lambda(B), \lambda(A), A \approx B, \Gamma \Rightarrow \Delta}{\lambda(A), \lambda(A), A \approx B, A \approx B, \Gamma \Rightarrow \Delta}\,cut}{\lambda(A), A \approx B, \Gamma \Rightarrow \Delta}\,L_{ctr}$$

Note that the sequent $\lambda(A), A \approx B \Rightarrow \lambda(B)$ is provable in $\mathbf{lG3}_{\mathsf{SCI}}$ for arbitrary formula $\lambda(A)$ since $\mathbf{lG3}_{\mathsf{SCI}}$ is complete with respect to SCI semantics and the aforementioned sequent is true under every SCI-valuation. So the left premiss of *cut* is provable and plays no role, whereas the right premiss is the premiss of L_{sub}. ■

Derivations which establish derivability of L_{sub} are typically complicated and consist of multiple applications of $L^3_{\approx}$ as is witnessed by the following example (for the sake of conciseness we omit formulas which are rewritten in the application of rules $L^2_\approx$ and $L^3_\approx$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(p\wedge q)\approx(p\wedge q)\Rightarrow(p\wedge q)\approx(p\wedge q)}{\Rightarrow(p\wedge q)\approx(p\wedge q)}\,L^1_\approx \qquad (p\wedge z)\approx(r\to s)\Rightarrow(p\wedge z)\approx(r\to s)}{((p\wedge z)\approx(r\to s))\approx((p\wedge q)\approx(p\wedge q))\Rightarrow(p\wedge z)\approx(r\to s)}\,L^2_\approx}{((p\wedge q)\approx(p\wedge q))\approx((p\wedge z)\approx(r\to s))\Rightarrow(p\wedge z)\approx(r\to s)}\,L_{sym}}{(p\wedge q)\approx(p\wedge z),(p\wedge q)\approx(r\to s)\Rightarrow(p\wedge z)\approx(r\to s)}\,L^3_\approx}{p\approx p,(p\wedge q)\approx(r\to s),q\approx z\Rightarrow(p\wedge z)\approx(r\to s)}\,L^3_\approx}{(p\wedge q)\approx(r\to s),q\approx z\Rightarrow(p\wedge z)\approx(r\to s)}\,L^1_\approx$$

5.2. Right System

We now proceed to show derivability of the right-sided equivalents of the aforementioned rules. They are constructed according to the right strategy as presented in Section 2. The right variant of the *modus ponens* rule for identity $A\wedge(A\approx B)\to B$:

$$\dfrac{\Gamma\Rightarrow\Delta,B,A \quad \Gamma\Rightarrow\Delta,B,A\approx B}{\Gamma\Rightarrow\Delta,B}\,R_{mp}$$

is derivable in $\mathbf{rG3}_{\mathsf{SCI}}$, albeit only if assuming cut, left weakening and contraction:

$$\dfrac{\dfrac{\Gamma\Rightarrow\Delta,B,A \qquad \dfrac{\dfrac{\Gamma\Rightarrow\Delta,B,A\approx B}{A,\Gamma\Rightarrow\Delta,B,A\approx B}\,L_w}{A,\Gamma\Rightarrow\Delta,B}\,R^2_\approx}{\Gamma,\Gamma\Rightarrow\Delta,\Delta,B,B}\,cut}{\Gamma\Rightarrow\Delta,B}\,L_{ctr},R_{ctr}$$

The right symmetry rule:

$$\dfrac{\Gamma\Rightarrow\Delta,B\approx A}{\Gamma\Rightarrow\Delta,A\approx B}\,R_{sym}$$

is derivable in $\mathbf{rG3}_{\mathsf{SCI}}$ by (where X stands for $(B\approx B)\approx(A\approx B)$):

$$\dfrac{\dfrac{}{\Rightarrow B\approx B}\,R^1_\approx \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma\Rightarrow\Delta,B\approx A}{\Gamma\Rightarrow\Delta,A\approx B,X,B\approx A}\,R_w}{B\approx B,\Gamma\Rightarrow\Delta,A\approx B,X,B\approx A}\,L_w \qquad \dfrac{}{\Gamma\Rightarrow\Delta,A\approx B,X,B\approx B}\,R^1_\approx}{B\approx B,\Gamma\Rightarrow\Delta,A\approx B,(B\approx B)\approx(A\approx B)}\,R^3_\approx}{B\approx B,\Gamma\Rightarrow\Delta,A\approx B}\,R^2_\approx}{\Gamma\Rightarrow\Delta,A\approx B}\,cut$$

