The Creation of Twist by Reconnection of Flux Tubes

E.R. Priest¹ · D.W. Longcope²

Received: 5 December 2019 / Accepted: 1 March 2020 / Published online: 25 March 2020
© The Author(s) 2020

Abstract A fundamental process in a plasma is the magnetic reconnection of one pair of flux tubes (such as solar coronal loops) to produce a new pair. During this process magnetic helicity is conserved, but mutual helicity can be transformed to self-helicity, so that the new tubes acquire twist. However, until recently, when Wright (*Astrophys. J.* **878**, 102, 2019) supplied a solution, the partition of self-helicity between the two tubes was an outstanding puzzle. Here we examine Wright's result in detail and apply it to a variety of cases. The simplest case, which Wright himself used to illustrate the result, is that of thin ribbons or flux sheets. We first explicitly apply his method to the usually expected standard case (when the tubes approach one another without twisting before reconnection) and confirm his result is valid for flux sheaths and tubes as well as sheets.

For the reconnection of sheets, it is shown that the orientation of the sheets needs to be chosen carefully. For flux sheaths and tubes, Wright's results are demonstrated to hold for the standard case. There is both a local and a global aspect to the effect of reconnection. The local effect of reconnection is to produce an equipartition of the added self-helicity (and therefore of twist), but the extra global effect of the location and orientation of the feet of the sheet, shell or tube in general adds different amounts of magnetic helicity to the two structures.

It is important, as Wright realized, to account for any twist or writhe already existing in the fluxes prior to reconnection. Here we show explicitly that, if a section of a flux sheet is twisted by a multiple of π with its ends held fixed and is then reconnected with another sheet, then the effect of the reconnection is to add that multiple of π to one sheet and subtract it from the other, while conserving the total helicity. If, on the other hand, the central part of a flux sheath is twisted before reconnection by any angle, then the effect of reconnection is to add that amount of twist to one sheath and subtract it from the other, while conserving the total helicity. Thus, for the local part of the process in both sheets and sheaths, there is no longer helicity equipartition.

✉ E.R. Priest

¹ School of Mathematics and Statistics, University of St. Andrews, Fife KY16 9SS, Scotland, UK

² Dept. of Physics, Montana State University, Bozeman, MT 59717, USA

Finally, we apply Wright's results explicitly to flux tubes, and in particular to determine the twist that is acquired by an erupting flux rope due to reconnection during an eruptive solar flare or coronal mass ejection.

Keywords Sun: solar flares · Coronal mass ejections · Erupting prominences · Sun: magnetic topology · Magnetic reconnection · Helicity

1. Introduction

The reconnection of neighbouring magnetic flux tubes in the solar atmosphere is a very common event. It is central on a large scale to the creation of twist in eruptive flux tubes, which show up as erupting filaments in eruptive solar flares and coronal mass ejections. It also has been proposed to take place during so-called *zipper reconnection* as an explanation for the creation of flare ribbons (Priest and Longcope, 2017). Furthermore, it is highly likely to be at the core of coronal heating events, both in X-ray bright points (Priest, Parnell, and Martin, 1994; Parnell, Priest, and Titov, 1994; Parnell and Priest, 1995) and also in nanoflares due to multiple current sheets produced by braiding of a uniform field (Parker, 1972; Galsgaard and Nordlund, 1996) or of a field produced by the magnetic carpet (Priest, Heyvaerts, and Title, 2002) or by relaxation of a braided field (Wilmot-Smith, Hornig, and Pontin, 2009). Indeed, Close *et al.* (2004) used quiet-Sun magnetograms from the Michelson Doppler Imager (MDI) instrument on the Solar and Heliospheric Observatory (SOHO) to track the motion of individual magnetic fragments and deduced that the time for all the field lines in the quiet Sun to change their connections is only 1.4 hours.

Magnetic helicity is a topological invariant (Woltjer, 1958; Moffatt, 1969; Arnol'd, 1974), which is crucial to three-dimensional reconnection, since it is conserved to a very high degree during the reconnection process and can be used to determine the lowest-energy equilibrium magnetic field configuration (Taylor, 1974). Fundamental advances in understanding magnetic helicity were made by Berger and Field (1984), who introduced a gauge-independent measure of magnetic helicity in a volume not bounded by a magnetic surface. They also realized that magnetic helicity arises partly from the internal structure within a flux tube (such as twist and kinking) and partly from external relations between tubes (i.e. linking and knotting), and that reconnection can transfer internal helicity into external helicity and vice versa. These two types of magnetic helicity were given the names *mutual helicity* (H_m) and *self-helicity* (H_s) by Wright and Berger (1989), who also applied the ideas to flux transfer events, showing how reconnection can generate twist in a flux tube.

