Statistics and Computing (2022) 32:100
https://doi.org/10.1007/s11222-022-10162-7

ORIGINALPAPER

Robust approach for comparing two dependent normal populations through Wald-type tests based on Rényi's pseudodistance estimators

Elena Castilla[1] · **María Jaenada**[2] · **Nirian Martín**[3] · **Leandro Pardo**[2]

Received: 2 February 2022 / Accepted: 27 September 2022 / Published online: 25 October 2022

Abstract

Since the two seminal papers by Fisher (Biometrika 10:507–521, 1915; Metron 1:1–32, 1921) were published, the test under a fixed value correlation coefficient null hypothesis for the bivariate normal distribution constitutes an important statistical problem. In the framework of asymptotic robust statistics, it remains being a topic of great interest to be investigated. For this and other tests, focused on paired correlated normal random samples, Rényi's pseudodistance estimators are proposed, their asymptotic distribution is established and an iterative algorithm is provided for their computation. From them the Wald-type test statistics are constructed for different problems of interest and their influence function is theoretically studied. For testing null correlation in different contexts, an extensive simulation study and two real data based examples support the robust properties of our proposal.

Keywords Correlation test · Influence function · Rényi pseudodistance · Robustness · Wald-type test

1 Introduction

In parametric estimation the role of divergence measures is very intuitive: minimizing a suitable divergence measure between the data and the assumed model in order to estimate the unknown parameters. These estimators are called "minimum divergence estimators" (MDEs). There is a growing body of literature that recognizes the importance of MDEs on the basis of their robustness, without a significant loss of efficiency, in comparison with the maximum likelihood estimator (MLE). Beran (1977) showed that the minimum Hellinger distance estimator that minimizes Hellinger distance between the modelled parametric density and its non-parametric estimator is robust against small perturbation in the underlying model. Other interesting results in relation to the MDEs can be seen in Tamura and Boos (1986), Simpson (1989a, b), Lindsay (1994), Pardo (2006), Basu et al. (2011), Broniatowski et al. (1961) and references therein.

In the case of continuous models, it is convenient to consider families of divergence measures for which non-parametric estimators of the unknown density function are needed. For instance, the theory developed by the cited paper of Beran needs a non-parametric estimator of the unknown density function. From this perspective, the density power divergence (DPD) family, leading to the minimum density power divergence estimators (MDPDEs), is a good example. For more details see Basu et al. (2011). However, there is another important family of divergence measures which neither needs non-parametric estimators, the Rényi's pseudodistances (RPDs). This family of pseudodistances will be considered in this paper.

Elena Castilla, María Jaenada, Nirian Martín and Leandro Pardo have contributed equally to this work.

✉ María Jaenada
mjaenada@ucm.es

Elena Castilla
elena.castilla@urjc.es

Nirian Martín
nimartin@ucm.es

Leandro Pardo
lpardo@mat.ucm.es

[1] Department of Applied Mathematics, Materials Science and Engineering, and Electronic Technology, Rey Juan Carlos University, 28933 Mostoles, Spain

[2] Department of Statistics and Operations Research, Complutense University of Madrid, 28040 Madrid, Spain

[3] Department of Financial and Actuarial Economics and Statistics, Complutense University of Madrid, 28003 Madrid, Spain

Let $X_1, \ldots, X_n$ be a random sample of size n from a population X, having true and unknown density function g, modelled by a parametric family of densities $f_{\boldsymbol{\theta}}$ with $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^p$. The RPD between the densities $f_{\boldsymbol{\theta}}$ and g is given, for a tuning parameter $\alpha > 0$, by

$$\begin{aligned} R_\alpha\left(f_{\boldsymbol{\theta}}, g\right) = & \frac{1}{\alpha+1} \log \int_{-\infty}^{+\infty} f_{\boldsymbol{\theta}}^{\alpha+1}(x) dx \\ & + \frac{1}{\alpha(\alpha+1)} \log \int_{-\infty}^{+\infty} g^{\alpha+1}(x) dx \\ & - \frac{1}{\alpha} \log \int_{-\infty}^{+\infty} f_{\boldsymbol{\theta}}^{\alpha}(x) g(x) dx. \end{aligned} \tag{1}$$

The RPD was considered for the first time in Jones et al. (2001). Fujisawa and Eguchi (2008) used the RPD under the name of γ-cross entropy. Due to the resemblance with the Rényi divergence (Rényi 1961), Broniatowski et al. (1961) named it RPD.

The RPD can be extended for $\alpha = 0$ taking continuous limits on the left yielding the expression

$$R_{\alpha=0}\left(f_{\boldsymbol{\theta}}, g\right) = \lim_{\alpha \downarrow 0} R_\alpha\left(f_{\boldsymbol{\theta}}, g\right) = \int_{-\infty}^{+\infty} g(x) \log \frac{g(x)}{f_\theta(x)} dx,$$

i.e., the RPD coincides with the Kullback–Leibler divergence (KLD) between g and $f_{\boldsymbol{\theta}}$, at $\alpha = 0$ (see Pardo 2006).

Broniatowski et al. (1961) established that the RPD is positive for any two densities and for all values of the tuning parameter $\alpha > 0$, $R_\alpha\left(f_{\boldsymbol{\theta}}, g\right) \geq 0$ and further $R_\alpha\left(f_{\boldsymbol{\theta}}, g\right) = 0$ if and only if $f_{\boldsymbol{\theta}} = g$. This property suggests the definition of the minimum RPD estimators (MRPDEs) as the minimizer of the RPD between the assumed distribution and the empirical distribution of the data. Therefore, the MRPDE for the unknown parameter $\boldsymbol{\theta}$, based on the random sample $X_1, \ldots, X_n$, $\widehat{\boldsymbol{\theta}}_{R,\alpha} = \widehat{\boldsymbol{\theta}}_{R,\alpha}(X_1, \ldots, X_n)$, is given, for a tuning parameter $\alpha > 0$, by

$$\widehat{\boldsymbol{\theta}}_{R,\alpha} = \arg\sup_{\boldsymbol{\theta} \in \Theta} \sum_{i=1}^{n} w_\alpha(\boldsymbol{\theta}) f_{\boldsymbol{\theta}}^{\alpha}(X_i), \tag{2}$$

where the weight is defined as $w_\alpha(\boldsymbol{\theta}) = \kappa_\alpha^{-\frac{\alpha}{\alpha+1}}(\boldsymbol{\theta})$ with

$$\kappa_\alpha(\boldsymbol{\theta}) = \mathrm{E}[f_{\boldsymbol{\theta}}^{\alpha}(X)] = \int_{-\infty}^{+\infty} f_{\boldsymbol{\theta}}^{\alpha+1}(x) dx. \tag{3}$$

Note that the value $\alpha = 0$ was defined as the KLD and hence, the MRPDE coincides with the MLE at $\alpha = 0$.

The estimating equations, based on (2), are given by

$$\sum_{i=1}^{n} \boldsymbol{\Psi}_\alpha(x_i; \boldsymbol{\theta}) = \mathbf{0}_p, \tag{4}$$

where $\mathbf{0}_p$ is the null column vector of dimension p and

$$\begin{aligned} \boldsymbol{\Psi}_\alpha(x_i; \boldsymbol{\theta}) &= f_{\boldsymbol{\theta}}^{\alpha}(x_i) \left(\boldsymbol{u}_{\boldsymbol{\theta}}(x_i) - \boldsymbol{c}_\alpha(\boldsymbol{\theta})\right), \\ \boldsymbol{u}_{\boldsymbol{\theta}}(x_i) &= \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}(x_i) = \frac{\frac{\partial}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}(x_i)}{f_{\boldsymbol{\theta}}(x_i)}, \\ \boldsymbol{c}_\alpha(\boldsymbol{\theta}) &= \frac{\frac{\partial}{\partial \boldsymbol{\theta}} \log \kappa_\alpha(\boldsymbol{\theta})}{\alpha+1} = \frac{\boldsymbol{\xi}_\alpha(\boldsymbol{\theta})}{\kappa_\alpha(\boldsymbol{\theta})} \\ &= \left(c_{\alpha,1}(\boldsymbol{\theta}), \ldots, c_{\alpha,p}(\boldsymbol{\theta})\right)^T, \end{aligned} \tag{5}$$

where $\kappa_\alpha(\boldsymbol{\theta})$ is given by (3) and

$$\begin{aligned} \boldsymbol{\xi}_\alpha(\boldsymbol{\theta}) &= \frac{1}{\alpha+1} \frac{\partial}{\partial \boldsymbol{\theta}} \kappa_\alpha(\boldsymbol{\theta}) = \mathrm{E}[f_{\boldsymbol{\theta}}^{\alpha}(X) \boldsymbol{u}_{\boldsymbol{\theta}}(X)] \\ &= \int_{-\infty}^{+\infty} f_{\boldsymbol{\theta}}^{\alpha+1}(x) \boldsymbol{u}_{\boldsymbol{\theta}}(x) dx. \end{aligned} \tag{6}$$

The MRPDE is an M-estimator and thus it asymptotic distribution and influence function (IF) can be obtained based on the asymptotic theory of the M-estimators. Broniatowski et al. (1961) studied the asymptotic properties and robustness of the MRPDEs. In relation with the asymptotic distribution they got

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{R,\alpha} - \boldsymbol{\theta}_0) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} \mathcal{N}\left(\mathbf{0}_p, \boldsymbol{V}_\alpha(\boldsymbol{\theta}_0)\right), \tag{7}$$

where $\boldsymbol{\theta}_0$ is the true unknown value of $\boldsymbol{\theta}$ and

$$\boldsymbol{V}_\alpha(\boldsymbol{\theta}) = \boldsymbol{S}_\alpha^{-1}(\boldsymbol{\theta}) \boldsymbol{K}_\alpha(\boldsymbol{\theta}) \boldsymbol{S}_\alpha^{-1}(\boldsymbol{\theta}), \tag{8}$$

with

$$\boldsymbol{S}_\alpha(\boldsymbol{\theta}) = -\mathrm{E}\left[\frac{\partial \boldsymbol{\Psi}_\alpha^T(X; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right], \tag{9}$$

$$\boldsymbol{K}_\alpha(\boldsymbol{\theta}) = \mathrm{E}\left[\boldsymbol{\Psi}_\alpha(X; \boldsymbol{\theta}) \boldsymbol{\Psi}_\alpha^T(X; \boldsymbol{\theta})\right]. \tag{10}$$

The new result given in Sect. 2 provides a simplified version which is very useful in practice.

At the same time (Broniatowski et al. 1961) established that the IF of the functional of the MRPDE of $\boldsymbol{\theta}$, $\boldsymbol{T}_\alpha$, is given by $\mathcal{IF}(x, \boldsymbol{T}_\alpha, F_{\boldsymbol{\theta}}) = \boldsymbol{S}_\alpha^{-1}(\boldsymbol{\theta}) \boldsymbol{\Psi}_\alpha(x, \boldsymbol{\theta})$. In aforementioned paper an application was presented to the multiple regression model (MRM) with random covariates. Toma and Leoni-Aubin (2013) used RP in order to define new robustness and efficiency measures. In the same vein, Castilla et al. (2020a) introduced Wald-type tests based on the minimum RPD estimators for the MRM and its extension for Generalized Linear models was presented in Jaenada and Pardo (2021, 2022). Further, Castilla et al. (2020b) studied the MRPDE for the linear regression model in the ultra-high dimensional set-up.

2 Simplified version of the asymptotic variance-covariance matrix of Rényi's pseudodistance estimators

This is a short but very important section as it establishes for the first time new and short expressions of $\boldsymbol{S}_\alpha(\boldsymbol{\theta})$ and $\boldsymbol{K}_\alpha(\boldsymbol{\theta})$, given in (9) and (10), in terms of a scalar $\kappa_\alpha(\boldsymbol{\theta})$, a vector $\boldsymbol{c}_\alpha(\boldsymbol{\theta})$, and a matrix $\boldsymbol{J}_\alpha(\boldsymbol{\theta})$, whose calculation of any distribution is exactly the same as the one developed for MDPDEs, so the complexity of the construction of the theory based on MRPDEs is not higher than the MDPDEs.

Theorem 1 *The expression of the variance-covariance matrix in the asymptotic distribution, (7), is given by (8) where*

$$\boldsymbol{S}_\alpha(\boldsymbol{\theta}) = \boldsymbol{J}_\alpha(\boldsymbol{\theta}) - \kappa_\alpha(\boldsymbol{\theta})\boldsymbol{c}_\alpha(\boldsymbol{\theta})\boldsymbol{c}_\alpha^T(\boldsymbol{\theta}), \tag{11}$$

$$\boldsymbol{K}_\alpha(\boldsymbol{\theta}) = \boldsymbol{S}_{2\alpha}(\boldsymbol{\theta}) + \kappa_{2\alpha}(\boldsymbol{\theta})\left(\boldsymbol{c}_{2\alpha}(\boldsymbol{\theta}) - \boldsymbol{c}_\alpha(\boldsymbol{\theta})\right)\left(\boldsymbol{c}_{2\alpha}(\boldsymbol{\theta}) - \boldsymbol{c}_\alpha(\boldsymbol{\theta})\right)^T, \tag{12}$$

with

$$\boldsymbol{J}_\alpha(\boldsymbol{\theta}) = \mathrm{E}[f_{\boldsymbol{\theta}}^\alpha(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\boldsymbol{u}_{\boldsymbol{\theta}}^T(X)] = \int_{-\infty}^{+\infty} f_{\boldsymbol{\theta}}^{\alpha+1}(x)\boldsymbol{u}_{\boldsymbol{\theta}}(x)\boldsymbol{u}_{\boldsymbol{\theta}}^T(x)dx, \tag{13}$$

and the expressions of $\kappa_\alpha(\boldsymbol{\theta})$ and $\boldsymbol{c}_\alpha(\boldsymbol{\theta})$ were given by (3) and (5) respectively.

Proof See "Appendix A". □

3 Minimum Rényi pseudodistance estimators for two dependent populations with normal distribution

In the previous results univariate case was considered, but it is straightforward to extend it for the multivariate set-up. In this paper we are considering the bidimensional normal distribution model, and so in the following the role of x is replaced by (x, y) and all the integrals are in $\mathbb{R}^2$. In addition, we are going to get Wald-type test statistics for testing different composite null hypothesis regarding the model parameters.

Let (X, Y) be a bidimensional normal model with density function

$$f_{\boldsymbol{\theta}}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]\right\}, \tag{14}$$

$\sigma_1, \sigma_2 > 0$, $\mu_1, \mu_2 \in \mathbb{R}$ and $-1 < \rho < 1$, and we shall denote by

$$\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)^T \tag{15}$$

the model parameters belonging to the parameter space $\Theta = \mathbb{R}^2 \times \mathbb{R}_+^2 \times (-1, 1)$.

We are interested, on the basis of a random sample of size n, $(X_1, Y_1), \ldots, (X_n, Y_n)$, in obtaining the MRPDE for $\boldsymbol{\theta}$, as well as the asymptotic distribution. Further, we aim to develop Wald-type tests, in the bidimensional normal model, based on MRPDE. Some preliminary results from which proofs the reader could find many clues were presented in Martín (2020).

Proposition 2 *For the bidimensional normal model, (14), the vector of score functions is given by*

$$\boldsymbol{u}_{\boldsymbol{\theta}}(x, y) = (u_{\mu_1}(x, y), u_{\mu_2}(x, y), u_{\sigma_1}(x, y), u_{\sigma_2}(x, y), u_\rho(x, y))^T, \tag{16}$$

where

$$u_{\mu_1}(x, y) = \frac{1}{\sigma_1(1-\rho^2)}\left[\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}\right],$$

$$u_{\mu_2}(x, y) = \frac{1}{\sigma_2(1-\rho^2)}\left[\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right],$$

$$u_{\sigma_1}(x, y) = -\frac{1}{\sigma_1} - \frac{1}{\sigma_1(1-\rho^2)} \times \left[\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2} - \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right],$$

$$u_{\sigma_2}(x, y) = -\frac{1}{\sigma_2} - \frac{1}{\sigma_2(1-\rho^2)} \times \left[\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2} - \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right],$$

$$u_\rho(x, y) = \frac{1}{(1-\rho^2)}\left[\rho + \frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}\right] - \frac{\rho}{(1-\rho^2)^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}\right].$$

Proposition 3 *For the bidimensional normal model, (14), the expressions of (6) and (3) are given by*

$$\boldsymbol{c}_\alpha(\boldsymbol{\theta}) = (c_\alpha(\mu_1), c_\alpha(\mu_2), c_\alpha(\sigma_1), c_\alpha(\sigma_2), c_\alpha(\rho))^T = \begin{pmatrix} \boldsymbol{c}_{1,\alpha}(\boldsymbol{\theta}) \\ \boldsymbol{c}_{2,\alpha}(\boldsymbol{\theta}) \end{pmatrix}, \tag{17}$$

where

$$\boldsymbol{c}_{1,\alpha}(\boldsymbol{\theta}) = \boldsymbol{0}_2, \quad \boldsymbol{c}_{2,\alpha}(\boldsymbol{\theta}) = \frac{\alpha}{\alpha+1}\boldsymbol{D}^{-1}_{2,\sigma_1,\sigma_2}\begin{pmatrix} -1 \\ -1 \\ \frac{\rho}{1-\rho^2} \end{pmatrix}$$
$$= \frac{\alpha}{\alpha+1}\begin{pmatrix} -\frac{1}{\sigma_1} \\ -\frac{1}{\sigma_2} \\ \frac{\rho}{1-\rho^2} \end{pmatrix},$$

with

$$\boldsymbol{D}_{2,\sigma_1,\sigma_2} = \text{diag}\{\sigma_1, \sigma_2, 1\}, \tag{18}$$

and

$$\kappa_\alpha(\boldsymbol{\theta}) = \frac{1}{k^\alpha(\boldsymbol{\theta})\,(\alpha+1)}, \tag{19}$$

with

$$k(\boldsymbol{\theta}) = 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}. \tag{20}$$

In the following theorem we shall present the expressions of the matrices $\boldsymbol{K}_\alpha(\boldsymbol{\theta})$ and $\boldsymbol{S}_\alpha(\boldsymbol{\theta})$, defined in (12) and (11). But first, it is necessary to provide the following result.

Proposition 4 *For the bidimensional normal model, (14), we have the following results concerning with the integrals of the cross product for the score functions*

$$\boldsymbol{J}_\alpha(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{J}_{1,\alpha}(\boldsymbol{\theta}) & \boldsymbol{0}_{2\times 3} \\ \boldsymbol{0}_{3\times 2} & \boldsymbol{J}_{2,\alpha}(\boldsymbol{\theta}) \end{pmatrix},$$

where

$$\boldsymbol{J}_{1,\alpha}(\boldsymbol{\theta}) = \frac{1}{k^\alpha(\boldsymbol{\theta})(\alpha+1)^2}\boldsymbol{D}^{-1}_{1,\sigma_1,\sigma_2}\boldsymbol{J}_1(\rho)\boldsymbol{D}^{-1}_{1,\sigma_1,\sigma_2},$$
$$\boldsymbol{J}_1(\rho) = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix},$$
$$\boldsymbol{J}_{2,\alpha}(\boldsymbol{\theta}) = \frac{1}{k^\alpha(\boldsymbol{\theta})(\alpha+1)^3}\boldsymbol{D}^{-1}_{2,\sigma_1,\sigma_2}\boldsymbol{J}_{2,\alpha}(\rho)\boldsymbol{D}^{-1}_{2,\sigma_1,\sigma_2},$$
$$\boldsymbol{J}_{2,\alpha}(\rho) = \frac{1}{1-\rho^2}\begin{pmatrix} \alpha^2-\rho^2(\alpha^2+1)+2 & \alpha^2-\rho^2(\alpha^2+1) & -\rho(\alpha^2+1) \\ \alpha^2-\rho^2(\alpha^2+1) & \alpha^2-\rho^2(\alpha^2+1)+2 & -\rho(\alpha^2+1) \\ -\rho(\alpha^2+1) & -\rho(\alpha^2+1) & \frac{\rho^2(\alpha^2+1)+1}{1-\rho^2} \end{pmatrix}, \tag{21}$$

with

$$\boldsymbol{D}_{1,\sigma_1,\sigma_2} = \text{diag}\{\sigma_1, \sigma_2\}, \tag{22}$$

$\boldsymbol{D}_{2,\sigma_1,\sigma_2}$ *is given by* (18) *and* $k(\boldsymbol{\theta})$ *by* (20).

Theorem 5 *For the bidimensional normal model,* (14)*, we have the following results concerning with the expectations of the estimating equations*

$$\boldsymbol{S}_\alpha(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{S}_{1,\alpha}(\boldsymbol{\theta}) & \boldsymbol{0}_{2\times 3} \\ \boldsymbol{0}_{3\times 2} & \boldsymbol{S}_{2,\alpha}(\boldsymbol{\theta}) \end{pmatrix},$$

with

$$\boldsymbol{S}_{1,\alpha}(\boldsymbol{\theta}) = \boldsymbol{J}_{1,\alpha}(\boldsymbol{\theta}),$$
$$\boldsymbol{S}_{2,\alpha}(\boldsymbol{\theta}) = \frac{1}{k^\alpha(\boldsymbol{\theta})(\alpha+1)^3}\boldsymbol{D}^{-1}_{2,\sigma_1,\sigma_2}\boldsymbol{S}_{2,1}(\rho)\boldsymbol{D}^{-1}_{2,\sigma_1,\sigma_2},$$

and

$$\boldsymbol{S}_{2,1}(\rho) = \frac{1}{1-\rho^2}\begin{pmatrix} 2-\rho^2 & -\rho^2 & -\rho \\ -\rho^2 & 2-\rho^2 & -\rho \\ -\rho & -\rho & \frac{1+\rho^2}{1-\rho^2} \end{pmatrix}. \tag{23}$$

On the other hand

$$\boldsymbol{K}_\alpha(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{K}_{1,\alpha}(\boldsymbol{\theta}) & \boldsymbol{0}_{2\times 3} \\ \boldsymbol{0}_{3\times 2} & \boldsymbol{K}_{2,\alpha}(\boldsymbol{\theta}) \end{pmatrix}$$

where

$$\boldsymbol{K}_{1,\alpha}(\boldsymbol{\theta}) = \boldsymbol{J}_{1,2\alpha}(\boldsymbol{\theta}),$$
$$\boldsymbol{K}_{2,\alpha}(\boldsymbol{\theta}) = \frac{1}{k^{2\alpha}(\boldsymbol{\theta})(2\alpha+1)^3(\alpha+1)^2}\boldsymbol{D}^{-1}_{2,\sigma_1,\sigma_2}\boldsymbol{K}_{2,\alpha}(\rho)\boldsymbol{D}^{-1}_{2,\sigma_1,\sigma_2},$$

with

$$\boldsymbol{K}_{2,\alpha}(\rho) = (\alpha+1)^2\boldsymbol{S}_{2,1}(\rho) + \alpha^2\boldsymbol{S}_{2,2}(\rho), \tag{24}$$

and

$$\boldsymbol{S}_{2,2}(\rho) = \frac{1}{1-\rho^2}\begin{pmatrix} 1-\rho^2 & 1-\rho^2 & -\rho \\ 1-\rho^2 & 1-\rho^2 & -\rho \\ -\rho & -\rho & \frac{\rho^2}{1-\rho^2} \end{pmatrix}. \tag{25}$$

For both, $\boldsymbol{J}_{1,\alpha}(\boldsymbol{\theta})$ *is given by* (21)*,* $\boldsymbol{D}_{2,\sigma_1,\sigma_2}$ *by* (18) *and* $k(\boldsymbol{\theta})$ *by* (20).

