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Distributivity for Upper Continuous and Strongly Atomic Lattices

Abstract. In the paper we introduce two conditions (D) and (D*) which are strengthenings of Birkhoff's conditions. We prove that an upper continuous and strongly atomic lattice is distributive if and only if it satisfies (D) and (D*). This result extends a theorem of R.P. Dilworth characterizing distributivity in terms of local distributivity and a theorem of M. Ward characterizing distributivity by means of covering diamonds.

Keywords: Distributive lattice, Strongly atomic lattice, Upper continuous lattice, Birkhoff's conditions, Covering conditions, Locally distributive lattice, Covering diamond.

1. Preliminaries

Let L be a lattice. L is *distributive* if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$. A lattice L is *modular* if $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in L$. Clearly, distributive lattices are modular.

Figure 1 presents Hasse diagrams of lattices considered in the paper. B_2 is a 4-element Boolean lattice, which is obviously distributive. Lattice M_3 is modular but non-distributive. On the other hand lattices N_5, S_7 and S_7^* are non-modular.

A well known Dedekind–Birkhoff Theorem characterizes modularity and distributivity in terms of forbidden sublattices.

Theorem 1. ([5], p. 59) *A lattice L is modular if and only if L does not contain a sublattice isomorphic to N_5. A lattice L is distributive if and only if L does not contain a sublattice isomorphic to N_5 nor M_3.*

If $x, y \in L$ we say that x is *covered* by y, $x \prec y$, if $x < y$ and there is no element $z \in L$ such that $x < z < y$. Every modular lattice L satisfies the so-called *upper* and *lower Birkhoff's conditions*:

$$(\forall x, y \in L)(x \wedge y \prec x, y \Rightarrow x, y \prec x \vee y), \tag{Bi}$$

$$(\forall x, y \in L)(x, y \prec x \vee y \Rightarrow x \wedge y \prec x, y). \tag{Bi*}$$

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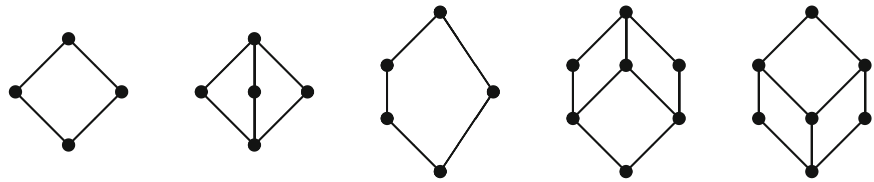

Figure 1. Lattices B_2, M_3, N_5, S_7 and S_7^*

We easily see that both conditions (Bi) and (Bi*) are true in B_2 and M_3, but false in N_5. The smallest lattice which satisfies (Bi) and violates (Bi*) is S_7. Similarly, the dual lattice S_7^* is the smallest lattice satisfying (Bi*) and violating (Bi).

G. Birkhoff proved that for lattices of finite length modularity is equivalent to the conjunction of (Bi) and (Bi*) (see [5], pp. 172–174). In [8] we extended Birkhoff's result for a larger class containing also lattices of infinite length. Recall that a lattice L is said to be *upper continuous* if L is complete and the following condition is satisfied for any $x \in L$ and for any chain $C \subseteq L$:

$$x \wedge \bigvee C = \bigvee \{x \wedge c : c \in C\}. \tag{UC}$$

A lattice L is called *strongly atomic* if

$$(\forall x, y \in L)\big(x < y \Rightarrow (\exists z \in L)(x \prec z \leq y)\big). \tag{SA}$$

Theorem 2. ([8], Proposition 4) *Let L be an upper continuous and strongly atomic lattice. Then L is modular if and only if L satisfies conditions* (Bi) *and* (Bi*).

Let us remind another consequences of modularity, the so-called *upper* and *lower covering conditions*:

$$(\forall x, y, z \in L)(x \prec y \Rightarrow x \vee z \preceq y \vee z), \tag{UCC}$$

$$(\forall x, y, z \in L)(x \prec y \Rightarrow x \wedge z \preceq y \wedge z). \tag{LCC}$$

It is easy to see that (UCC) implies (Bi), and (LCC) implies (Bi*), but the converses do not hold in general. Although, in the class of upper continuous and strongly atomic lattices the conjunction of (UCC) and (LCC) is equivalent to the conjunction of (Bi) and (Bi*) (this is a consequence of Theorem 2), the use of (UCC) and (LCC) is much more comfortable in practice. Other properties closely related to Birkhoff's conditions are discussed in [9, 10].

