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Normal Modal Logics Determined by Aligned Clusters

Abstract. We consider the family of logics from NExt(**KTB**) which are determined by linear frames with reflexive and symmetric relation of accessibility. The condition of linearity in such frames was first defined in the paper [9]. We prove that the cardinality of the logics under consideration is uncountably infinite.

Keywords: Brouwer modal logic, Kripke frames, Characteristic formulas, Normal extensions.

1. Introduction

Since the emergence of Kripke semantics, the semantical analysis of propositional modal logics has achieved a great success for modal logics with the *transitivity axiom*, or at least a *weak transitivity axiom*. In contrast to these rich harvests, modal logics without weak transitivity axioms seem to remain almost untouched, and a further investigation must be needed in order to open a next door of the study of modal logics.

The Brouwer logic **KTB** is a normal extension of the minimal normal modal logic **K** by adding the following axioms:

$$T := \Box p \to p$$
$$B := p \to \Box\Diamond p$$

Semantically, it is determined by the class of reflexive and symmetric frames (admitting non-transitivity). Hence, **KTB** is said to be a non-transitive logic. Adding transitivity gives us the Lewis logic **S5**. The feature of transitivity (or, at least weak transitivity) for frames is very desirable by modal logicians. Thus, the logics located in the interval **S4**–**S5** are intensively studied. Also, for weak transitive logics there are known some important results mostly connected with Kripke incompleteness (see [6–8,11,12]). In contrast to these two families of logics, the family of non-transitive logics has not been thoroughly examined yet.

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In this paper we deal with non-transitive logics and continue research initiated in the paper [9]. Actually, we extract from the whole family NEXT(**KTB**) a sub-family of logics determined by frames having linear shape. Our motivation for such a choice has two sources. One is the logic **S4.3** := **S4** $\oplus$ (3), where:

$$(3) \ := \Box(\Box p \to q) \vee \Box(\Box q \to p).$$

It is complete with respect to linearly quasi ordered frames (xRy or yRx for any distinct $x, y \in W$). They are usually presented as chains of *clusters*. A cluster in a Kripke frame $\langle W, R\rangle$ is a maximal subset $C \subseteq W$ such that for all $x, y \in C$ xRy. In a reflexive and transitive frame, all clusters turn out to be disjoint. The famous results for **S4.3** and its normal extensions are the following (see, for example [1]):

THEOREM 1.1. (Bull's Theorem) *Every normal modal logic extending* **S4.3** *has the finite model property (f.m.p).*

THEOREM 1.2. (Fine's Theorem) *Every normal modal logic extending* **S4.3** *is finitely axiomatizable (and hence—decidable).*

The second source for our motivation comes from a normal modal logic **KTBAlt(3)** := **KTB** $\oplus$ alt_3, where:

$$(alt_3) := \Box p \vee \Box(p \to q) \vee \Box((p \wedge q) \to r)) \vee \Box((p \wedge q \wedge r) \to s).$$

This logic is determined by the class of reflexive and symmetric frames forming, either chains of points, or circles of points. It is proved in [2,3] that all logics from NEXT(**KTBAlt(3)**), have also very strong properties.

THEOREM 1.3. (Byrd and Ullrich [2] and Byrd [3]) *Every normal modal logic extending* **KTBAlt(3)** *has the finite model property and is finitely axiomatizable (and hence—decidable).*

It is easily seen by the above theorem that the cardinality of the class NEXT(**KTBAlt(3)**) is only countably infinite. This means that this is rather a nice subclass of modal logics in NEXT(**KTB**).

It is, here, worth comparing the above result with those of Bull's and Fine's. For modal logics from NEXT(**S4.3**), all clusters are disjoint in a frame for those logics, because of transitivity, and so every frame for them can be uniquely represented as a chain of clusters. However, in connected KTB-frames, clusters are not always disjoint. Thus a representation of frames for logics in NEXT(**KTB**) must be a little different. In a reflexive and symmetric Kripke frame, some clusters have non-empty intersection that plays a role of a link between them. In spite of this big difference, it is helpful to consider clusters in frames for logics in NExt(**KTB**). It has to be emphasized here that (alt_3) permits the existence of two-element clusters, at most. There is space here for extending their class of logics to a wider and still a gentle one.

