

JOANNA GOLIŃSKA-PILAREK

On the Minimal Non-Fregean Grzegorzczuk Logic

To the Memory of Andrzej Grzegorzczuk

Abstract. The paper concerns Grzegorzczuk's non-Fregean logics that are intended to be a formal representation of the *equimeaning relation* defined on descriptions. We argue that the main Grzegorzczuk logics discussed in the literature are too strong and we propose a new logical system, MGL, which satisfies Grzegorzczuk's fundamental requirements. We present a sound and complete semantics for MGL and we prove that it is decidable. Finally, we show that many non-classical logics are extensions of MGL, which makes it a generic non-Fregean logic.

Keywords: Non-Fregean logic, Descriptive equivalence, Equimeaning, Extensionality.

Introduction

There are plenty of logical systems discussed in the literature which are focused on different aspects of formal representation of real world problems. Over the years, the question as to which logic is adequate has been a matter of lively discussion. One of the fundamental controversies is the philosophical status of classical two-valued logic, and in particular the paradoxes of material implication and equivalence. In order to remedy the dissatisfactory aspects of classical logic, many logical systems have been introduced, such as modal, intuitionistic, relevant, non-Fregean, and paraconsistent logics, among others.

In [3] Andrzej Grzegorzczuk introduced yet another logical system intended to cover the fundamental properties of descriptive equivalence of sentences. His main assumption is that the human language is used primarily to create descriptions of reality and logical connectives are the main tool to represent them formally. In order to be able to speak about interactions between descriptions—as Grzegorzczuk argues—a logical language

Presented by **Heinrich Wansing**; *Received* February 21, 2015

should contain at least the following four connectives: negation (\neg), conjunction (\wedge), disjunction (\vee), and descriptive equivalence (\equiv). According to Grzegorzczuk, neither classical propositional logic nor any of its non-classical extensions can serve as an adequate formal language of descriptions. As stated in [3], classical logic “restricts itself to considering only one, admittedly the most important, parameter of the content of a claim, namely its *truth value*” which is the source of many paradoxes of material implication and equivalence. In particular the classical equivalence cannot properly express the reality of descriptions. Indeed, the expressions “ $2 + 2 = 4$ ” and “Warsaw is the capital of Poland” are classically equivalent, but clearly they do not describe the same. As pointed out by Grzegorzczuk in [3, p. 477]:

Such combined propositions, established as equivalences according to classical formal logic, do not lead to anything interesting. The paradoxical nature of the classical concept of equivalence arises because we may expect equivalence to open the possibility of linking the content of one claim with that of the other. We would like equivalent sentences not only to be equally true, but also TO SPEAK ABOUT THE SAME SUBJECT. It seems (from a philosophical point of view) that claims that are not connected by a common subject cannot be treated as fully equivalent.

Thus classical logic cannot be accepted as a basis of a proper logic of descriptions, and hence Grzegorzczuk defines his new propositional logic from scratch (for more details on Grzegorzczuk’s motivations see [1]). He rejects all the classical tautologies except the law of contradiction and adds axioms reflecting intended properties of the descriptive equivalence, which are:

- (LD1) \equiv represents an equivalence relation and satisfies the extensionality property (equal descriptions can be substituted for each other).
- (LD2) \equiv joins some of the boolean combinations of descriptions, like: associativity, commutativity, and idempotency of conjunction and disjunction, distributivity of conjunction (resp. disjunction) over disjunction (resp. conjunction), involution of negation that additionally satisfies de Morgan laws.

The new propositional logic obtained in this way has been called *the Logic of Descriptions* and denoted by LD. As shown in [1], the logic LD is indeed a new propositional logic as it cannot be thought of as a subsystem or extension of any of the known non-classical logics. In the last years the logic LD has been extensively studied. A sound and complete semantics for LD

introduced in [1] shows that the logic LD is *non-Fregean* in a sense that it rejects the main assumption of classical Fregean logic, according to which sentences with the same truth value have the same denotations. This implies that LD has much in common with the non-Fregean logic SCI introduced by Roman Suszko in [4] (cf. [5,6]).

Recall that SCI is the logic obtained by expanding the language of classical propositional logic with the connective \equiv (called in Suszko's logic *the identity connective*) and adding to the axiomatization of classical logic four axioms for \equiv . Even though the philosophical motivations for introducing logics LD of Grzegorzczuk and SCI of Suszko were similar, i.e., the necessity of separating logical values from denotations, these logics differ considerably in construction. Suszko's logic SCI is an extension of classical logic, while Grzegorzczuk's logic LD is built from the ground up. Suszko accepts the classical implication, Grzegorzczuk removes it from the language (so the classical modus ponens rule is not present in the logic LD). Contrary to SCI, in the logic LD the connective \equiv is not an additional operator, but rather it *supersedes* the classical implication and equivalence.¹ It means that the extensionality property cannot be expressed in Grzegorzczuk's logic in terms of the classical implication as it is in Suszko's logic SCI. As a consequence, the logics LD and SCI are completely different realizations of the similar philosophical assumptions on the ontological status of sentences. The difference is evidently manifested on the formal level: there are theorems (resp. models) of one of the logics which are not theorems (resp. models) of the other. Hence, LD and SCI are uncomparable. More details on differences between LD and SCI can be found in [3].

Further results presented in [1] showed that the logic LD has many peculiar and strange formal properties, which led to some modifications of LD studied in detail in the paper [2]. Nevertheless, the philosophical interpretation of some axioms of Grzegorzczuk's logics remains unclear and controversial. In fact, after many discussions at the seminar devoted to the logic LD, professor Grzegorzczuk admitted that LD may be too strong and he was inclined to amend his axioms, but he insisted that (LD1) is the fundamental property of descriptive equivalence which a proper logic must reflect. The

¹Grzegorzczuk was firmly convinced that the descriptive equivalence is much more primitive and better reflects the human way of thinking than classical implication or equivalence which he described as coarse, derivative and too paradoxical. To justify his position Grzegorzczuk has formulated a complete philosophical theory of acquisition of logical connectives that makes use of the results from empirical sciences and linguistics. Unfortunately, professor Grzegorzczuk did not finish this work before his death in 2014.

question whether any of the proposed logics is a good realization of this fundamental assumption of Grzegorzczuk on descriptive equivalence is still open. In this paper we argue that the main three logics of Grzegorzczuk are too strong. We propose a very weak logical system, which satisfies (LD1) and can serve as a generic non-Fregean logic that can be extended to non-Fregean logics constructed in Grzegorzczuk or Suszko style.

The paper is organized as follows. In Sect. 1 we present Grzegorzczuk's non-Fregean logics, their axiomatizations and some formal properties. Section 2 is concerned with a discussion on philosophical difficulties with an interpretation of some of Grzegorzczuk's axioms. In Sect. 3 we present a very weak Grzegorzczuk's non-Fregean logic, MGL. We construct denotational semantics for MGL and we show its soundness and completeness. Furthermore, we show that MGL is decidable. In Sect. 4 we argue that MGL can be seen as a generic logic by showing that many non-classical logics are just extensions of MGL. Conclusions and prospects of future work are presented in Sect. 5.

1. Grzegorzczuk's Non-Fregean Logics

In this section, we present the basics of Grzegorzczuk non-Fregean approach. We start with some general definitions, and then we present in detail the three representative Grzegorzczuk's logics.

By a logic we would mean a formal system determined by a language, set of axioms, rules of inference, and the provability relation. All the logics considered in the paper are propositional.

DEFINITION 1.1. (*Formulas*) The set $\text{FOR}_{\mathbf{L}}$ of \mathbf{L} -formulas is the smallest set including propositional variables from a countable set \mathbb{V} and closed with respect to all the propositional operations from a finite set $\text{OP}_{\mathbf{L}} \supseteq \{\neg, \wedge, \vee\}$. \dashv

DEFINITION 1.2. (*Axioms and rules*) Distinguished \mathbf{L} -formulas play the role of \mathbf{L} -axioms. The \mathbf{L} -rules have the following general form:

$$(\mathbf{L}\text{-rule}) \quad \frac{\phi_1, \dots, \phi_n}{\psi_1, \dots, \psi_k}$$

where $n, k \geq 1$ and $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_k$ are \mathbf{L} -formulas. An \mathbf{L} -formula ψ follows from the formulas ϕ_1, \dots, ϕ_n by $\frac{\phi_1, \dots, \phi_n}{\psi_1, \dots, \psi_k}$ if $\psi \in \{\psi_1, \dots, \psi_k\}$. \dashv

DEFINITION 1.3. (*Provability*) Let X be a set of \mathbf{L} -formulas. A formula φ is said to be \mathbf{L} -provable ($\vdash_{\mathbf{L}} \varphi$ for short) whenever there exists a finite sequence $\varphi_1, \dots, \varphi_n$ of \mathbf{L} -formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and each φ_i ,

$i \in \{1, \dots, n\}$, is an L-axiom or follows from earlier formulas in the sequence by one of the L-rules of inference. A formula which is L-provable will be referred to as an L-theorem. A formula φ is said to be L-provable from X ($X \vdash_L \varphi$ for short) whenever there exists a finite sequence $\varphi_1, \dots, \varphi_n$ of formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and each φ_i , $i \in \{1, \dots, n\}$, is an L-axiom or a formula from X or follows from earlier formulas in the sequence by one of the L-rules of inference. \dashv

DEFINITION 1.4. (*Logic*) A logic L is a pair (FOR_L, \vdash_L) , where FOR_L is the set of L-formulas as defined in Definition 1.1, and \vdash_L is the provability relation on sets of L-formulas as defined in Definition 1.3. \dashv

Therefore, each logic is uniquely determined by its set of propositional operations, axioms, and rules.