The right transitivity rule:

$$\dfrac{\Gamma \Rightarrow \Delta, A \approx B, A \approx C \quad \Gamma \Rightarrow \Delta, B \approx C, A \approx C}{\Gamma \Rightarrow \Delta, A \approx C}\ R_{trans}$$

is derivable in $\mathbf{rG3}_{\mathsf{SCI}}$ by (where X stands for $(B \approx B) \approx (A \approx C)$ and Z stands for $B \approx B$):

$$\dfrac{\dfrac{}{\Rightarrow B \approx B}\ R^1_{\approx} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A \approx C, A \approx B}{Z, \Gamma \Rightarrow \Delta, A \approx C, X, A \approx B}\ L_w, R_w}{Z, \Gamma \Rightarrow \Delta, A \approx C, X, B \approx A}\ R_{sym} \quad \dfrac{\Gamma \Rightarrow \Delta, A \approx C, B \approx C}{Z, \Gamma \Rightarrow \Delta, A \approx C, X, B \approx C}\ L_w, R_w}{B \approx B, \Gamma \Rightarrow \Delta, A \approx C, X}\ R^3_{\approx}}{B \approx B, \Gamma \Rightarrow \Delta, A \approx C}\ R^2_{\approx}}{\Gamma \Rightarrow \Delta, A \approx C}\ L_{\approx}$$

Now we shall consider substitution in $\mathbf{rG3}_{\mathsf{SCI}}$. By applying the scheme R shown in Section 2 to the following axiom:

$$\lambda(A) \wedge A \approx B \to \lambda(B)$$

we obtain the right-sided equivealent of the rule L_{sub}:

$$\dfrac{\Gamma \Rightarrow \Delta, \lambda(B), \lambda(A) \quad \Gamma \Rightarrow \Delta, \lambda(B), A \approx B}{\Gamma \Rightarrow \Delta, \lambda(B)}\ R_{sub}$$

Proposition 3. *Assuming the rules of cut and contraction, the rule R_{sub} is derivable in* $\mathbf{rG3}_{\mathsf{SCI}}$.

Proof. The following derivation:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \lambda(B), \lambda(A) \qquad \dfrac{\Gamma \Rightarrow \Delta, \lambda(B), A \approx B \quad A \approx B, \lambda(A) \Rightarrow \lambda(B)}{\lambda(A), \Gamma \Rightarrow \Delta, \lambda(B), \lambda(B)}\ cut}{\Gamma, \Gamma \Rightarrow \Delta, \Delta, \lambda(B), \lambda(B), \lambda(B)}\ cut}{\Gamma \Rightarrow \Delta, \lambda(B)}\ L_{ctr}, R_{ctr}$$

establishes derivability of R_{sub}. Note that the rightmost premiss has to be provable due to the fact that $\mathbf{rG3}_{\mathsf{SCI}}$ is complete with respect to SCI-semantics. ■

6. Final Remarks

In this paper we have introduced two systems for the logic SCI, following the strategy described by Negri and von Plato [5]. We have showed that the cut rule is admissible in the system obtained by an application of the left strategy. We have also presented a conjecture according to which the rule of cut is not admissible in the right system. These calculi are not analytic (even if the cut rule is putted aside) in the sense that not all formulas occurring in a derivation for a given formula A have to be subformulas of A, nevertheless the structural rules of weakening and contraction are admissible in them.

Since our completeness proof is not direct and relays on Hilbert-style system for SCI, we have not developed a systematic decidability procedure for the introduced calculi—we consider it as one of the main open problems, which we intend to tackle in the future.

A detailed comparison between systems presented here and other systems for SCI would require a decidability procedure based on $\mathbf{lG3}_{\mathsf{SCI}}$ or $\mathbf{rG3}_{\mathsf{SCI}}$ and a definition of a canonical derivation. At the moment we are not in a possession of decidability procedure, therefore we will stay on the level of a few basic observations concerning the relation to the system based on dual tableaux [9]. As we mentioned, this formalisation of SCI is based on one specific rule:

$$\begin{array}{c} \lambda(A) \\ \hline A \approx B, \lambda(A) \quad \mid \quad \lambda(B), \lambda(A) \end{array}$$

where $\lambda(A)$ is a formula, which has A as its subformula and $\lambda(B)$ is obtained from $\lambda(A)$, by replacing some occurrences of formula A in $\lambda(A)$ by a formula B. We proved that the sequent version of this rule, obtained from the axiom

$$\lambda(A) \wedge A \approx B \rightarrow \lambda(B) \tag{2}$$

by the right strategy is derivable in the system $\mathbf{rG3}_{\mathsf{SCI}}$ (without cut-rule) and the rule obtained from axiom 2 by the left strategy is derivable in $\mathbf{rG3}_{\mathsf{SCI}} + cut$.

This observation is not supposed to mean that any of the systems introduced are better suited for proof search than dual tableaux system. This only means that presented systems are general enough to simulate derivations performed in dual tableaux formalisation of SCI.

Moreover we have shown that there is an asymmetry between the left and right strategy in the case of SCI. We suppose that this difference (which is reflected in non-admissibility of the cut rule in the right system) is caused by non-atomic character of SCI axioms. The question of possible extensions of Negri strategy remains open.

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