2. The Main Result

In this section we give an analogous to Theorem 2 characterization of distributivity in the class of upper continuous and strongly atomic lattices. Consider the following conditions:

$$(\forall x, y \in L)(x \wedge y \prec x, y \Rightarrow [x \wedge y, x \vee y] \cong B_2), \tag{D}$$

$$(\forall x, y \in L)(x, y \prec x \vee y \Rightarrow [x \wedge y, x \vee y] \cong B_2). \tag{D*}$$

Obviously, (D) and (D*) are consequences of distributivity, (D) implies (Bi), and (D*) implies (Bi*). Moreover, on the ground of Birkhoff's conditions, (D) is equivalent to (D*). In [7] we observed that if L is a lattice of finite length then the conjunction of (D) and (D*) implies the distributivity of L. Now we extend this result to the class of upper continuous and strongly atomic lattices.

The following simple observation we will use repeatedly.

Fact. *Let L be an arbitrary lattice and $v, a, b, c, u \in L$. If $a \wedge b = b \wedge c = a \wedge c = v$ and $a \vee b = b \vee c = a \vee c = u$ and $b \neq u$, then $\{v, a, b, c, u\} \cong M_3$.*

Theorem 3. *Let L is an upper continuous and strongly atomic lattice. Then L is distributive if and only if L satisfies conditions* (D) *and* (D*).

Proof. The implication ($\Rightarrow$) is obvious. To prove the implication ($\Leftarrow$) assume that L satisfies conditions (D) and (D*), and suppose that L is not distributive. Clearly, L satisfies (Bi) and (Bi*), so, by Theorem 2, L is modular, henceforth by Theorem 1, there exists a sublattice $\{v, a, b, c, u\}$ of a lattice L isomorphic to M_3.

Let $P = [a, u] \times [c, u]$. For every $(x, y) \in P$ put $M(x, y) = \{x \wedge b \wedge y, x, b, y, u\}$ and consider the set T defined as follows:

$$T = \{(x, y) \in P : M(x, y) \cong M_3\},$$

(see the first picture in Figure 2). T is partially ordered by the relation

$$(x, y) \leq (x', y') \quad \Leftrightarrow \quad x \leq x' \text{ and } y \leq y'.$$

Since $(a, c) \in T$, T is nonempty. Let $\{(x_i, y_i) : i \in I\}$ be a chain in T. To show that $(T, \leq)$ satisfies the premise of the Kuratowski–Zorn Lemma we need to prove that $(\overline{x}, \overline{y}) \in T$, where $\overline{x} = \bigvee_{i \in I} x_i$ and $\overline{y} = \bigvee_{i \in I} y_i$. Obviously:

$$(\overline{x}, \overline{y}) \in P \quad \text{and} \quad \overline{x} \vee b = b \vee \overline{y} = \overline{x} \vee \overline{y} = u. \tag{1}$$

To prove

$$\overline{x} \wedge b = b \wedge \overline{y} \tag{2}$$

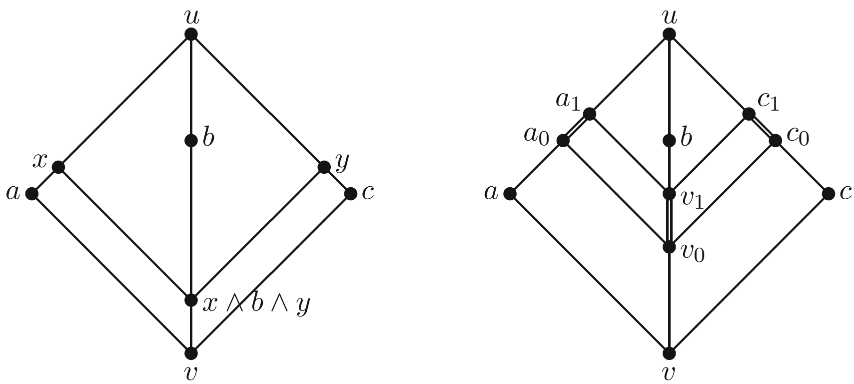

Figure 2. The illustration of the proof of Theorem 3

we employ (UC) calculating as follows:

$$\overline{x} \wedge b = b \wedge \bigvee_{i\in I} x_i = \bigvee_{i\in I}(b \wedge x_i) = \bigvee_{i\in I}(b \wedge y_i) = b \wedge \bigvee_{i\in I} y_i = b \wedge \overline{y}.$$