DEFINITION 1.5 (*(Strong) Admissibility*). Let L be a logic and let ϕ_i, ψ_j be L-formulas, for all $i, j \geq 1$. A rule of the form $\frac{\phi_1, \dots, \phi_n}{\psi_1, \dots, \psi_k}$, for $n, k \geq 1$, is said to be L-admissible whenever the following condition is satisfied:

If ϕ_1, \dots, ϕ_n are L-provable, then ψ_1, \dots, ψ_k are L-provable.

A rule of the form $\frac{\phi_1, \dots, \phi_n}{\psi_1, \dots, \psi_k}$, for $n, k \geq 1$, is said to be strongly L-admissible whenever for every set X of L-formulas:

If ϕ_1, \dots, ϕ_n are L-provable from X , then ψ_1, \dots, ψ_k are L-provable from X . \dashv

Clearly, strong L-admissibility implies L-admissibility.

By the L-validity of an L-formula we mean its truth in all possible semantic interpretations, provided that the semantics for L exists.

DEFINITION 1.6. (*Correctness*) Let L be a logic and let ϕ_i, ψ_j be formulas, for all $i, j \geq 1$. A rule of the form $\frac{\phi_1, \dots, \phi_n}{\psi_1, \dots, \psi_k}$, for $n, k \geq 1$, is said to be L-correct whenever the following condition is satisfied:

If ϕ_1, \dots, ϕ_n are L-valid, then ψ_1, \dots, ψ_k are L-valid. \dashv

Grzegorzczuk's non-Fregean logics discussed in this paper belong to a family of non-Fregean propositional logics with the connective \equiv . All the logics share a common language, but differ in axioms.

DEFINITION 1.7 (*Language of Grzegorzczuk's non-Fregean logics*). The language of Grzegorzczuk's non-Fregean logics is determined by the set of propositional operations of negation \neg , disjunction \vee , conjunction \wedge , and descriptive equivalence \equiv . We also admit the following three shorthand notations:

- $(p \rightarrow q) \stackrel{\text{df}}{=} (\neg p \vee q)$ (classical implication)
- $(p \leftrightarrow q) \stackrel{\text{df}}{=} (p \rightarrow q) \wedge (q \rightarrow p)$ (classical equivalence)
- $(p \Rightarrow q) \stackrel{\text{df}}{=} (p \equiv (p \wedge q))$ (descriptive implication)

⊥

Below we present axioms and rules of the three basic Grzegorzczuk's non-Fregean logics which are: the *Logic of Descriptions* LD, introduced in [3] and widely discussed in [1], the *Logic of Equimeaning* LE and the *Logic of Descriptions with Suszko's Axioms* LDS, both introduced in [2]. All these logics share most of their axioms and all of their rules. Their common axioms are the following:

- (Ax0) $\neg(p \wedge \neg p)$
- (Ax1) $p \equiv p$
- (Ax2) $\neg\neg p \equiv p$
- (Ax3) $p \equiv (p \wedge p)$
- (Ax4) $p \equiv (p \vee p)$
- (Ax5) $(p \wedge q) \equiv (q \wedge p)$
- (Ax6) $(p \vee q) \equiv (q \vee p)$
- (Ax7) $(p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r)$
- (Ax8) $(p \vee (q \vee r)) \equiv ((p \vee q) \vee r)$
- (Ax9) $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$
- (Ax10) $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$
- (Ax11) $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
- (Ax12) $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
- (Ax13) $(p \equiv q) \equiv (q \equiv p)$
- (Ax14) $(p \equiv q) \equiv (\neg p \equiv \neg q)$

The rules have the following forms:

$$\begin{array}{ll}
 \text{(MPE)} \quad \frac{\varphi \equiv \psi, \varphi}{\psi} & \text{(Sub)} \quad \frac{\varphi(p_0, \dots, p_n)}{\varphi(p_0/\psi_0, \dots, p_n/\psi_n)} \\
 \\
 (\wedge_1) \quad \frac{\varphi, \psi}{\varphi \wedge \psi} & (\wedge_2) \quad \frac{\varphi \wedge \psi}{\varphi, \psi}
 \end{array}$$

As in the classical case, the rule (Sub) may be applied only to axioms. This restriction is not essential when no additional assumptions are used in the proof.

Logics LD, LE, and LDS differ on the last three axioms. They have the following form in the original logic LD:

$$(Ax15)_{LD} \quad (p \equiv q) \Rightarrow ((p \equiv r) \equiv (q \equiv r))$$

$$(Ax16)_{LD} \quad (p \equiv q) \Rightarrow ((p \wedge r) \equiv (q \wedge r))$$

$$(Ax17)_{LD} \quad (p \equiv q) \Rightarrow ((p \vee r) \equiv (q \vee r))$$

The last three axioms of LE are:

$$(Ax15)_{LE} \quad ((p \equiv q) \wedge (p \equiv r)) \equiv ((p \equiv q) \wedge (q \equiv r))$$

$$(Ax16)_{LE} \quad ((p \equiv q) \wedge (p \wedge r)) \equiv ((p \equiv q) \wedge (q \wedge r))$$

$$(Ax17)_{LE} \quad ((p \equiv q) \wedge (p \vee r)) \equiv ((p \equiv q) \wedge (q \vee r))$$

In the logic LDS the three axioms are descriptive copies of specific axioms of the *Sentential Calculus of Identity* introduced by Suszko in [4].

$$(Ax15)_{LDS} \quad ((p \equiv q) \wedge (r \equiv s)) \Rightarrow ((p \equiv r) \equiv (q \equiv s))$$

$$(Ax16)_{LDS} \quad ((p \equiv q) \wedge (r \equiv s)) \Rightarrow ((p \wedge r) \equiv (q \wedge s))$$

$$(Ax17)_{LDS} \quad ((p \equiv q) \wedge (r \equiv s)) \Rightarrow ((p \vee r) \equiv (q \vee s))$$

The intended meaning of axioms (Ax0) and (Ax1) is clear: (Ax0) is the classical law of contradiction and (Ax1) expresses reflexivity of the descriptive equivalence. Axioms (Ax2)–(Ax14) reflect some properties of Boolean combinations of descriptions: associativity, commutativity, and idempotency of conjunction and disjunction ((Ax3)–(Ax8)), distributivity of conjunction (resp. disjunction) over disjunction (resp. conjunction) ((Ax9) and (Ax10)), involutiveness of negation that additionally satisfies de Morgan laws ((Ax2), (Ax11), (Ax12)). The last five axioms are intended to express the extensionality and the condition that \equiv represents an equivalence relation. Problems with an interpretation of these axioms is discussed in the next section.

A sound and complete semantics for LD has been presented in [1]. Models of LD are non-Fregean structures based on the *Grzegorzczuk algebras* which are de Morgan bisemilattices satisfying some additional conditions (see [1]). In [2] it is shown that LD-models can be weakened to a paraconsistent version, thus the logic LD is contradiction-tolerant: there are formulas ϕ and ψ such that $(\phi \wedge \neg\phi) \not\vdash_{LD} \psi$.

The following proposition proved in [1] shows that \equiv is not the classical equivalence:

PROPOSITION 1.8. *The following formulas are not provable in LD:*

- $(\varphi \equiv \psi) \leftrightarrow (\varphi \leftrightarrow \psi)$
- $(\varphi \Rightarrow \psi) \leftrightarrow (\varphi \rightarrow \psi)$
- $(\varphi \equiv \varphi) \equiv (\psi \equiv \psi)$
- $(\varphi \vee (\varphi \wedge \psi)) \equiv \varphi$
- $(\varphi \wedge (\varphi \vee \psi)) \equiv \varphi$
- $(\varphi \vee \neg\varphi) \equiv (\psi \vee \neg\psi)$
- $(\varphi \wedge \neg\varphi) \Rightarrow \psi$
- $\varphi \Rightarrow (\psi \vee \neg\psi)$
- $\neg(\varphi \equiv \neg\varphi)$

Further results from [1] show that negation and disjunction behave in LD in a non-classical way: the formula $p \equiv \neg p$ is unsatisfiable in any LD-model, $(p \equiv \neg p) \vee \neg(p \equiv \neg p)$ is valid in all LD-models, but $\neg(p \equiv \neg p)$ is not LD-provable.

In the paper [2] it has been proved that the last three axioms of LD are essentially different from their corresponding forms in logics LE and LDS:

PROPOSITION 1.9. 1. $(A \times 15)_{LD}$ is not provable in LE and LDS.

2. The formula $(p \equiv q) \equiv ((p \equiv q) \wedge (q \equiv q))$ is provable in LE but not in LD.
3. The formula $(p \equiv q) \Rightarrow [(p \equiv (p \equiv (p \equiv p))) \equiv (q \equiv (q \equiv (q \equiv q)))]$ is provable in LDS, but not in LD.