Proof See "Appendix B". □

Inverting the diagonal blocks of $\boldsymbol{S}_\alpha(\boldsymbol{\theta})$, we obtain

$$\boldsymbol{S}^{-1}_{1,\alpha}(\boldsymbol{\theta}) = k^\alpha(\boldsymbol{\theta})\,(\alpha+1)^2\,\boldsymbol{D}_{1,\sigma_1,\sigma_2}\boldsymbol{J}^{-1}_1(\rho)\boldsymbol{D}_{1,\sigma_1,\sigma_2}, \tag{26}$$

where

$$\boldsymbol{J}^{-1}_1(\rho) = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

and

$$S_{2,\alpha}^{-1}(\boldsymbol{\theta}) = k^{\alpha}(\boldsymbol{\theta})\,(\alpha+1)^3\,\boldsymbol{D}_{2,\sigma_1,\sigma_2}\boldsymbol{S}_{2,1}^{-1}(\rho)\boldsymbol{D}_{2,\sigma_1,\sigma_2}, \tag{27}$$

where

$$\boldsymbol{S}_{2,1}^{-1}(\rho) = \frac{1}{2}\begin{pmatrix} 1 & \rho^2 & \rho(1-\rho^2) \\ \rho^2 & 1 & \rho(1-\rho^2) \\ \rho(1-\rho^2) & \rho(1-\rho^2) & 2(1-\rho^2)^2 \end{pmatrix}. \tag{28}$$

Therefore, (8) is given by

$$\boldsymbol{V}_{\alpha}\,(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{V}_{1,\alpha}(\boldsymbol{\theta}) & \boldsymbol{0}_{2\times 3} \\ \boldsymbol{0}_{3\times 2} & \boldsymbol{V}_{2,\alpha}(\boldsymbol{\theta}) \end{pmatrix}, \tag{29}$$

where

$$\begin{aligned} \boldsymbol{V}_{1,\alpha}(\boldsymbol{\theta}) &= \boldsymbol{S}_{1,\alpha}^{-1}(\boldsymbol{\theta})\,\boldsymbol{K}_{1,\alpha}(\boldsymbol{\theta})\,\boldsymbol{S}_{1,\alpha}^{-1}(\boldsymbol{\theta}) \\ &= \frac{(\alpha+1)^4}{(2\alpha+1)^2}\boldsymbol{D}_{1,\sigma_1,\sigma_2}\boldsymbol{J}_1^{-1}(\rho)\boldsymbol{D}_{1,\sigma_1,\sigma_2}, \qquad (30) \\ \boldsymbol{V}_{2,\alpha}(\boldsymbol{\theta}) &= \boldsymbol{S}_{2,\alpha}^{-1}(\boldsymbol{\theta})\,\boldsymbol{K}_{2,\alpha}(\boldsymbol{\theta})\,\boldsymbol{S}_{2,\alpha}^{-1}(\boldsymbol{\theta}) \\ &= \frac{(\alpha+1)^4}{(2\alpha+1)^3}\boldsymbol{D}_{2,\sigma_1,\sigma_2}\boldsymbol{V}_{2,\alpha}(\rho)\,\boldsymbol{D}_{2,\sigma_1,\sigma_2}, \qquad (31) \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{V}_{2,\alpha}(\rho) &= \boldsymbol{S}_{2,1}^{-1}(\rho)\boldsymbol{K}_{2,\alpha}(\rho)\boldsymbol{S}_{2,1}^{-1}(\rho) \\ &= (\alpha+1)^2\boldsymbol{S}_{2,1}^{-1}(\rho) + \alpha^2\boldsymbol{S}_{2,1}^{-1}(\rho)\boldsymbol{S}_{2,2}(\rho)\boldsymbol{S}_{2,1}^{-1}(\rho). \end{aligned} \tag{32}$$

Based on the previous results we have the following Theorem.

Theorem 6 *For the bidimensional normal model,* (14)*, the MRPDE for* $\boldsymbol{\theta}$*,*

$$\widehat{\boldsymbol{\theta}}_{R,\alpha} = (\widehat{\mu}_{1,R,\alpha}, \widehat{\mu}_{2,R,\alpha}, \widehat{\sigma}_{1,R,\alpha}, \widehat{\sigma}_{2,R,\alpha}, \widehat{\rho}_{R,\alpha})^T, \tag{33}$$

is obtained as a solution of

$$\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}\,(\boldsymbol{u}_{\boldsymbol{\theta}}(X_i,Y_i) - \boldsymbol{c}_{\alpha}(\boldsymbol{\theta})) = \boldsymbol{0}_5,$$

with

$$\begin{aligned} w_{i,\boldsymbol{\theta}} = \exp\Bigg\{ \frac{1}{2(1-\rho^2)}\Bigg[\left(\frac{X_i-\mu_1}{\sigma_1}\right)^2 + \left(\frac{Y_i-\mu_2}{\sigma_2}\right)^2 \\ -2\rho\frac{X_i-\mu_1}{\sigma_1}\frac{Y_i-\mu_2}{\sigma_2}\Bigg]\Bigg\}, \end{aligned} \tag{34}$$

$\boldsymbol{u}_{\boldsymbol{\theta}}(X_i, Y_i)$ *is given in Proposition* 2 *and* $\boldsymbol{c}_{\alpha}(\boldsymbol{\theta})$ *in Proposition* 3*. The corresponding asymptotic distribution is*

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{R,\alpha} - \boldsymbol{\theta}_0) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} \mathcal{N}\left(\boldsymbol{0}_p, \boldsymbol{V}_{\alpha}\,(\boldsymbol{\theta}_0)\right), \tag{35}$$

where $\boldsymbol{\theta}_0$ *is the true unknown value of* (15) *and* $\boldsymbol{V}_{\alpha}\,(\boldsymbol{\theta})$ *was given in* (29)*.*

The following algorithm is useful for computing the MRPDE of $\boldsymbol{\theta}$ given in Theorem 6. It works iteratively for a sequence of increasing values of the tuning parameter, $\alpha \in \{\alpha_k\}_{k=0}^{K}$ with $\alpha_k = \frac{k}{K}$, having a very simple iterative scheme and converging rapidly to the final optimal value. As the MLEs have an explicit expression, the tuning parameter $\alpha_0 = 0$ initializes the iterations. Herein the following parameter transformation is considered

$$\boldsymbol{\vartheta} = \left(\mu_1, \mu_2, \zeta_1^2, \zeta_2^2, \rho\right)^T \quad \text{where}$$
$$\sigma_j^2 = (\alpha+1)\zeta_j^2, \quad j = 1, 2.$$

The strength of the algorithm is its simplicity for estimating in a chained way and with the semi-explicit expressions given in the inner iterations, with expression which mimics the MLEs as weighted version (see the corresponding proof given in "Appendix D"). The updating recursive elements comprise only of the weights, as the name Iteratively Reweighted Moments Algorithm suggests. The semi-explicit expressions given in the inner iterations of Algorithm 1 are particular cases of (6) and (7) of Toma and Leoni-Aubin (2015) with N equals 2. However, our proposed algorithm is different from the one proposed in the Monte-Carlo simulations of Toma and Leoni-Aubin (2015) in two features. First, their algorithm does not consider outer iterations as the estimation is initialized always with the MLE and second, they do not consider the reparameterization of the variance-covariance matrix. Both are crucial features, first one to save iterations when a grid of the tuning parameter is considered, taking into account the smoothness of the objective function. The second one justifies our proposed Iteratively Reweighted Moments Algorithm to be an EM algorithm. It is well-known that some M-estimators are obtained from estimating equations associated to scale mixture of normal distributions, and their corresponding iterative reweighted algorithm is an EM algorithm. Under weak conditions, each step of IRLS increases the objective function and the solution of the iterative reweighted algorithm converges to a local maximum of the likelihood function. The minimum pseudodistance distance estimators for bivariate normal populations falls within this class of M-estimators. From Algorithm 1 and taking into account the Bayesian interpretation of the EM algorithm (see for example Sect. 12.2 in Little and Rubin (2019), we obtain the following way of understanding the weights update or the E step. Let us consider the sample $(X_i^{(k)}, Y_i^{(k)}, Z_i^{(k)})$, $i = 1, \ldots, n$, with $(X_i^{(k)}, Y_i^{(k)})$ being a bivariate normal distribution with parameters $\boldsymbol{\vartheta}_k = (\mu_1, \mu_2, \zeta_{1,k}^2, \zeta_{2,k}^2, \rho)^T$, where $\zeta_{j,k}^2 = \frac{1}{\alpha_k+1}\sigma_j^2$, $j = 1, 2$, and $Z_i^{(k)}$ being an auxiliary random variable (a latent random variable for the usual EM algorithm with missing data). Assuming that $Z_i^{(k)}$ is a random variable degenerated at $\frac{\alpha_k}{\alpha_k+1}$ (prior distribution), it holds $\left((X_i^{(k)}, Y_i^{(k)}) \middle| Z_i^{(k)} = z\right)$ behaves as a bivariate normal distribution with parameters $(\mu_1, \mu_2, \frac{\zeta_{1,k}^2}{z}, \frac{\zeta_{2,k}^2}{z}, \rho)^T$, while the weight is the expectation of the posterior distribution (line 13 of Algorithm 1),

$$\widehat{\varpi}_{i,\alpha_k} = E\left[Z_i^{(k)} \middle| (X_i^{(k)}, Y_i^{(k)})\right], \quad i = 1, \ldots, n.$$

The estimator of $\boldsymbol{\vartheta}_k$ is updated in the M step (line 14 of Algorithm 1). While parameters $\zeta_{j,k}^2$, $j = 1, 2$, produce a shrinkage effect on the original variance parameters σ_j^2, $j = 1, 2$, the estimates associated with small values the weights, $\widehat{\varpi}_{i,\alpha_k}^{(k)}$, $i = 1, \ldots, n$, produce a down-weighting effect on estimates of $\boldsymbol{\vartheta}_k$ to prevent from high value of the Mahalanobis distance (line 11) for the i-th observation and k-th tuning parameter α_k, $k = 1, \ldots, n$.

It is clear that due to the invariance property of the MRPDEs, it holds

$$\widehat{\sigma}_{j,R,\alpha} = \sqrt{\alpha+1}\widehat{\zeta}_{j,R,\alpha}, \quad j = 1, 2.$$

4 Wald-type tests based on Rényi's pseudodistance estimators

Based on the asymptotic distribution of the MRPDE for $\boldsymbol{\theta}$, $\widehat{\boldsymbol{\theta}}_{R,\alpha}$, given in Theorem 6, we present Wald-type tests for testing composite null hypothesis regarding bidimensional normal model parameters.

The restricted parameter space $\Theta_0 \subset \Theta = \mathbb{R}^2 \times \mathbb{R}_+^2 \times (-1, 1)$, is often defined by a set of r restrictions of the form

$$\boldsymbol{m}(\boldsymbol{\theta}) = \boldsymbol{0}_r, \tag{36}$$

where $\boldsymbol{\theta}$ is (15) and $\boldsymbol{m} : \Theta \to \mathbb{R}^r$ (see Serfling 1980). Assume that the $5 \times r$ matrix

$$\boldsymbol{M}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{m}^T(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \tag{37}$$

exists and is continuous in $\boldsymbol{\theta}$, and $\text{rank}(\boldsymbol{M}(\boldsymbol{\theta})) = r$, where $r \leq 5$. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample of size n from a distribution modelled by the bidimensional normal model probability density function $f_{\boldsymbol{\theta}}(x, y)$, where $\boldsymbol{\theta} \in \Theta$. Our interest is in testing the hypothesis

$$H_0 : \boldsymbol{\theta} \in \Theta_0 \quad \text{against} \quad H_1 : \boldsymbol{\theta} \notin \Theta_0, \tag{38}$$

where $\Theta_0 = \{\boldsymbol{\theta} \in \Theta : \boldsymbol{m}(\boldsymbol{\theta}) = \boldsymbol{0}\}$.

Definition 1 Let $\widehat{\boldsymbol{\theta}}_{R,\alpha}$ be the MDPDE of $\boldsymbol{\theta}$. The family of proposed Wald-type test statistics for testing the null hypothesis in (38) is given by

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\boldsymbol{m}^T(\widehat{\boldsymbol{\theta}}_{R,\alpha})\left(\boldsymbol{M}^T(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{V}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{M}(\widehat{\boldsymbol{\theta}}_{R,\alpha})\right)^{-1}\boldsymbol{m}(\widehat{\boldsymbol{\theta}}_{R,\alpha}), \tag{39}$$

where the matrix $\boldsymbol{V}_\alpha$ is as in (29) and the functions $\boldsymbol{m}(\cdot)$ and $\boldsymbol{M}(\cdot)$ are defined in (36) and (37).

Theorem 7 *The asymptotic null distribution of the proposed Wald-type test statistics given in (39) is chi-square with r degrees of freedom, χ_r^2.*

Proof See "Appendix C". □

We will reject the null hypothesis in (36) if $W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) > \chi_{r,\varsigma}^2$, where $\chi_{r,\varsigma}^2$ is the upper percentage point of order ς of the χ_r^2 distribution. Based on Definition 1 and Theorem 7, the following subsections are devoted to derive a variety hypothesis tests for the bidimensional normal model. The proofs of the stated formulas are given in "Appendix F".

Case 1 (Comparing means of two dependent populations with normal distribution).
If we are interested in testing

$$H_0 : \mu_1 = \mu_2, \tag{40}$$

the corresponding Wald-type test statistics based on MRPDEs is

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^2}{(\alpha+1)^4} \frac{(\widehat{\mu}_{1,R,\alpha} - \widehat{\mu}_{2,R,\alpha})^2}{(\widehat{\sigma}_{1,R,\alpha} - \widehat{\sigma}_{2,R,\alpha})^2 + 2(1-\widehat{\rho}_{R,\alpha})\widehat{\sigma}_{1,R,\alpha}\widehat{\sigma}_{2,R,\alpha}}, \tag{41}$$

and its asymptotic distribution is a chi-squared distribution with one degree of freedom under (40).

In Case 1 a non-standard Behrens–Fisher problem is covered, i.e., a comparison of the means of two populations which may possess not only different variances, but also a non-null correlation. It is of great interest to be aware that formulating the same problem as a paired test constructed taking the difference of both populations, $V = X - Y$, as a single population problem for testing $H_0 : \mu_V = 0$, with an

Algorithm 1 Iteratively Reweighted Moments Algorithm for MRPDE of $\boldsymbol{\vartheta}$

Input: $K, \{\alpha_k = \frac{k}{K}\}_{k=0}^K, \{(X_i, Y_i)\}_{i=1}^n, \xi$;

$k \leftarrow 0, r_{k=0} \leftarrow 0$ ▷ Inizialization

$\mathrm{vec}\{\widehat{\varpi}_{i,\alpha_{k=0}=0,(r_{k=0}=0)}\}_{i=1}^n = \mathbf{1}_n$

Compute $\widehat{\boldsymbol{\vartheta}}_{\alpha_{k=0}=0,(r_{k=0}=0)}$:

$$\widehat{\mu}_{1,R,\alpha_{k=0}=0,(r_{k=0}=0)} = \frac{\sum_{i=1}^n X_i}{n}, \quad \widehat{\zeta}^2_{1,R,\alpha_{k=0}} = \frac{\sum_{i=1}^n (X_i - \widehat{\mu}_{1,R,\alpha_0=0,(r_{k=0}=0)})^2}{n},$$

$$\widehat{\mu}_{2,R,\alpha_{k=0}=0,(r_{k=0}=0)} = \frac{\sum_{i=1}^n Y_i}{n}, \quad \widehat{\zeta}^2_{2,R,\alpha_{k=0}} = \frac{\sum_{i=1}^n (Y_i - \widehat{\mu}_{2,R,\alpha_0=0,(r_{k=0}=0)})^2}{n},$$

$$\widehat{\rho}_{R,\alpha_{k=0}=0,(r_{k=0}=0)} = \frac{\sum_{i=1}^n \frac{X_i - \widehat{\mu}_{1,R,\alpha_{k=0}=0,(r_{k=0}=0)}}{\widehat{\zeta}_{1,R,\alpha_{k=0}=0,(r_{k=0}=0)}} \frac{Y_i - \widehat{\mu}_{2,R,\alpha_0=0,(r_{k=0}=0)}}{\widehat{\zeta}_{2,R,\alpha_{k=0}=0,(r_{k=0}=0)}}}{n};$$

while $k < K$ **do** ▷ Outer loop starts

  $k \leftarrow k+1, r_k \leftarrow 0$

  $\mathrm{vec}\{\widehat{\varpi}_{i,\alpha_k,(r_k=0)}\}_{i=1}^n \leftarrow \mathrm{vec}\{\widehat{\varpi}_{i,\alpha_{k-1},r_{k-1}}\}_{i=1}^n$

  $\widehat{\boldsymbol{\vartheta}}_{\alpha_k,(r_k=0)} = \widehat{\boldsymbol{\vartheta}}_{\alpha_{k-1},r_{k-1}}$ ▷ Inner loop starts

  repeat

    $\mathrm{vec}\{\widehat{X}_{i,\alpha_k,(r_k)}\}_{i=1}^n \leftarrow \mathrm{vec}\left\{\frac{X_i - \widehat{\mu}_{1,R,\alpha_k,(r_k)}}{\widehat{\zeta}_{1,R,\alpha_k,(r_k)}}\right\}_{i=1}^n$

    $\mathrm{vec}\{\widehat{Y}_{i,\alpha_k,(r_k)}\}_{i=1}^n \leftarrow \mathrm{vec}\left\{\frac{Y_i - \widehat{\mu}_{2,R,\alpha_k,(r_k)}}{\widehat{\zeta}_{2,R,\alpha_k,(r_k)}}\right\}_{i=1}^n$

    $\mathrm{vec}\{d^2_{i,\alpha_k,(r_k)}\}_{i=1}^n \leftarrow \mathrm{vec}\left\{\frac{\widehat{X}^2_{i,\alpha_k,(r_k)} + \widehat{Y}^2_{i,\alpha_k,(r_k)} - 2\widehat{\rho}_{1,R,\alpha_k,(r_k)}\widehat{X}_{i,\alpha_k,(r_k)}\widehat{Y}_{i,\alpha_k,(r_k)}}{1 - \widehat{\rho}^2_{1,R,\alpha_k,(r_k)}}\right\}_{i=1}^n$

    $r_k \leftarrow r_k + 1,$

    $\mathrm{vec}\{\widehat{\varpi}_{i,\alpha_k,(r_k)}\}_{i=1}^n \leftarrow \mathrm{vec}\{\exp^{-\frac{\alpha_h}{\alpha_h+1}}(\frac{1}{2}d^2_{i,\alpha_k,(r_k-1)})\}_{i=1}^n$

    Compute $\widehat{\boldsymbol{\vartheta}}_{\alpha_k,(r_k)}$:

$$t_{\alpha_k,(r_k)} = \sum_{i=1}^n \widehat{\varpi}_{i,\alpha_k,(r_k)},$$

$$\widehat{\mu}_{1,R,\alpha_k,(r_k)} = \frac{\sum_{i=1}^n \widehat{\varpi}_{i,\alpha_k,(r_k)} X_i}{t_{\alpha_k,(r_k)}}, \quad \widehat{\zeta}^2_{1,R,\alpha_k,(r_k)} = \frac{\sum_{i=1}^n \widehat{\varpi}_{i,\alpha_k,(r_k)}(X_i - \widehat{\mu}_{1,R,\alpha_k,(r_k)})^2}{t_{\alpha_k,(r_k)}},$$

$$\widehat{\mu}_{2,R,\alpha_k,(r_k)} = \frac{\sum_{i=1}^n \widehat{\varpi}_{i,\alpha_k,(r_k)} Y_i}{t_{\alpha_k,(r_k)}}, \quad \widehat{\zeta}^2_{2,R,\alpha_k,(r_k)} = \frac{\sum_{i=1}^n \widehat{\varpi}_{i,\alpha_k,(r_k)}(Y_i - \widehat{\mu}_{2,R,\alpha_k,(r_k)})^2}{t_{\alpha_k,(r_k)}},$$

$$\widehat{\rho}_{R,\alpha_k,(r_k)} = \frac{\sum_{i=1}^n \widehat{\varpi}_{i,\alpha_k,(r_k)} \frac{X_i - \widehat{\mu}_{1,R,\alpha_k,(r_k)}}{\widehat{\zeta}_{1,R,\alpha_k,(r_k)}} \frac{Y_i - \widehat{\mu}_{2,R,\alpha_k,(r_k)}}{\widehat{\zeta}_{2,R,\alpha_k,(r_k)}}}{t_{\alpha_k,(r_k)}}$$

  until $\left\|\widehat{\boldsymbol{\vartheta}}_{\alpha_k,(r_k)} - \widehat{\boldsymbol{\vartheta}}_{\alpha_k,(r_k-1)}\right\|_2 < \xi$ ▷ Inner loop ends

end while ▷ Outer loop ends

Output: $\{\widehat{\boldsymbol{\vartheta}}_{\alpha_k} = \widehat{\boldsymbol{\vartheta}}_{\alpha_k,(r_k)}\}_{k=0}^K$.

unknown variance σ_V^2, the same value of the Wald-type test statistics

$$W_{n,\alpha}(\widehat{\mu}_{V,R,\alpha}, \widehat{\sigma}_{V,R,\alpha}) = n\frac{(2\alpha+1)^2}{(\alpha+1)^4}\frac{\widehat{\mu}^2_{V,R,\alpha}}{\widehat{\sigma}^2_{V,R,\alpha}}$$

is obtained ($W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = W_{n,\alpha}(\widehat{\mu}_{V,R,\alpha}, \widehat{\sigma}_{V,R,\alpha})$) from the invariance property of the Rényi's pseudodistance estimators, since $\mu_V = \mu_1 - \mu_2$, $\sigma_V^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = (\sigma_1 - \sigma_2)^2 + 2(1-\rho)\sigma_1\sigma_2$.

The most efficient classic procedure to address this problem is the paired t-test, i.e

$$T_V = \sqrt{n}\frac{\overline{V}_n}{S_{V,n-1}},$$

where

$$\overline{V}_n = \overline{X}_n - \overline{Y}_n,$$

$$S^2_{V,n-1} = \frac{1}{n-1}\sum_{i=1}^n (V_i - \overline{V}_n)^2 = S^2_{X,n-1} + S^2_{Y,n-1} - 2S_{XY,n-1},$$

$$S^2_{X,n-1} = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2 = \frac{n}{n-1}\widehat{\sigma}^2_{1,R,\alpha=0},$$

$$S^2_{Y,n-1} = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \overline{Y}_n)^2 = \frac{n}{n-1}\widehat{\sigma}^2_{2,R,\alpha=0},$$

$$S_{XY,n-1} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)(Y_i - \overline{Y}_n) = \frac{n}{n-1}\widehat{\rho}_{R,\alpha=0}\widehat{\sigma}_{2,R,\alpha=0}\widehat{\sigma}_{1,R,\alpha=0}.$$

Its exact distribution is a Student-t with $n-1$ degrees of freedom, t_{n-1}.

Case 2 (Comparing variances of two dependent populations with normal distribution). If we are interested in testing

$$H_0 : \sigma_1 = \sigma_2. \tag{42}$$

the corresponding Wald-type test statistics based on MRPDEs is

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^3}{(\alpha+1)^6}\frac{(\widehat{\sigma}_{1,R,\alpha} - \widehat{\sigma}_{2,R,\alpha})^2}{\beta_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})}, \tag{43}$$

where

$$\beta_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = \left[\frac{1}{4}\left(\frac{\alpha}{\alpha+1}\right)^2 + \frac{1}{2}\right](\widehat{\sigma}_{1,R,\alpha} - \widehat{\sigma}_{2,R,\alpha})^2 + (1-\widehat{\rho}^2_{R,\alpha})\widehat{\sigma}_{1,R,\alpha}\widehat{\sigma}_{2,R,\alpha}. \tag{44}$$

The asymptotic distribution of (43) is a chi-squared distribution with 1 degree of freedom under (42).

Case 3 (Fixing a value of the for correlation coefficient of two normal populations).
If we are interested in testing

$$H_0 : \rho = \rho_0, \tag{45}$$

the corresponding Wald-type test statistics based on MRPDEs is

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^3}{(\alpha+1)^6}\frac{(\widehat{\rho}_{R,\alpha} - \rho_0)^2}{(1-\widehat{\rho}^2_{R,\alpha})^2}, \tag{46}$$

and its asymptotic distribution is a chi-squared distribution with 1 degree of freedom under (45).