In this paper we will consider a more general condition of *linearity* in reflexive and symmetric frames. We allow for the existence of n-element clusters for any $n \in \mathbb{N}$. The appropriate requirements are defined in [9] and in [10] (see also the next section). Then, the logic determined by such a class of frames is axiomatized as follows: $\mathbf{KTB.3'A} := \mathbf{KTB} \oplus 3' \oplus A$ where:

$$(3') := \Box p \vee \Box(\Box p \to \Box q) \vee \Box((\Box p \wedge \Box q) \to r),$$

$$(A) := \Box((\Box p \wedge q) \to r) \vee \Box((\Box q \wedge r) \to s) \vee \Box((\Box r \wedge s \wedge \Diamond\neg s) \to p) \vee \vee\Box((\Box s \wedge p \wedge \Diamond\neg p) \to q).$$

A theorem similar to Theorems 1.1 and 1.3, is also proved for logics above **KTB.3′A** in [9], (see also [10]).

Theorem 1.4. *Every normal modal logic extending* **KTB.3′A** *has the finite model property.*

We see that all logics from those three families NExt(**KTB.3′A**), NExt(**S4.3**) and NExt(**KTBAlt(3)**) have the f.m.p. Thus, a question about decidability of logics from the first family arises. It depends on the answer of the following problem from [9]:

Problem 1. *What is the cardinality of the class* NExt(**KTB.3′A**)*?*

In this paper we will solve this problem.

2. Preliminaries

In this section we remind the basic definitions from [9]. We apply a frame-theoretic approach here.

Definition 2.1. Relation R is called a tolerance if it is reflexive and symmetric.

Definition 2.2. A non-empty subset $U \subseteq W$ is called a block of the tolerance R, if U is a maximal subset with $U \times U \subseteq R$ (if $U \subseteq V$ and $V \times V \subseteq R$, then $U = V$).

Note that the two notions *cluster* and *block of tolerance* coincide. But we prefer to use the second one since, in our case, clusters sometimes have non-empty intersections. Then we define:

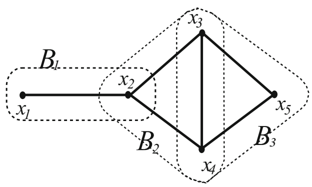

Diagram 1. A frame with linearly ordered blocks

Definition 2.3. We say that a frame $\langle W, R\rangle$ consists of linearly ordered blocks if the following two conditions hold:

$(L1)$ $B_1 \cap B_2 \cap B_3 = \emptyset$,

$(L2)$ $(B_1 \cap B_2 \neq \emptyset \ \&\ B_2 \cap B_3 \neq \emptyset) \ \Rightarrow \ (B_1 \cap B_2) \cup (B_2 \cap B_3) = B_2$

for any three distinct blocks B_1, B_2, B_3

Below, we give two examples.

Example 2.4. Suppose $W := \{x_1, x_2, x_3, x_4, x_5\}$ and R is symmetric and reflexive, and additionally the following points are related (and only these points): x_1Rx_2, x_2Rx_3, x_2Rx_4, x_3Rx_4, x_3Rx_5, x_4Rx_5 (see Diagram 1). Then the tolerance has three blocks: $B_1 = \{x_1, x_2\}$, $B_2 = \{x_2, x_3, x_4\}$, $B_3 = \{x_3, x_4, x_5\}$. They are linearly ordered.

Example 2.5. Suppose $W := \{x_1, x_2, x_3, x_4, x_5\}$, R is symmetric and reflexive, and additionally the following points are related (and only these points): x_1Rx_2, x_2Rx_3, x_2Rx_4, x_2Rx_5, x_3Rx_5, x_3Rx_4 (see Diagram 2). Then the tolerance has three blocks: $B_1 = \{x_1, x_2\}$, $B_2 = \{x_2, x_3, x_4\}$, $B_3 = \{x_2, x_3, x_5\}$. They are not linearly ordered since $B_1 \cap B_2 \cap B_3 = \{x_2\}$.