Self-helicity therefore refers to the twisting and kinking of a flux tube, while mutual helicity represents the linkage between different flux tubes. In turn, the self-helicity of an isolated flux tube of magnetic flux F , say, consists of the sum of its writhe helicity and twist helicity (Berger, 1999; Berger and Prior, 2006), namely,

$$H_s = \mathcal{W}F^2 + \frac{\Phi}{2\pi}F^2, \quad (1)$$

where the writhe ($\mathcal{W}$) measures the winding and kinking of the axis of the tube, while the twist (Φ) represents the mean twist of field lines in the flux tube about that axis. $\mathcal{W}$ is measured in units such that it equals unity for one complete writhe, while $\Phi = 2\pi$ for one complete twist. Whereas the self-helicity of a flux tube is a topological invariant under ideal motions, neither the writhe nor the twist is invariant. Thus, a tube with high twist but very

Figure 1 A flux tube with (a) large twist and small writhe transforming via an ideal motion into one with (b) small twist and large writhe.

little writhe may be transformed by an ideal motion into a tube with small twist and large writhe, as indicated in Figure 1.

Suppose a plane surface is threaded by N thin flux tubes of magnetic flux F_i and that each footpoint possesses a uniform rotation at rate ω_i together with a translation. Then a useful expression for the rate at which the magnetic helicity changes is

$$\frac{dH}{dt} = -\frac{1}{2\pi} \left(\sum_{i=1}^N \omega_i F_i^2 + \sum_{i=1}^N \sum_{j=1}^N \frac{d\theta_{ij}}{dt} F_i F_j \right), \quad (2)$$

where $d\theta_{ij}/dt$ represents the rate of change of the angle between the footpoints i and j (Berger and Field, 1984). The first term represents the rate at which twist is injected into each footpoint (the self-helicity) and the second term represents the braiding of the thin flux tubes (their mutual helicity).

Woltjer (1958) suggested the lowest-energy state that conserves magnetic helicity is a linear force-free field, and its role has been studied in laboratory plasmas (Taylor, 1974) and dynamo theory (Moffatt, 1978). Later, Heyvaerts and Priest (1984) suggested the importance of magnetic helicity in the solar corona, both in coronal heating and coronal mass ejections, a role that has been further developed by others, such as Pevtsov, Canfield, and Metcalf (1995), Canfield and Pevtsov (1998), and Pevtsov *et al.* (2008). An important development was the proposal of a gauge-invariant relative magnetic helicity

$$H = \int \mathbf{A} \cdot \mathbf{B} - \mathbf{A}_0 \cdot \mathbf{B}_0 dV, \quad (3)$$

where $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ is a reference potential field with the same normal component on the boundary as the magnetic field $\mathbf{B}$ (Berger and Field, 1984), $\mathbf{A}$ is the vector potential.

Our aim here is to apply Wright (2019)'s recent innovative suggestion for calculating the changes in magnetic helicity in two initially untwisted coronal loops when they reconnect.

First, we set up the model and calculate the effect of magnetic helicity conservation on the twists that are produced in the reconnected loops as mutual helicity is converted into self-helicity (Section 2).

Then we introduce Wright's proposal for a second relation between the twists in the reconnected loops (Section 3).

Next, we consider the reconnection of coronal loops modelled as magnetic flux sheets (Section 4) or flux shells (Section 5) or flux tubes (Section 6). By explicit calculation, we confirm that Wright's technique is valid in each case and gives the simplest and most likely partition of helicities between the reconnected tubes, but that other partitions are possible when pre-reconnection twist or writhe is taken into account.

Finally, we use Wright's technique to estimate the twist produced in an eruptive flare (Section 7).