The classic Wald and Rao test statistics are given by

$$W_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0}) = n\frac{(\widehat{\rho}_{R,\alpha=0} - \rho_0)^2}{(1-\widehat{\rho}^2_{R,\alpha=0})^2},$$

$$R_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0}) = n\frac{(\widehat{\rho}_{R,\alpha=0} - \rho_0)^2}{(1-\rho_0\widehat{\rho}_{R,\alpha=0})^2},$$

where

$$\widehat{\rho}_{R,\alpha=0} = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}\sqrt{\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2}} = R_{XY},$$

but

$$W'_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0}) = n\frac{(\widehat{\rho}_{R,\alpha=0} - \rho_0)^2}{(1-\rho_0^2)^2}$$

convergences more rapidly to the chi-square distribution with 1 degree of freedom (see Anderson 2003). The extension of $W'_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0})$ to

$$W'_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^3}{(\alpha+1)^6}\frac{(\widehat{\rho}_{R,\alpha} - \rho_0)^2}{(1-\rho_0^2)^2}, \tag{47}$$

is directly obtained from the same proof of $W_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0})$, since in (F14) $\widehat{\rho}_{R,\alpha=0}$ can be replaced by ρ_0. In the particular case of fixing $\rho_0 = 0$ under the null, (45), it holds $W'_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0}) = R_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0})$

Case 4 (Comparing means and variances of two dependent populations with normal distribution).
If we are interested in testing

$$H_0 : \mu_1 = \mu_2 \text{ and } \sigma_1 = \sigma_2, \tag{48}$$

the corresponding Wald-type test statistics based on MRPDEs is

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^2}{(\alpha+1)^4} \times \left(\frac{(\widehat{\mu}_{1,R,\alpha} - \widehat{\mu}_{2,R,\alpha})^2}{(\widehat{\sigma}_{1,R,\alpha} - \widehat{\sigma}_{1,R,\alpha})^2 + 2(1-\widehat{\rho}^2_{R,\alpha})\widehat{\sigma}_{1,R,\alpha},\widehat{\sigma}_{2,R,\alpha}} + \frac{(2\alpha+1)\,(\widehat{\sigma}^\alpha_{1,R} - \widehat{\sigma}^\alpha_{2,R})^2}{(\alpha+1)^2\,\beta_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})}\right), \tag{49}$$

and its asymptotic distribution is a chi-squared distribution with 2 degrees of freedom under (48).

Case 5 (Fixing a value for covariance of two normal populations).
If we are interested in testing

$$H_0 : \sigma_1\sigma_2\rho = \sigma_{12,0}, \tag{50}$$

where $\sigma_{12,0} \in \mathbb{R}$, the corresponding Wald-type test statistics based on MRPDEs is

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^3}{(\alpha+1)^4} \times \frac{\left(\widehat{\sigma}_{1,R,\alpha}\widehat{\sigma}_{2,R,\alpha}\widehat{\rho}_{R,\alpha} - \sigma_{12,0}\right)^2}{\widehat{\sigma}^2_{1,R,\alpha}\widehat{\sigma}^2_{2,R,\alpha}\left[(\alpha+1)^2\left(\widehat{\rho}^2_{R,\alpha}+1\right) + \frac{\alpha^2}{2}\widehat{\rho}^2_{R,\alpha}\right]}. \tag{51}$$

The asymptotic distribution of (51) is a chi-squared distribution with 1 degree of freedom under (50).

Case 6 (Fixing values for means of two dependent populations with normal distribution).
If we are interested in testing

$$H_0 : \mu_1 = \mu_{1,0} \text{ and } \mu_2 = \mu_{2,0}, \tag{52}$$

the corresponding Wald-type test statistics based on MRPDEs is

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^2}{(\alpha+1)^4}\left[\frac{\left(\frac{\widehat{\mu}_{1,R,\alpha}-\mu_{1,0}}{\widehat{\sigma}_{1,R,\alpha}} - \frac{\widehat{\mu}_{2,R,\alpha}-\mu_{2,0}}{\widehat{\sigma}_{2,R,\alpha}}\right)^2}{1-\widehat{\rho}^2_{R,\alpha}} + \frac{2(1-\widehat{\rho}_{R,\alpha})\left(\frac{\widehat{\mu}_{1,R,\alpha}-\mu_{1,0}}{\widehat{\sigma}_{1,R,\alpha}}\right)\left(\frac{\widehat{\mu}_{2,R,\alpha}-\mu_{2,0}}{\widehat{\sigma}_{2,R,\alpha}}\right)}{1-\widehat{\rho}^2_{R,\alpha}}\right] \tag{53}$$

The asymptotic distribution of (53) is a chi squared distribution with 2 degrees of freedom under (52).

Case 7 (Fixing values for variances and covariance of two dependent populations with normal distribution).
If we are interested in testing

$$H_0 : \sigma_1 = \sigma_{1,0}, \sigma_2 = \sigma_{2,0}, \sigma_1\sigma_2\rho = \sigma_{12,0}, \tag{54}$$

the corresponding Wald-type test statistics based on MRPDEs is

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = n\frac{(2\alpha+1)^3}{(\alpha+1)^4}\boldsymbol{w}^T_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{V}^{-1}_{2,\alpha}(\widehat{\rho}_{R,\alpha})\boldsymbol{w}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha}), \tag{55}$$

where

$$\boldsymbol{w}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = \boldsymbol{D}^{-1}_{2,\widehat{\sigma}_{1,R,\alpha},\widehat{\sigma}_{2,R,\alpha}}\left(\boldsymbol{M}^T_{22}(\boldsymbol{\theta})\right)^{-1}\boldsymbol{m}(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = \begin{pmatrix} 1-\frac{\sigma_{1,0}}{\widehat{\sigma}_{1,R,\alpha}} \\ 1-\frac{\sigma_{2,0}}{\widehat{\sigma}_{2,R,\alpha}} \\ \widehat{\rho}_{R,\alpha} - \frac{\sigma_{12,0}}{\widehat{\sigma}_{1,R,\alpha}\widehat{\sigma}_{2,R,\alpha}} - \widehat{\rho}_{R,\alpha}\left(2-\frac{\sigma_{1,0}}{\widehat{\sigma}_{1,R,\alpha}}-\frac{\sigma_{2,0}}{\widehat{\sigma}_{2,R,\alpha}}\right)\end{pmatrix},$$

with

$$\boldsymbol{V}^{-1}_{2,\alpha}(\widehat{\rho}_{R,\alpha}) = (\alpha+1)^2\boldsymbol{S}_{2,1}(\widehat{\rho}_{R,\alpha})\left[\boldsymbol{S}_{2,1}(\widehat{\rho}_{R,\alpha}) + \left(\frac{\alpha}{\alpha+1}\right)^2\boldsymbol{S}_{2,2}(\widehat{\rho}_{R,\alpha})\right]^{-1}\boldsymbol{S}_{2,1}(\widehat{\rho}_{R,\alpha}),$$

$\boldsymbol{S}_{2,1}(\widehat{\rho}_{R,\alpha})$ given by (23) and $\boldsymbol{S}_{2,2}(\widehat{\rho}_{R,\alpha})$ by (25). The asymptotic distribution of (55) is a chi squared with 3 degrees of freedom under (54).

5 Study of the influence function

In the precedent sections we have developed the MRPDE for $\boldsymbol{\theta}$ in the bidimensional normal model, as well as Wald-type tests based on MRPDE, as a robust alternative to the MLE and classic Wald-type tests. In this section, we will theoretically justify the robustness of the proposed estimators through the study of its Influence Function (IF). The IF (Hampel et al. 1986) for any estimator defined in terms of an statistical functional $\boldsymbol{T}(F)$ from the true distribution F, is defined as

$$\mathcal{IF}(t, \boldsymbol{T}, F) = \lim_{\varepsilon\downarrow 0}\frac{\boldsymbol{T}((1-\varepsilon)F+\varepsilon\Delta_t) - \boldsymbol{T}(F)}{\varepsilon} = \left.\frac{\partial \boldsymbol{T}(F_\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0^+}, \tag{56}$$

with ε being the contamination proportion and Δ_t being the degenerate distribution at the contamination point t. Thus, the IF, as a function of t, measures the standardized asymptotic bias caused by the infinitesimal contamination at the point t. The maximum of this IF over t indicates the extent of bias due to contamination and so smaller its value, the more robust the estimator is. Note that, in this context, the statistical functional $\boldsymbol{T}_\alpha$ corresponding to the MRPDE is defined as the minimizer of $R_\alpha(f_{\boldsymbol{\theta}}, g)$ in (1).

IF for MLE in the bidimensional normal model has been widely studied in literature. For example, Devlin et al. (1975) presented the IF for the Pearson's correlation coefficient ρ. A proof was given years later by Chernick (1982). IFs for the mean and variance can be found, in Radhakrishnan and Kshirsagar (1981) and Isogai (1989), among others.

In Broniatowski et al. (1961) (Theorem 5), the IF of Renyi's pseudodistances-based estimators was provided in a general form and particularized to some particular models. Castilla et al. (2021) generalized this result to the case of independent not identically distributed observations. Based on these results, in Theorem 8 we present the IF associated to the MRPDE of $\boldsymbol{\theta}$ the bidimensional normal model. A detailed proof of the following result is provided in "Appendix E".

Theorem 8 *Let us consider the bidimensional normal model (14). The IF associated to the MRPDE of $\boldsymbol{\theta}$ is given by*

$$\mathcal{IF}((x,y)^T, \boldsymbol{T}_\alpha, F_{\boldsymbol{\theta}}) = (\mathcal{IF}_\alpha(\mu_1), \mathcal{IF}_\alpha(\mu_2), \mathcal{IF}_\alpha(\sigma_1), \mathcal{IF}_\alpha(\sigma_2), \mathcal{IF}_\alpha(\rho))^T,$$

where

$$\mathcal{IF}_\alpha(\mu_1) = (\alpha+1)^2 w_{\boldsymbol{\theta}}^{-\alpha/(1-\rho^2)}(x,y)(x-\mu_1), \tag{57}$$

$$\mathcal{IF}_\alpha(\mu_2) = (\alpha+1)^2 w_{\boldsymbol{\theta}}^{-\alpha/(1-\rho^2)}(x,y)(y-\mu_2), \tag{58}$$

$$\mathcal{IF}_\alpha(\sigma_1) = \frac{(\alpha+1)^3}{2} w_{\boldsymbol{\theta}}^{-\alpha}(x,y)\frac{\sigma_1}{1-\rho^2} \times \left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - (1-\rho^2)(1+2\rho^2)\frac{1}{\alpha+1}\right], \tag{59}$$

$$\mathcal{IF}_\alpha(\sigma_2) = \frac{(\alpha+1)^3}{2} w_{\boldsymbol{\theta}}^{-\alpha}(x,y)\frac{\sigma_2}{1-\rho^2} \times \left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - (1-\rho^2)(1+2\rho^2)\frac{1}{\alpha+1}\right], \tag{60}$$

$$\mathcal{IF}_\alpha(\rho) = (\alpha+1)^3\, w_{\boldsymbol{\theta}}^{-\alpha}(x,y)\left\{-\frac{\rho}{2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right] + \frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}\right\}, \tag{61}$$

with

$$w_{\boldsymbol{\theta}}(x,y) = \exp\left\{\frac{1}{2(1-\rho^2)}\left[(\frac{x-\mu_1}{\sigma_1})^2 + (\frac{y-\mu_2}{\sigma_2})^2 - 2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}\right]\right\}.$$

Remark 1 In particular, for $\alpha = 0$ (MLE),

$$\mathcal{IF}_{\alpha=0}(\mu_1) = x - \mu_1,$$
$$\mathcal{IF}_{\alpha=0}(\mu_2) = y - \mu_2,$$
$$\mathcal{IF}_{\alpha=0}(\sigma_1) = \frac{\sigma_1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right] - \frac{\sigma_1}{2}(1+2\rho^2),$$
$$\mathcal{IF}_{\alpha=0}(\sigma_2) = \frac{\sigma_2}{2(1-\rho^2)}\left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right] - \frac{\sigma_1}{2}(1+2\rho^2),$$
$$\mathcal{IF}_{\alpha=0}(\rho) = -\frac{\rho}{2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right] + \frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}.$$

The IF (62) is bounded for positive values of the parameter α, $\alpha > 0$, and is unbounded at the MLE, $\alpha = 0$.

Once we have computed the IF for the minimum RP estimators, we can define and study the IF for the Wald-type test statistics defined in (39). As noted by Castilla et al. (2021), when the corresponding IF is identically zero and is therefore necessary to consider the second order IF of the proposed Wald-type tests functional W_α.

Theorem 9 *Let us consider the bidimensional normal model (14). The second order IF of the proposed Wald-type test functionals for testing simple null hypothesis in (38) is given by*

$$\mathcal{IF}_2((x,y)^T, W_\alpha, F_{\boldsymbol{\theta}}) = 2(\mathcal{IF}((x,y)^T, \boldsymbol{T}_\alpha, F_{\boldsymbol{\theta}}))^T \boldsymbol{M}(\boldsymbol{\theta}) \left(\boldsymbol{M}^T(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})\right)^{-1} \boldsymbol{M}^T(\boldsymbol{\theta})\mathcal{IF}((x,y)^T, \boldsymbol{T}_\alpha, F_{\boldsymbol{\theta}}), \tag{62}$$

where $\mathcal{IF}((x,y)^T, \boldsymbol{T}_\alpha, F_{\boldsymbol{\theta}})$ is given in Theorem 8.

Note that the second-order IF of the proposed Wald-type tests is a quadratic function of the corresponding IF of the MRPDE. Therefore, the boundedness of the IF of MRPDE at $\alpha > 0$ also indicates the boundedness of the IF of the Wald-type test functionals, implying its robustness against contamination. Figures 1 and 2 plot the IF of the MRPDEs for μ_1, σ_1 and ρ (Fig. 1) and Wald-type test statistic (Fig 2) with $\alpha = 0$ (MLE) and $\alpha = 0.3$, for the multivariate normal distribution with true parameter $\boldsymbol{\theta} = (1, 2, 1, 1.5, 0.3)^T$. As illustrated, a positive value of the tuning parameter $\alpha = 0.3$ results in robust estimators and statistic with bounded IFs, whereas the MLE and classical Wald-type test statistic ($\alpha = 0$) lack of robustness.

6 Simulation study

It is well-known that the Morgan-Pitman test is best unbiased and best invariant test statistic for testing equality of variances (see Morgan 1939; Pitman 1939; Hsu 1940). The idea behind the Morgan-Pitman test allows us to include for testing equality of variances not only the family of test statistics given in Case 2 but also the family of test statistics given in Case 3 for the data $(u_1, v_1), \ldots, (u_n, v_n)$, where $U = X + Y$ and $V = X - Y$ are transformed variables. As $\mathrm{Cov}[U, V] = \sigma_1^2 - \sigma_2^2$, testing H_0: $\sigma_1 = \sigma_2$ given in (42) (Case 2), matches H_0: $\rho_{UV} = 0$, from (45) with $\rho_0 = 0$ (Case 3). We have compared the performance of the MRPEs for the model parameters $\gamma_{R,\alpha}$ and $\rho_{R,\alpha}$ as well the performance of the two Wald-type test statistic families for different values of the tuning parameter α. Results are presented in Tables 1–4 of the main paper and G1-G12 of the appendix. First (43) in Tables 2, 10, 14 and 18, rewritten as

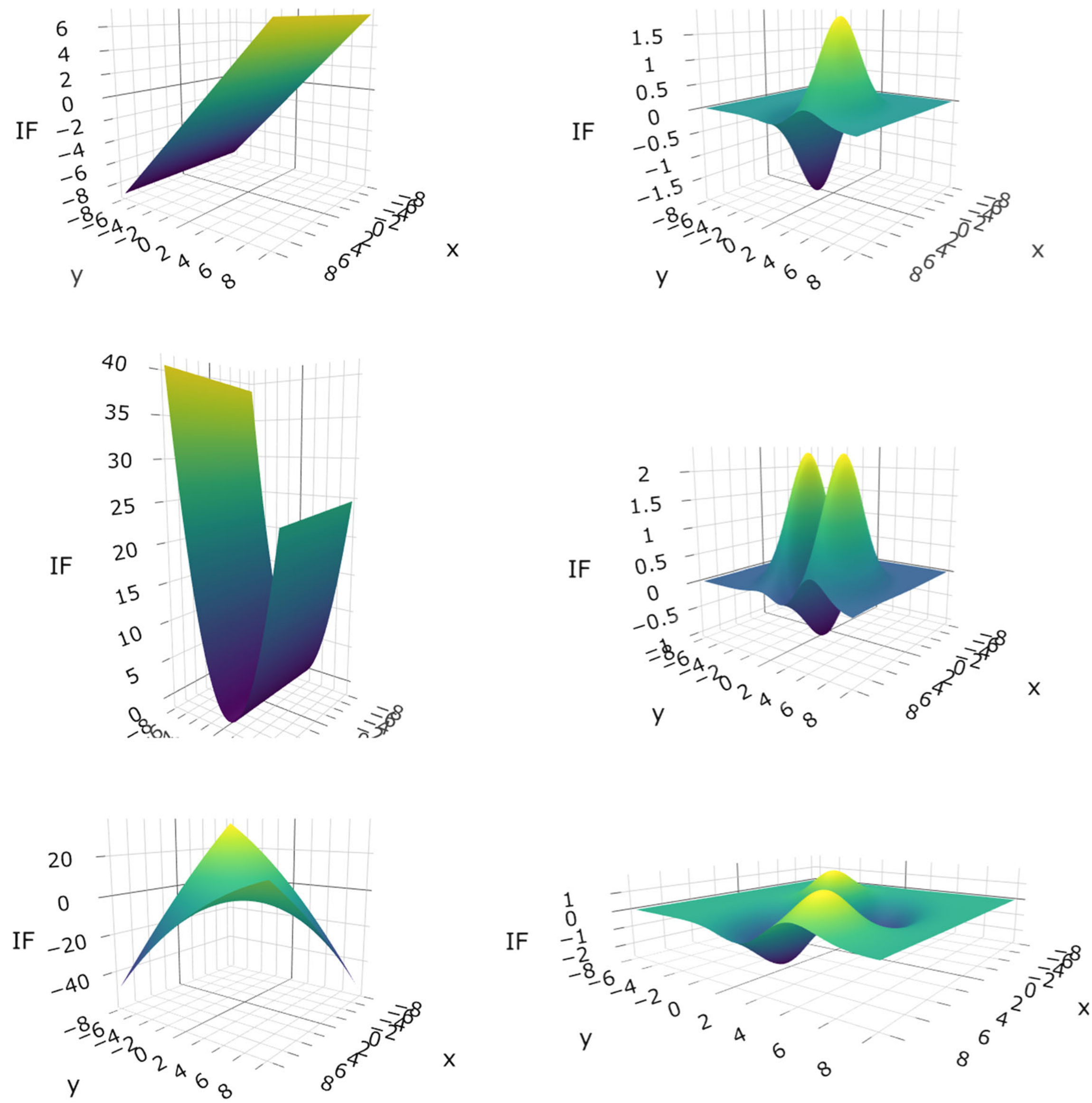
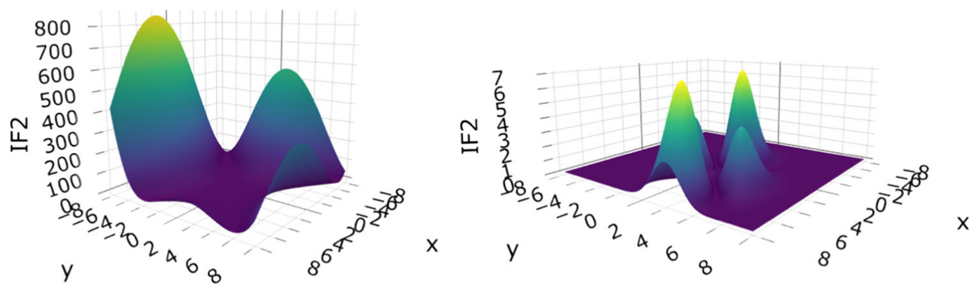


Fig. 1 $\mathcal{IF}_\alpha(\mu_1)$ (above), $\mathcal{IF}_\alpha(\sigma_1)$ (middle) and $\mathcal{IF}_\alpha(\rho)$ (below) for $\alpha = 0$ (left) and $\alpha = 0.3$ (right), with $\boldsymbol{\theta} = (1, 2, 1, 1.5, 0.3)^T$

Fig. 2 $\mathcal{IF}_2((x, y)^T, W_\alpha, F_{\boldsymbol{\theta}})$ for testing H_0: $\sigma_1 = \sigma_2$, with $\alpha = 0$ (left) and $\alpha = 0.3$ (right), and $\boldsymbol{\theta} = (1, 2, 1, 1.5, 0.3)^T$



$$W_{n,\alpha}(\widehat{\gamma}_{R,\alpha},\widehat{\rho}_{R,\alpha}) = n\frac{(2\alpha+1)^3}{(\alpha+1)^6}\frac{(\widehat{\gamma}_{R,\alpha}-1)^2}{\beta_\alpha(\widehat{\gamma}_{R,\alpha},\widehat{\rho}_{R,\alpha})}, \quad (63)$$

where

$$\widehat{\gamma}_{R,\alpha} = \frac{\widehat{\sigma}_{1,R,\alpha}}{\widehat{\sigma}_{2,R,\alpha}},$$
$$\beta_\alpha(\widehat{\gamma}_{R,\alpha},\widehat{\rho}_{R,\alpha}) = \frac{1}{4}\left[\left(\frac{\alpha}{\alpha+1}\right)^2+2\right](\widehat{\gamma}_{R,\alpha}-1)^2 + (1-\widehat{\rho}^2_{R,\alpha})\widehat{\gamma}_{R,\alpha}. \quad (64)$$

and second (47) in Tables 4, 12, 16 and 20, rewritten as

$$W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha}) = n\frac{(2\alpha+1)^3}{(\alpha+1)^6}\widehat{\rho}^2_{UV,R,\alpha}. \quad (65)$$

A third one, (46), was also considered but do not present the results here as the corresponding results were very bad in comparison with (47). In addition, the exact Morgan-Pitman test,

$$T_{MP} = \widehat{\rho}_{UV,R,\alpha=0}\sqrt{\frac{n-2}{1-\widehat{\rho}^2_{UV,R,\alpha=0}}}, \quad (66)$$

is taken into account, whose exact distribution is a Student t with $n-2$ degrees of freedom (t_{n-2}) under H_0, with

$$\widehat{\rho}_{UV,R,\alpha=0} = \frac{\sum_{i=1}^n(U_i-\bar{U}_n)(V_i-\bar{V}_n)}{\sqrt{\sum_{i=1}^n(U_i-\bar{U}_n)^2}\sqrt{\sum_{i=1}^n(V_i-\bar{V}_n)^2}} = R_{UV},$$

being the Pearson correlation coefficient, i.e. the MLE of ρ_{UV}. Furthermore, we included the simulated significance level of the Morgan Pitman test described in (66) in the aforemetioned tables and calculated by simulation $\text{MSE}(\widehat{\gamma}_{R,\alpha}) = \left|\widehat{\gamma}_{R,\alpha}-1\right|$ in Tables 1, 9, 13 and 17 as well as $\text{MSE}(\widehat{\gamma}_{R,\alpha}) = \left|\widehat{\rho}_{R,\alpha}\right|$ in Tables 3, 11, 15 and 19.

So as to evaluate the performance of the proposed Wald-type tests, we considered the bidimensional normal model (14) for the true parameters values $\mu_1=\mu_2=0, \sigma_1=\sigma_2=1$ and the different correlations between the normal variables $\rho \in \{0, 0.3, 0.6, 0.9\}$. Additionally, in order to evaluate the robustness of the Wald-type tests, we analysed ten different scenarios of contamination:

- Pure data
- Slightly contaminated data : We replace a 5%, 10% and 20% of the samples by a bidimensional normal distribution, substituting the true parameter values $\sigma'_1=\sigma'_2=1$ by $\sigma'_1=\sigma'_2=\sqrt{3}$.
- Contaminated data : We replace a 5%, 10% and 20% of the samples by a bidimensional Student t distribution with $d=5$ degrees of freedom.
- Heavily contaminated data : We replace a 5%, 10% and 20% of the samples by a bidimensional normal distribution, substituting the true parameter value $\sigma'_2=1$ by $\sigma'_2=5$.