Now we will show that

$$b \wedge \overline{y} = \overline{x} \wedge \overline{y}. \tag{3}$$

Observe firstly, that for $i, j \in I$ there is $k \in I$ such that $x_i \wedge y_j \leq x_k \wedge y_k$. Indeed, fix $i, j \in I$; without loss of generality we can assume that $(x_i, y_i) \leq (x_j, y_j)$, then putting $k = j$ we get $x_i \wedge y_j \leq x_k \wedge y_k$. Henceforth, applying (UC) we compute:

$$\overline{x} \wedge \overline{y} = \overline{x} \wedge \bigvee_{j\in I} y_j = \bigvee_{j\in I}(\overline{x} \wedge y_j) = \bigvee_{j\in I}\big(y_j \wedge \bigvee_{i\in I} x_i\big) = \bigvee_{j\in I}\big(\bigvee_{i\in I}(y_j \wedge x_i)\big) =$$
$$= \bigvee_{k\in I}(y_k \wedge x_k) = \bigvee_{k\in I}(b \wedge y_k) = b \wedge \bigvee_{k\in I} y_k = b \wedge \overline{y}.$$

Directly by (3), we obtain

$$\overline{x} \wedge \overline{y} = \overline{x} \wedge b \wedge \overline{y}. \tag{4}$$

By (1)–(4) and $b < u$, the Fact provides that $\{\overline{x} \wedge b \wedge \overline{y}, \overline{x}, b, \overline{y}, u\} \cong M_3$, and consequently $(\overline{x}, \overline{y}) \in T$.

According to the Kuratowski–Zorn Lemma, there is a maximal pair (a_0, c_0) in T. Hence, putting $v_0 = a_0 \wedge b \wedge c_0$, we have $\{v_0, a_0, b, c_0, u\} \cong M_3$ (see the second picture in Figure 2).

Let us observe that $v_0 \not\prec b$. Indeed, if $v_0 \prec b$ then, by (UCC), $a_0 \prec u$ and $c_0 \prec u$, so by (D*), we get $[v_0, u] = [a_0 \wedge c_0, a_0 \vee c_0] \cong B_2$, which is impossible. Henceforth, by (SA), there exists v_1 such that $v_0 \prec v_1 < b$.

Defining $a_1 = a_0 \vee v_1$ and $c_1 = c_0 \vee v_1$, by (UCC), we easily achieve that $a_0 \prec a_1$ and $c_0 \prec c_1$. We will show that $(a_1, c_1) \in T$ which will contradict the maximality of (a_0, c_0).

Obviously, $(a_1, c_1) \in P$ and $a_1 \vee b = b \vee c_1 = a_1 \vee c_1 = u$. Moreover, by modularity, we get

$$a_1 \wedge b = (v_1 \vee a_0) \wedge b = v_1 \vee (a_0 \wedge b) = v_1 \vee v_0 = v_1.$$

Similarly we obtain $b \wedge c_1 = v_1$. To prove $a_1 \wedge c_1 = v_1$, firstly we show that:

$$a_1 \wedge c_0 = v_0 \quad \text{ or } \quad a_0 \wedge c_1 = v_0. \tag{5}$$

To the contrary, suppose that $a_1 \wedge c_0 \neq v_0$ and $a_0 \wedge c_1 \neq v_0$. Then (LCC) applied to $a_0 \prec a_1$ and $c_0 \prec c_1$ provides $v_0 \prec a_1 \wedge c_0$ and $v_0 \prec a_0 \wedge c_1$. But $(a_1 \wedge c_0) \wedge (a_0 \wedge c_1) = v_0$, therefore, by (D), we get

$$[(a_1 \wedge c_0) \wedge (a_0 \wedge c_1), (a_1 \wedge c_0) \vee (a_0 \wedge c_1)] \cong B_2.$$

Now, using modularity, we calculate:

$$\begin{aligned}(a_1 \wedge c_0) \vee (a_0 \wedge c_1) = \big((a_1 \wedge c_0) \vee a_0\big) \wedge c_1 = \big(a_1 \wedge (c_0 \vee a_0)\big) \wedge c_1 = \\ = (a_1 \wedge u) \wedge c_1 = a_1 \wedge c_1,\end{aligned}$$

therefore $[v_0, a_1 \wedge c_1] \cong B_2$, and hence $[v_0, a_1 \wedge c_1] = \{v_0, a_1 \wedge c_0, a_0 \wedge c_1, a_1 \wedge c_1\}$. On the other hand, $v_1 \in [v_0, a_1 \wedge c_1]$. The contradiction completes the proof of (5).