Figure 2 (a) The schematic connections of an initial pair of untwisted flux tubes AB and CD, viewed from above and lying above a polarity inversion line (PIL). (b) The effect of reconnection is to create an overlying twisted flux rope AD (denoted by R) together with underlying flux rope BC, denoted by U. (c) and (d) show perspective views, while (e) and (f) show the same situation as (c) and (d) for sheets that lie in a direction parallel to AB.

2. Effect of Magnetic Helicity Conservation on the Reconnection of Two Coronal Loops

Consider a pair of flux tubes side by side, straddling a polarity inversion line indicated by a dotted line in the Figure 2. The footpoints of one of the flux tubes are located at points A and B, while those of the other tube are at C and D. A perspective view is shown in Figure 2c. We suppose that reconnection creates a long twisted flux rope denoted by R and joining A to D, together with an underlying flux tube denoted by U and joining B to C, as shown in Figures 2b and d. The distance BD is denoted by L, while the perpendicular distance between AC and BD (namely, AE) is w and the shear distance EB is s (see Figure 2a).

We assume that initially both flux tubes have flux F and are untwisted, so that they possess zero self-helicity. The initial mutual helicity (H_m^i) is, however, non-zero when there is shear – i.e. when $s \neq 0$. It has the form

$$H_m^i = \frac{\theta_2 - \theta_1}{\pi} F^2, \quad (4)$$

where the angles θ_1 and θ_2 are indicated in Figure 2a.

The final magnetic helicity after reconnection, however, has both self-helicity (H_s^f) and mutual helicity (H_m^f). The mutual helicity has the form

$$H_m^f = -\frac{2\theta_3}{\pi} F^2, \quad (5)$$

where the angle θ_3 is indicated in Figure 2b

The final self-helicity has contributions from the overlying and underlying flux ropes and may be written

$$H_s^f = \frac{\Phi_R + \Phi_U}{2\pi} F^2, \quad (6)$$

where Φ_R is the twist in the overlying flux rope, while Φ_U is the twist in the underlying flux.

We need two conditions to determine the twists Φ_R and Φ_U produced by reconnection, the first of which is magnetic helicity conservation, so that the initial mutual helicity will equal the sum of the final mutual and self-helicities, namely,

$$H_m^i = H_m^f + H_s^f \quad (7)$$

or

$$\frac{1}{2}(\Phi_R + \Phi_U) = \theta_2 - \theta_1 + 2\theta_3. \quad (8)$$

For the second condition, Priest, Longcope, and Janvier (2016) and Priest and Longcope (2017) suggested helicity equipartition as a hypothesis when applying the theory to the creation of twist in an eruptive flare. This hypothesis was motivated by considering the region in which the reconnection occurs. In the local neighbourhood of the reconnection region, there is no obvious difference between the two reconnection by-products, so it seemed reasonable to assume they each receive the same self-helicity. The flaw in this reasoning is that helicity is not a local quantity, so even a local process like reconnection can produce unbalanced results. Indeed, Wright (2019) derived an alternative condition instead, which for our configuration may be written

$$\Phi_R = \pi + 2(\theta_3 - \theta_1), \quad (9)$$

as proved in the next section.

3. The Twist of Sheets with Different Orientation

The method of Wright (2019) for calculating the self-helicity of the reconnected fluxes is as follows: (i) start with equal fluxes that are untwisted and have no writhe (Figure 2c), and suppose that they lean over to reconnect, as shown in Figure 2d; (ii) consider the local region around the reconnection site, and allow a contribution to the self-helicity equivalent to a half twist for each of the reconnected fluxes; (iii) the total self-helicity of a given flux can be determined by identifying the footpoint rotations that are required to leave the flux untwisted and with no writhe via the self-helicity removed by these footpoint rotations; (iv) if there is twist or writhe in the fluxes prior to reconnection, this must be accounted for in addition to the calculation in steps (i)–(iii).

Wright (2019) demonstrates the method by applying it explicitly to flux in the form of sheets like those in Figure 2e that are aligned parallel to AB and CD. Suppose that the sheets

Figure 3 (a) The flux rope AD created by reconnection when the feet are rotated to be normal to AD, so that the twist is π . (b) The flux rope AD when the feet are unrotated to their original positions aligned along AB and CD.

lean over to reconnect, as shown in Figure 2f. Then the problem is to determine the twist and therefore the self-helicity of the flux rope AD in Figure 2f. Here we first consider flux sheets and reproduce Wright (2019)'s analysis for sheets of different orientations, before discussing more details of flux sheet reconnection (Section 4) and extending it to sheaths (Section 5) and tubes (Section 6).