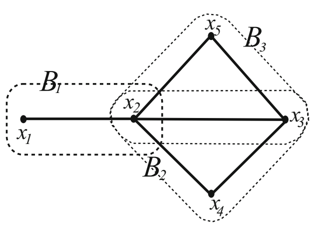

Diagram 2. A frame with blocks, which are not linearly ordered

The class of reflexive and symmetric frames with linearly ordered blocks will be marked by $\mathcal{LOB}$. We may consider two types of frames from this class: open and closed. In an open frame we can distinguish the first and the last

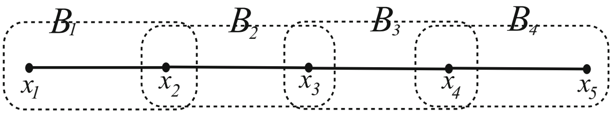

Diagram 3. An open frame from the class $\mathcal{LOB}$

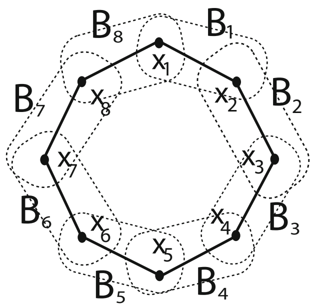

Diagram 4. A closed frame from the class $\mathcal{LOB}$

blocks of the tolerance. Each of them sees only one block. Examples of such open frames from $\mathcal{LOB}$ are presented in Diagrams 1 and 3. In closed frames each block sees two other distinct blocks of tolerance. See Diagram 4.

In this paper we will deal with open frames, only. We briefly recall the definition of a p-morphism between Kripke frames.

Definition 2.6. Let $\mathfrak{F}_1 = \langle W_1, R_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2\rangle$ be Kripke frames. A map $f : W_1 \to W_2$ is a p-morphism from $\mathfrak{F}_1$ to $\mathfrak{F}_2$, if it satisfies the following conditions:

- (p1) f is from W_1 onto W_2,
- (p2) for all $x, y \in W_1$, xR_1y implies $f(x)R_2f(y)$,
- (p3) for each $x \in W_1$ and for each $a \in W_2$, if $f(x)R_2a$ then there exists $y \in W_1$ such that xR_1y and $f(y) = a$.

3. The Existence of a Contiuum in NExt(KTB.3′A)

In this section, we show that there exists a continuum of normal modal logics in $\mathrm{NExt}(\mathbf{KTB.3'A})$. We utilize an infinite sequence $\mathcal{S} = \{\mathcal{F}_k\}_{k\geq 1}$ of Kripke frames in $\mathcal{LOB}$ and the characteristic formulas for such frames, to prove that the sequence $\{\mathbf{L}(\mathcal{F}_k)\}_{k\geq 1}$ of logics of the frames determines

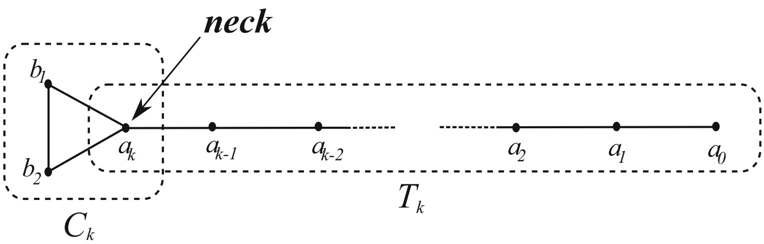

Diagram 5. The frame $\mathcal{F}_k$

a class of infinite mutually incomparable logics, and that every different subclass of $\mathcal{S}$ defines a different logic.