According to (5), without loss of generality we can assume that $a_1 \wedge c_0 = v_0$. Then, using modularity, we compute:

$$a_1 \wedge c_1 = a_1 \wedge (c_0 \vee v_1) = (a_1 \wedge c_0) \vee v_1 = v_0 \vee v_1 = v_1.$$

Finally, since $b < u$ the Fact provides that $\{v_1, a_1, b, c_1, u\} \cong M_3$ and therefore $(a_1, c_1) \in T$. This contradiction completes the proof. ∎

3. Concluding Remarks

If L is a complete lattice, then for any $x \in L$ define $x^+ = \sup\{p \in L : x \prec p\}$. L is said to be *upper locally distributive* if:

$$[x, x^+] \text{ is a Boolean sublattice of } L \tag{ULD}$$

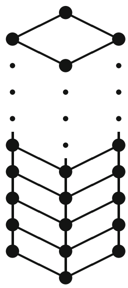

Figure 3. The upper continuous and strongly atomic lattice which satisfies (D^*) but violates (LLD)

for any $x < 1$ (see [4], Definition 6.1). Obviously, (ULD) implies (D). The converse implication is true for strongly atomic and upper continuous lattices (see [4], Corollary after Theorem 6.1[1]).

Dually one can define *lower local distributivity*, (LLD). As before, (LLD) easily implies (D^*), but the converse is not true even in the class of upper continuous and strongly atomic lattices (see Figure 3).

Dilworth proved (see [3], Corollary 1.4) that for a lattice with unit element in which every interval satisfies the ascending and descending chain conditions distributivity is equivalent to the conjunction of conditions (ULD) and (LLD). Our Theorem 3 gives the following strengthening of this result.

Corollary 1. *Let L be an upper continuous and strongly atomic lattice. Then L is distributive if and only if L satisfies both* (ULD) *and* (LLD).

Let us point out other application of Theorem 3. A sublattice $\{v, a, b, c, u\}$ of a given lattice L is called a *covering diamond of* L if $\{v, a, b, c, u\} \cong M_3$ and moreover $v \prec a, b, c \prec u$ (see [6], pp. 111, 326). In [11] M. Ward proved that a modular, non-distributive lattice satisfying the ascending chain condition contains a covering diamond. By Theorem 3, we get an extension of the Ward's result:

Corollary 2. *Let L be an upper continuous, strongly atomic, modular but non-distributive lattice. Then L contains a covering diamond.*

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[1]The original Dilworth's claim is: for an upper continuous and strongly atomic lattice L the following conditions are equivalent: (1) L is upper locally distributive, and (2) for every set of four distinct elements $a, p_1, p_2, p_3 \in L$ for which $a \prec p_1, p_2, p_3$, the interval $[a, p_1 \vee p_2 \vee p_3]$ is an eight-element Boolean lattice (see also [2], p. 53). Note that this formulation is incorrect, since a pentagon satisfies (2) but not (1). In fact, (1) and (2) are equivalent in modular lattices. Moreover, (D) implies (2) in general, and (2) implies (D) in modular lattices (comp. also [10], Corollary 7.1.4).

PROOF. Since L is non-distributive then, by Theorem 3, (D) or (D*) is violated. Without loss of generality, assume that (D) is false, therefore there are $x, y \in L$ such that $x \wedge y \prec x, y$ but $[x \wedge y, x \vee y] \ncong B_2$. Then there exists $z \in L$ such that $z \in [x \wedge y, x \vee y]$ but $z \notin \{x \wedge y, x, y, x \vee y\}$. However L is modular, so by (Bi), we obtain $x, y \prec x \vee y$, and hence we easily show that $x \vee y = y \vee z = x \vee z$ and $x \wedge y = y \wedge z = x \wedge z$, so the Fact provides that $\{x \wedge y, x, y, z, x \vee y\} \cong M_3$. Moreover, $x \wedge y \prec z \prec x \vee y$, i.e. $\{x \wedge y, x, y, z, x \vee y\}$ is a covering diamond. ■

COROLLARY 3. *Let L be an upper continuous, strongly atomic, and modular lattice. Then L is distributive if and only if every interval $[x, y] \subseteq L$ of finite length is distributive.*

PROOF. The implication ($\Rightarrow$) is trivial. To prove the implication ($\Leftarrow$) assume that L is non-distributive. By Corollary 2, L contains a covering diamond $\{v, a, b, c, u\}$. Hence the interval $[v, u]$ is non-distributive and contains a maximal chain of the length equal 2. However, L as a modular lattice satisfies the Jordan–Dedekind chain condition, so by the Croisot–Szasz Theorem (see [10], Theorem 1.9.1), every maximal chain in $[v, u]$ has the length equal 2. ■

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