3.1. Sheets Oriented Parallel to AB

First of all, suppose the feet of the flux tube AD are rotated to lie along lines perpendicular to AD, as shown in Figure 3a, in which case its twist would simply be π . However, in order to transform from the configuration of Figure 2f into that of Figure 3a, we would need to rotate footpoint A through an angle of $-\pi/2 + \theta_A$ and footpoint D through an angle of $\pi/2 + \theta_D$, as indicated in Figure 3a. In this figure the angles $\theta_A = \theta_D = \theta_3 - \theta_1$, so that the extra twist that is added in going from Figure 3a to Figure 2f is $\theta_A + \theta_D = 2(\theta_3 - \theta_1)$. Thus, in total, the required twist in the flux rope R stretching between A and D in Figures 2d and 2f is

$$\Phi_R = \pi + 2(\theta_3 - \theta_1), \quad (10)$$

and its appearance is sketched in Figure 3b.

A similar argument gives the twist in the underlying flux tube U as

$$\Phi_U = 2(\theta_3 + \theta_2) - \pi, \quad (11)$$

which may be proved as follows. We need the feet of the flux tube to be oriented parallel to AB, but suppose to start with that they were rotated to lie along lines perpendicular to BC (Figure 4a), in which case its twist would simply be 0. Now, transform this into the configuration of Figure 2f by rotating both footpoints B and C clockwise through an angle of $(\pi/2 - \theta_B)$. However, from the geometry of the parallelogram ABDC, it can be seen that

$$\theta_B + \theta_2 + \theta_3 = \pi, \quad (12)$$

and so

$$\pi/2 - \theta_B = \theta_2 + \theta_3 - \pi/2. \quad (13)$$

Thus, in total, the twist in the underlying flux tube U stretching between B and C is $2(\theta_3 + \theta_2) - \pi$, as required.

One consequence of Equations 10 and 11 for the twists is that magnetic helicity is conserved during the reconnection process, since Equation 8 is indeed satisfied. Another is that Φ_R lies between π and 2π , while Φ_U lies between 0 and π , with $\Phi_R + \Phi_U$ lying between 0 and 2π .

Figure 4 (a) The flux rope BC created by reconnection. Assume to start with that the feet B and C are oriented in a direction normal to BC, so that the twist vanishes. Then rotate both footpoints B and C through angles $(\pi/2 - \theta_B)$ as shown, to align them parallel to AB. (b) The flux rope AD created by reconnection. Assume the feet A and D are oriented in a direction normal to AD, and then rotate both footpoints through angles $(\theta_3 - \theta_1)$ to align them perpendicular to AB.

Figure 5 (a) An initial arcade of coronal loops, modelled as a pair of sheets AB and CD oriented in a direction normal to AB. (b) The sheet AD created by reconnection when the feet are rotated to be normal to AD, and when the feet are unrotated to their original positions.

3.2. Sheets Oriented in a Direction Perpendicular to AB

Is the method of Wright robust when applied to sheets? Does the orientation of the sheets matter? To address this question we consider alternative scenarios where the sheets are oriented in different directions. We start with the flux tubes in Figures 2c and d and suppose they are flattened out to form sheets whose feet lie perpendicular to AB, as shown in Figure 5. Then is the twist the same in this alternative as in the case when the sheets are aligned parallel to AB (Figures 2e and f)?

In this case, when the rope AD has its feet rotated to be normal to AD (Figure 4b), the twist is π . In order to recover the tube AD with their original positions, we have to rotate both A and D in the same direction by $(\theta_3 - \theta_1)$, and so the twist becomes $\Phi_R = \pi + 2(\theta_3 - \theta_1)$, as before.

The reason for this result being unchanged is that, in going from a tube that is flattened in the AB direction to one flattened in a direction perpendicular to AB, you rotate A by $\pi/2$ in

Figure 6 An untwisted magnetic flux tube whose axis lies in a vertical plane and which consists of four fibrils joining the footpoints that are shown.

a clockwise direction and B by $\pi/2$ in a counterclockwise direction, and so do not change the net twist.