For each $k \geq 1$, the frame $\mathcal{F}_k := \langle W_k, R_k \rangle$ is defined as follows:

$$W_k := T_k \cup C_k, where T_k := \{a_i \mid 0 \leq i \leq k\}, \text{ and } C_k := \{b_1, b_2, a_k\},$$
$$R_k := \{(x,x) \mid x \in T_k\} \cup \{(a_i, a_{i+1}), (a_{i+1}, a_i) \text{ for } 0 \leq i \leq k-1\}$$
$$\cup\{(x,y) \mid x, y \in C_k\}.$$

In $\mathcal{F}_k$, T_k is a *tail* part, that consists of an undirected chain of $k+1$ reflexive points, whereas C_k is a three-point-*cluster* part, and these two parts are connected by a point $a_k \in T_k \cap C_k$. This point will play a significant role in our proof, and so we call this point a_k a *neck*.

For each $\mathcal{F}_k$ $(k \geq 1)$, we define a *characteristic formula* δ_k. Characteristic formulas were first introduced for intuitionistic logic (and Heyting algebras) by V.Jankov [5]; for modal logics they were modified by K. Fine [4]. First of all, we prepare a finite set P_k of propositional variables, that correspond to points in W_k. That is, we associate p_i with a point $a_i \in T_k$ for each i $(0 \leq i \leq k)$ and p_{k+1} for b_1 and p_{k+2} for b_2. Then, the *diagram* Δ_k of this frame $\mathcal{F}_k$ is defined as:

$$\Delta_k := \{p_x \to \Diamond p_y \mid xRy\} \cup \{p_x \to \neg\Diamond p_y \mid \neg(xRy)\} \cup \{p_x \to \neg p_y \mid x \neq y\}$$
$$\cup\{\bigvee_{x \in W_k} p_x\}$$

Then the characteristic formula δ_k for the frame $\mathcal{F}_k$ is just the conjunction of this diagram, that is, $\delta_k := \bigwedge \Delta_k$. Here we use the formula $\sigma_k := \Box^{k+2}\delta_k \wedge p_0$. The following lemma is crucial for our task.

Lemma 3.1. *For any $m, n \geq 1$, σ_m is satisfiable in $\mathcal{F}_n$ if and only if $m = n$.*

Proof. $(\Longleftarrow)$

If $m = n$, we define a valuation V_0 on $\mathcal{F}_n$ as: $V_0(p_i) := \{a_i\}$ for $0 \leq i \leq m$, and $V_0(p_{m+j}) := \{b_j\}$ for $j = 1, 2$. Then it is obvious that σ_m is satisfiable at the point a_0 in a model $\langle \mathcal{F}_n, V_0 \rangle$.
$(\Longrightarrow)$
Suppose that σ_m is satisfiable in $\mathcal{F}_n$ and $m > n$. Formula σ_m includes the following sub-formulas: $p_i \rightarrow \Diamond p_{i+1}$ for $i := 0, 1, 2, \ldots, m+1$. The range of σ_m is the whole frame $\mathcal{F}_n$ because the frame $\mathcal{F}_n$ consist of $n + 3$ points with $n < m$. Then obviously there is at least one point in $\mathcal{F}_n$, at which two distinct variables p_i and p_j must be true. Since σ_m includes also sub-formulas $p_i \rightarrow \neg p_j$ for $i \neq i$ and $i, j := 0, 1, 2, \ldots, m+2$ then we see that for any valuation in this case the formula σ_m is not satisfiable. Then we get a contradiction.
Suppose then that σ_m is satisfiable in $\mathcal{F}_n$ and $m < n$.
One may notice that a_m is the only point in $\mathcal{F}_m$ that is related by R_m to three different points except for itself, that is, b_1, b_2 and a_{m-1} in $\mathcal{F}_m$. Therefore we find that p_m is true at nowhere else but at a_n in $\mathcal{F}_n$. Then, variables p_{m+1} and p_{m+2} can be satisfied at b_1, b_2 in $\mathcal{F}_n$. For variables for the tail part in $\mathcal{F}_m$, p_{m-1} must be true at a_{n-1}, p_{m-2} must be true at a_{n-2}, and finally we reach the fact that p_0 must be true at the point a_{n-m}, in the middle of the tail in $\mathcal{F}_n$ since $m < n$. Hence, we see that the range of formula σ_m is $m + 2$ in both directions from the point a_{n-m}.
Case 1. $n - m \leq m$. To match the valuation in the other part of the tail we may choose for the next point a_{n-m-1} either p_0 or p_1. It is because in σ_m we have the sub-formulas $p_0 \rightarrow \Diamond p_0$, $p_0 \rightarrow \Diamond p_1$ and $p_0 \rightarrow \neg\Diamond p_i$, for $i \neq 0, 1$.
Sub-case 1a. Suppose that we choose p_0. Since in σ_m there are also sub-formulas $p_i \rightarrow \Diamond p_{i+1}$, for $i = 0, 1, \ldots, m+2$ then at the next points a_k's with $n - m - 2 \geq k \geq 0$ we set the variables $p_1, p_2, \ldots, p_{n-m-1}$ true. At the last point a_0 in $\mathcal{F}_n$ we valuate variable p_{n-m-1}. Since in this case the range of σ_m is the whole frame $\mathcal{F}_n$ then at a_0 we should have the formula $p_{n-m-1} \rightarrow \Diamond p_{n-m}$ true. But it is impossible, so we get a contradiction.
Sub-case 1b. Suppose we take p_1 and $n - m < m$. As above at the next points a_k's with $n - m - 2 \geq k \geq 0$ we valuate variables $p_2, p_3, \ldots, p_{n-m-1}$. At the last point a_0 in $\mathcal{F}_n$ we valuate variable p_{n-m}. Again, the range of σ_m is the whole frame $\mathcal{F}_n$. Then at a_0 we should have true the formula $p_{n-m} \rightarrow \Diamond p_{n-m+1}$. But it is impossible, so we get a contradiction.
Sub-case 1c. Suppose we take p_1 and $n - m = m$. Again at the points a_k's with $n - m - 2 \geq k \geq 0$ we valuate variables $p_2, p_3, \ldots, p_{n-m-1}$. At the last point a_0 in $\mathcal{F}_n$ we valuate variable p_{n-m}. But then $p_{n-m} = p_m$. We may notice that in σ_m we have sub-formulas $p_m \rightarrow \Diamond p_{m+1}$ and $p_m \rightarrow \Diamond p_{m+2}$. But at a_0 it is impossible to valuate that formulas and we get a contradiction.