We repeated the same schema for a nominal type I error, $\varsigma=0.05$, for different sample sizes $n \in \{15, 25, 50, 100\}$, but in the main document only the case of $n=25$ is presented (the remaining sizes are included in the "Appendix G"). We report, for the different values of the tuning parameter $\alpha \in \{0, 0.1, 0.2, 0.3, 0.5, 0.7\}$, the simulated mean square error (MSE) committed in the estimation of $\gamma=\sigma_1/\sigma_2$ and ρ as well as the simulated significance level of the tests, computed as the number of times the null hypothesis is rejected out of the total simulated samples R = 15,000 (Tables 1, 2).

With pure data, as expected, the MSEs and closeness of the simulated significance level of both asymptotic tests, (63) and (65), to the nominal significance level, $\varsigma=0.05$, is improved as the sample size, n, increases. For MSEs under contamination $\alpha \in \{0.1, 0.2\}$ tuning parameters outperform the MSEs with $\alpha=0$ but the greatest improvement under contamination is for the simulated significance levels of $W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha})$, given in (65), when $\alpha=0.2$, since it is always better than any other, included the well-known Morgan-Pitman test, for all the considered scenarios (Table 3).

For MSEs under contamination, MRPDEs with $\alpha \geq 0.1$ outperform the MSEs with $\alpha=0$ (the MLE) for all the considered scenarios. Furthermore, the effect of contamination is accented for low sample sizes. The greater contamination, the greater the optimal choice of the tuning parameter is, although moderate values of α over 0.2 generally offer a suitable trade-off between efficiency and robustness. Conversely, contamination is quite difficult to measure in real-life datasets, and therefore an optimality criterion for the choice of the best tuning parameter is of great practical interest. Several criteria have been proposed in the robustness literature for choosing optimal values of the DPD tuning parameter, that can be straightforward extended for the RP divergence. Warwick and Jones (2005) proposed a useful data-based procedure based on the minimization of an estimate of the asymptotic MSE, given by

$$\widehat{\text{MSE}}(\alpha) = (\widehat{\boldsymbol{\theta}}_{R,\alpha}-\boldsymbol{\theta}_P)^T(\widehat{\boldsymbol{\theta}}_{R,\alpha}-\boldsymbol{\theta}_P) + \frac{1}{n}\text{tr}(\boldsymbol{V}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})), \quad (67)$$

where the tr denotes the trace of the matrix $\boldsymbol{V}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})$ given in (8) and $\boldsymbol{\theta}_P$ is a pilot estimator used for assessing the bias. Naturally, the previous criterion was considerably pilot-

Table 1 Simulated mean square error of the MRPDE for ratio of variances, $\widehat{\gamma}_{R,\alpha}$, when $n = 25$

ρ	α	Pure	Slightly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.169	0.175	0.182	0.190	0.173	0.179	0.186	0.279	0.380	0.522
	0.1	0.170	0.173	0.178	0.184	0.172	0.175	0.178	0.209	0.281	0.432
	0.2	0.176	0.177	0.182	0.187	0.178	0.179	0.181	0.187	0.220	0.320
	0.3	0.187	0.187	0.192	0.198	0.188	0.189	0.192	0.191	0.209	0.267
	0.5	0.223	0.224	0.229	0.236	0.228	0.227	0.230	0.226	0.234	0.264
	0.7	0.307	0.327	0.313	0.320	0.304	0.384	0.313	0.330	0.317	0.351
0.3	0	0.161	0.169	0.175	0.182	0.164	0.168	0.176	0.278	0.376	0.523
	0.1	0.162	0.167	0.170	0.176	0.162	0.164	0.169	0.204	0.273	0.430
	0.2	0.168	0.171	0.174	0.180	0.168	0.168	0.173	0.181	0.210	0.313
	0.3	0.178	0.181	0.184	0.189	0.177	0.177	0.183	0.184	0.199	0.256
	0.5	0.213	0.216	0.219	0.225	0.214	0.213	0.218	0.218	0.226	0.252
	0.7	0.290	0.290	0.304	0.329	0.298	0.289	0.295	0.290	0.312	0.353
0.6	0	0.133	0.142	0.146	0.150	0.139	0.142	0.147	0.268	0.371	0.522
	0.1	0.134	0.140	0.142	0.146	0.137	0.139	0.140	0.178	0.252	0.417
	0.2	0.139	0.143	0.145	0.149	0.141	0.143	0.143	0.154	0.181	0.288
	0.3	0.147	0.151	0.154	0.157	0.149	0.152	0.152	0.156	0.169	0.227
	0.5	0.176	0.179	0.184	0.189	0.176	0.181	0.183	0.183	0.191	0.220
	0.7	0.241	0.248	0.253	0.254	0.237	0.245	0.244	0.250	0.268	0.289
0.9	0	0.074	0.077	0.079	0.081	0.075	0.077	0.081	0.240	0.367	0.525
	0.1	0.074	0.076	0.077	0.078	0.075	0.075	0.077	0.099	0.162	0.348
	0.2	0.077	0.078	0.079	0.080	0.077	0.077	0.078	0.082	0.098	0.170
	0.3	0.081	0.082	0.084	0.084	0.081	0.081	0.083	0.085	0.092	0.120
	0.5	0.097	0.098	0.101	0.102	0.099	0.097	0.099	0.101	0.107	0.121
	0.7	0.131	0.131	0.136	0.143	0.137	0.130	0.135	0.136	0.148	0.204

Table 2 Simulated significance level for testing equal variances through $W_{n,\alpha}(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha})$ given by (63) and the Morgan-Pitman test, when $n = 25$

ρ	α	Pure	Slightly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.059	0.070	0.081	0.093	0.067	0.074	0.089	0.352	0.585	0.852
	0.1	0.058	0.063	0.072	0.079	0.062	0.064	0.069	0.172	0.362	0.699
	0.2	0.059	0.064	0.069	0.075	0.063	0.064	0.066	0.094	0.180	0.438
	0.3	0.064	0.068	0.071	0.079	0.067	0.067	0.072	0.081	0.124	0.276
	0.5	0.085	0.091	0.094	0.103	0.093	0.089	0.097	0.100	0.115	0.182
	0.7	0.144	0.148	0.155	0.167	0.154	0.146	0.159	0.157	0.173	0.217
	MP	0.051	0.062	0.071	0.082	0.059	0.065	0.080	0.341	0.572	0.844
0.3	0	0.061	0.075	0.082	0.092	0.062	0.072	0.086	0.365	0.596	0.860
	0.1	0.060	0.066	0.071	0.078	0.056	0.060	0.069	0.179	0.361	0.708
	0.2	0.061	0.064	0.067	0.075	0.058	0.058	0.068	0.097	0.181	0.433
	0.3	0.065	0.066	0.070	0.079	0.062	0.063	0.073	0.081	0.123	0.271
	0.5	0.089	0.093	0.097	0.104	0.088	0.088	0.098	0.101	0.122	0.181
	0.7	0.146	0.152	0.156	0.172	0.148	0.143	0.154	0.158	0.181	0.223
	MP	0.052	0.063	0.071	0.082	0.054	0.062	0.075	0.352	0.584	0.852

Table 2 continued

ρ	α	Pure	Slightly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0.6	0	0.057	0.072	0.081	0.090	0.067	0.075	0.090	0.394	0.631	0.883
	0.1	0.056	0.064	0.071	0.075	0.060	0.064	0.070	0.175	0.368	0.713
	0.2	0.058	0.064	0.068	0.072	0.060	0.062	0.070	0.095	0.167	0.430
	0.3	0.063	0.069	0.074	0.077	0.065	0.067	0.074	0.085	0.116	0.265
	0.5	0.088	0.093	0.102	0.109	0.088	0.091	0.099	0.101	0.120	0.183
	0.7	0.148	0.148	0.160	0.171	0.143	0.151	0.155	0.157	0.179	0.226
	MP	0.047	0.062	0.069	0.076	0.057	0.064	0.079	0.380	0.616	0.875
0.9	0	0.064	0.078	0.084	0.091	0.068	0.077	0.092	0.465	0.719	0.932
	0.1	0.061	0.068	0.073	0.075	0.062	0.065	0.073	0.125	0.282	0.647
	0.2	0.060	0.067	0.073	0.072	0.063	0.063	0.071	0.075	0.111	0.280
	0.3	0.066	0.069	0.079	0.077	0.068	0.068	0.075	0.073	0.094	0.162
	0.5	0.089	0.093	0.105	0.110	0.097	0.092	0.102	0.100	0.118	0.146
	0.7	0.147	0.151	0.165	0.171	0.157	0.153	0.163	0.158	0.185	0.217
	MP	0.051	0.064	0.069	0.077	0.057	0.063	0.078	0.448	0.706	0.927

Table 3 Simulated mean square error of the MRPDE for correlation coefficient, $\widehat{\rho}_{R,\alpha}$, when $n = 25$

ρ	α	Pure	Slightly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.165	0.171	0.177	0.185	0.169	0.174	0.181	0.317	0.448	0.627
	0.1	0.167	0.169	0.174	0.180	0.168	0.170	0.174	0.225	0.322	0.515
	0.2	0.172	0.173	0.177	0.183	0.173	0.174	0.177	0.193	0.238	0.371
	0.3	0.181	0.182	0.187	0.192	0.183	0.183	0.187	0.192	0.218	0.297
	0.5	0.213	0.214	0.218	0.224	0.217	0.215	0.219	0.220	0.234	0.272
	0.7	0.270	0.272	0.277	0.284	0.275	0.273	0.278	0.277	0.291	0.320
0.3	0	0.165	0.173	0.178	0.185	0.168	0.173	0.180	0.325	0.453	0.636
	0.1	0.166	0.171	0.174	0.180	0.166	0.169	0.173	0.227	0.321	0.520
	0.2	0.171	0.175	0.178	0.184	0.171	0.173	0.177	0.194	0.236	0.370
	0.3	0.181	0.184	0.187	0.192	0.181	0.182	0.187	0.194	0.216	0.294
	0.5	0.213	0.217	0.220	0.225	0.213	0.213	0.220	0.223	0.234	0.271
	0.7	0.270	0.275	0.278	0.287	0.271	0.270	0.277	0.280	0.293	0.322
0.6	0	0.163	0.172	0.177	0.183	0.169	0.172	0.179	0.342	0.477	0.664
	0.1	0.165	0.170	0.173	0.178	0.168	0.169	0.172	0.225	0.324	0.534
	0.2	0.171	0.175	0.177	0.181	0.172	0.174	0.175	0.192	0.230	0.370
	0.3	0.180	0.183	0.186	0.191	0.181	0.184	0.184	0.193	0.212	0.290
	0.5	0.212	0.215	0.218	0.225	0.212	0.215	0.218	0.221	0.232	0.271
	0.7	0.270	0.272	0.280	0.287	0.268	0.275	0.276	0.279	0.294	0.323
0.9	0	0.166	0.173	0.178	0.183	0.168	0.172	0.180	0.382	0.542	0.729
	0.1	0.167	0.171	0.174	0.176	0.167	0.167	0.172	0.199	0.283	0.514
	0.2	0.172	0.175	0.178	0.179	0.172	0.171	0.175	0.179	0.201	0.294
	0.3	0.181	0.183	0.187	0.189	0.182	0.180	0.184	0.186	0.198	0.235
	0.5	0.215	0.216	0.223	0.228	0.217	0.214	0.219	0.220	0.232	0.254
	0.7	0.285	0.286	0.296	0.307	0.289	0.284	0.294	0.294	0.311	0.344

Table 4 Simulated significance level for testing null correlation coefficient through $W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha})$ given by (65) and the Morgan-Pitman test, when $n = 25$

ρ	α	Pure	Slightly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.053	0.065	0.074	0.085	0.062	0.069	0.082	0.345	0.576	0.847
	0.1	0.053	0.058	0.065	0.072	0.057	0.059	0.063	0.162	0.351	0.689
	0.2	0.053	0.058	0.063	0.068	0.057	0.058	0.061	0.087	0.170	0.425
	0.3	0.058	0.060	0.064	0.071	0.060	0.061	0.065	0.073	0.115	0.264
	0.5	0.076	0.082	0.085	0.095	0.085	0.080	0.090	0.092	0.106	0.168
	0.7	0.132	0.137	0.143	0.154	0.141	0.135	0.146	0.144	0.160	0.203
	MP	0.051	0.062	0.071	0.082	0.059	0.065	0.080	0.341	0.572	0.844
0.3	0	0.055	0.067	0.074	0.084	0.056	0.065	0.079	0.357	0.588	0.855
	0.1	0.055	0.059	0.063	0.072	0.051	0.054	0.062	0.170	0.350	0.699
	0.2	0.055	0.058	0.060	0.067	0.051	0.051	0.062	0.088	0.170	0.422
	0.3	0.058	0.059	0.062	0.072	0.055	0.056	0.066	0.073	0.113	0.258
	0.5	0.080	0.084	0.088	0.094	0.079	0.080	0.088	0.091	0.111	0.168
	0.7	0.134	0.138	0.143	0.157	0.137	0.132	0.141	0.144	0.167	0.208
	MP	0.052	0.063	0.071	0.082	0.054	0.062	0.075	0.352	0.584	0.852
0.6	0	0.049	0.064	0.072	0.080	0.060	0.067	0.082	0.386	0.621	0.878
	0.1	0.048	0.057	0.063	0.066	0.053	0.056	0.064	0.164	0.353	0.705
	0.2	0.049	0.056	0.060	0.064	0.052	0.055	0.060	0.087	0.155	0.417
	0.3	0.055	0.060	0.065	0.068	0.057	0.058	0.065	0.075	0.104	0.251
	0.5	0.077	0.082	0.089	0.098	0.077	0.082	0.087	0.089	0.106	0.168
	0.7	0.135	0.134	0.148	0.159	0.130	0.138	0.142	0.146	0.166	0.211
	MP	0.047	0.062	0.069	0.076	0.057	0.064	0.079	0.380	0.616	0.875
0.9	0	0.055	0.067	0.072	0.081	0.060	0.067	0.081	0.453	0.710	0.929
	0.1	0.052	0.059	0.063	0.064	0.053	0.055	0.063	0.112	0.268	0.635
	0.2	0.051	0.057	0.061	0.063	0.052	0.053	0.060	0.064	0.098	0.265
	0.3	0.055	0.059	0.065	0.066	0.057	0.057	0.064	0.064	0.082	0.148
	0.5	0.080	0.084	0.094	0.100	0.088	0.082	0.091	0.089	0.106	0.140
	0.7	0.152	0.158	0.171	0.182	0.161	0.157	0.168	0.164	0.191	0.236
	MP	0.051	0.064	0.069	0.077	0.057	0.063	0.078	0.448	0.706	0.927

dependent, as the pilot invariably draws the optimal estimator towards itself. Basak et al. (2021) improved the method by iteratively updating the pilot with the optimal estimate obtained, and so the process was repeated until there was no further change in the optimal estimate. The iterative algorithm empirically shown to alleviate the pilot dependency of the optimal choice. Then, Basak et al. algorithm could be adopted in real life problems for choosing the best MRPDE of the bivariate normal distribution, iteratively minimizing formula (67). The MLE or MRPDE with moderate values of the tuning parameter $\alpha = 0.2$ can be used as initial pilot estimators.

Further, applying transformed Wald-type test statistics, $W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha})$ given in (65) entails a clear improvement in terms of significance levels under contamination. In this case, we would empirically recommend the choice of $\alpha = 0.2$ since it is better than any other, included the well-known Morgan-Pitman test, in most of the considered scenarios (Table 4).

7 Illustrative examples

7.1 Cork data set: comparing means or variances

Originally studied in Rao (1948), there is a well-known and publicity available real data set, the cork data set. It is included in several R packages (R Core Team 2020), in particular in `agridat` as a `box.cork` data. The data report the weights of cork boring of the trunk of 28 trees in the north, east, west and south sides. Rao pointed out that there exist positive correlation between the reported pairs of 4 variables, and sometimes it is assumed that they follow a normal distribution. Four-dimensional normality is an

Table 5 Normality tests for three versions of the corn data set

Normality test	4-Dimensional		2-Dimensional		log 2-Dimensional	
	Value	p Value	Value	p Value	Value	p Value
1.- `Doornik-Hansen`	16.123	0.041	9.833	0.043	2.490	0.646
2.- `Henze–Zirkler`	0.999	0.011	1.236	0.003	0.784	0.053
3.- `Royston`	12.161	0.003	11.784	0.002	5.564	0.047
4.- `E-statistic`	1.276	0.007	1.473	0.001	0.931	0.053
5a.- `Mardia: Skewness`	20.890	0.404	10.231	0.037	2.622	0.623
5b.- `Mardia: Kurtosis`	-0.398	0.690	0.899	0.369	-0.795	0.427

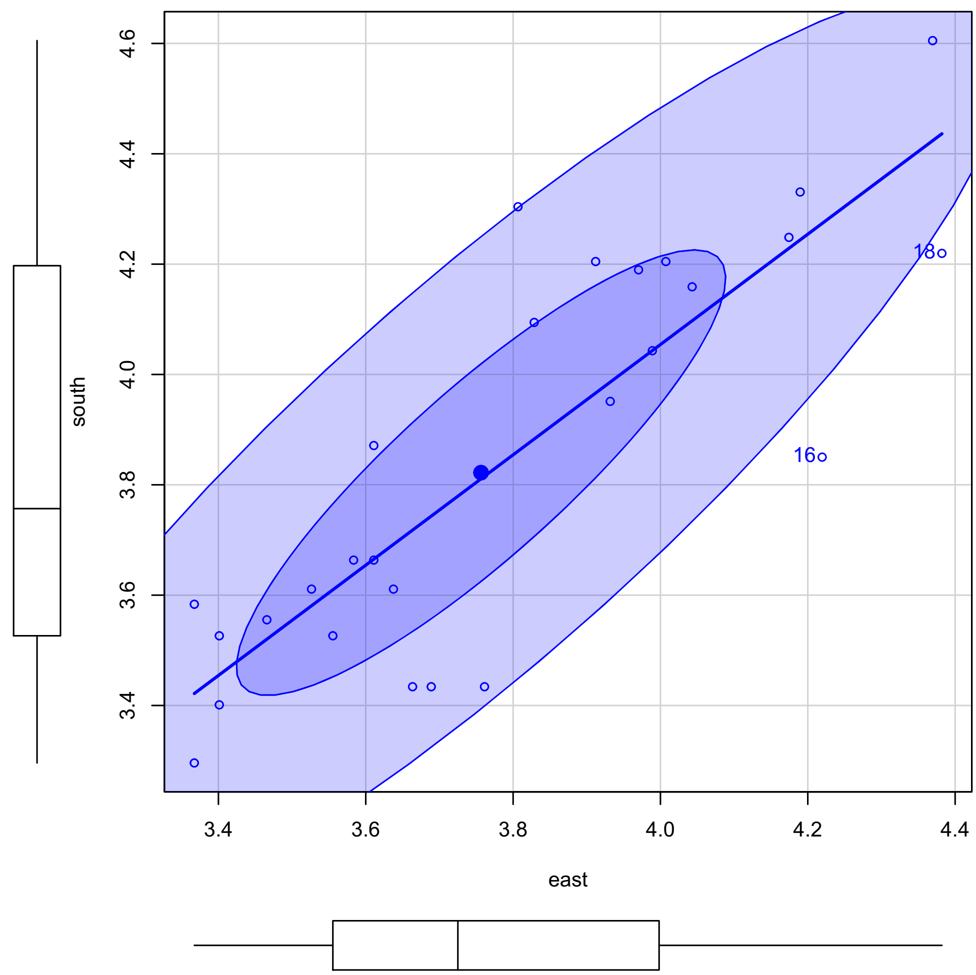


Fig. 3 Scatter-plot of east and south variables, with confidence ellipses, for log-transformed cork data set

arguable issue since using the R package MVN, in four out of five tests could multivariate normality be rejected with significance level 0.05, as shown in Table 5 (left hand side). We focussed on the two variables devoted to east and south sides respectively, as in Wilcox (2015, 2016), performed two-dimensional normal tests and this time all the tests rejected according to Table 5 (central columns). We did not study the robustness of an estimator and test statistic as in Wilcox (2015), in the sense of being resistant to data coming from distribution different from the bivariate normal as required for the original data. Our proposed estimator and test statistic are robust in the sense of being resistant to outliers once normality is being assumed. Having this in mind, the data were transformed using the base e logarithm and as shown in Table 5 (right hand side) and in four out five tests could not be multivariate normality rejected. In addition, outliers were studied for transformed data through the scatter-plot

Table 6 Classic exact tests of equal means or equal variances for the log-transformed corn data set with respect to full data or ouliers deleted data

Classic exact t-test	Full data		Outliers deleted data	
	Value	p Value	Value	p Value
`Paired t-test` (equal means), T_V	-1.454	0.157	-2.233	0.035
`Morgan-Pitman test` (equal variances), T_{MP}	-1.656	0.110	-3.033	0.005

with confidence ellipses shown in Fig. 3, concluding that observations 18 and 16 were suspicious to be outliers.

Taking the root of the Wald-type test statistics given in Cases 1, 2 and 3, as well as the Wald-type test given in Sect. 6 based on transformed variables, here we are going to provide alternative but equivalent expressions in practice, having the same p value. From the original data, $(x_1, y_1), \ldots, (x_n, y_n)$, the z-type test statistic for equal variances and based on MRPDEs, is given by

$$Z_{n,\alpha}(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha}) = \operatorname{sign}(\widehat{\gamma}_{R,\alpha} - 1)\sqrt{W_{n,\alpha}(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha})} = \sqrt{n}\left(\frac{\sqrt{2\alpha+1}}{\alpha+1}\right)^3 \frac{\widehat{\gamma}_{R,\alpha} - 1}{\sqrt{\beta_\alpha(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha})}}, \tag{68}$$

with $\beta_\alpha(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha})$ given by (64), has a standard normal asymptotic distribution. From the transformed data $(u_1, v_1), \ldots, (u_n, v_n)$, where $U = X + Y$ and $V = X - Y$, the z-type test statistic for equal variances and based on MRPDEs, is given by

$$Z'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha}) = \operatorname{sign}(\widehat{\rho}_{UV,R,\alpha})\sqrt{W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha})} = \sqrt{n}\left(\frac{\sqrt{2\alpha+1}}{\alpha+1}\right)^3 \widehat{\rho}_{UV,R,\alpha}. \tag{69}$$

Notice that the paired t-test for the same test (null correlation), with exact distribution t_{n-2}, is the Morgan-Pitman test T_{MP} given by (66). The z-type test statistic for equal means and based on MRPDEs and the transformed data $v_1, \ldots, v_n$, where $V = X - Y$, is given by

$$Z_{n,\alpha}(\widehat{\mu}_{V,R,\alpha}, \widehat{\sigma}_{V,R,\alpha}) = \operatorname{sign}(\widehat{\mu}_{V,R,\alpha})\sqrt{W_{n,\alpha}(\widehat{\mu}_{V,R,\alpha}, \widehat{\sigma}_{V,R,\alpha})} = \sqrt{n}\frac{2\alpha+1}{(\alpha+1)^2}\frac{\widehat{\mu}_{V,R,\alpha}}{\widehat{\sigma}_{V,R,\alpha}}. \tag{70}$$

Notice that the paired t-test for the same test (equal means), with exact distribution t_{n-1}, is given by

$$T_V = \sqrt{\frac{n-1}{n}} Z_{n,\alpha=0}(\widehat{\mu}_{V,R,\alpha}, \widehat{\sigma}_{V,R,\alpha}).$$

The results of the classic exact tests, T_V for testing (40) or T_{MP} for (42), with two-sided alternative, are summarized in Table 6. The decision, with 0.05 significance level, is opposite for both versions of the data, since the null hypothesis cannot be rejected for the full data set, while it is rejected for the outliers deleted data.