3.3. Sheets Oriented in Other Directions

So what happens when the flux sheets are oriented in other directions? Consider an untwisted flux tube with no writhe whose axis lies in a vertical plane. Suppose it consists of four fibrils (each of flux F_f), as shown in Figure 6 joining footpoints A_1 to B_1 , A_2 to B_2 , A_3 to B_3 , and A_4 to B_4 . Consider what happens when it is flattened in the direction A_2A_4 and B_2B_4 to produce a sheet perpendicular the vertical plane in which the axis of the tube lies. Then the change in magnetic helicity produced by motions of A_1 , A_3 , B_1 , and B_3 towards the centres O_A and O_B (see Figure 7) are given from Equation 2 by

$$\Delta H = -\frac{F_f^2}{2\pi} \sum_{i,j} \Delta\theta_{i,j}, \quad (14)$$

where $\theta_{i,j}$ represents the changes in orientation of each of the 8 fibril footpoints with respect to all the others.

However, if the separation L of the two feet of the flux tube is much greater than the radius of the flux tube, then the only important contributions arise from the footpoints A_1 , A_2 , A_3 , and A_4 with respect to each other and from B_1 , B_2 , B_3 , and B_4 with respect to each other. Thus, it can be seen from Figure 7a that

$$\begin{aligned} \Delta H &= -\frac{F_f^2}{2\pi} [\Delta\theta_{A_1A_4A_2} + \Delta\theta_{A_1A_2A_4} + \Delta\theta_{A_3A_4A_2} + \Delta\theta_{A_3A_2A_4} \\ &\quad + \Delta\theta_{B_1B_4B_2} + \Delta\theta_{B_1B_2B_4} + \Delta\theta_{B_3B_4B_2} + \Delta\theta_{B_3B_2B_4}] \\ &= -\frac{F_f^2}{2\pi} [+ \pi/4 - \pi/4 - \pi/4 + \pi/4 - \pi/4 + \pi/4 + \pi/4 - \pi/4] = 0, \end{aligned} \quad (15)$$

and so the flattening of the tube in this way does not change the magnetic helicity.

Next suppose instead (Figure 7b) that the flattening occurs in the direction A_1A_3 to produce a sheet lying in the same plane as the axis of the flux tube (Figure 6). Then the change in magnetic helicity produced by motions of A_2 , A_4 , B_2 , and B_4 to the centres O_A and O_B is given from Equation 2 by

$$\begin{aligned} \Delta H &= -\frac{F_f^2}{2\pi} [\Delta\theta_{A_4A_3A_1} + \Delta\theta_{A_4A_1A_3} + \Delta\theta_{A_2A_3A_1} + \Delta\theta_{A_2A_1A_3} \\ &\quad + \Delta\theta_{B_4B_3B_1} + \Delta\theta_{B_4B_1B_3} + \Delta\theta_{B_2B_3B_1} + \Delta\theta_{B_2B_1B_3}] \end{aligned}$$

Figure 7 (a) Suppose a flux tube is flattened to create a flux sheet by moving the footpoints A_1 and A_3 , to align with A_2 , A_4 , while the footpoints B_1 and B_3 align with B_2 , B_4 . (b) Flattening in a direction perpendicular to the previous case. (c) Flattening in an arbitrary direction.

$$= -\frac{F_f^2}{2\pi} [+\pi/4 - \pi/4 - \pi/4 + \pi/4 - \pi/4 + \pi/4 + \pi/4 - \pi/4] = 0, \quad (16)$$

and so the flattening of the tube in this way again does not change the magnetic helicity.

Finally, suppose the flattening is at an arbitrary angle θ to the first case, and consider four fibrils with feet at A_5 , A_6 , A_7 , and A_8 that map to points B_5 , B_6 , B_7 , and B_8 , as shown in Figure 7c. Then the change in helicity is given by

$$\Delta H = -\frac{F_f^2}{2\pi} [\Delta\theta_{A_8 A_5 A_7} + \Delta\theta_{A_8 A_7 A_5} + \Delta\theta_{A_6 A_5 A_7} + \Delta\theta_{A_6 A_7 A_5} \\ + \alpha_{B_5 B_8} + \alpha_{B_6 B_7} + \alpha_{B_7 B_8} + \alpha_{B_5 B_6}],$$