The above proposition implies that the last three axioms of logics LD, LE, and LDS express the extensionality in a different way. The further question is whether any of the trio adequately realizes the intended extensionality principle for descriptions. We discuss this issue in the next section.

2. Extensionality

All the Grzegorzczuk's logics have been defined to represent fundamental properties of a relation *having the same meaning* defined on sentences. On the formal side this relation is represented by the operation \equiv called the *descriptive equivalence* or the *equimeaning operation*. Thus, possible intended readings of an expression " $\varphi \equiv \psi$ " are: φ and ψ have equal meaning, φ and ψ have the same content, φ and ψ describe the same reality, φ and ψ are equal descriptions of the world, φ and ψ denote the same situations.

According to Grzegorzczuk, formal properties of the operation \equiv should guarantee that it represents an equivalence relation. Furthermore, it is expected that a logic designed to adequately reflect properties of *equimeaning* should meet the following *Extensionality Principle*:

Sentences that have equal meaning (have the same content or are the same descriptions) are *interchangeable* in all possible contexts.

Is any of the logics LD, LE, and LDS a good realization of these requirements? Observe that last three axioms of LD, LE, and LDS can be reformulated as:

$$\begin{aligned} (LD)_{\equiv} & (p \equiv q) \equiv [(p \equiv q) \wedge ((p \# r) \equiv (q \# r))] \\ (LE)_{\equiv} & ((p \equiv q) \wedge (p \# r)) \equiv ((p \equiv q) \wedge (q \# r)) \end{aligned}$$

$$(LDS)_{\equiv} [(p \equiv q) \wedge (r \equiv s)] \equiv [((p \equiv q) \wedge (r \equiv s)) \wedge ((p\#r) \equiv (q\#s))]$$

where $\# \in \{\equiv, \wedge, \vee\}$.

As we mentioned in the previous section, LD, LE, and LDS are formally uncomparable. But what is the philosophical difference between these three logics? What do the last three axioms actually express? Is it the case that in all these logics the operation \equiv is a formal representation of the equimeaning relation?

It is quite clear what the axioms (Ax1)–(Ax12) mean if we read them in terms of *having the same meaning*-relation. They just express some of its elementary properties. For instance, the axiom (Ax2) can be read as: ‘ p ’ means the same as ‘it is not the case that not p ’. Although some or all of the axioms (Ax1)–(Ax12) can be thought of as too strong (e.g., it is not obvious that changing the order in a conjunction leads to sentences with the same content), their interpretation as ‘the sentence on the left means the same as the sentence on the right’ is intuitive, natural, and unquestionable. The problem arises when we try to do the same with the other axioms.

Note that in all the axioms (Ax1)–(Ax12) the connective \equiv occurs once and it binds formulas in which \equiv does not occur explicitly. It is not the case with the axioms (Ax13), (Ax14), (LD) $_{\equiv}$, (LE) $_{\equiv}$, and (LDS) $_{\equiv}$, in which the operation \equiv is *nested*. The axioms (Ax13) and (Ax14) are not so problematic as the last three axioms. In particular, the axiom (Ax13) we can read as:

The case that p has the same meaning as q
means the same as

The case that q has the same meaning as p .

Such a reading of (Ax13) is understandable and even easily acceptable. The same concerns the axiom (Ax14).

Unfortunately, the last three axioms cause much more difficulties. Let us consider, for instance, the axiom (Ax16) $_{LD}$. Assume that in our theory the propositions “ $2 + 2 = 4$ ” and “ $1 + 1 + 1 + 1 = 4$ ” have the same meaning. Then, the following is an instance of (Ax16) $_{LD}$:

(1) The case that $2 + 2 = 4$ means the same as $1 + 1 + 1 + 1 = 4$
means the same as:

(2) The case that $2 + 2 = 4$ means the same as $1 + 1 + 1 + 1 = 4$, and in addition, the case that ($2 + 2 = 4$ and each square is a rectangle) means the same as the case that ($1 + 1 + 1 + 1 = 4$ and each square is a rectangle).

The equimeaning of (1) and (2) is not so evident as in the previous example, but can be explained as:

The concept of equimeaning brings into play the information that sentences having the same meaning are interchangeable in the conjunction context. That is: if the sentences ‘ $2 + 2 = 4$ ’ and ‘ $1 + 1 + 1 + 1 = 4$ ’ ACTUALLY mean the same, then the meaning of (1) already contains in itself the meaning of (2).

The problem, however, is that axioms should be universal and hold in every case, in particular when propositional variables are substituted with sentences which are not true or do not have the same meaning. It seems that the justification of $(\text{Ax16})_{\text{LD}}$ given above works because we used sentences which can be easily seen as having the same meaning. Nevertheless, the last three axioms are extremely hard to accept if propositional variables are substituted with sentences that do not have the same meaning.

By way of example, consider now the axiom $(\text{Ax16})_{\text{LE}}$ of the logic LE:

$$((p \equiv q) \wedge (p \wedge r)) \equiv ((p \equiv q) \wedge (q \wedge r)).$$

This axiom forces us to accept that:

(3) The case that $2 + 2 = 4$ means the same as $2 + 2 = 5$, and in addition, $2 + 2 = 4$ and $1 = 1$

means the same as

(4) The case that $2 + 2 = 4$ means the same as $2 + 2 = 5$, and in addition, $2 + 2 = 5$ and $1 = 1$

Such a consequence is highly controversial. It is very difficult or even impossible to find any reason for accepting that (3) and (4) say the same thing. They are, of course, equivalent, but cannot have the same meaning. For better intuition, let us denote “ $2 + 2 = 4$ means the same as $2 + 2 = 5$ ”, “ $2 + 2 = 4$ and $1 = 1$ ”, “ $2 + 2 = 5$ and $1 = 1$ ” by False_1 , True , and False_2 , respectively. Then, the following is an instance of the axiom $(\text{Ax16})_{\text{LE}}$:

$$\text{False}_1 \wedge \text{True} \quad \text{means the same as} \quad \text{False}_1 \wedge \text{False}_2$$

The axiom $(\text{Ax16})_{\text{LE}}$ forces us to accept such propositions only because the conjunctions on both sides of \equiv share the first component. A similar problem would appear if we tried to interpret other axioms among $(\text{LD})_{\equiv}$, $(\text{LE})_{\equiv}$, and $(\text{LDS})_{\equiv}$.

The reason of all these problematic cases is that the axioms $(LD)_{\equiv}$, $(LE)_{\equiv}$, and $(LDS)_{\equiv}$ are intended to express the extensionality property which is a *meta-property*, but they make it in the object language instead of the meta-language. In fact, to guarantee the extensionality it would suffice to assume that whenever the left side of an axiom holds then so does the right side. To express this we need a rule rather than an axiom. Although axioms $(LD)_{\equiv}$, $(LE)_{\equiv}$, and $(LDS)_{\equiv}$ enable us to prove all required formal properties of the equimeaning relation, they also express some strong additional properties, which are even difficult to capture in the natural language. Since axioms $(LD)_{\equiv}$, $(LE)_{\equiv}$, and $(LDS)_{\equiv}$ are too strong, none of the logics LD, LE, LDS is a good realization of the Extensionality Principle. Hence, the best solution is to replace the axioms with a rule that appropriately expresses the extensionality.

Following this line of thought, in the next section we will define a new logic MGL—the minimal Grzegorzczuk’s logic in the non-Fregean style. As we will show, the Extensionality Principle is represented in the logic MGL in an adequate way. Furthermore, since we are searching for a generic logic, we will reject all the axioms that express non-trivial properties of \equiv , in particular, the axioms (Ax2)–(Ax14), which are not theorems of Suszko’s logic SCI. In this way, we will obtain a logic which is as weak as possible and can be seen as a general logical framework for studying and comparing all the non-Fregean logics under consideration.

3. The Minimal Grzegorzczuk’s Non-Fregean Logic

The *minimal Grzegorzczuk’s non-Fregean logic*, MGL, is meant to be a minimal formal system satisfying the fundamental assumptions of Grzegorzczuk’s approach, according to which \equiv represents an equivalence relation on descriptions and satisfies the Extensionality Principle. At this point, it should be emphasized that the main motivation for logic MGL—apart from fulfilling Grzegorzczuk’s fundamental assumption on the extensionality of the descriptive equivalence—is to have a *generic* non-Fregean logic that can be used as a general framework for representing, investigating, implementing, and comparing theories with incompatible languages and/or semantics. We will argue that the logic MGL is able to play such a role.

Therefore, the logic MGL is not meant to be a logic representing a specific philosophical theory, but it should rather be seen as a starting point for investigating one’s favourite non-Fregean theory of descriptions. As a consequence, the logic MGL does not include any specific axioms for the

classical connectives, except the law of contradiction and the rules which are common for all the logics in question. This concerns in particular the disjunction connective. Actually, the connectives \wedge and \vee are not treated in LD in the same way. Both are present in the language, but their status in deduction is different: there are rules for the conjunction, while nothing is said about reasoning with disjunction. Although in LD, due to its specific axioms, the formula $p \vee \neg p$ is a theorem, the disjunction does not behave in a classical way, in particular it cannot be treated as a dual form of conjunction (for details see [1]). The logic MGL inherits this peculiar property of LD. It must contain \vee in the language, since otherwise we could not interpret logics that contain the disjunction. However, as MGL aspires to be a generic non-Fregean logic, it cannot assume the symmetry between \wedge and \vee , between the law of contradiction (which is an axiom of LD and MGL) and the law of excluded middle, which is rejected in intuitionistic logics, among others. In this way the logic MGL may also serve as a base for studying non-Fregean intuitionistic logics.