The advantage of these new expressions, (68)–(70), is that one-sided tests can be considered, apart from the two sided ones (as in the example given in Sect. 7.2). Based on (68)–(70), in Figs. 4 and 5 the values of the test statistics (left hand side) and the values of the estimates of parameters used to construct the test statistics (right hand sides) are shown, in solid lines the ones associated with the full log-transformed cork data set and in dashed lines the ones associated with the outliers deleted log-transformed cork data set. All the left hand side figures suggest rejecting the null, equal means or variances, as an appropriate decision with 0.05 significance level.

7.2 Lactate levels data set: fixing a positive correlation coefficient

Hutson (2019) studied the one sided test (45), $H_0 : \rho = \rho_0$ vs. $H_0 : \rho > \rho_0$, where $\rho_0 = 0$, for lactate levels measured in the blood and the cerebrospinal fluid on 13 female subjects. The study was done with a newly proposed robust test statistic, but in the sense of being resistant to data coming from distribution different from the bivariate normal. Using the R package `MVN`, in none of tests could multivariate normality be rejected with significance level 0.05, as shown in Table 7. Again, we highlight that our proposed estimators and test statistics are robust in the sense of being resistant to outliers once normality is being assumed. Deleting the two most influential observations, i.e. taking observations 1 and 7 as influential (rather than outliers), the sample Pearson correlation is modified from 0.572 for the full data to 0.471 for the influential observations deleted data (see Fig. 6) and according to Table 8, using the Morgan-Pitman exact test statistic, the decision of being accepted a positive correlation with 0.05 significance level is modified to not being possible to be accepted. With Fig. 7 we try to test whether with the Z-test statistic based on MRPDE of ρ, (70), a positive correlation could be accepted for the lactate levels data and

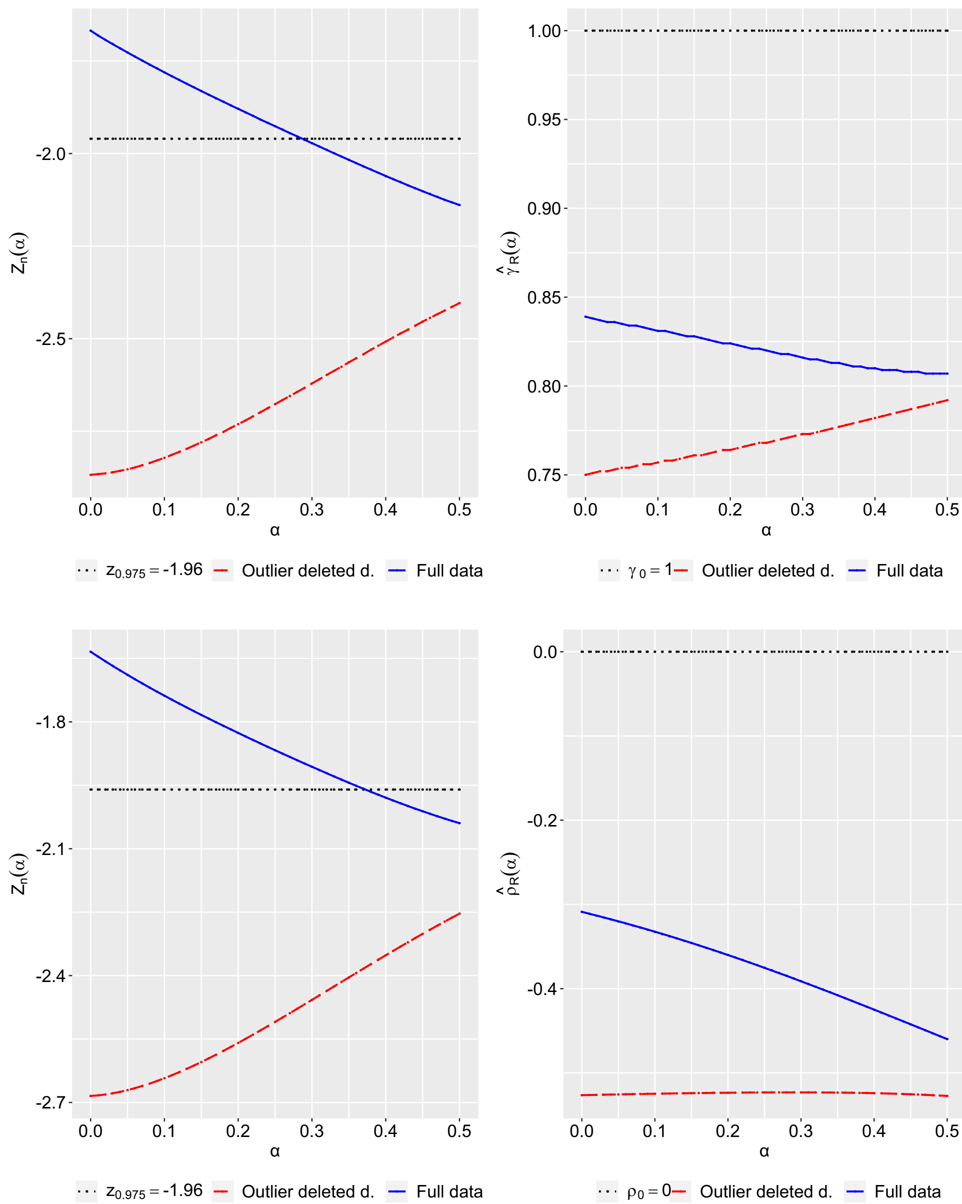


Fig. 4 Wald-type tests (left) and estimates (right) for log-transformed cork data set: Case 2 (above) and Case 3 (below)

actually it suggest as desirable decision nor being possible to reject it.

8 Concluding remarks

In practice, it is very important finding out a robust estimator and test statistic which does not loose much efficiency. Under the null hypothesis $\rho = 0$, the Morgan-Pitman exact test is the most efficient one but among the classic asymptotic tests there are several versions we should know. However, we would like to highlight that in comparative studies of recent papers the most competitive one, the Rao test given in Case 3 (Sect. 4), $R_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0})$, is not often being recognized. Since the MRPDEs are regulated through a positive α tuning parameter, being the tuning parameter $\alpha = 0$ the cornerstone as being the most efficient one out of all possible values of $\alpha \geq 0$. In case of having a poor efficiency for the asymptotic test statistic with the null tuning parameter of the MRPDEs, the test statistics constructed with the other values of the tuning parameters will increase such lack of efficiency and the obtained robustness could not compensate such drawback. This is just what happened with the Wald-type test statistic $W_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0})$, given in Case 3 (Sect. 4) for testing $\rho = 0$. Further, we used a modified version of the Wald-type test statistic $W'_{n,\alpha=0}(\widehat{\boldsymbol{\theta}}_{R,\alpha=0})$, given in Case 3 (Sect. 4), which matches the Rao test statistic only when $\rho_0 = 0$, and has provided for $\alpha = 0$ magnificent results in efficiency and

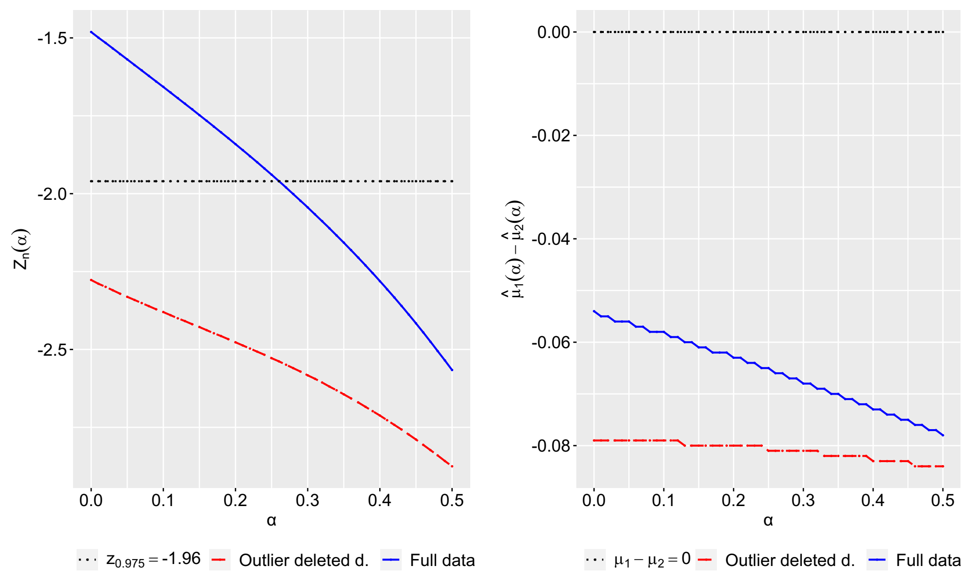


Fig. 5 Wald-type tests in Case 1 (left) and mean diference estimates (right) for log-transformed cork data set

Table 7 Normality tests for lactate levels data set

Normality test	Value	*p* Value
1.- Doornik–Hansen	1.436	0.838
2.- Henze–Zirkler	0.352	0.461
3.- Royston	0.923	0.642
4.- E-statistic	0.620	0.509
5a.- Mardia: Skewness	1.814	0.770
5b.- Mardia: Kurtosis	−0.774	0.439

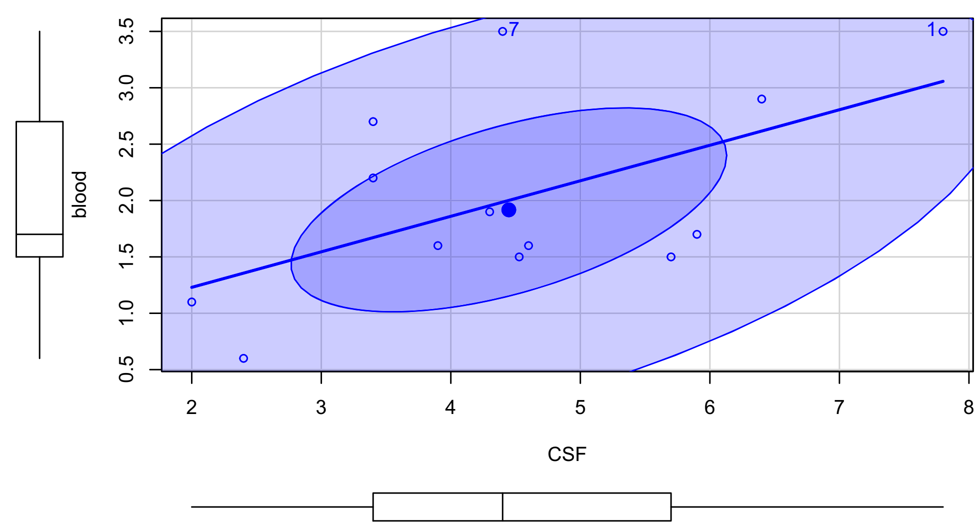


Fig. 6 Scatter-plot of CSF and blood variables, with confidence ellipses, for lactate levels data set

Table 8 Classic exact test of uncorrelation for the lactate levels data set with respect to full data or influential observations deleted data

Classic exact t-test	Full data		Infl. obs. deleted data	
	Value	p Value	Value	p Value
One-sided positive correlation with T_{MP}	2.313	0.020	1.601	0.072

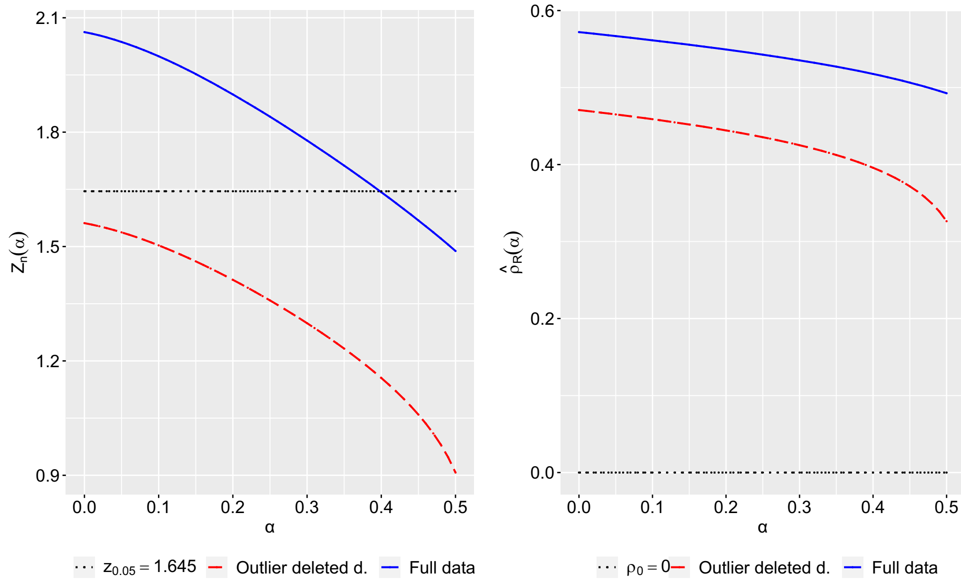


Fig. 7 Wald-type tests (left) and estimates (right) for the lactate levels data set

also for $\alpha > 0$ strong robust properties. The improvements and properties are shown by simulation for the specific null hypothesis $\rho = 0$, but proven in the framework of the developed general theory.

Acknowledgements This research is supported by the Spanish Grants PGC2018-095194-B-100 and FPU/018240 (M. Jaenada). E. Castilla, M.Jaenada, N. Martín and L. Pardo are members of the Instituto de Matematica Interdisciplinar, Complutense University of Madrid.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Human and animals rights Not applicable.

Informed consent Each named author has substantially contributed to conducting the underlying research and drafting this manuscript. All the authors are aware of the current submission.

Appendix A Proof of Theorem 1

$$
\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\theta}} \boldsymbol{\Psi}_{\boldsymbol{\alpha}}^{\boldsymbol{T}}(X;\boldsymbol{\theta}) \\
&\quad= \frac{\partial}{\partial \boldsymbol{\theta}} f_{\boldsymbol{\theta}}^{\alpha}(X)\left(\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X) - \boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta})\right) \\
&\qquad+ f_{\boldsymbol{\theta}}^{\alpha}(X)\left(\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X) - \frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta})\right) \\
&\quad= \alpha f_{\boldsymbol{\theta}}^{\alpha}(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\left(\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X) - \boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta})\right) \\
&\qquad+ f_{\boldsymbol{\theta}}^{\alpha}(X)\left(\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X) - \frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta})\right) \\
&\quad= \alpha f_{\boldsymbol{\theta}}^{\alpha}(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X) - \alpha f_{\boldsymbol{\theta}}^{\alpha}(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta}) \\
&\qquad+ f_{\boldsymbol{\theta}}^{\alpha}(X)\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X) - f_{\boldsymbol{\theta}}^{\alpha}(X)\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta}),
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\frac{\partial}{\partial \boldsymbol{\theta}} \boldsymbol{\Psi}_{\boldsymbol{\alpha}}^{\boldsymbol{T}}(X;\boldsymbol{\theta})\right] \\
&\quad= \alpha E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X)\right] - \alpha E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\right]\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta}) \\
&\qquad+ E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X)\right] - E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\right]\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta}) \\
&\quad= \alpha E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X)\right] - \alpha E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\right]\boldsymbol{c}_{\alpha}(\boldsymbol{\theta})\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta}) \\
&\qquad+ E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X)\right] - E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\right]\frac{\partial}{\partial \boldsymbol{\theta}}\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta}) \\
&\quad= \alpha E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\boldsymbol{u}_{\boldsymbol{\theta}}(X)\boldsymbol{u}_{\boldsymbol{\theta}}^{T}(X)\right] - \alpha E\left[f_{\boldsymbol{\theta}}^{\alpha}(X)\right]\boldsymbol{c}_{\alpha}(\boldsymbol{\theta})\boldsymbol{c}_{\alpha}^{T}(\boldsymbol{\theta})
\end{aligned}
$$

$$
\begin{aligned}
&+(\alpha+1)E\left[f_{\boldsymbol\theta}^{\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\right]\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)\\
&-(\alpha+1)E\left[f_{\boldsymbol\theta}^{\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\boldsymbol u_{\boldsymbol\theta}^{T}(X)\right]\\
&=E\left[f_{\boldsymbol\theta}^{\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\right]\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)-E\left[f_{\boldsymbol\theta}^{\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\boldsymbol u_{\boldsymbol\theta}^{T}(X)\right]\\
&=E\left[f_{\boldsymbol\theta}^{\alpha}(X)\right]\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)-E\left[f_{\boldsymbol\theta}^{\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\boldsymbol u_{\boldsymbol\theta}^{T}(X)\right],
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol S_{\alpha}(\boldsymbol\theta)&=-E\left[\tfrac{\partial}{\partial\boldsymbol\theta}\boldsymbol\Psi_{\boldsymbol\alpha}^{T}(x;\boldsymbol\theta)\right]=E\left[f_{\boldsymbol\theta}^{\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\boldsymbol u_{\boldsymbol\theta}^{T}(X)\right]\\
&\quad-E\left[f_{\boldsymbol\theta}^{\alpha}(X)\right]\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)\\
&=\boldsymbol J_{\alpha}(\boldsymbol\theta)-\kappa_{\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta),
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol K_{\alpha}(\boldsymbol\theta)\\
&=\mathrm{E}\left[\left(f_{\boldsymbol\theta}^{\alpha}(X)\left(\boldsymbol u_{\boldsymbol\theta}(X)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)\right)\left(f_{\boldsymbol\theta}^{\alpha}(X)\left(\boldsymbol u_{\boldsymbol\theta}^{T}(X)-\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)\right)\right)\right]\\
&=\mathrm{E}\left[f_{\boldsymbol\theta}^{2\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\boldsymbol u_{\boldsymbol\theta}^{T}(X)\right]+\mathrm{E}\left[f_{\boldsymbol\theta}^{2\alpha}(X)\right]\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)\\
&\quad-\mathrm{E}\left[f_{\boldsymbol\theta}^{2\alpha}(X)\boldsymbol u_{\boldsymbol\theta}(X)\right]\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\mathrm{E}\left[f_{\boldsymbol\theta}^{2\alpha}(X)\boldsymbol u_{\boldsymbol\theta}^{T}(X)\right]\\
&=\boldsymbol J_{2\alpha}(\boldsymbol\theta)+\kappa_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)\\
&\quad-\kappa_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)-\kappa_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{2\alpha}^{T}(\boldsymbol\theta)\\
&=\boldsymbol S_{2\alpha}(\boldsymbol\theta)+\kappa_{2\alpha}(\boldsymbol\theta)\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)^{T},
\end{aligned}
$$

where

$$
\begin{aligned}
&\boldsymbol S_{2\alpha}(\boldsymbol\theta)=\boldsymbol J_{2\alpha}(\boldsymbol\theta)-\kappa_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{2\alpha}^{T}(\boldsymbol\theta)\\
&\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)^{T}\\
&\quad=\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)-\boldsymbol c_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{\alpha}^{T}(\boldsymbol\theta)\\
&\quad\quad-\boldsymbol c_{\alpha}(\boldsymbol\theta)\boldsymbol c_{2\alpha}^{T}(\boldsymbol\theta)+\boldsymbol c_{2\alpha}(\boldsymbol\theta)\boldsymbol c_{2\alpha}^{T}(\boldsymbol\theta).
\end{aligned}
$$

Appendix B Proof of Theorem 5

We shall follow 1 as well as Propositions 2, 3, 4. About the first partition, since $\boldsymbol c_{1,\alpha}(\boldsymbol\theta)=\boldsymbol 0_2$ it is trivial that

$$
\begin{aligned}
\boldsymbol S_{1,\alpha}(\boldsymbol\theta)&=\boldsymbol J_{1,\alpha}(\boldsymbol\theta),\\
\boldsymbol K_{1,\alpha}(\boldsymbol\theta)&=\boldsymbol J_{1,2\alpha}(\boldsymbol\theta).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\boldsymbol S_{2,\alpha}(\boldsymbol\theta)&=\boldsymbol J_{2,\alpha}(\boldsymbol\theta)-\kappa_{\alpha}(\boldsymbol\theta)\boldsymbol c_{2,\alpha}(\boldsymbol\theta)\boldsymbol c_{2,\alpha}^{T}(\boldsymbol\theta)\\
&=\frac{1}{k^{\alpha}(\boldsymbol\theta)(\alpha+1)^3}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\boldsymbol J_{2,\alpha}(\rho)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\\
&\quad-\frac{\alpha^2}{k^{\alpha}(\boldsymbol\theta)(\alpha+1)^3}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\boldsymbol S_{2,2}(\rho)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\\
&=\frac{1}{k^{\alpha}(\boldsymbol\theta)(\alpha+1)^3}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\\
&\quad\times\left[\boldsymbol J_{2,\alpha}(\rho)-\alpha^2\boldsymbol S_{2,2}(\rho)\right]\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
&\boldsymbol J_{2,\alpha}(\rho)-\alpha^2\boldsymbol S_{2,2}(\rho)\\
&=\frac{1}{1-\rho^2}\left[\begin{pmatrix}\alpha^2-\rho^2(\alpha^2+1)+2 & \alpha^2-\rho^2(\alpha^2+1) & -\rho(\alpha^2+1)\\ \alpha^2-\rho^2(\alpha^2+1) & \alpha^2-\rho^2(\alpha^2+1)+2 & -\rho(\alpha^2+1)\\ -\rho(\alpha^2+1) & -\rho(\alpha^2+1) & \frac{\rho^2(\alpha^2+1)+1}{1-\rho^2}\end{pmatrix}\right.\\
&\quad\left.-\alpha^2\begin{pmatrix}1-\rho^2 & 1-\rho^2 & -\rho\\ 1-\rho^2 & 1-\rho^2 & -\rho\\ -\rho & -\rho & \frac{\rho^2}{1-\rho^2}\end{pmatrix}\right]\\
&=\frac{1}{1-\rho^2}\begin{pmatrix}2-\rho^2 & -\rho^2 & -\rho\\ -\rho^2 & 2-\rho^2 & -\rho\\ -\rho & -\rho & \frac{\rho^2+1}{1-\rho^2}\end{pmatrix}=\boldsymbol S_{2,1}(\rho).
\end{aligned}
$$

In addition,

$$
\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)=\frac{\alpha}{(2\alpha+1)(\alpha+1)}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\begin{pmatrix}-1\\-1\\\frac{\rho}{1-\rho^2}\end{pmatrix}
$$

$$
\begin{aligned}
&\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)^{T}\\
&\quad=\frac{\alpha^2}{(2\alpha+1)^2(\alpha+1)^2}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\boldsymbol S_{2,2}(\rho)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol K_{\alpha}(\boldsymbol\theta)&=\boldsymbol S_{2\alpha}(\boldsymbol\theta)+\kappa_{2\alpha}(\boldsymbol\theta)\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)\left(\boldsymbol c_{2\alpha}(\boldsymbol\theta)-\boldsymbol c_{\alpha}(\boldsymbol\theta)\right)^{T}\\
&=\frac{1}{k^{2\alpha}(\boldsymbol\theta)(2\alpha+1)^3}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\boldsymbol S_{2,1}(\rho)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\\
&\quad+\kappa_{2\alpha}(\boldsymbol\theta)\frac{\alpha^2}{(2\alpha+1)^2(\alpha+1)^2}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\boldsymbol S_{2,2}(\rho)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\\
&=\frac{1}{k^{2\alpha}(\boldsymbol\theta)(2\alpha+1)^3}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\boldsymbol S_{2,1}(\rho)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\\
&\quad+\frac{\alpha^2}{k^{2\alpha}(\boldsymbol\theta)(2\alpha+1)^3(\alpha+1)^2}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\boldsymbol S_{2,2}(\rho)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\\
&=\frac{1}{k^{2\alpha}(\boldsymbol\theta)(2\alpha+1)^3}\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}\left((\alpha+1)^2\boldsymbol S_{2,1}(\rho)\right.\\
&\quad\left.+\alpha^2\boldsymbol S_{2,2}(\rho)\right)\boldsymbol D_{2,\sigma_1,\sigma_2}^{-1}.
\end{aligned}
$$