where $\alpha_{B_5 B_8}$, $\alpha_{B_6 B_7}$, $\alpha_{B_7 B_8}$, $\alpha_{B_5 B_6}$ are the angles through which $B_5 B_8$, $B_6 B_7$, $B_7 B_8$, and $B_5 B_6$, respectively, rotate. Thus,

$$\Delta H = -\frac{F_f^2}{2\pi} [-\pi/4 + \pi/4 + \pi/4 - \pi/4 - (\pi - 2\theta) - (\pi - 2\theta)] = 8\theta, \quad (17)$$

which in general is non-zero and so flattening in an arbitrary direction adds magnetic helicity to the tube. Note that this amount vanishes when $\theta = 0$ and it is a multiple of 2π when $\theta = \pi/2$. We therefore conclude that flattening in the way we have suggested can add self-helicity prior to reconnection which would need to be taken into account in addition to the self-helicity associated with reconnection. The exception is flattening parallel or perpendicular to AB when no self-helicity is added.

This section has been concerned with describing the self-helicity of a sheet with general orientations of its two feet. We have seen that, when the orientations of a sheet at both ends are θ relative to the line joining the feet, and θ is not equal to 0 or $\pi/2$, then the sheet is twisted, and its self-helicity needs to be accounted for. On the other hand, Wright (private correspondence) has noticed that, if the orientations of the two feet are instead θ and $-\theta$, then the sheet is untwisted and so it has no self-helicity.

4. Reconnection of Flux Sheets

We have seen in the previous section that the way in which magnetic helicity is distributed between two sheets by reconnection depends on two factors, namely, the reconnection process itself and the relative location of the footpoints. We regard the former as a local process and the latter as a global process. In this section, let us isolate the local reconnection effect, by neglecting the effects of the locations of the feet and the orientations of the flux sheet that we considered in Section 3 – i.e. supposing the sheets are of unlimited length, so that there is no reflection at the feet of the torsional Alfvén waves that communicate the effect of the reconnection away from the reconnection site. We shall find that the local process produces an equipartition of helicity by distributing the helicity that is released equally between the two flux sheets. The simplest way to see this is to suppose two sheets are initially crossing and then reconnect.

4.1. Local Effect of Reconnection – Equipartition of Helicity

Consider two untwisted flux sheets, each of flux F , intersecting at an arbitrary angle and having their footpoints at arbitrary locations (Figure 8a). Suppose the footpoints are flattened in directions normal to the field lines and are passive in the sense of having no effect on the helicity.

Suppose each of the flux sheets consists of N fibrils of flux F/N (as shown in Figure 8a for $N = 4$). Then it may be proved that, in the limit as $N \rightarrow \infty$, there is equipartition of self-helicity, so that each tube acquires a twist of π . The proof of this is given in the appendix of Wright (2019) and we summarise his argument as follows.

The initial mutual helicity for such a tube of flux F crossing another tube of flux F is simply

$$H_m^i = F^2. \quad (18)$$

Alternatively, this may be shown by summing the effect of the multiple fibrils. Thus, the mutual helicity of fibril a_1 with b_1 , both having flux F/N , is $(F/N)^2$. But a_1 crosses N other fibrils and so it contributes $N(F/N)^2$ to the total mutual helicity. The same is true for each of the N other fibrils $a_1, a_2, \dots, a_N$, and so in total they all give a mutual helicity of

$$H_m^i = NN(F/N)^2 = F^2, \quad (19)$$

Figure 8 (a) A pair of intersecting flux tubes each consisting of four fibrils a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 . (b) The result after fibrils a_1b_1, a_2b_2, a_3b_3 , and a_4b_4 have reconnected.

which is the same as before.

Now suppose each of $a_1b_1, a_2b_2, \dots, a_Nb_N$ reconnects, as shown in Figure 8b for the case of $N = 4$, so that the upper and lower parts of the initial tubes are no longer linked. Consider the upper flux bundle. Its initial magnetic helicity from the a_1b_1 crossover is $(F/N)^2$, and so after reconnection assume a fraction α_1 , say, of this is given as self-helicity to the upper flux rope and $1 - \alpha_1$ is given to the lower flux rope. The upper flux rope consists of $\frac{1}{2}(N^2 - N)$ crossings, and so its mutual helicity will be $\frac{1}{2}(N^2 - N)(F/N)^2$, making a total helicity of

$$H_{upper} = \frac{1}{2}(N^2 - N)(F/N)^2 + \sum \alpha_i (F/N)^2. \quad (20)$$

In the limit as $N \rightarrow \infty$, this reduces to

$$H_{upper} = \frac{1}{2}F^2, \quad (21)$$

as required.