Case 2. $n - m > m$. As in Case 1 we have to match the valuation in the other part of the tail and we may choose for the next point a_{n-m-1}, that either p_0 or p_1 is true.
Sub-case 2a. Suppose we choose p_0. Analogously to sub-case 1a we have to valuate variables $p_1, p_2, \ldots, p_m$ in the next points $a_{n-m-2}, a_{n-m-3}, \ldots, a_{n-2m-1}$. Since the range of formula σ_m is $m+2$ in both directions from the point a_{n-m}, then in the point a_{n-2m-1} we valuate p_m. In formula σ_m we have $p_m \to \Diamond p_{m+1}$ and $p_m \to \Diamond p_{m+2}$, so we should at the next point valuate both p_{m+1} and p_{m+2}. If $n - 2m - 1 = 0$ then we get immediately a contradiction. If $n - 2m - 1 > 0$ then there is a next to a_{n-2m-1} point a_{n-2m-2}. So we should valuate at a_{n-2m-2} both p_{m+1} and p_{m+2}. But a_{n-2m-2} lies within the range of σ_m so we must take into account formulas $p_i \to \neg p_j$ for $i \neq j$, $i, j := 0, 1, \ldots, m + 2$. Hence we get a contradiction.
Sub-case 2b. Suppose we choose p_1. Analogously as in Sub-case 1b we have to valuate variables $p_2, p_3, \ldots, p_m$ in the next points $a_{n-m-2}, a_{n-m-3}, \ldots, a_{n-2m}$. Then at a_{n-2m-1} we valuate both p_{m+1} and p_{m+2}. As before a_{n-2m-1} lies within the range of σ_m so we must take into account formulas $p_i \to \neg p_j$ for $i \neq j$, $i, j := 0, 1, \ldots, m + 2$. Hence we get a contradiction. ■

Now we are in a position to show our main theorem.