Appendix C Proof of Theorem 7

Let $\boldsymbol\theta_0\in\Theta_0$ be the true value of $\boldsymbol\theta$. Using a Taylor series expansion we get

$$
\begin{aligned}
\boldsymbol m(\widehat{\boldsymbol\theta}_{R,\alpha})&=\boldsymbol m(\boldsymbol\theta_0)+\boldsymbol M^{T}(\boldsymbol\theta_0)(\widehat{\boldsymbol\theta}_{R,\alpha}-\boldsymbol\theta_0)+o_p\left(\|\widehat{\boldsymbol\theta}_{R,\alpha}-\boldsymbol\theta_0\|\right)\\
&=\boldsymbol M^{T}(\boldsymbol\theta_0)(\widehat{\boldsymbol\theta}_{R,\alpha}-\boldsymbol\theta_0)+o_p\left(\|\widehat{\boldsymbol\theta}_{R,\alpha}-\boldsymbol\theta_0\|\right),\qquad\text{(C1)}
\end{aligned}
$$

because from Eq. (36) we have $\boldsymbol m(\boldsymbol\theta_0)=\boldsymbol 0_r$. Now, under H_0,

$$
n^{1/2}(\widehat{\boldsymbol\theta}_{R,\alpha}-\boldsymbol\theta_0)\underset{n\to\infty}{\overset{\mathcal L}{\longrightarrow}}\mathcal N(\boldsymbol 0_p,\boldsymbol V_{\alpha}(\boldsymbol\theta_0)).
$$

Therefore, from Eq. (C1) we get, under H_0,

$$
n^{1/2}\boldsymbol m(\widehat{\boldsymbol\theta}_{R,\alpha})\underset{n\to\infty}{\overset{\mathcal L}{\longrightarrow}}\mathcal N(\boldsymbol 0_r,\boldsymbol M^{T}(\boldsymbol\theta_0)\boldsymbol V_{\alpha}(\boldsymbol\theta_0)\boldsymbol M(\boldsymbol\theta_0)).
$$

As $\operatorname{rank}(\boldsymbol{M}(\boldsymbol{\theta}))=r$, we get

$$n\boldsymbol{m}^T(\widehat{\boldsymbol{\theta}}_{R,\alpha})\left(\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{V}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0)\right)^{-1}\boldsymbol{m}(\widehat{\boldsymbol{\theta}}_{R,\alpha})\xrightarrow[n\to\infty]{\mathcal{L}}\chi_r^2.$$

Now $\boldsymbol{M}^T(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{V}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{M}(\widehat{\boldsymbol{\theta}}_{R,\alpha})$ is a consistent estimator of

$$\boldsymbol{M}^T(\boldsymbol{\theta}_0)\boldsymbol{V}_\alpha(\boldsymbol{\theta}_0)\boldsymbol{M}(\boldsymbol{\theta}_0).$$

Hence, under H_0,

$$W_{n,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha})\xrightarrow[n\to\infty]{\mathcal{L}}\chi_r^2.$$

Appendix D Proof of the formulas of the inner iterations of Algorithm 1

Taking into account Theorem 6 and the components of $\boldsymbol{u}_{\boldsymbol{\theta}}(x,y)-\boldsymbol{c}_\alpha(\boldsymbol{\theta})$, given by

$$\begin{aligned}
&u_{\mu_1}(x,y)-c_\alpha(\mu_1)=\tfrac{1}{\sigma_1(1-\rho^2)}\left[\tfrac{x-\mu_1}{\sigma_1}-\rho\left(\tfrac{y-\mu_2}{\sigma_2}\right)\right],\\
&u_{\mu_2}(x,y)-c_\alpha(\mu_2)=\tfrac{1}{\sigma_2(1-\rho^2)}\left[\tfrac{y-\mu_2}{\sigma_2}-\rho\left(\tfrac{x-\mu_1}{\sigma_1}\right)\right],\\
&u_{\sigma_1}(x,y)-c_\alpha(\sigma_1)\\
&\quad=-\tfrac{1}{\sigma_1}\left\{\tfrac{1}{\alpha+1}+\tfrac{1}{1-\rho^2}\left[\rho\left(\tfrac{x-\mu_1}{\sigma_1}\right)\left(\tfrac{y-\mu_2}{\sigma_2}\right)-\left(\tfrac{x-\mu_1}{\sigma_1}\right)^2\right]\right\},\\
&u_{\sigma_2}(x,y)-c_\alpha(\sigma_2)\\
&\quad=-\tfrac{1}{\sigma_2}\left\{\tfrac{1}{\alpha+1}+\tfrac{1}{1-\rho^2}\left[\rho\left(\tfrac{x-\mu_1}{\sigma_1}\right)\left(\tfrac{y-\mu_2}{\sigma_2}\right)-\left(\tfrac{y-\mu_2}{\sigma_2}\right)^2\right]\right\},\\
&u_\rho(x,y)-c_\alpha(\rho)\\
&\quad=\tfrac{1}{(1-\rho^2)^2}\left\{(1+\rho^2)\left(\tfrac{x-\mu_1}{\sigma_1}\right)\left(\tfrac{y-\mu_2}{\sigma_2}\right)\right.\\
&\qquad\left.-\rho\left[\tfrac{1-\rho^2}{\alpha+1}+\left(\tfrac{x-\mu_1}{\sigma_1}\right)^2+\left(\tfrac{y-\mu_2}{\sigma_2}\right)^2\right]\right\},
\end{aligned}$$

the estimating equations are

$$\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i-\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i=0, \tag{D2}$$

$$\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i-\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i=0, \tag{D3}$$

$$\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}+\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i-\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i^2=0, \tag{D4}$$

$$\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}+\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i-\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i^2=0, \tag{D5}$$

$$\begin{aligned}&(1+\rho^2)\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i+\rho\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\\
&\quad-\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i^2-\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i^2=0,\end{aligned} \tag{D6}$$

$$\begin{aligned}&w_{i,\boldsymbol{\theta}}=\exp\left\{\tfrac{1}{2(1-\rho^2)}\left[\widetilde{X}_i^2+\widetilde{Y}_i^2-2\rho\widetilde{X}_i\widetilde{Y}_i\right]\right\},\\
&\widetilde{X}_i=\tfrac{X_i-\mu_1}{\sigma_1},\qquad \widetilde{Y}_i=\tfrac{Y_i-\mu_2}{\sigma_2}.\end{aligned}$$

Since $\rho\in(-1,1)$, from (D2)–(D3) it holds

$$\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i=0 \tag{D7}$$

$$\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i=0, \tag{D8}$$

from (D4)–(D5)

$$\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i^2=\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i^2, \tag{D9}$$

with

$$\begin{aligned}&\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i^2=\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}+\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i,\\
&\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i^2=\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}+\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i.\end{aligned}$$

Replacing both in (D6) we get

$$\begin{aligned}&(1+\rho^2)\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i+\rho\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\\
&\quad-2\rho\left(\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}+\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i\right)=0,\end{aligned}$$

i.e.

$$\rho=\frac{(\alpha+1)\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i}{\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}}. \tag{D10}$$

Hence

$$\begin{aligned}&\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}+\rho\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i=\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i^2,\\
&\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}+\frac{\rho^2}{\alpha+1}\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}=\sum_{i=1}^n w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i^2,\end{aligned}$$

$$\frac{1}{\alpha+1}\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha} = \sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i^2, \tag{D11}$$

and

$$\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha} + \rho\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{X}_i\widetilde{Y}_i = \sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i^2,$$

$$\frac{1-\rho^2}{\alpha+1}\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha} + \frac{\rho^2}{\alpha+1}\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha} = \sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i^2,$$

$$\frac{1}{\alpha+1}\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha} = \sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}\widetilde{Y}_i^2. \tag{D12}$$

Finally,

$$\mu_1 = \frac{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha} X_i}{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}}, \quad \mu_2 = \frac{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha} Y_i}{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}},$$

$$\frac{\sigma_1^2}{\alpha+1} = \frac{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}(X_i-\mu_1)^2}{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}},$$

$$\frac{\sigma_2^2}{\alpha+1} = \frac{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}(Y_i-\mu_2)^2}{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}},$$

$$\rho = \frac{(\alpha+1)\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}\frac{X_i-\mu_1}{\sigma_1}\frac{Y_i-\mu_2}{\sigma_2}}{\sum_{i=1}^{n} w_{i,\boldsymbol{\theta}}^{-\alpha}},$$

from which are derived the main formulas of the inner iterations of the Iteratively Reweighted Moments Algorithm.

Appendix E Proof of Theorem 8

From Theorem 5 in Broniatowski et al. (1961), the IF associated to the MRPDE of $\boldsymbol{\theta}$ is given by

$$\begin{aligned}&\mathcal{IF}((x,y)^T, \boldsymbol{T}_\alpha, F_{\boldsymbol{\theta}})\\ &\quad = \boldsymbol{S}_\alpha^{-1}(\boldsymbol{\theta})\, f_{\boldsymbol{\theta}}^{\alpha}(x,y)\,[\boldsymbol{u}_{\boldsymbol{\theta}}(x,y) - \boldsymbol{c}_\alpha(\boldsymbol{\theta})],\end{aligned}$$

where

$$\boldsymbol{S}_\alpha^{-1}(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{S}_{1,\alpha}^{-1}(\boldsymbol{\theta}) & \boldsymbol{0}_{2\times 3} \\ \boldsymbol{0}_{3\times 2} & \boldsymbol{S}_{2,\alpha}^{-1}(\boldsymbol{\theta}) \end{pmatrix}$$

is the inverse of the matrix, $\boldsymbol{S}_\alpha(\boldsymbol{\theta})$, defined in Theorem 5. The expressions of $\boldsymbol{S}_{1,\alpha}^{-1}(\boldsymbol{\theta})$ and $\boldsymbol{S}_{2,\alpha}^{-1}(\boldsymbol{\theta})$ are given in (26) and (27) respectively and the ones of $\boldsymbol{u}_{\boldsymbol{\theta}}(x,y) - \boldsymbol{c}_\alpha(\boldsymbol{\theta})$ in (D2)–(D5). On one hand

$$\begin{pmatrix}\mathcal{IF}_\alpha(\mu_1)\\ \mathcal{IF}_\alpha(\mu_2)\end{pmatrix} = \frac{k^\alpha(\boldsymbol{\theta})\,(\alpha+1)^2}{1-\rho^2} f_{\boldsymbol{\theta}}^{\alpha}(x,y)\begin{pmatrix}\sigma_1 & \sigma_1\rho\\ \sigma_2\rho & \sigma_2\end{pmatrix} \times \begin{pmatrix}\frac{x-\mu_1}{\sigma_1} - \rho\left(\frac{y-\mu_2}{\sigma_2}\right)\\ \frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\end{pmatrix},$$

where

$$\begin{pmatrix}\sigma_1 & \sigma_1\rho\\ \sigma_2\rho & \sigma_2\end{pmatrix}\begin{pmatrix}\frac{x-\mu_1}{\sigma_1} - \rho\left(\frac{y-\mu_2}{\sigma_2}\right)\\ \frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\end{pmatrix} = (1-\rho^2)\begin{pmatrix}x-\mu_1\\ y-\mu_2\end{pmatrix},$$

and

$$\begin{aligned}f_{\boldsymbol{\theta}}^{\alpha}(x,y) &= \frac{w_{\alpha,\boldsymbol{\theta}}^{-\alpha/(1-\rho^2)}(x,y)}{k^\alpha(\boldsymbol{\theta})}\\ &= \frac{1}{k^\alpha(\boldsymbol{\theta})}\exp\left\{-\frac{\alpha}{2(1-\rho^2)}\left[(\tfrac{x-\mu_1}{\sigma_1})^2 + (\tfrac{y-\mu_2}{\sigma_2})^2 \right.\right.\\ &\quad\left.\left. -2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}\right]\right\},\end{aligned}$$

hence we obtain (57)–(58). On the other hand, we have

$$\begin{pmatrix}\mathcal{IF}_\alpha(\sigma_1)\\ \mathcal{IF}_\alpha(\sigma_2)\end{pmatrix} = k^\alpha(\boldsymbol{\theta})\frac{(\alpha+1)^3}{2} f_{\boldsymbol{\theta}}^{\alpha}(x,y)\begin{pmatrix}\sigma_1 & \sigma_1\rho^2 & \sigma_1\rho(1-\rho^2)\\ \sigma_2\rho^2 & \sigma_2 & \sigma_2\rho(1-\rho^2)\end{pmatrix} \times \begin{pmatrix} -\left\{\frac{1}{\alpha+1} + \frac{1}{1-\rho^2}\left[\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} - \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\}\\ -\left\{\frac{1}{\alpha+1} + \frac{1}{1-\rho^2}\left[\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} - \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}\\ \frac{1}{(1-\rho^2)^2}\left\{(1+\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) -\rho\left[\frac{1-\rho^2}{\alpha+1} + \left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}\end{pmatrix},$$

where

$$\begin{aligned}&\begin{pmatrix}\sigma_1 & \sigma_1\rho^2 & \sigma_1\rho(1-\rho^2)\\ \sigma_2\rho^2 & \sigma_2 & \sigma_2\rho(1-\rho^2)\end{pmatrix} \cdot \begin{pmatrix} -\left\{\frac{1}{\alpha+1} + \frac{1}{1-\rho^2}\left[\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} - \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\}\\ -\left\{\frac{1}{\alpha+1} + \frac{1}{1-\rho^2}\left[\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} - \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}\\ \frac{1}{(1-\rho^2)^2}\left\{(1+\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) -\rho\left[\frac{1-\rho^2}{\alpha+1} + \left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}\end{pmatrix}\\ &= \begin{pmatrix}\frac{\sigma_1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - (1-\rho^2)(1+2\rho^2)\frac{1}{\alpha+1}\right]\\ \frac{\sigma_2}{1-\rho^2}\left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - (1-\rho^2)(1+2\rho^2)\frac{1}{\alpha+1}\right]\end{pmatrix}.\end{aligned}$$

Hence, we obtain (59)–(60). Finally,

$$\mathcal{IF}_\alpha(\rho) = k^\alpha(\boldsymbol{\theta})\frac{(\alpha+1)^3}{2} f_{\boldsymbol{\theta}}^{\alpha}(x,y)\left(\rho(1-\rho^2)\ \ \rho(1-\rho^2)\ \ 2(1-\rho^2)^2\right)$$

$$\times \begin{pmatrix} -\left\{\frac{1}{\alpha+1}+\frac{1}{1-\rho^2}\left[\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}-\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\} \\ -\left\{\frac{1}{\alpha+1}+\frac{1}{1-\rho^2}\left[\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}-\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \\ \frac{1}{(1-\rho^2)^2}\left\{(1+\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right. \\ \left.-\rho\left[\frac{1-\rho^2}{\alpha+1}+\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \end{pmatrix}$$

where

$$\begin{aligned} &\left(\rho(1-\rho^2)\ \rho(1-\rho^2)\ 2(1-\rho^2)^2\right) \\ &\cdot\begin{pmatrix} -\left\{\frac{1}{\alpha+1}+\frac{1}{1-\rho^2}\left[\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}-\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\} \\ -\left\{\frac{1}{\alpha+1}+\frac{1}{1-\rho^2}\left[\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}-\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \\ \frac{1}{(1-\rho^2)^2}\left\{(1+\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right. \\ \left.-\rho\left[\frac{1-\rho^2}{\alpha+1}+\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \end{pmatrix} \\ &= -\rho\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]+2\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right), \end{aligned}$$

from which it follows (61).

Appendix F Proofs of cases of testing problems (Sect. 4)

Case 1 (Comparing means of two dependent populations with normal distributions).

If we consider the function

$$m\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)=\mu_1-\mu_2,$$

the null hypothesis can be given by $m\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)=0$. In this case

$$\begin{aligned} &\boldsymbol{M}^T\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)=\left(1\ -1\ 0\ 0\ 0\right), \\ &\left(\boldsymbol{M}^T(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{V}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{M}(\widehat{\boldsymbol{\theta}}_{R,\alpha})\right)^{-1}=\left(\left(1\ -1\right)\boldsymbol{V}_{1,\alpha}(\widehat{\boldsymbol{\theta}}_{R,\alpha})\begin{pmatrix}1\\-1\end{pmatrix}\right)^{-1} \\ &=\frac{(2\alpha+1)^2}{(\alpha+1)^4}\left(\widehat{\sigma}^2_{1,R,\alpha}-2\widehat{\rho}_{R,\alpha}\widehat{\sigma}_{1,R,\alpha}\widehat{\sigma}_{2,R,\alpha}+\widehat{\sigma}^2_{2,R,\alpha}\right)^{-1} \\ &=\frac{(2\alpha+1)^2}{(\alpha+1)^4}\left[(\widehat{\sigma}_{1,R,\alpha}-\widehat{\sigma}_{2,R,\alpha})^2+2(1-\widehat{\rho}_{R,\alpha})\widehat{\sigma}_{1,R,\alpha},\widehat{\sigma}_{2,R,\alpha}\right], \end{aligned}$$

where $\boldsymbol{V}_{1,\alpha}(\cdot)$ is given by (30). Therefore, (41) is obtained.

Case 2 (Comparing variances of two dependent populations with normal distributions).

If we consider the function

$$m\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)=\sigma_1-\sigma_2,$$

the null hypothesis can be given by $m\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)=0$. In this case

$$\boldsymbol{M}^T\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)=\left(0\ 0\ 1\ -1\ 0\right)$$

and taking

$$\begin{aligned} &\boldsymbol{M}^T\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right) \\ &\quad=\left(\sigma_1\ -\sigma_2\ 0\right)\boldsymbol{V}_{2,\alpha}(\rho)\begin{pmatrix}\sigma_1\\-\sigma_2\\0\end{pmatrix}=\frac{(\alpha+1)^4}{(2\alpha+1)^3}b_\alpha(\boldsymbol{\theta}), \end{aligned}$$

we denote

$$\begin{aligned} b_\alpha(\boldsymbol{\theta})&=(\alpha+1)^2b_{1,\alpha}(\boldsymbol{\theta})+\alpha^2b_{2,\alpha}(\boldsymbol{\theta}) \\ &=\frac{2\left(\alpha+1\right)^2+\alpha^2}{4}(\sigma_1-\sigma_2)^2+(\alpha+1)^2\left(1-\rho^2\right)\sigma_1\sigma_2 \\ &=\frac{1}{4}\left(\alpha+1\right)^2\left\{\left[2+\left(\frac{\alpha}{\alpha+1}\right)^2\right](\sigma_1-\sigma_2)^2\right. \\ &\quad\left.+4(1-\rho^2)\sigma_1\sigma_2\right\}, \end{aligned} \tag{F13}$$

where

$$\begin{aligned} b_{1,\alpha}(\boldsymbol{\theta})&=\frac{1}{2}\left(\sigma_1\ -\sigma_2\right)\begin{pmatrix}1&\rho^2\\\rho^2&1\end{pmatrix}\begin{pmatrix}\sigma_1\\-\sigma_2\end{pmatrix} \\ &=\frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)-\rho^2\sigma_1\sigma_2\right) \\ &=\frac{1}{2}(\sigma_1-\sigma_2)^2+(1-\rho^2)\sigma_1\sigma_2, \end{aligned}$$

$$\begin{aligned} b_{2,\alpha}(\boldsymbol{\theta})&=\frac{1}{4}\frac{1}{1-\rho^2}\left(\sigma_1\ -\sigma_2\ 0\right) \\ &\quad\times\begin{pmatrix}1&\rho^2&\rho(1-\rho^2)\\\rho^2&1&\rho(1-\rho^2)\\\rho(1-\rho^2)&\rho(1-\rho^2)&2(1-\rho^2)^2\end{pmatrix} \\ &\quad\times\begin{pmatrix}1-\rho^2&1-\rho^2&-\rho\\1-\rho^2&1-\rho^2&-\rho\\-\rho&-\rho&\frac{\rho^2}{1-\rho^2}\end{pmatrix} \\ &\quad\times\begin{pmatrix}1&\rho^2&\rho(1-\rho^2)\\\rho^2&1&\rho(1-\rho^2)\\\rho(1-\rho^2)&\rho(1-\rho^2)&2(1-\rho^2)^2\end{pmatrix}\begin{pmatrix}\sigma_1\\-\sigma_2\\0\end{pmatrix} \\ &=\frac{1}{4}\left(\sigma_1\ -\sigma_2\right)\mathbf{1}_2\mathbf{1}_2^T\begin{pmatrix}\sigma_1\\-\sigma_2\end{pmatrix}=\frac{1}{4}(\sigma_1-\sigma_2)^2. \end{aligned}$$

Finally, denoting $b_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})=(\alpha+1)^2\beta_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})$, we get (43).

Case 3 (Fixing a value of the correlation coefficient of two dependent populations with normal distributions).

If we consider the function

$$m\left(\mu_1,\mu_2,\sigma_1,\sigma_2,\rho\right)=\rho-\rho_0,$$

the null hypothesis can be given by $m(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = 0$. In this case $\boldsymbol{M}^T(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and we have

$$\left(\boldsymbol{M}^T(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{V}_\alpha(\widehat{\boldsymbol{\theta}}_{R,\alpha})\boldsymbol{M}(\widehat{\boldsymbol{\theta}}_{R,\alpha})\right)^{-1} = \frac{(2\alpha+1)^3}{(\alpha+1)^6}\frac{1}{(1-\widehat{\rho}_{R,\alpha}^2)^2}. \tag{F14}$$

Therefore, we get (46).

Case 4 (Comparing means and variances of two dependent populations with normal distribution).
If we consider the function

$$\boldsymbol{m}^T(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = (\mu_1 - \mu_2, \sigma_1 - \sigma_2),$$

the null hypothesis can be written by $\boldsymbol{m}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \boldsymbol{0}_2$. In this case,

$$\boldsymbol{M}^T(\widehat{\boldsymbol{\theta}}_{R,\alpha}) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

and

$$\boldsymbol{M}^T(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta}) = \frac{(\alpha+1)^4}{(2\alpha+1)^2}\begin{pmatrix} \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 & 0 \\ 0 & \frac{(\alpha+1)^2}{2\alpha+1}\beta_\alpha(\boldsymbol{\theta}) \end{pmatrix},$$

with $\beta_\alpha(\boldsymbol{\theta})$ given in (F13). Therefore, we get (49).

Case 5 (Fixing a value for covariance of two normal populations).
If we consider the function

$$m(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \sigma_1\sigma_2\rho - \sigma_{12,0},$$

the null hypothesis can be written as $m(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = 0$ and

$$\begin{aligned}
&\boldsymbol{M}^T(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta}) \\
&= \sigma_1^2\sigma_2^2\begin{pmatrix} \rho & \rho & 1 \end{pmatrix}\boldsymbol{D}_{1,\sigma_1,\sigma_2}\boldsymbol{V}_{2,\alpha}(\rho)\boldsymbol{D}_{1,\sigma_1,\sigma_2}\begin{pmatrix} \rho \\ \rho \\ 1 \end{pmatrix} \\
&= \sigma_1^2\sigma_2^2\left[(\alpha+1)^2(\widehat{\rho}_{R,\alpha}^2 + 1) + \frac{\alpha^2}{2}\widehat{\rho}_{R,\alpha}^2\right],
\end{aligned}$$

where $\boldsymbol{V}_{2,\alpha}(\rho)$ is given by (32).Therefore, we get (51).