The language of MGL is defined as in Definition 1.7. The axioms of MGL are (Ax0) and (Ax1):

$$(Ax0) \quad \neg(p \wedge \neg p) \quad \text{and} \quad (Ax1) \quad p \equiv p$$

The rules are (MPE), (Sub), (\wedge_1), (\wedge_2) presented in Sect. 1, and the *extensionality rule* of the following form:

$$(ext) \quad \frac{\varphi \equiv \psi}{\alpha(\varphi) \equiv \alpha(\varphi/\psi)}$$

where φ, ψ are arbitrary formulas, α is a formula in which φ is a subformula, and $\alpha(\varphi/\psi)$ is a formula obtained from α by replacing some or all occurrences of φ with ψ .

The notions of *MGL-provability* of a formula, *MGL-theorem*, and (strong) *MGL-admissibility* of a rule are defined in a standard way as in Definitions 1.3 and 1.5.

PROPOSITION 3.1. *For all formulas $\varphi, \psi, \theta, \chi$ and $\# \in \{\wedge, \vee, \equiv\}$, the following rules are strongly MGL-admissible:*

$$\begin{array}{ll} (sym) \quad \frac{\varphi \equiv \psi}{\psi \equiv \varphi} & (tran) \quad \frac{\varphi \equiv \psi, \psi \equiv \theta}{\varphi \equiv \theta} \\ (neg) \quad \frac{\varphi \equiv \psi}{\neg\varphi \equiv \neg\psi} & (com) \quad \frac{\varphi \equiv \psi, \theta \equiv \chi}{(\varphi\#\theta) \equiv (\psi\#\chi)} \end{array}$$

PROOF. Let X be a set of MGL-formulas and assume that φ and ψ are MGL-formulas such that $\varphi \equiv \psi$ is MGL-provable from X . Then, the following proves $\psi \equiv \varphi$ in MGL from X :

1. $\varphi \equiv \psi$ MGL-provable from X
2. $\varphi \equiv \varphi$ MGL-theorem (by (Ax1))
3. $(\varphi \equiv \varphi) \equiv (\psi \equiv \varphi)$ (ext) to 1. for $\alpha(\varphi) := (\varphi \equiv \varphi)$
4. $\psi \equiv \varphi$ (MPE) to 2. and 3.

Therefore, (sym) is strongly MGL-admissible. Now, we show that if $\varphi \equiv \psi$ is MGL-provable from X , then so is $\neg\varphi \equiv \neg\psi$.

1. $\varphi \equiv \psi$ MGL-provable from X
2. $\neg\varphi \equiv \neg\psi$ (ext) to 1. for $\alpha(\varphi) := \neg\varphi$

Thus, (neg) is strongly MGL-admissible.

Let φ, ψ, θ be formulas such that $\varphi \equiv \psi$ and $\psi \equiv \theta$ are MGL-provable from X . Then, $\varphi \equiv \theta$ is MGL-provable from X as the following proof shows:

1. $\varphi \equiv \psi$ MGL-provable from X
2. $\psi \equiv \theta$ MGL-provable from X
3. $(\varphi \equiv \theta) \equiv (\psi \equiv \theta)$ (ext) to 1. for $\alpha(\varphi) := (\varphi \equiv \theta)$
4. $(\psi \equiv \theta) \equiv (\varphi \equiv \theta)$ (sym) to 3.
5. $\varphi \equiv \theta$ (MPE) to 2. and 4.

Hence, the rule (tran) is strongly MGL-admissible.

Now, assume that $\varphi \equiv \psi$ and $\theta \equiv \chi$ are MGL-provable from X . We will show that $(\varphi\#\theta) \equiv (\psi\#\chi)$ is MGL-provable from X , for any $\# \in \{\wedge, \vee, \equiv\}$.

1. $\varphi \equiv \psi$ MGL-provable from X
2. $\theta \equiv \chi$ MGL-provable from X
3. $(\varphi\#\theta) \equiv (\psi\#\theta)$ (ext) to 1. for $\alpha(\varphi) := (\varphi\#\theta)$
4. $(\psi\#\theta) \equiv (\psi\#\chi)$ (ext) to 2. for $\alpha(\theta) := (\psi\#\theta)$
5. $(\varphi\#\theta) \equiv (\psi\#\chi)$ (tran) to 3. and 4.

Therefore, the rule (com) is strongly MGL-admissible. ■

Now, we define semantics for the logic MGL.

DEFINITION 3.2 (MGL-structure). An MGL-structure is a tuple $(U, \sim, \circ, \otimes, \oplus, D)$ such that U and D are non-empty sets satisfying $D \subseteq U$, \sim is a unary

operation on U , and \circ, \otimes, \oplus are binary operations on U such that for all $a, b \in U$ the following hold:

$$(a \circ b) \in D \text{ if and only if } a = b,$$

$$(a \otimes b) \in D \text{ if and only if } a \in D \text{ and } b \in D,$$

$$\sim(a \otimes \sim a) \in D. \quad \dashv$$

Note that MGL-structures are *paraconsistent*, in the sense that there may exist $a \in U$ such that both a and $\sim a$ belong to D . A trivial example of an MGL-structure is a structure based on the two-element Boolean algebra. Now, we will present more interesting example of an MGL-structure. Let $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D)$ be such that $U = \{0, 1, 2\}$, $D = \{2\}$, and operations \sim, \circ, \otimes , and \oplus are defined for all $a, b \in U$ as follows:

$$\begin{aligned} \sim a &= \begin{cases} 2, & \text{if } a = 0 \\ 0, & \text{otherwise} \end{cases} & a \circ b &= \begin{cases} 2, & \text{if } a = b \\ 0, & \text{otherwise} \end{cases} \\ a \otimes b &= \begin{cases} 2, & \text{if } a = b = 2 \\ 0, & \text{otherwise} \end{cases} & a \oplus b &= \begin{cases} 2, & \text{if } a = 2 \text{ or } b = 2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Clearly, \mathcal{M} satisfies all the conditions of MGL-structures. Note also that the operation \sim does not behave in \mathcal{M} as the classical negation: $1 \notin D$ but also $\sim 1 = 0 \notin D$. Moreover, \mathcal{M} does not satisfy the semantic version of the excluded middle law, that is $a \oplus \sim a$ does not hold for all $a \in U$: $(1 \oplus \sim 1) = (1 \oplus 0) = 0 \notin D$. On the other hand, it satisfies the semantic version of the contradiction law, because $\sim(a \otimes \sim a) \in D$, for all $a \in U$.

A *valuation* on an MGL-structure $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D)$ is any mapping $v: \mathbb{V} \rightarrow U$ such that for all formulas φ and ψ the following conditions are satisfied:

- $v(\neg\varphi) = \sim v(\varphi)$,
- $v(\varphi \wedge \psi) = v(\varphi) \otimes v(\psi)$,
- $v(\varphi \equiv \psi) = v(\varphi) \circ v(\psi)$,
- $v(\varphi \vee \psi) = v(\varphi) \oplus v(\psi)$.

A formula φ is said to be *satisfied in an MGL-structure \mathcal{M} by a valuation v* , $\mathcal{M}, v \models \varphi$, if and only if $v(\varphi) \in D$. It is *true in \mathcal{M}* , $\mathcal{M} \models \varphi$, whenever it is satisfied in \mathcal{M} by all the valuations on \mathcal{M} , and it is *MGL-valid* if it is true in all MGL-structures.

An MGL-formula φ is said to be a *semantic MGL-consequence* of a set X of MGL-formulas, $X \models_{\text{MGL}} \varphi$ for short, whenever for every MGL-structure \mathcal{M} and for every valuation v in \mathcal{M} , if $\mathcal{M}, v \models \psi$, for all $\psi \in X$, then $\mathcal{M}, v \models \varphi$. Clearly, semantic MGL-consequences of the empty set of formulas are just MGL-valid formulas.

As pointed out in [1], structures like MGLs can be thought of as variants of Suszko’s non-Fregean structures. We may think of the universe U as consisting of denotations of sentences. Usually the elements of U are referred to as *situations* or *states of affairs*. The set D of *factual situations* consists of the correlates of true sentences. The correlates of false sentences, i.e., sentences whose negations are true, can be referred to as *counterfactual situations*. However, there may be sentences which are neither true or false. Their correlates can be referred to as *undetermined situations*.