THEOREM 3.2. (1) *For subclasses $\mathcal{C}, \mathcal{D} \subseteq \mathcal{S}$, if $\mathcal{C} \neq \mathcal{D}$, then* $\mathbf{L}(\mathcal{C}) \neq \mathbf{L}(\mathcal{D})$.
(2) *There exists a continuum of normal modal logics in* NEXT(**KTB.3′A**).

PROOF. (1): Suppose $\mathcal{C} \not\subseteq \mathcal{D}$. Then, $\mathcal{F}_m \in \mathcal{C}$ and $\mathcal{F}_m \notin \mathcal{D}$ for some $\mathcal{F}_m \in \mathcal{S}$. Then, by the above lemma, $\neg\sigma_m \in \mathbf{L}(\mathcal{D})$ and $\neg\sigma_m \notin \mathbf{L}(\mathcal{C})$. Hence we have $\mathbf{L}(\mathcal{D}) \not\subseteq \mathbf{L}(\mathcal{C})$.
(2): It follows from (1). ■

4. Conclusions and Problems

We proved that the cardinality of NEXT(**KTB.3′A**) is uncountably infinite. This fact distinguishes the logic **KTB.3′A** from **S4.3** and **KTBAlt(3)** as well. It occurred that the logic **KTB.3′A** has continuum normal extensions, all of which are Kripke complete and have the f.m.p.

It is well known that the axiom alt_3 is an instance of the following general axiom alt_n: $(n \geq 0)$

$$(alt_n) := \Box p_1 \vee \Box(p_1 \to p_2) \vee \Box((p_1 \wedge p_2) \to p_3)) \vee \\ \cdots \vee \Box((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to p_{n+1}).$$

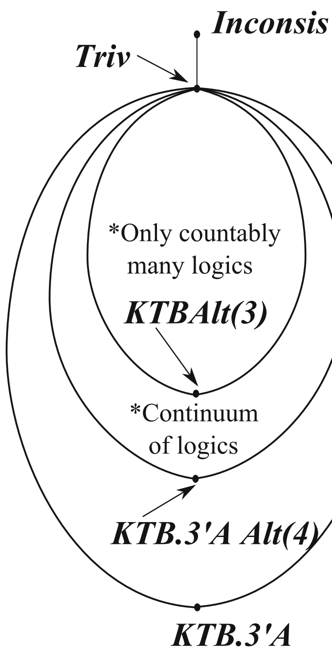

Diagram 6. The lattice NEXT(**KTB.3′A**)

This axiom characterizes the class of frames in which every point can see at most n points. Here, let us take a closer look at our construction of frames in Lemma 3.1. Then it is not very hard to see that for any n, every point in the frame $\mathcal{F}_n$ can see at most four points. Indeed, only the neck of each frame can see four points including itself. This means that all members of the class $\{\mathcal{F}_n\}_{n\geq 1}$ are frames for $\mathbf{KTBAlt}(\mathbf{4}) := \mathbf{KTB} \oplus alt_4$. Hence we have proved the following stronger fact.

THEOREM 4.1. *There exists a continuum of normal modal logics in* NEXT($\mathbf{KTB.3'A} \oplus \mathbf{alt_4}$). ■

The above theorem, of course, implies that the cardinality of the class NEXT($\mathbf{KTBAlt}(\mathbf{4})$) is uncountably infinite, which shows us a sharp boundary located between $\mathbf{KTBAlt}(\mathbf{3})$ and $\mathbf{KTBAlt}(\mathbf{4})$. In this sense, the logic $\mathbf{KTBAlt}(\mathbf{3})$ sits on a special position in the lattice NEXT(**KTB**).

The lattice of NEXT(**KTB.3′A**) is so intriguing that it requires further investigations. Our future work will concern the following problems:

1. Existence of splitting logics,
2. Local finiteness,

3. Algebraic counterpart of Kripke frames for **KTB.3′A**.

It would be also very interesting to generalize the axiom (3′) together with (A) (analogously like from alt_3 to alt_n) and obtain a syntactical characterization of reflexive and symmetric frames in which each cluster is in accessibility relation with a bounded number of other clusters. Then we could investigate logics determined by frames in a shape of net.

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