Case 6 (Fixing values for means of two dependent populations with normal distribution).
If we consider the function

$$\boldsymbol{m}^T(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = (\mu_1 - \mu_{1,0}, \mu_2 - \mu_{2,0}),$$

the null hypothesis can be written by $m(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = 0$. It is clear that

$$\boldsymbol{M}^T(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we get (53) since

$$\begin{aligned}
&\left(\boldsymbol{M}^T(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})\right)^{-1} = \boldsymbol{V}_{1,\alpha}^{-1}(\boldsymbol{\theta}) \\
&= \left(\frac{(\alpha+1)^4}{(2\alpha+1)^2}\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)^{-1} \\
&= \frac{(2\alpha+1)^2}{(1-\rho^2)(\alpha+1)^4}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{m}^T(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)\left(\boldsymbol{M}^T(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})\right)^{-1}\boldsymbol{m}(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \\
&= \frac{(2\alpha+1)^2}{(\alpha+1)^4}\left[\frac{(\widehat{\mu}_{1,R,\alpha} - \mu_{1,0})^2\widehat{\sigma}_{2,R,\alpha}^2}{\widehat{\sigma}_{1,R,\alpha}^2\widehat{\sigma}_{2,R,\alpha}^2(1-\widehat{\rho}_{R,\alpha}^2)}\right. \\
&\quad \left. + \frac{-2\widehat{\rho}_{R,\alpha}(\widehat{\mu}_{1,R,\alpha} - \mu_{1,0})(\widehat{\mu}_{2,R,\alpha} - \mu_{2,0})\widehat{\sigma}_{1,R,\alpha}\widehat{\sigma}_{2,R,\alpha} + (\widehat{\mu}_{2,R,\alpha} - \mu_{2,0})^2\widehat{\sigma}_{1,R,\alpha}^2}{\widehat{\sigma}_{1,R,\alpha}^2\widehat{\sigma}_{2,R,\alpha}^2(1-\widehat{\rho}_{R,\alpha}^2)}\right] \\
&= \frac{(2\alpha+1)^2}{(\alpha+1)^4}\left[\frac{\left(\frac{\widehat{\mu}_{1,R,\alpha} - \mu_{1,0}}{\widehat{\sigma}_{1,R,\alpha}}\right)^2 - 2\widehat{\rho}_{R,\alpha}\left(\frac{\widehat{\mu}_{1,R,\alpha} - \mu_{1,0}}{\widehat{\sigma}_{1,R,\alpha}}\right)\left(\frac{\widehat{\mu}_{2,R,\alpha} - \mu_{2,0}}{\widehat{\sigma}_{2,R,\alpha}}\right)}{1-\widehat{\rho}_{R,\alpha}^2}\right. \\
&\quad \left. + \frac{\left(\frac{\widehat{\mu}_{2,R,\alpha} - \mu_{2,0}}{\widehat{\sigma}_{2,R,\alpha}}\right)^2}{1-\widehat{\rho}_{R,\alpha}^2}\right] \\
&= \frac{(2\alpha+1)^2}{(\alpha+1)^4}\left[\frac{\left(\frac{\widehat{\mu}_{1,R,\alpha} - \mu_{1,0}}{\widehat{\sigma}_{1,R,\alpha}} - \frac{\widehat{\mu}_{2,R,\alpha} - \mu_{2,0}}{\widehat{\sigma}_{2,R,\alpha}}\right)^2}{1-\widehat{\rho}_{R,\alpha}^2}\right. \\
&\quad \left. + \frac{+2(1-\widehat{\rho}_{R,\alpha})\left(\frac{\widehat{\mu}_{1,R,\alpha} - \mu_{1,0}}{\widehat{\sigma}_{1,R,\alpha}}\right)\left(\frac{\widehat{\mu}_{2,R,\alpha} - \mu_{2,0}}{\widehat{\sigma}_{2,R,\alpha}}\right)}{1-\widehat{\rho}_{R,\alpha}^2}\right]
\end{aligned}$$

Case 7 (Fixing values for variances and covariance of two dependent populations with normal distribution).
If we consider the function

$$\boldsymbol{m}^T(\boldsymbol{\theta}) = (\sigma_1 - \sigma_{1,0}, \sigma_2 - \sigma_{2,0}, \sigma_1\sigma_2\rho - \sigma_{12,0}),$$

the null hypothesis can be written by $\boldsymbol{m}(\boldsymbol{\theta}) = \boldsymbol{0}_3$. Therefore,

$$\boldsymbol{M}^T(\boldsymbol{\theta}) = (\boldsymbol{0}_{3\times 2}, \boldsymbol{M}_{22}^T(\boldsymbol{\theta})), \quad \boldsymbol{M}_{22}^T(\boldsymbol{\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_2\rho & \sigma_1\rho & \sigma_1\sigma_2 \end{pmatrix}$$

and we denote by

$$\boldsymbol{M}^T(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta}) = \boldsymbol{M}_{22}^T(\boldsymbol{\theta})\boldsymbol{V}_{2,\alpha}(\boldsymbol{\theta})\boldsymbol{M}_{22}(\boldsymbol{\theta}),$$

and

$$\begin{aligned}
&\left(\boldsymbol{M}^T(\boldsymbol{\theta})\boldsymbol{V}_\alpha(\boldsymbol{\theta})\boldsymbol{M}(\boldsymbol{\theta})\right)^{-1} = \left(\boldsymbol{M}_{22}^T(\boldsymbol{\theta})\boldsymbol{V}_{2,\alpha}(\boldsymbol{\theta})\boldsymbol{M}_{22}(\boldsymbol{\theta})\right)^{-1} \\
&= \boldsymbol{M}_{22}^{-1}(\boldsymbol{\theta})
\end{aligned}$$

Table 9 Simulated mean square error of the MRPDE for ratio of variances, $\widehat{\gamma}_{R,\alpha}$, when $n = 15$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.225	0.232	0.239	0.249	0.229	0.234	0.245	0.305	0.372	0.499
	0.1	0.227	0.231	0.236	0.243	0.229	0.232	0.239	0.268	0.316	0.438
	0.2	0.237	0.239	0.244	0.251	0.239	0.241	0.245	0.250	0.276	0.355
	0.3	0.258	0.259	0.266	0.270	0.259	0.262	0.264	0.264	0.280	0.328
	0.5	0.347	0.360	0.361	0.368	0.353	0.363	0.353	0.364	0.366	0.396
	0.7	0.596	0.580	0.761	0.597	0.612	0.558	0.597	0.575	0.620	0.709
0.3	0	0.214	0.226	0.230	0.237	0.220	0.225	0.234	0.299	0.371	0.496
	0.1	0.215	0.224	0.227	0.232	0.219	0.223	0.229	0.258	0.311	0.434
	0.2	0.224	0.232	0.235	0.240	0.228	0.231	0.236	0.242	0.267	0.350
	0.3	0.244	0.251	0.254	0.261	0.247	0.251	0.256	0.255	0.269	0.322
	0.5	0.322	0.335	0.338	0.351	0.330	0.338	0.342	0.337	0.387	0.387
	0.7	0.510	0.555	0.557	0.615	0.550	0.573	0.550	0.556	0.627	0.641
0.6	0	0.182	0.188	0.193	0.200	0.185	0.189	0.195	0.275	0.362	0.494
	0.1	0.184	0.187	0.190	0.195	0.185	0.187	0.191	0.223	0.285	0.415
	0.2	0.192	0.193	0.197	0.201	0.192	0.193	0.196	0.205	0.233	0.317
	0.3	0.209	0.210	0.212	0.217	0.208	0.208	0.213	0.215	0.231	0.281
	0.5	0.280	0.285	0.291	0.294	0.282	0.279	0.290	0.299	0.298	0.332
	0.7	0.468	0.460	0.485	0.496	0.460	0.468	0.455	0.468	0.512	0.513
0.9	0	0.098	0.103	0.105	0.109	0.101	0.103	0.106	0.230	0.339	0.492
	0.1	0.098	0.102	0.104	0.107	0.101	0.102	0.104	0.131	0.188	0.336
	0.2	0.102	0.106	0.107	0.110	0.105	0.106	0.107	0.112	0.131	0.205
	0.3	0.112	0.115	0.116	0.119	0.114	0.114	0.115	0.118	0.128	0.166
	0.5	0.149	0.153	0.155	0.161	0.153	0.151	0.157	0.159	0.169	0.195
	0.7	0.247	0.263	0.248	0.247	0.251	0.249	0.299	0.271	0.280	0.311

Table 10 Simulated significance level for testing equal variances through $W_{n,\alpha}(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha})$ given by (63) and the Morgan-Pitman test, when $n = 15$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.064	0.071	0.085	0.091	0.071	0.074	0.089	0.270	0.426	0.687
	0.1	0.062	0.065	0.075	0.082	0.067	0.066	0.077	0.169	0.290	0.563
	0.2	0.067	0.068	0.075	0.085	0.069	0.067	0.077	0.103	0.167	0.354
	0.3	0.080	0.080	0.087	0.097	0.083	0.081	0.088	0.098	0.131	0.249
	0.5	0.141	0.147	0.151	0.162	0.145	0.145	0.149	0.158	0.172	0.235
	0.7	0.253	0.263	0.267	0.280	0.255	0.257	0.269	0.273	0.288	0.338
	MP	0.049	0.057	0.066	0.073	0.058	0.058	0.071	0.250	0.405	0.666
0.3	0	0.064	0.080	0.084	0.097	0.074	0.079	0.090	0.270	0.441	0.692
	0.1	0.062	0.074	0.076	0.086	0.068	0.071	0.080	0.167	0.299	0.567
	0.2	0.064	0.075	0.077	0.085	0.069	0.074	0.078	0.108	0.169	0.363
	0.3	0.078	0.084	0.090	0.099	0.082	0.086	0.091	0.100	0.136	0.255
	0.5	0.138	0.147	0.150	0.164	0.142	0.145	0.151	0.157	0.174	0.236
	0.7	0.250	0.255	0.268	0.277	0.257	0.260	0.267	0.265	0.290	0.326
	MP	0.049	0.063	0.065	0.074	0.057	0.060	0.071	0.249	0.417	0.672

$$\boldsymbol{D}^{-1}_{2,\widehat{\sigma}_{1,R,\alpha},\widehat{\sigma}_{2,R,\alpha}} \boldsymbol{V}^{-1}_{2,\alpha}(\rho)\, \boldsymbol{D}^{-1}_{2,\widehat{\sigma}_{1,R,\alpha},\widehat{\sigma}_{2,R,\alpha}} \left(\boldsymbol{M}^{T}_{22}(\boldsymbol{\theta})\right)^{-1}, \qquad \text{and}$$

$$\boldsymbol{m}^{T}(\widehat{\boldsymbol{\theta}}_{R,\alpha})$$

Table 10 continued

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0.6	0	0.070	0.081	0.083	0.100	0.075	0.080	0.092	0.286	0.474	0.735
	0.1	0.068	0.074	0.076	0.089	0.071	0.074	0.081	0.163	0.303	0.582
	0.2	0.073	0.077	0.081	0.088	0.073	0.076	0.079	0.106	0.166	0.353
	0.3	0.085	0.088	0.092	0.099	0.087	0.087	0.092	0.102	0.134	0.248
	0.5	0.146	0.150	0.158	0.163	0.148	0.149	0.154	0.162	0.179	0.237
	0.7	0.260	0.262	0.271	0.280	0.255	0.262	0.265	0.273	0.293	0.335
	MP	0.050	0.061	0.065	0.077	0.056	0.061	0.071	0.262	0.449	0.711
0.9	0	0.068	0.084	0.088	0.107	0.082	0.086	0.093	0.342	0.547	0.805
	0.1	0.067	0.080	0.080	0.097	0.078	0.080	0.081	0.136	0.248	0.528
	0.2	0.070	0.082	0.083	0.099	0.079	0.082	0.083	0.092	0.124	0.276
	0.3	0.083	0.095	0.093	0.111	0.092	0.093	0.094	0.101	0.113	0.193
	0.5	0.143	0.155	0.158	0.172	0.156	0.154	0.155	0.162	0.181	0.225
	0.7	0.257	0.268	0.273	0.285	0.270	0.273	0.268	0.279	0.300	0.345
	MP	0.047	0.061	0.063	0.081	0.060	0.061	0.068	0.312	0.519	0.786

Table 11 Simulated mean square error of the MRPDE for correlation coefficient, $\widehat{\rho}_{R,\alpha}$, when $n = 15$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.217	0.224	0.230	0.239	0.221	0.224	0.233	0.338	0.431	0.596
	0.1	0.219	0.223	0.227	0.234	0.221	0.223	0.229	0.284	0.353	0.516
	0.2	0.227	0.230	0.233	0.240	0.229	0.230	0.234	0.253	0.290	0.403
	0.3	0.244	0.247	0.251	0.257	0.246	0.247	0.250	0.258	0.282	0.352
	0.5	0.306	0.313	0.314	0.325	0.309	0.312	0.314	0.320	0.332	0.374
	0.7	0.409	0.420	0.421	0.434	0.412	0.417	0.423	0.426	0.438	0.476
0.3	0	0.217	0.226	0.231	0.238	0.221	0.226	0.234	0.338	0.439	0.601
	0.1	0.218	0.225	0.228	0.234	0.220	0.224	0.229	0.282	0.357	0.518
	0.2	0.226	0.232	0.235	0.240	0.228	0.231	0.235	0.253	0.292	0.404
	0.3	0.243	0.248	0.252	0.258	0.244	0.249	0.252	0.259	0.282	0.354
	0.5	0.305	0.311	0.316	0.325	0.307	0.309	0.315	0.319	0.334	0.373
	0.7	0.410	0.413	0.424	0.432	0.412	0.417	0.422	0.421	0.441	0.468
0.6	0	0.218	0.226	0.231	0.240	0.222	0.226	0.232	0.347	0.462	0.628
	0.1	0.220	0.225	0.228	0.235	0.222	0.224	0.228	0.279	0.360	0.527
	0.2	0.228	0.232	0.235	0.240	0.230	0.231	0.234	0.252	0.289	0.400
	0.3	0.246	0.250	0.251	0.256	0.247	0.246	0.251	0.259	0.281	0.348
	0.5	0.311	0.314	0.319	0.324	0.311	0.311	0.317	0.322	0.335	0.376
	0.7	0.415	0.423	0.432	0.439	0.416	0.417	0.424	0.431	0.448	0.478
0.9	0	0.216	0.226	0.230	0.239	0.223	0.228	0.232	0.377	0.507	0.688
	0.1	0.217	0.225	0.227	0.235	0.223	0.226	0.227	0.258	0.325	0.503
	0.2	0.226	0.233	0.234	0.243	0.231	0.233	0.233	0.239	0.261	0.354
	0.3	0.244	0.251	0.253	0.264	0.250	0.251	0.252	0.255	0.267	0.319
	0.5	0.323	0.333	0.338	0.353	0.333	0.333	0.333	0.340	0.355	0.397
	0.7	0.459	0.472	0.476	0.497	0.468	0.473	0.471	0.480	0.497	0.539

$$= (\widehat{\sigma}_{1,R,\alpha} - \sigma_{1,0}, \widehat{\sigma}_{2,R,\alpha} - \sigma_{2,0}, \widehat{\sigma}_{1,R,\alpha}\widehat{\sigma}_{2,R,\alpha}\widehat{\rho}_{R,\alpha} - \sigma_{12,0}).$$

Therefore, we get (55).

Appendix G Complementary tables for simulation (Sect. 6)

See Tables 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.

Table 12 Simulated significance level for testing null correlation coefficient through $W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha})$ given by (65) and the Morgan-Pitman test, when $n = 15$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.054	0.061	0.072	0.079	0.063	0.063	0.076	0.061	0.072	0.079
	0.1	0.052	0.055	0.063	0.070	0.057	0.056	0.066	0.055	0.063	0.070
	0.2	0.056	0.055	0.064	0.072	0.058	0.056	0.064	0.055	0.064	0.072
	0.3	0.068	0.065	0.075	0.083	0.070	0.067	0.074	0.065	0.075	0.083
	0.5	0.122	0.128	0.133	0.143	0.127	0.128	0.130	0.128	0.133	0.143
	0.7	0.229	0.238	0.240	0.254	0.228	0.231	0.241	0.238	0.240	0.254
	MP	0.049	0.057	0.066	0.073	0.058	0.058	0.071	0.250	0.405	0.666
0.3	0	0.053	0.067	0.070	0.081	0.061	0.065	0.077	0.067	0.070	0.081
	0.1	0.051	0.062	0.063	0.070	0.057	0.060	0.066	0.062	0.063	0.070
	0.2	0.052	0.062	0.064	0.071	0.058	0.060	0.065	0.062	0.064	0.071
	0.3	0.065	0.071	0.074	0.085	0.067	0.073	0.076	0.071	0.074	0.085
	0.5	0.119	0.127	0.131	0.144	0.124	0.126	0.132	0.127	0.131	0.144
	0.7	0.224	0.230	0.243	0.253	0.231	0.233	0.241	0.230	0.243	0.253
	MP	0.049	0.063	0.065	0.074	0.057	0.060	0.071	0.249	0.417	0.672
0.6	0	0.056	0.065	0.069	0.083	0.062	0.066	0.076	0.065	0.069	0.083
	0.1	0.054	0.060	0.062	0.072	0.058	0.059	0.066	0.060	0.062	0.072
	0.2	0.058	0.060	0.065	0.072	0.060	0.061	0.064	0.060	0.065	0.072
	0.3	0.069	0.071	0.075	0.082	0.072	0.072	0.076	0.071	0.075	0.082
	0.5	0.127	0.129	0.138	0.142	0.129	0.130	0.133	0.129	0.138	0.142
	0.7	0.233	0.241	0.249	0.258	0.231	0.236	0.242	0.241	0.249	0.258
	MP	0.050	0.061	0.065	0.077	0.056	0.061	0.071	0.262	0.449	0.711
0.9	0	0.051	0.067	0.069	0.087	0.065	0.067	0.072	0.067	0.069	0.087
	0.1	0.051	0.062	0.063	0.076	0.061	0.062	0.063	0.062	0.063	0.076
	0.2	0.053	0.063	0.064	0.078	0.063	0.062	0.063	0.063	0.064	0.078
	0.3	0.067	0.077	0.076	0.094	0.076	0.075	0.076	0.077	0.076	0.094
	0.5	0.147	0.160	0.164	0.185	0.159	0.155	0.158	0.160	0.164	0.185
	0.7	0.296	0.313	0.317	0.347	0.304	0.312	0.309	0.313	0.317	0.347
	MP	0.047	0.061	0.063	0.081	0.060	0.061	0.068	0.312	0.519	0.786

Table 13 Simulated mean square error of the MRPDE for ratio of variances, $\widehat{\gamma}_{R,\alpha}$, when $n = 50$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.115	0.123	0.128	0.131	0.121	0.123	0.131	0.278	0.403	0.552
	0.1	0.117	0.120	0.123	0.126	0.119	0.118	0.121	0.160	0.251	0.440
	0.2	0.121	0.123	0.125	0.128	0.122	0.122	0.124	0.134	0.168	0.296
	0.3	0.127	0.129	0.130	0.133	0.127	0.127	0.130	0.133	0.148	0.217
	0.5	0.144	0.145	0.147	0.151	0.144	0.144	0.148	0.147	0.153	0.179
	0.7	0.168	0.171	0.173	0.178	0.171	0.168	0.175	0.172	0.176	0.193
0.3	0	0.110	0.117	0.121	0.124	0.114	0.119	0.124	0.274	0.402	0.552
	0.1	0.111	0.114	0.116	0.119	0.112	0.114	0.116	0.155	0.242	0.438
	0.2	0.115	0.116	0.118	0.121	0.116	0.117	0.118	0.127	0.158	0.290
	0.3	0.121	0.122	0.123	0.127	0.121	0.122	0.124	0.126	0.139	0.210
	0.5	0.137	0.137	0.139	0.143	0.138	0.137	0.140	0.139	0.143	0.171
	0.7	0.159	0.161	0.163	0.168	0.161	0.161	0.164	0.162	0.166	0.184

Table 13 continued

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0.6	0	0.093	0.097	0.102	0.106	0.097	0.100	0.105	0.270	0.402	0.553
	0.1	0.094	0.095	0.098	0.101	0.095	0.096	0.097	0.134	0.224	0.430
	0.2	0.097	0.098	0.100	0.102	0.098	0.098	0.099	0.108	0.138	0.262
	0.3	0.101	0.102	0.104	0.106	0.102	0.103	0.103	0.107	0.121	0.180
	0.5	0.114	0.115	0.118	0.120	0.116	0.116	0.116	0.118	0.125	0.146
	0.7	0.134	0.135	0.139	0.141	0.136	0.136	0.137	0.137	0.144	0.157
0.9	0	0.050	0.053	0.056	0.057	0.052	0.054	0.057	0.263	0.401	0.555
	0.1	0.051	0.052	0.054	0.054	0.051	0.052	0.053	0.068	0.128	0.365
	0.2	0.053	0.053	0.054	0.055	0.053	0.053	0.054	0.056	0.066	0.138
	0.3	0.055	0.056	0.057	0.057	0.055	0.055	0.056	0.058	0.062	0.086
	0.5	0.063	0.063	0.064	0.065	0.063	0.062	0.063	0.064	0.067	0.077
	0.7	0.073	0.074	0.075	0.076	0.074	0.074	0.074	0.075	0.079	0.088

Table 14 Simulated significance level for testing equal variances through $W_{n,\alpha}(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha})$ given by (63) and the Morgan-Pitman test, when $n = 50$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.054	0.072	0.081	0.089	0.063	0.071	0.091	0.535	0.809	0.980
	0.1	0.054	0.061	0.066	0.068	0.057	0.056	0.063	0.206	0.485	0.885
	0.2	0.055	0.058	0.061	0.064	0.057	0.053	0.061	0.100	0.215	0.593
	0.3	0.055	0.059	0.061	0.065	0.058	0.055	0.062	0.079	0.131	0.348
	0.5	0.062	0.064	0.071	0.076	0.065	0.063	0.071	0.072	0.097	0.182
	0.7	0.079	0.082	0.090	0.096	0.083	0.081	0.091	0.088	0.107	0.151
	MP	0.050	0.067	0.077	0.084	0.061	0.068	0.086	0.529	0.805	0.979
0.3	0	0.057	0.067	0.077	0.087	0.063	0.074	0.089	0.542	0.817	0.981
	0.1	0.057	0.058	0.065	0.071	0.055	0.059	0.063	0.206	0.485	0.890
	0.2	0.056	0.056	0.061	0.066	0.054	0.056	0.061	0.099	0.205	0.592
	0.3	0.058	0.060	0.060	0.066	0.057	0.057	0.063	0.080	0.119	0.345
	0.5	0.065	0.065	0.067	0.073	0.067	0.066	0.070	0.076	0.088	0.175
	0.7	0.082	0.081	0.086	0.092	0.082	0.082	0.090	0.090	0.099	0.146
	MP	0.052	0.063	0.072	0.081	0.059	0.070	0.084	0.536	0.814	0.980
0.6	0	0.058	0.067	0.079	0.092	0.066	0.079	0.090	0.580	0.847	0.986
	0.1	0.056	0.057	0.068	0.072	0.058	0.059	0.061	0.197	0.487	0.891
	0.2	0.056	0.056	0.065	0.066	0.058	0.057	0.060	0.093	0.199	0.565
	0.3	0.056	0.059	0.064	0.069	0.058	0.058	0.060	0.074	0.119	0.315
	0.5	0.065	0.066	0.072	0.073	0.066	0.064	0.067	0.075	0.090	0.161
	0.7	0.080	0.083	0.090	0.094	0.084	0.080	0.084	0.090	0.104	0.143
	MP	0.053	0.062	0.074	0.085	0.060	0.074	0.084	0.574	0.843	0.985
0.9	0	0.054	0.068	0.085	0.088	0.064	0.075	0.090	0.679	0.907	0.996
	0.1	0.054	0.058	0.069	0.071	0.056	0.060	0.064	0.113	0.322	0.806
	0.2	0.054	0.055	0.066	0.066	0.057	0.056	0.061	0.066	0.103	0.339
	0.3	0.056	0.055	0.067	0.066	0.060	0.058	0.062	0.061	0.079	0.164
	0.5	0.067	0.067	0.074	0.074	0.068	0.064	0.070	0.070	0.083	0.112
	0.7	0.084	0.082	0.090	0.095	0.084	0.083	0.085	0.089	0.100	0.125
	MP	0.048	0.062	0.079	0.081	0.057	0.070	0.084	0.673	0.904	0.996