The following can be easily proved:

PROPOSITION 3.3. *Let φ and ψ be formulas of a non-Fregean Grzegorzczuk logic. Then, for every MGL-structure $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D)$ and for every valuation v on \mathcal{M} , if $v(\varphi) = v(\psi)$, then for every formula α in which φ is a subformula the following holds: $v(\alpha(\varphi)) = v(\alpha(\varphi/\psi))$.*

We define the MGL-correctness of a rule as in Definition 1.6. A rule $\frac{\varphi_1, \dots, \varphi_n}{\psi_1, \dots, \psi_k}$, $n, k \geq 1$, is said to be *strongly MGL-correct* whenever for every MGL-structure \mathcal{M} and for every valuation v in \mathcal{M} , if $\mathcal{M}, v \models \varphi_i$, for all $i \in \{1, \dots, n\}$, then $\mathcal{M}, v \models \psi_j$, for all $j \in \{1, \dots, k\}$. Clearly, strong MGL-correctness implies MGL-correctness.

PROPOSITION 3.4. *All the MGL-rules except (Sub) are strongly MGL-correct. The rule (Sub) is MGL-correct.*

PROOF. The MGL-correctness of the rule (Sub) and the strong MGL-correctness of the rules (\wedge_1) and (\wedge_2) are easy to prove. So by way of example, we will show the strong MGL-correctness of the rules (MPE) and (ext). Let $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D)$ be an MGL-structure and v a valuation on \mathcal{M} .

Assume $\varphi \equiv \psi$ and φ are satisfied in \mathcal{M} by v . Then, $v(\varphi \equiv \psi) \in D$ and $v(\varphi) \in D$. Since $v(\varphi \equiv \psi) \in D$, we have also $v(\varphi) \circ v(\psi) \in D$, so $v(\varphi) = v(\psi)$. Therefore, $v(\psi) \in D$, which means that $\mathcal{M}, v \models \psi$. Hence, the rule (MPE) is strongly MGL-correct.

Now, assume that $\mathcal{M}, v \models \varphi \equiv \psi$. Then, $v(\varphi) = v(\psi)$. Let $\alpha(\varphi)$ be a formula in which φ is a subformula. Thus, since $v(\varphi) = v(\psi)$ and by Proposition 3.3, $v(\alpha(\varphi)) = v(\alpha(\varphi/\psi))$, so $v(\alpha(\varphi) \equiv \alpha(\varphi/\psi)) \in D$. Therefore, $\mathcal{M}, v \models \alpha(\varphi) \equiv \alpha(\varphi/\psi)$, which means that the rule (ext) is strongly MGL-correct. ■

PROPOSITION 3.5. (Strong Soundness of MGL) *Let X be a set of MGL-formulas and φ be a single MGL-formula. Then, $X \vdash_{\text{MGL}} \varphi$ implies $X \models_{\text{MGL}} \varphi$.*

PROOF. First, observe that both MGL-axioms are MGL-valid. Indeed, for every MGL-structure $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D)$, for every valuation v in \mathcal{M} , and for every propositional variable $p \in \mathbb{V}$, it holds that $\sim(v(p) \otimes \sim v(p)) = v(\neg(p \wedge \neg p)) \in D$. Thus, the axiom (Ax0) is MGL-valid. Moreover, $v(p) = v(p)$, so $v(p \equiv p) \in D$. Therefore, the axiom (Ax1) is MGL-valid.

Assume $X \vdash_{\text{MGL}} \varphi$. Let \mathcal{M} be an MGL-structure and v be a valuation in \mathcal{M} such that $\mathcal{M}, v \models \psi$, for all $\psi \in X$. Since φ is MGL-provable from X , there is a finite sequence $\varphi_1, \dots, \varphi_n$ of MGL-formulas such that $\varphi_n = \varphi$ and for all $k \leq n$, φ_k is either an MGL-axiom or $\varphi_k \in X$ or φ_k is obtained from earlier formulas in the sequence by one of the MGL-rules of inference. Clearly, all φ_k , for $k \in \{1, \dots, n\}$, which are MGL-axioms or formulas from X must be satisfied in \mathcal{M} by v . Note also that the rule (Sub) can be applied only to MGL-axioms, so each application of this rule must result with a formula that is satisfied in \mathcal{M} by v . Furthermore, due to the strong MGL-correctness of the other rules given in Proposition 3.4, each application of an MGL-rule different than (Sub) to formulas φ_k 's in the sequence results with a formula satisfied in \mathcal{M} by v . Hence, for each formula φ_k in the sequence, $\mathcal{M}, v \models \varphi_k$, and so $\mathcal{M}, v \models \varphi$. Therefore, $X \models_{\text{MGL}} \varphi$. ■

In order to prove completeness, we will define a relation \mathcal{R} on the set of all MGL-formulas. Let X be a fixed set of MGL-formulas. Then:

$\varphi \mathcal{R} \psi$ if and only if $\varphi \equiv \psi$ is MGL-provable from X (i.e., $X \vdash_{\text{MGL}} \varphi \equiv \psi$).

PROPOSITION 3.6. *Let X be a fixed set of MGL-formulas. The relation \mathcal{R} is an equivalence relation on the set of all formulas. Moreover, \mathcal{R} is compatible with all connectives of the logic MGL.*

PROOF. Let φ, ψ, ϑ be arbitrary formulas. By the substitution rule applied to the axiom (Ax1), $\varphi \equiv \varphi$ is provable in MGL, so in particular it is MGL-provable from X . Thus, \mathcal{R} is reflexive. Assume that $\varphi \mathcal{R} \psi$. Then, $\varphi \equiv \psi$ is MGL-provable from X , so by the rule (sym) the formula $\psi \equiv \varphi$ is provable in MGL from X . Hence, $\psi \mathcal{R} \varphi$, which means that \mathcal{R} is symmetric. Now, assume that $\varphi \mathcal{R} \psi$ and $\psi \mathcal{R} \vartheta$. Then, $\varphi \equiv \psi$ and $\psi \equiv \vartheta$ are provable in MGL from X , so by the rule (tran) the formula $\varphi \equiv \vartheta$ is provable in MGL from X . Therefore, $\varphi \mathcal{R} \vartheta$, and hence \mathcal{R} is transitive.

Now, we show that \mathcal{R} is compatible with all connectives. Indeed, if $\varphi \mathcal{R} \varphi'$, then $\varphi \equiv \varphi'$ is provable in MGL from X , so by the rule (\neg), is so $\neg \varphi \equiv \neg \varphi'$. Thus, $\neg \varphi \mathcal{R} \neg \varphi'$, so \mathcal{R} is compatible with \neg . Let $\# \in \{\wedge, \vee, \equiv\}$. Assume $\varphi \mathcal{R} \psi$ and $\vartheta \mathcal{R} \chi$, that is $\varphi \equiv \psi$ and $\vartheta \equiv \chi$ are provable in MGL from X . Then, due

to the rule (com), the formula $(\varphi\#\vartheta) \equiv (\psi\#\chi)$ is provable in MGL from X , thus $(\varphi\#\vartheta)\mathcal{R}(\psi\#\chi)$. Hence, \mathcal{R} is also compatible with any $\# \in \{\wedge, \vee, \equiv\}$. ■

The above proposition enables us to define the quotient MGL-structure.

DEFINITION 3.7. (*Canonical structure*) Let X be a set of MGL-formulas. The *canonical MGL-structure with respect to X* has the form $\mathcal{M}^X = (U, \sim, \circ, \otimes, \oplus, D)$ where:

- $U = \{|\varphi|_{\mathcal{R}} \mid \varphi \text{ is any formula}\}$, that is, U is the set of equivalence classes of \mathcal{R} on the set of all formulas,
- $D = \{|\varphi|_{\mathcal{R}} \mid \varphi \text{ is provable in MGL from } X\}$, that is, D is the set of equivalence classes of \mathcal{R} on the set of all formulas which are MGL-provable from X ,
- For all $|\varphi|_{\mathcal{R}}, |\psi|_{\mathcal{R}} \in U$:

$$\begin{aligned} \sim|\varphi|_{\mathcal{R}} &\stackrel{\text{df}}{=} \neg\varphi|_{\mathcal{R}}, & |\varphi|_{\mathcal{R}} \circ |\psi|_{\mathcal{R}} &\stackrel{\text{df}}{=} |\varphi \equiv \psi|_{\mathcal{R}}, \\ |\varphi|_{\mathcal{R}} \otimes |\psi|_{\mathcal{R}} &\stackrel{\text{df}}{=} |\varphi \wedge \psi|_{\mathcal{R}}, & |\varphi|_{\mathcal{R}} \oplus |\psi|_{\mathcal{R}} &\stackrel{\text{df}}{=} |\varphi \vee \psi|_{\mathcal{R}}. \end{aligned}$$

Due to Definition 3.7 and the rule (MPE) we have the following:

FACT 3.8. *Let X be a set of MGL-formulas, and let $\mathcal{M}^X = (U, \sim, \circ, \otimes, \oplus, D)$ be the canonical structure with respect to X . Then, the set D is well defined, that is for all MGL-formulas φ and ψ , if $|\varphi|_{\mathcal{R}} \in D$ and $\varphi\mathcal{R}\psi$, then $|\psi|_{\mathcal{R}} \in D$.*

PROPOSITION 3.9. *Let X be a set of MGL-formulas. The canonical structure with respect to X , $\mathcal{M}^X = (U, \sim, \circ, \otimes, \oplus, D)$, is an MGL-structure.*

PROOF. Clearly, U and D are non-empty sets satisfying $D \subseteq U$. The definition of the structure \mathcal{M}^X implies that for any formula φ , $|\varphi|_{\mathcal{R}} \in D$ if and only if φ is provable in MGL from X . Let $|\varphi|_{\mathcal{R}}, |\psi|_{\mathcal{R}} \in U$. Then, $|\varphi|_{\mathcal{R}} \circ |\psi|_{\mathcal{R}} = |\varphi \equiv \psi|_{\mathcal{R}} \in D$ iff $\varphi \equiv \psi$ is provable in MGL from X iff $\varphi\mathcal{R}\psi$ iff $|\varphi|_{\mathcal{R}} = |\psi|_{\mathcal{R}}$. Thus, $|\varphi|_{\mathcal{R}} \circ |\psi|_{\mathcal{R}} \in D$ if and only if $|\varphi|_{\mathcal{R}} = |\psi|_{\mathcal{R}}$. In a similar way, we show the condition for \otimes . We have: $|\varphi|_{\mathcal{R}} \otimes |\psi|_{\mathcal{R}} = |\varphi \wedge \psi|_{\mathcal{R}} \in D$ iff $\varphi \wedge \psi$ is provable in MGL from X iff (due to the rules (\wedge_1) and (\wedge_2)) φ and ψ are provable in MGL from X iff $|\varphi|_{\mathcal{R}} \in D$ and $|\psi|_{\mathcal{R}} \in D$. Hence, we have proved that $|\varphi|_{\mathcal{R}} \otimes |\psi|_{\mathcal{R}} \in D$ if and only if $|\varphi|_{\mathcal{R}} \in D$ and $|\psi|_{\mathcal{R}} \in D$.

Due to the axiom (Ax0), for any formula φ , $\neg(\varphi \wedge \neg\varphi)$ is an MGL-theorem, so in particular $\neg(\varphi \wedge \neg\varphi)$ is MGL-provable from the set X . Hence, we get $\sim(|\varphi|_{\mathcal{R}} \otimes \sim|\varphi|_{\mathcal{R}}) \in D$. Hence, \mathcal{M}^X is an MGL-structure. ■

PROPOSITION 3.10. (Strong Completeness of MGL) *Let X be a set of MGL-formulas and φ be a single MGL-formula. Then, $X \models_{\text{MGL}} \varphi$ implies $X \vdash_{\text{MGL}} \varphi$.*

PROOF. Suppose $X \not\models_{\text{MGL}} \varphi$. Let $\mathcal{M}^X = (U, \sim, \circ, \otimes, \oplus, D)$ be the canonical structure with respect to X and let $v: \mathbb{V} \rightarrow U$ be the function defined by $v(\psi) = |\psi|_{\mathcal{R}}$. It is easy to prove that v is a valuation on \mathcal{M}^X . Thus, by the assumption and the definition of \mathcal{M}^X , $v(\varphi) = |\varphi|_{\mathcal{R}} \notin D$, so $\mathcal{M}^X, v \not\models_{\text{MGL}} \varphi$. On the other hand, clearly, $X \vdash_{\text{MGL}} \psi$, for every $\psi \in X$, which means that $\mathcal{M}^X, v \models_{\text{MGL}} \psi$, for every $\psi \in X$. Hence, $X \not\models_{\text{MGL}} \varphi$. ■

Hence, we have proved that MGL is strongly complete with respect to the class of MGL-structures. Propositions 3.5 and 3.10 imply:

THEOREM 3.11. (Strong Soundness and Completeness of MGL) *Let X be a set of MGL-formulas and φ be a single MGL-formula. Then:*

$$X \vdash_{\text{MGL}} \varphi \text{ if and only if } X \models_{\text{MGL}} \varphi.$$

As a corollary we obtain:

THEOREM 3.12. (Soundness and Completeness of MGL) *For every MGL-formula φ :*

$$\varphi \text{ is MGL-provable if and only if } \varphi \text{ is MGL-valid.}$$

In what follows, MGL-structures will be referred to as MGL-models. By the size of an MGL-structure $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D)$ we will mean the size (cardinality) of its universe U . The cardinality of a set X will be denoted by $|X|$.

Let $\text{Sb}(\varphi)$ be the set of all subformulas of a formula φ defined recursively as:

$$\text{Sb}(p) = \{p\}$$

$$\text{Sb}(\neg\psi) = \text{Sb}(\psi) \cup \{\neg\psi\}$$

$$\text{Sb}(\psi\#\psi') = \text{Sb}(\psi) \cup \text{Sb}(\psi') \cup \{\psi\#\psi'\}, \text{ for any } \# \in \{\circ, \wedge, \vee\}.$$

Clearly, $\text{Sb}(\varphi)$ is finite, for all formulas φ .

PROPOSITION 3.13. *For every formula φ , if there are an MGL-model \mathcal{M} and a valuation v in \mathcal{M} such that $\mathcal{M}, v \not\models \varphi$, then there are a finite MGL-model \mathcal{M}' of size at most $|\text{Sb}(\varphi)| + 2$ and a valuation v' in \mathcal{M}' such that $\mathcal{M}', v' \not\models \varphi$.*

PROOF. Let φ be a formula, $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D)$ an MGL-model, and let v be a valuation on \mathcal{M} such that $\mathcal{M}, v \not\models \varphi$. Now, we define a new MGL-structure $\mathcal{M}' = (U', \sim', \circ', \otimes', \oplus', D')$ such that:

- $U' = \{v(\psi) \in U \mid \psi \in \mathbf{Sb}(\varphi)\} \cup \{1, 0\}$, for $1, 0 \notin U$,
- $D' = \{v(\psi) \in D \mid \psi \in \mathbf{Sb}(\varphi)\} \cup \{1\}$,
- For all $a, b \in U'$, operations $\sim', \circ', \otimes', \oplus'$ are defined as:

$$\begin{aligned} \sim' a &\stackrel{\text{df}}{=} \begin{cases} \sim a & \text{if } a \in U \text{ and } \sim a \in U' \setminus \{1, 0\} \\ 1, & \text{if } a \in \{1, 0\} \text{ or both } a \in U \text{ and } \sim a \notin U' \setminus \{1, 0\} \end{cases} \\ a \circ' b &\stackrel{\text{df}}{=} \begin{cases} a \circ b & \text{if } a \circ b \in U' \setminus \{1, 0\} \\ 1, & \text{if } a \circ b \notin U' \setminus \{1, 0\} \text{ and } a = b \\ 0, & \text{otherwise} \end{cases} \\ a \otimes' b &\stackrel{\text{df}}{=} \begin{cases} a \otimes b & \text{if } a \otimes b \in U' \setminus \{1, 0\} \\ 1, & \text{if } a \otimes b \notin U' \setminus \{1, 0\} \text{ and } a, b \in D' \\ 0, & \text{otherwise} \end{cases} \\ a \oplus' b &\stackrel{\text{df}}{=} \begin{cases} a \oplus b & \text{if } a \oplus b \in U' \setminus \{1, 0\} \\ 1, & \text{if } a \oplus b \notin U' \setminus \{1, 0\} \text{ and } (a \in D' \text{ or } b \in D') \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The structure \mathcal{M}' defined above is an MGL-model. Indeed, by the definition U', D' are non-empty and $D' \subseteq U'$. Observe also that $U' \setminus \{1, 0\} \subseteq U$ and $D' \setminus \{1\} \subseteq D$.

Now, we will prove that \mathcal{M}' satisfies all the other required constraints. First, we prove that $a \circ' b \in D'$ iff $a = b$, for all $a, b \in U'$. So let $a, b \in U'$ be such that $a \circ' b \notin U' \setminus \{1, 0\}$. Then, $a \circ' b \in D'$ iff $a \circ' b = 1$ iff $a = b$. Now, let $a, b \in U'$ be such that $a \circ' b \in U' \setminus \{1, 0\}$. Then the definition of \mathcal{M}' implies that $a, b \in U$ and $(a \circ' b) = (a \circ b) \in U$. Since \mathcal{M} is an MGL-model, the following holds: $a \circ' b \in D'$ iff $a \circ b \in D$ iff $a = b$. Hence, $a \circ' b \in D'$ iff $a = b$, for all $a, b \in U'$.

Now, let $a, b \in U'$ be such that $a \otimes' b \notin U' \setminus \{1, 0\}$. Then, by the definition of \mathcal{M}' , $a \otimes' b \in D'$ iff $a \otimes' b = 1$ iff $a, b \in D'$. If $a, b \in U'$ are such that $a \otimes' b \in U' \setminus \{1, 0\}$, then $a \otimes' b = a \otimes b \in U$. As \mathcal{M} is an MGL-model, the following holds: $a \otimes' b = a \otimes b \in D$ iff $a, b \in D$ iff (since $a, b \in U'$) $a, b \in D'$. Therefore, $a \otimes' b \in D'$ iff $a, b \in D'$, for all $a, b \in U'$.

Now, we show that $\sim'(a \otimes' \sim' a) \in D'$, for all $a \in U'$. Indeed, by the definition of \sim' , either $\sim'(a \otimes' \sim' a) = 1 \in D'$ or $\sim'(a \otimes' \sim' a) \in U' \setminus \{1, 0\}$. The latter means that $\sim'(a \otimes' \sim' a) = \sim(a \otimes \sim a)$. Since \mathcal{M} is an MGL-model, $\sim(a \otimes \sim a) \in D$, hence $\sim'(a \otimes' \sim' a) \in D$. So then $\sim'(a \otimes' \sim' a) \in D'$. Therefore, we have proved that \mathcal{M}' is an MGL-model.