Table 15 Simulated mean square error of the MRPDE for correlation coefficient, $\widehat{\rho}_{R,\alpha}$, when $n = 50$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.114	0.121	0.126	0.130	0.119	0.122	0.129	0.322	0.479	0.665
	0.1	0.115	0.119	0.121	0.124	0.117	0.117	0.120	0.174	0.288	0.525
	0.2	0.119	0.121	0.123	0.126	0.120	0.121	0.123	0.139	0.184	0.345
	0.3	0.125	0.127	0.129	0.132	0.126	0.126	0.129	0.136	0.157	0.245
	0.5	0.141	0.142	0.145	0.148	0.141	0.142	0.146	0.147	0.157	0.193
	0.7	0.164	0.166	0.168	0.173	0.166	0.164	0.170	0.169	0.177	0.201
0.3	0	0.115	0.121	0.125	0.128	0.118	0.123	0.129	0.326	0.487	0.673
	0.1	0.116	0.118	0.120	0.123	0.117	0.118	0.120	0.175	0.285	0.532
	0.2	0.120	0.121	0.122	0.125	0.120	0.121	0.122	0.138	0.179	0.346
	0.3	0.125	0.126	0.127	0.131	0.126	0.126	0.128	0.135	0.153	0.244
	0.5	0.141	0.142	0.143	0.147	0.142	0.142	0.145	0.146	0.154	0.192
	0.7	0.164	0.164	0.166	0.171	0.165	0.165	0.168	0.167	0.174	0.200
0.6	0	0.114	0.120	0.126	0.130	0.119	0.124	0.129	0.348	0.518	0.704
	0.1	0.116	0.118	0.121	0.124	0.117	0.119	0.120	0.171	0.289	0.552
	0.2	0.119	0.121	0.123	0.125	0.121	0.122	0.122	0.136	0.177	0.338
	0.3	0.125	0.126	0.128	0.130	0.126	0.127	0.127	0.134	0.154	0.232
	0.5	0.141	0.142	0.145	0.146	0.142	0.142	0.143	0.146	0.156	0.186
	0.7	0.163	0.165	0.169	0.171	0.165	0.165	0.166	0.169	0.178	0.196
0.9	0	0.114	0.121	0.127	0.129	0.118	0.123	0.129	0.411	0.588	0.767
	0.1	0.115	0.118	0.122	0.123	0.116	0.118	0.120	0.141	0.228	0.539
	0.2	0.119	0.121	0.123	0.125	0.120	0.121	0.122	0.124	0.139	0.242
	0.3	0.125	0.126	0.128	0.130	0.125	0.126	0.127	0.128	0.135	0.171
	0.5	0.141	0.141	0.144	0.146	0.141	0.141	0.142	0.143	0.149	0.164
	0.7	0.164	0.164	0.169	0.171	0.165	0.165	0.166	0.167	0.174	0.188

Table 16 Simulated significance level for testing null correlation coefficient through $W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha})$ given by (65) and the Morgan-Pitman test, when $n = 50$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.051	0.068	0.078	0.086	0.062	0.069	0.088	0.532	0.807	0.979
	0.1	0.052	0.058	0.063	0.065	0.055	0.053	0.061	0.199	0.480	0.883
	0.2	0.052	0.055	0.058	0.061	0.054	0.051	0.058	0.096	0.210	0.588
	0.3	0.052	0.057	0.057	0.062	0.056	0.052	0.060	0.075	0.126	0.343
	0.5	0.059	0.061	0.067	0.072	0.062	0.059	0.067	0.069	0.092	0.176
	0.7	0.074	0.078	0.085	0.092	0.079	0.075	0.085	0.084	0.101	0.145
	MP	0.050	0.067	0.077	0.084	0.061	0.068	0.086	0.529	0.805	0.979
0.3	0	0.053	0.064	0.074	0.083	0.060	0.071	0.086	0.538	0.815	0.980
	0.1	0.054	0.055	0.062	0.069	0.051	0.056	0.060	0.201	0.480	0.888
	0.2	0.054	0.053	0.058	0.063	0.051	0.054	0.058	0.095	0.200	0.586
	0.3	0.055	0.056	0.057	0.064	0.054	0.054	0.058	0.076	0.114	0.339
	0.5	0.062	0.061	0.063	0.070	0.063	0.061	0.067	0.072	0.084	0.168
	0.7	0.077	0.077	0.080	0.088	0.078	0.077	0.084	0.085	0.094	0.139
	MP	0.052	0.063	0.072	0.081	0.059	0.070	0.084	0.536	0.814	0.980

Table 16 continued

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0.6	0	0.054	0.063	0.076	0.086	0.061	0.076	0.086	0.575	0.845	0.985
	0.1	0.052	0.054	0.064	0.067	0.055	0.056	0.058	0.191	0.481	0.889
	0.2	0.052	0.053	0.060	0.062	0.055	0.053	0.056	0.088	0.192	0.558
	0.3	0.053	0.055	0.061	0.064	0.055	0.054	0.055	0.070	0.114	0.306
	0.5	0.060	0.061	0.067	0.069	0.060	0.060	0.062	0.069	0.085	0.155
	0.7	0.074	0.077	0.083	0.087	0.078	0.074	0.078	0.084	0.098	0.134
	MP	0.053	0.062	0.074	0.085	0.060	0.074	0.084	0.574	0.843	0.985
0.9	0	0.050	0.064	0.080	0.082	0.059	0.071	0.085	0.674	0.905	0.996
	0.1	0.050	0.053	0.064	0.065	0.051	0.055	0.059	0.108	0.315	0.804
	0.2	0.051	0.051	0.061	0.061	0.053	0.052	0.056	0.060	0.096	0.331
	0.3	0.051	0.051	0.062	0.061	0.056	0.052	0.058	0.056	0.073	0.155
	0.5	0.061	0.060	0.067	0.067	0.062	0.059	0.064	0.063	0.076	0.104
	0.7	0.077	0.075	0.083	0.088	0.079	0.076	0.079	0.082	0.093	0.118
	MP	0.048	0.062	0.079	0.081	0.057	0.070	0.084	0.673	0.904	0.996

Table 17 Simulated mean square error of the MRPDE for ratio of variances, $\widehat{\gamma}_{R,\alpha}$, when $n = 100$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.081	0.086	0.089	0.093	0.085	0.089	0.094	0.288	0.429	0.569
	0.1	0.082	0.084	0.085	0.088	0.083	0.084	0.085	0.133	0.241	0.451
	0.2	0.085	0.086	0.087	0.089	0.086	0.086	0.086	0.100	0.141	0.292
	0.3	0.089	0.090	0.090	0.092	0.090	0.089	0.089	0.095	0.114	0.197
	0.5	0.099	0.100	0.100	0.103	0.100	0.099	0.099	0.102	0.108	0.140
	0.7	0.111	0.113	0.113	0.116	0.112	0.111	0.112	0.114	0.117	0.135
0.3	0	0.077	0.082	0.086	0.088	0.080	0.084	0.090	0.288	0.427	0.569
	0.1	0.078	0.079	0.082	0.084	0.078	0.080	0.081	0.128	0.234	0.449
	0.2	0.081	0.081	0.083	0.084	0.080	0.082	0.082	0.094	0.134	0.284
	0.3	0.084	0.085	0.086	0.087	0.084	0.085	0.085	0.090	0.109	0.190
	0.5	0.093	0.094	0.096	0.097	0.093	0.094	0.095	0.096	0.104	0.135
	0.7	0.105	0.107	0.108	0.110	0.105	0.107	0.106	0.108	0.113	0.130
0.6	0	0.064	0.070	0.072	0.074	0.068	0.071	0.076	0.289	0.427	0.571
	0.1	0.065	0.068	0.069	0.070	0.066	0.067	0.068	0.109	0.210	0.443
	0.2	0.067	0.069	0.070	0.070	0.068	0.068	0.069	0.079	0.112	0.255
	0.3	0.070	0.072	0.073	0.073	0.071	0.071	0.072	0.076	0.091	0.161
	0.5	0.077	0.080	0.081	0.081	0.079	0.079	0.080	0.081	0.088	0.114
	0.7	0.087	0.090	0.091	0.092	0.089	0.089	0.090	0.091	0.095	0.111
0.9	0	0.035	0.038	0.039	0.040	0.037	0.039	0.041	0.286	0.428	0.570
	0.1	0.035	0.037	0.037	0.038	0.036	0.037	0.037	0.049	0.106	0.387
	0.2	0.036	0.037	0.038	0.039	0.037	0.037	0.038	0.040	0.049	0.112
	0.3	0.038	0.039	0.039	0.040	0.039	0.039	0.039	0.040	0.044	0.064
	0.5	0.042	0.043	0.044	0.045	0.043	0.043	0.044	0.044	0.046	0.055
	0.7	0.048	0.049	0.049	0.050	0.049	0.049	0.050	0.050	0.052	0.058

Table 18 Simulated significance level for testing equal variances through $W_{n,\alpha}(\widehat{\gamma}_{R,\alpha}, \widehat{\rho}_{R,\alpha})$ given by (63) and the Morgan-Pitman test, when $n = 100$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.053	0.068	0.080	0.087	0.067	0.080	0.096	0.748	0.963	1.000
	0.1	0.054	0.059	0.063	0.067	0.057	0.056	0.061	0.280	0.684	0.983
	0.2	0.053	0.058	0.059	0.062	0.057	0.055	0.057	0.120	0.303	0.795
	0.3	0.054	0.058	0.060	0.063	0.056	0.055	0.058	0.086	0.166	0.500
	0.5	0.057	0.059	0.063	0.066	0.057	0.058	0.059	0.069	0.097	0.227
	0.7	0.062	0.064	0.069	0.072	0.065	0.062	0.063	0.072	0.089	0.153
	MP	0.052	0.066	0.077	0.085	0.065	0.078	0.094	0.746	0.962	1.000
0.3	0	0.051	0.069	0.079	0.086	0.063	0.077	0.095	0.760	0.964	1.000
	0.1	0.051	0.057	0.063	0.069	0.050	0.057	0.057	0.280	0.678	0.988
	0.2	0.052	0.055	0.058	0.066	0.052	0.054	0.054	0.117	0.291	0.793
	0.3	0.052	0.055	0.058	0.065	0.051	0.055	0.056	0.083	0.161	0.494
	0.5	0.053	0.057	0.060	0.068	0.055	0.059	0.058	0.068	0.096	0.223
	0.7	0.059	0.065	0.065	0.073	0.061	0.063	0.065	0.070	0.087	0.148
	MP	0.049	0.067	0.077	0.084	0.060	0.074	0.093	0.759	0.964	1.000
0.6	0	0.049	0.071	0.084	0.088	0.066	0.078	0.098	0.803	0.975	1.000
	0.1	0.047	0.059	0.068	0.070	0.055	0.059	0.062	0.267	0.661	0.987
	0.2	0.049	0.058	0.065	0.063	0.054	0.056	0.056	0.102	0.262	0.773
	0.3	0.049	0.059	0.063	0.062	0.056	0.057	0.056	0.076	0.146	0.464
	0.5	0.053	0.059	0.066	0.068	0.059	0.061	0.058	0.066	0.095	0.207
	0.7	0.058	0.065	0.070	0.074	0.063	0.066	0.065	0.070	0.089	0.144
	MP	0.047	0.069	0.081	0.085	0.063	0.076	0.096	0.801	0.974	1.000
0.9	0	0.051	0.070	0.079	0.091	0.068	0.078	0.096	0.881	0.994	1.000
	0.1	0.050	0.058	0.062	0.072	0.055	0.057	0.060	0.131	0.423	0.948
	0.2	0.050	0.058	0.059	0.065	0.054	0.056	0.059	0.068	0.118	0.443
	0.3	0.050	0.059	0.058	0.065	0.054	0.056	0.059	0.063	0.084	0.201
	0.5	0.056	0.062	0.058	0.067	0.057	0.057	0.063	0.064	0.071	0.116
	0.7	0.062	0.068	0.066	0.075	0.062	0.064	0.070	0.069	0.073	0.106
	MP	0.047	0.067	0.076	0.088	0.065	0.075	0.093	0.880	0.993	1.000

Table 19 Simulated mean square error of the MRPDE for correlation coefficient, $\widehat{\rho}_{R,\alpha}$, when $n = 100$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.081	0.086	0.089	0.092	0.085	0.089	0.093	0.334	0.510	0.685
	0.1	0.082	0.084	0.085	0.087	0.083	0.083	0.084	0.144	0.274	0.538
	0.2	0.085	0.086	0.086	0.088	0.085	0.085	0.085	0.104	0.154	0.337
	0.3	0.088	0.089	0.090	0.092	0.089	0.089	0.089	0.098	0.121	0.221
	0.5	0.098	0.099	0.099	0.102	0.099	0.098	0.098	0.103	0.112	0.152
	0.7	0.110	0.111	0.112	0.115	0.111	0.110	0.111	0.114	0.120	0.144
0.3	0	0.081	0.085	0.089	0.092	0.084	0.088	0.093	0.342	0.517	0.695
	0.1	0.082	0.083	0.085	0.087	0.081	0.083	0.084	0.143	0.274	0.545
	0.2	0.084	0.084	0.086	0.088	0.084	0.085	0.085	0.103	0.152	0.337
	0.3	0.088	0.088	0.090	0.091	0.087	0.089	0.089	0.097	0.120	0.220
	0.5	0.097	0.098	0.099	0.101	0.097	0.098	0.098	0.102	0.112	0.152
	0.7	0.109	0.110	0.112	0.114	0.109	0.111	0.110	0.113	0.120	0.143

Table 19 continued

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0.6	0	0.080	0.086	0.089	0.092	0.085	0.088	0.094	0.373	0.550	0.726
	0.1	0.081	0.084	0.086	0.087	0.083	0.083	0.084	0.140	0.271	0.569
	0.2	0.083	0.085	0.087	0.088	0.085	0.085	0.086	0.101	0.144	0.330
	0.3	0.087	0.089	0.091	0.091	0.089	0.089	0.089	0.096	0.116	0.208
	0.5	0.096	0.099	0.100	0.100	0.098	0.098	0.099	0.102	0.111	0.147
	0.7	0.108	0.111	0.113	0.114	0.111	0.110	0.112	0.114	0.120	0.141
0.9	0	0.079	0.086	0.088	0.092	0.085	0.089	0.094	0.446	0.626	0.786
	0.1	0.080	0.084	0.085	0.087	0.082	0.084	0.085	0.104	0.194	0.572
	0.2	0.083	0.085	0.086	0.088	0.084	0.085	0.086	0.088	0.104	0.205
	0.3	0.086	0.089	0.089	0.092	0.088	0.089	0.090	0.090	0.097	0.132
	0.5	0.096	0.099	0.099	0.102	0.098	0.098	0.100	0.099	0.103	0.118
	0.7	0.109	0.111	0.112	0.115	0.110	0.111	0.113	0.112	0.115	0.126

Table 20 Simulated significance level for testing null correlation coefficient through $W'_{n,\alpha}(\widehat{\rho}_{UV,R,\alpha})$ given by (65) and the Morgan-Pitman test, when $n = 100$

ρ	α	Pure	Slighly			Regular			Heavily		
			0.05	0.10	0.20	0.05	0.10	0.20	0.05	0.10	0.20
0	0	0.053	0.067	0.078	0.086	0.066	0.079	0.095	0.747	0.962	1.000
	0.1	0.052	0.058	0.062	0.066	0.056	0.055	0.060	0.278	0.682	0.983
	0.2	0.052	0.057	0.058	0.061	0.056	0.053	0.056	0.118	0.301	0.794
	0.3	0.053	0.056	0.058	0.061	0.055	0.053	0.057	0.085	0.164	0.497
	0.5	0.055	0.057	0.061	0.064	0.056	0.056	0.057	0.068	0.095	0.224
	0.7	0.061	0.063	0.067	0.070	0.063	0.060	0.062	0.070	0.087	0.150
	MP	0.052	0.066	0.077	0.085	0.065	0.078	0.094	0.746	0.962	1.000
0.3	0	0.049	0.068	0.077	0.085	0.061	0.076	0.093	0.759	0.964	1.000
	0.1	0.050	0.055	0.061	0.068	0.049	0.056	0.055	0.276	0.675	0.988
	0.2	0.051	0.053	0.057	0.065	0.050	0.053	0.052	0.116	0.289	0.791
	0.3	0.050	0.054	0.056	0.063	0.050	0.054	0.054	0.081	0.159	0.490
	0.5	0.052	0.056	0.059	0.066	0.053	0.057	0.056	0.066	0.094	0.220
	0.7	0.057	0.063	0.064	0.071	0.059	0.060	0.063	0.067	0.085	0.145
	MP	0.049	0.067	0.077	0.084	0.060	0.074	0.093	0.759	0.964	1.000
0.6	0	0.048	0.070	0.082	0.086	0.064	0.077	0.096	0.801	0.974	1.000
	0.1	0.046	0.058	0.066	0.068	0.053	0.058	0.061	0.264	0.659	0.987
	0.2	0.047	0.056	0.062	0.060	0.053	0.054	0.054	0.099	0.259	0.771
	0.3	0.047	0.058	0.060	0.060	0.054	0.056	0.054	0.074	0.143	0.460
	0.5	0.051	0.057	0.063	0.065	0.057	0.059	0.056	0.063	0.093	0.204
	0.7	0.055	0.061	0.068	0.071	0.061	0.063	0.062	0.068	0.086	0.140
	MP	0.047	0.069	0.081	0.085	0.063	0.076	0.096	0.801	0.974	1.000
0.9	0	0.048	0.068	0.076	0.089	0.065	0.076	0.094	0.880	0.994	1.000
	0.1	0.048	0.056	0.060	0.069	0.053	0.055	0.058	0.128	0.418	0.947
	0.2	0.047	0.056	0.056	0.063	0.052	0.054	0.057	0.066	0.115	0.438
	0.3	0.048	0.056	0.056	0.062	0.052	0.053	0.057	0.061	0.081	0.197
	0.5	0.053	0.060	0.056	0.064	0.054	0.054	0.059	0.061	0.068	0.112
	0.7	0.059	0.065	0.062	0.071	0.059	0.061	0.067	0.066	0.069	0.101
	MP	0.047	0.067	0.076	0.088	0.065	0.075	0.093	0.880	0.993	1.000

References

Anderson, T.W.: An Introduction to Multivariate Statistical Analysis, 3rd edn. Wiley, New York (2003)

Basak, S., Basu, A., Jones, M.C.: On the optimal density power divergence tuning parameter. J. Appl. Stat. **48**(3), 536–556 (2021)

Basu, A., Shioya, H., Park, C.: Statistical Inference: The Minimum Distance Approach. Chapman & Hall, Boca Raton (2011)

Beran, R.: Minimum Hellinger distance estimates for parametric models. Ann. Stat. **5**, 445–463 (1977)

Broniatowski, M., Toma, A., Vajda, I.: Decomposable pseudodistances and applications in statistical estimation. J. Stat. Plan. Inference **142**, 2574–2585 (1961)

Castilla, E., Martin, N., Muñoz, S., Pardo, L.: Robust Wald-type tests based on minimum Rényi pseudodistance estimators for the multiple linear regression model. J. Stat. Comput. Simul. **90**, 2655–2680 (2020a)

Castilla, E., Ghosh, A., Jaenada, M., Pardo, L.: On Regularization Methods Based on Renyi's Pseudodistances for Sparse High-Dimensional Linear Regression Models. arXiv:2007.15929 (2020b)

Castilla, E., Jaenada, M., Pardo, L.: Estimation and testing on independent not identically distributed observations based on Renyi's pseudodistances. IEEE Trans. Inf. Theory **68**(7) (2022)

Chernick, M.R.: The influence function and its application to data validation. Am. J. Math. Manag. Sci. **2**(4), 263–288 (1982)

Devlin, S.J., Gnanadesikan, R., Kettering, J.R.: Robust estimation and outlier detection with correlation coefficients. Biometrika **62**, 531–545 (1975)

Fisher, R.A.: Frequency-distribution of the values of the correlation coefficient in samples from an indefinitely large population. Biometrika **10**, 507–521 (1915)

Fisher, R.A.: On the "probable error" of a coefficient of correlation deduced from a small sample. Metron **1**, 1–32 (1921)

Fujisawa, H., Eguchi, S.: Robust parameter estimation with a small bias against contamination. J. Multivar. Anal. **99**, 2053–2081 (2008)

Hampel, F.R., Ronchetti, E.M., Rousseeuw, P.J., Stahel, W.A.: Robust Statistics: The Approach Based on Influence Functions. Wiley, New York (1986)

Hsu, C.T.: On samples from a normal bivariate population. Ann. Math. Stat. **11**, 410–426 (1940)

Hutson, A.D.: A robust Pearson correlation test for a general point null using a surrogate bootstrap distribution. PLoS ONE **14**(5), e0216287 (2019)

Isogai, T.: On using influence functions for testing multivariate normality. Ann. Inst. Stat. Math. **41**(1), 169–186 (1989)

Jaenada M., Pardo L.: The minimum Rényi's pseudodistances estimators for generalized linear models. In: Data Analysis and Related Applications: Theory and Practice, Proceeding of the ASMDA. Wiley, Athens (2021)

Jaenada, M., Pardo, L.: Robust statistical inference in generalized linear models based on minimum Rényi's pseudodistance estimators. Entropy **24**, 123 (2022)

Jones, M.C., Hjort, N.L., Harris, I.R., Basu, A.: A comparison of related density-based minimum divergence estimators. Biometrika **88**, 865–873 (2001)

Lindsay, B.G.: Efficiency versus robustness: the case for minimum Hellinger distance and related methods. Ann. Stat. **22**, 1081–1114 (1994)

Little, R.J., Rubin, D.B.: Statistical Analysis with Missing Data, 3rd edn. Wiley, New York (2019)

Martín, N.: Rao's Score Tests on Correlation Matrices. arXiv:2012.14238 (2020)

Morgan, W.A.: TA test for the significance of the difference between two variances in a sample from a normal bivariate population. Biometrika **31**, 13–19 (1939)

Pardo, L.: Statistical Inference Based on Divergence Measures. Chapman & Hall, Boca Raton (2006)

Pitman, E.: A note on normal correlation. Biometrika **31**, 9–12 (1939)

R Core Team: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna (2020)

Radhakrishnan, R., Kshirsagar, A.M.: Influence functions for certain parameters in multivariate analysis. Commun. Stati. A Theory Methods **10**, 515–529 (1981)

Rao, C.R.: Tests of significance in multivariate analysis. Biometrika **35**, 58–79 (1948)

Rényi, A.: On measures of entropy and information. In: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics, pp. 547–561. University of California Press (1961)

Serfling, R.J.: Approximation Theorems of Mathematical Statistics. Wiley, New York (1980)

Simpson, D.G.: Minimum Hellinger distance estimation for the analysis of count data. J. Am. Stat. Assoc. **82**(399), 802–807 (1989a)

Simpson, D.G.: Hellinger deviance tests: efficiency, breakdown points, and examples. J. Am. Stat. Assoc. **84**, 107–113 (1989b)

Tamura, R.N., Boos, D.D.: Minimum Hellinger distance estimation for multivariate location and covariance. J. Am. Stat. Assoc. **81**, 223–229 (1986)

Toma, A., Leoni-Aubin, S.: Optimal robust M-estimators using Rényi pseudodistances. J. Multivar. Anal. **115**, 259–273 (2013)

Toma, A., Leoni-Aubin, S.: Robust portfolio optimization using pseudodistances. PLoS ONE **10**(10), e0140546 (2015)

Warwick, J., Jones, M.C.: Choosing a robustness tuning parameter. J. Stat. Comput. Simul. **75**(7), 581–588 (2005)

Wilcox, R.: Comparing the variances of two dependent variables. J. Stat. Distrib. Appl. **2**, 7 (2015)

Wilcox, R.: Understanding and Applying Basic Statistical Methods Using R. Wiley, New York (2016)