Clearly, by the definition \mathcal{M}' is finite and of size at most $|\mathbf{Sb}(\varphi)| + 2$.

Now, let v' be a valuation in \mathcal{M}' such that $v'(p) = v(p)$, for all propositional variables $p \in \text{Sb}(\varphi)$. We will show that for all $\psi \in \text{Sb}(\varphi)$, the following holds:

$$(*) \quad v(\psi) = v'(\psi) \in U'.$$

For $\psi = p \in \text{Sb}(\varphi)$, $(*)$ holds by the definition of v' . Assume $(*)$ holds for $\psi', \psi_1, \psi_2 \in \text{Sb}(\varphi)$.

Let $\psi = \neg\psi' \in \text{Sb}(\varphi)$. Then, $v(\neg\psi') = \sim v(\psi')$. Clearly, $\sim v(\psi') \in U' \setminus \{1, 0\}$, since $\neg\psi' \in \text{Sb}(\varphi)$. By the induction hypothesis $v(\psi') = v'(\psi')$, so by the definition of the operation \sim' , we have $\sim v(\psi') = \sim' v'(\psi') = v'(\neg\psi')$. Hence, $v(\neg\psi) = v'(\neg\psi')$.

Let $\psi = (\psi_1 \equiv \psi_2) \in \text{Sb}(\varphi)$. Then, $v(\psi_1 \equiv \psi_2) = v(\psi_1) \circ v(\psi_2)$. Then, $v(\psi_1) \circ v(\psi_2) \in U' \setminus \{1, 0\}$, because $\psi_1 \equiv \psi_2 \in \text{Sb}(\varphi)$. By the induction hypothesis, $v(\psi_1) = v'(\psi_1)$ and $v(\psi_2) = v'(\psi_2)$. Thus, due to the definition of \circ' , we obtain: $v(\psi_1) \circ v(\psi_2) = v'(\psi_1) \circ' v'(\psi_2) = v'(\psi_1 \equiv \psi_2)$, which ends the proof of $(*)$ for ψ . For $\psi = (\psi_1 \wedge \psi_2)$ and $\psi = (\psi_1 \vee \psi_2)$ the proofs of $(*)$ are similar.

Now, due to $(*)$ it can be easily proved that for all formulas $\psi \in \text{Sb}(\varphi)$:

$$(**) \quad \mathcal{M}, v \models \psi \text{ if and only if } \mathcal{M}', v' \models \psi.$$

Indeed, by $(*)$ we have: $v(\psi) = v'(\psi) \in U'$, for all $\psi \in \text{Sb}(\varphi)$. Hence, $\mathcal{M}, v \models \psi$ iff $v(\psi) \in D$ iff $v'(\psi) \in D'$ iff $\mathcal{M}', v' \models \psi$.

Finally, the assumption $\mathcal{M}, v \not\models \varphi$ and $(**)$ imply that the model \mathcal{M}' and valuation v' do not satisfy the formula φ , i.e., $\mathcal{M}', v' \not\models \varphi$. ■

The above proposition implies the decidability of MGL:

THEOREM 3.14. *The logic MGL is decidable.*

PROOF. We need to show that given an MGL-formula there is a procedure that decides in finite number of steps whether the formula is MGL-valid or not. Let ϕ be a given formula and let \mathcal{K} be the class of all MGL-models $(U, \sim, \circ, \otimes, \oplus, D)$ such that $U \subseteq \{1, 2, \dots, k\}$ for some $k \leq |\text{Sb}(\phi)| + 2$. Clearly, every finite MGL-model of size at most $|\text{Sb}(\phi)| + 2$ is isomorphic to an element of the class \mathcal{K} . Given $\mathcal{M} \in \mathcal{K}$, let

$$\mathcal{V}_{\mathcal{M}} = \{v \upharpoonright \text{Sb}(\phi) \mid v \text{ is a valuation on } \mathcal{M}\}.$$

Clearly, \mathcal{K} and $\mathcal{V}_{\mathcal{M}}$ are finite, for any $\mathcal{M} \in \mathcal{K}$. We verify if φ is satisfied in $\mathcal{M} \in \mathcal{K}$ by valuations from $\mathcal{V}_{\mathcal{M}}$ until we find a counterexample. If there are $\mathcal{M} = (U, \sim, \circ, \otimes, \oplus, D) \in \mathcal{K}$ and $w \in \mathcal{V}_{\mathcal{M}}$ such that $w(\phi) \notin D$, then $w = v \upharpoonright \text{Sb}(\phi)$ for some v such that $\mathcal{M}, v \not\models \phi$, which shows that ϕ is not

MGL-valid. Otherwise by Proposition 3.13, ϕ is satisfied by all MGL-models and all the valuations, hence it is MGL-valid. ■

4. MGL as a Generic Grzegorzczuk’s Logic

The logic MGL is as weak as possible. It is manifested not only in the number of axioms but also in its possibility of extending to other non-Fregean logics. In what follows, whenever we write “L-logic” or “set of formulas” we mean a logic or formulas in the language of Grzegorzczuk’s non-Fregean logics as defined in Definition 1.7.

DEFINITION 4.1. Let S be a finite set of formulas and let R be a finite set of rules in the language of Grzegorzczuk’s non-Fregean logics. A formula φ is said to be MGL_S^R -provable ($\vdash_{MGL_S^R} \varphi$) whenever there exists a finite sequence $\varphi_1, \dots, \varphi_n$ of formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and each $\varphi_i, i \in \{1, \dots, n\}$, is either an MGL-axiom or a formula from S or it follows from earlier formulas in the sequence by one of the MGL- or R -rules of inference, where the rule (Sub) can also be applied to formulas from S . \dashv

Hence, we may think about MGL_S^R -provability as the provability in a logic obtained by an extension of MGL with additional rules from R and/or formulas from S treated as additional axioms.

DEFINITION 4.2 ((axiomatic) extension). A logic L is said to be an *extension* of MGL whenever there are finite sets S of formulas and R of rules such that for all formulas φ :

φ is L-provable if and only if φ is MGL_S^R -provable.

A logic L is an *axiomatic extension* of MGL whenever it is an extension of MGL for which $R = \emptyset$. \dashv

PROPOSITION 4.3. *The logics LD and LDS are axiomatic extensions of MGL.*

PROOF. First, we prove that the rule (ext) is LD-admissible. Let $\varphi, \psi, \alpha(\varphi)$, and $\alpha(\varphi/\psi)$ be formulas such that $\alpha(\varphi)$ is a formula in which φ is a subformula and $\alpha(\varphi/\psi)$ is obtained from $\alpha(\varphi)$ by replacing an occurrence of φ with ψ .

Assume $\varphi \equiv \psi$ is LD-provable. We show by induction on the complexity of α that

$$(*) \quad \alpha(\varphi) \equiv \alpha(\varphi/\psi) \text{ is LD-provable.}$$

If $\alpha(\varphi) = \varphi$, then by the assumption $\varphi \equiv (\varphi/\psi)$ is trivially LD-provable. Assume $(*)$ holds for formulas $\alpha'(\varphi)$, $\alpha_1(\varphi)$, $\alpha_2(\varphi)$. We will show that it holds for formulas $\neg\alpha'(\varphi)$ and $\alpha_1(\varphi)\#\alpha_2(\varphi)$, for $\# \in \{\circ, \wedge, \vee\}$.

LD-proof of $\neg\alpha'(\varphi) \equiv \neg\alpha'(\varphi/\psi)$

1. $\alpha'(\varphi) \equiv \alpha'(\varphi/\psi)$ induction hypothesis
2. $(\alpha'(\varphi) \equiv \alpha'(\varphi/\psi)) \equiv (\neg\alpha'(\varphi) \equiv \neg\alpha'(\varphi/\psi))$ (Sub) to (Ax14)
for $p/\alpha'(\varphi)$, $q/\alpha'(\varphi/\psi)$
3. $\neg\alpha'(\varphi) \equiv \neg\alpha'(\varphi/\psi)$ (MPE) to 1. and 2.

LD-proof of $(\alpha_1(\varphi)\#\alpha_2(\varphi)) \equiv (\alpha_1(\varphi/\psi)\#\alpha_2(\varphi/\psi))$

1. $\alpha_1(\varphi) \equiv \alpha_1(\varphi/\psi)$ induction hypothesis
2. $\alpha_2(\varphi) \equiv \alpha_2(\varphi/\psi)$ induction hypothesis
3. $(\alpha_1(\varphi) \equiv \alpha_1(\varphi/\psi)) \Rightarrow (\alpha_1(\varphi)\#\alpha_2(\varphi)) \equiv (\alpha_1(\varphi/\psi)\#\alpha_2(\varphi))$
(Sub) to (Axi)_{LD}, for $i = \begin{cases} 15, & \text{if } \# = \equiv \\ 16, & \text{if } \# = \wedge \\ 17, & \text{if } \# = \vee \end{cases}$
for $p/\alpha_1(\varphi)$, $q/\alpha_1(\varphi/\psi)$, $r/\alpha_2(\varphi)$
4. $(\alpha_2(\varphi) \equiv \alpha_2(\varphi/\psi)) \Rightarrow (\alpha_2(\varphi)\#\alpha_1(\varphi/\psi)) \equiv (\alpha_2(\varphi/\psi)\#\alpha_1(\varphi/\psi))$
(Sub) to (Axi)_{LD}, for $i = \begin{cases} 15, & \text{if } \# = \equiv \\ 16, & \text{if } \# = \wedge \\ 17, & \text{if } \# = \vee \end{cases}$
for $p/\alpha_2(\varphi)$, $q/\alpha_2(\varphi/\psi)$, $r/\alpha_1(\varphi/\psi)$
5. $(\alpha_1(\varphi)\#\alpha_2(\varphi)) \equiv (\alpha_1(\varphi/\psi)\#\alpha_2(\varphi))$ (MPE) to 1. and 3., then (\wedge_2)
6. $(\alpha_2(\varphi)\#\alpha_1(\varphi/\psi)) \equiv (\alpha_2(\varphi/\psi)\#\alpha_1(\varphi/\psi))$ (MPE) to 2. and 4., then (\wedge_2)
7. $(\alpha_1(\varphi/\psi)\#\alpha_2(\varphi)) \equiv (\alpha_2(\varphi)\#\alpha_1(\varphi/\psi))$
(Sub) to (Axi), for $i = \begin{cases} 13, & \text{if } \# = \equiv \\ 5, & \text{if } \# = \wedge \\ 6, & \text{if } \# = \vee \end{cases}$
for $p/\alpha_1(\varphi/\psi)$, $q/\alpha_2(\varphi)$
8. $(\alpha_1(\varphi)\#\alpha_2(\varphi)) \equiv (\alpha_2(\varphi)\#\alpha_1(\varphi/\psi))$ (tran) to 5. and 7.
9. $(\alpha_1(\varphi)\#\alpha_2(\varphi)) \equiv (\alpha_2(\varphi/\psi)\#\alpha_1(\varphi/\psi))$ (tran) to 8. and 6.

$$10. (\alpha_2(\varphi/\psi)\# \alpha_1(\varphi/\psi)) \equiv (\alpha_1(\varphi/\psi)\# \alpha_2(\varphi/\psi))$$

$$(\text{Sub}) \text{ to } (\text{Axi}), \text{ for } i = \begin{cases} 13, & \text{if } \# = \equiv \\ 5, & \text{if } \# = \wedge \\ 6, & \text{if } \# = \vee \end{cases}$$

for $p/\alpha_2(\varphi/\psi), q/\alpha_1(\varphi/\psi)$

$$11. (\alpha_1(\varphi)\# \alpha_2(\varphi)) \equiv (\alpha_1(\varphi/\psi)\# \alpha_2(\varphi/\psi))$$

(tran) to 9. and 10.

Therefore, the rule (ext) is LD-admissible. Since all MGL-axioms and all the other MGL-rules are also assumed in LD, we get

$$\text{LD} = \text{MGL}_{\mathcal{D}},$$

$$\text{for } \mathcal{D} = \{(\text{Axi}) \mid i \in \{2, \dots, 14\}\} \cup \{(\text{Axi})_{\text{LD}} \mid i \in \{15, 16, 17\}\}.$$

For the logic LDS, we proceed essentially in a similar way. Indeed, since the logics LD and LDS differ only on axioms 15, 16, 17, it suffices to show the induction step for $\alpha_1(\varphi)\# \alpha_2(\varphi)$, where $\# \in \{\circ, \wedge, \vee\}$.

LDS-proof of $(\alpha_1(\varphi)\# \alpha_2(\varphi)) \equiv (\alpha_1(\varphi/\psi)\# \alpha_2(\varphi/\psi))$

$$1. \alpha_1(\varphi) \equiv \alpha_1(\varphi/\psi) \quad \text{induction hypothesis}$$

$$2. \alpha_2(\varphi) \equiv \alpha_2(\varphi/\psi) \quad \text{induction hypothesis}$$

$$3. (\alpha_1(\varphi) \equiv \alpha_1(\varphi/\psi)) \wedge (\alpha_2(\varphi) \equiv \alpha_2(\varphi/\psi)) \quad (\wedge_1) \text{ to } 1 \text{ and } 2$$

$$4. [(\alpha_1(\varphi) \equiv \alpha_1(\varphi/\psi)) \wedge (\alpha_2(\varphi) \equiv \alpha_2(\varphi/\psi))] \Rightarrow$$

$$\Rightarrow (\alpha_1(\varphi)\# \alpha_2(\varphi)) \equiv (\alpha_1(\varphi/\psi)\# \alpha_2(\varphi/\psi))$$

$$(\text{Sub}) \text{ to } (\text{Axi})_{\text{LDS}}, \text{ for } i = \begin{cases} 15, & \text{if } \# = \equiv \\ 16, & \text{if } \# = \wedge \\ 17, & \text{if } \# = \vee \end{cases}$$

for $p/\alpha_1(\varphi), q/\alpha_1(\varphi/\psi), r/\alpha_2(\varphi), s/\alpha_2(\varphi/\psi)$

$$5. (\alpha_1(\varphi)\# \alpha_2(\varphi)) \equiv (\alpha_1(\varphi/\psi)\# \alpha_2(\varphi/\psi)) \quad (\text{MPE}) \text{ to } 3. \text{ and } 4., \text{ then } (\wedge_2)$$

Therefore, the rule (ext) is LDS-admissible, and hence we have:

$$\text{LDS} = \text{MGL}_{\mathcal{S}},$$

$$\text{for } \mathcal{S} = \{(\text{Axi}) \mid i \in \{2, \dots, 14\}\} \cup \{(\text{Axi})_{\text{LDS}} \mid i \in \{15, 16, 17\}\}. \quad \blacksquare$$

As we mentioned in the Introduction the logic LD is uncomparable with Suszko's non-Fregean logic SCI, which is obtained by expanding the language of classical propositional logic with the identity connective \equiv and adding to the axiomatization of classical logic the following four axioms, for $\# \in \{\vee, \wedge, \rightarrow, \leftrightarrow, \equiv\}$:

- (SCI1) $\varphi \equiv \varphi$,
 (SCI2) $(\varphi \equiv \psi) \rightarrow (\neg\varphi \equiv \neg\psi)$,
 (SCI3) $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$,
 (SCI4) $[(\varphi \equiv \psi) \wedge (\vartheta \equiv \xi)] \rightarrow [(\varphi\#\vartheta) \equiv (\psi\#\xi)]$.

Clearly, (Ax0) and (Ax1) are SCI-theorems. It is also easy to prove that the rules (MPE), (Sub), (\wedge_1), and (\wedge_2) are SCI-admissible. Furthermore, the SCI-admissibility of the rule (ext) can be proved as in the case of logic LDS. Hence, this implies that SCI is an extension of MGL and the same concerns any of the extensions of SCI, in particular any of those propositional non-Fregean logics that are equivalent to certain modal logics. Recall that the three best-known propositional extensions of SCI are: WB—the least Boolean SCI-theory, WT—an SCI-theory corresponding to the modal logic S4, and WH—an SCI-theory corresponding to the modal logic S5. Finally, we obtain:

The logic MGL is a generic Grzegorzczuk's non-Fregean logic which can serve as a base for other non-classical logics. In particular, the following logics are its extensions: Grzegorzczuk's logics LD and LDS, Suszko's logic SCI, WB, WT, WH, the last two of which are non-Fregean translations of the standard modal logics S4 and S5, respectively.

It is worth mentioning that the logic LE is beyond the expressive power of MGL. As shown in [2], a weak instance of the extensionality rule is not admissible in LE, so is the rule (ext). Hence, the logic LE is incomparable even with the minimal Grzegorzczuk's non-Fregean logic MGL. This is yet another justification for rejecting LE, but on the other hand it also means that MGL is not a trivial logic.

5. Conclusion

We have discussed philosophical interpretations of the three Grzegorzczuk's logics LD, LE, and LDS. We concluded that these logics do not realize the Extensionality Principle in an adequate intended way, since the readings and interpretation of some of their axioms are obscure and unintuitive. As a remedy a new Grzegorzczuk's non-Fregean logic MGL has been proposed. A sound and complete semantics for MGL was constructed and the logic has been proved to be decidable. Finally, we showed that some other crucial non-

classical logics are extensions of MGL, so it can be treated as a common base for logics with different languages, semantics, and philosophical constraints.

The results presented in the paper lead to some further questions. In particular, as the logic MGL is decidable, it is natural to search for a deduction system which would be a decision procedure for MGL. Furthermore, it would be interesting to complete the picture and determine whether intuitionistic, many-valued, relevant, and other paraconsistent logics are extensions of MGL.

Acknowledgements. This paper was supported by the Polish National Science Centre (NCN) grant *Logic of language experience* nr 2011/03/B/HS1/04580. The author is very grateful to the anonymous referees and Dr Taneli Huuskonen (University of Helsinki) for their valuable comments and suggestions which certainly improved the manuscript.

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J. GOLIŃSKA-PILAREK
Institute of Philosophy
University of Warsaw
Krakowskie Przedmieście 3
Warsaw, Poland
j.golinska@uw.edu.pl