Note that this is just a local argument for the twist added by reconnection, since it does not take account of the orientations of the feet, which also play a role, as in Figure 2. The reason that Figure 2 gives a different answer is that the flux tube may possess extra writhe or twist (and therefore helicity) between the feet and the reconnection location. In other words, the resulting twist depends both on the local effect of the reconnection and the global geometry of the feet in accordance with Section 2.

4.2. Local Reconnection of Flux Sheets Consisting of Two Fibrils

Consider two untwisted sheets each of flux F and each consisting of just two fibrils of flux $\frac{1}{2}F$ intersecting at right angles. Is the situation just considered in Figure 8 the only way to reconnect the two sheets? This question has been partially addressed by Wright and Berger (1989). Their Figure 2 considers configurations similar our Figure 9a-d, since the focus of their study was on reconnecting the fluxes to produce completely unlinked fluxes. For context, we reproduce their analysis here before extending it to the situation in Figure 9e and f.

In the initial state (Figure 9a), the mutual magnetic helicity may be regarded as that of two sheet crossings of flux F (namely, F^2) or as four fibril crossings of flux $\frac{1}{2}F$, namely,

$$H_m^i = 4(\frac{1}{2}F)^2 = F^2. \quad (22)$$

Figure 9 (a) A pair of flux sheets intersecting at right angles and each consisting of two fibrils a_1 , a_2 , and b_1 , b_2 . The result after reconnection of (b) fibrils a_1b_1 , a_2b_2 , (c) the upper crossover, and (d) the lower crossover. The situation (e) before and (f) after a_2b_1 and a_1b_2 have reconnected

First, suppose that a_1 reconnects with b_1 and a_2 with b_2 , as indicated by the large dots in Figure 9a, to give the situation in Figure 9b. When each of the crossing fibrils of mutual helicity $F^2/4$ reconnects, the two new fibrils each gain self-helicity $F^2/8$, so that the fibrils in the resulting upper flux rope have self-helicity $F^2/4$ and mutual helicity $F^2/4$, making a total of $\frac{1}{2}F^2$, while the same is true of the lower flux rope.

Next, suppose the two upper fibrils in Figure 9b reconnect to give Figure 9c, in which the self-helicity of the fibrils has been increased by $F^2/4$ to $\frac{1}{2}F^2$, while their mutual helicity has decreased to zero. Then, in Figure 9d, the same thing happens to the two lower fibrils, increasing their self-helicity also to $\frac{1}{2}F^2$.

Finally, suppose the upper and lower crossings (Figure 9e) are reconnected to give Figure 9f. In this case, the upper and lower fibrils are freed, but the two middle fibrils are still interlinked, so we have not succeeded in reconnecting the whole flux sheet. We conclude that there are several ways to reconnect the initial sheets (Figure 9a) to give the configurations shown in Figures 9b, c and d, but in each case, although the internal arrangement of fibrils is different, there is local self-helicity equipartition with the total self-helicities in the reconnected sheets both being the same ($\frac{1}{2}F^2$). We may regard this as the standard and natural situation.

However, it is possible to obtain an infinite number of different non-standard situations without local self-helicity equipartition, but differing by discrete values of self-helicity. Wright (2019) showed how flipping a flux sheet over at the reconnection site prior to reconnection will add $\pm 1/2$ turn of twist to the sections of sheet adjacent to the reconnection site. It is important to recognise that the helicity associated with this twist must be added to the $1/2$ turns that result from reconnection. The effect on the flux sheets, therefore, is to leave one reconnected sheet untwisted, whilst the other has a twist of one full turn. Clearly preconditioning the reconnected fluxes affects their final self-helicities.

Figure 10 The same situation as in Figure 9, except that the central part of $b_1 b_2$ is rotated before reconnection, with the result that a_1 now connects with b_2 instead of b_1 .

We now demonstrate this effect for two fibrils. Suppose that, in place of Figure 9a, the central part of the underlying tube is rotated through half a turn, say, while keeping the footpoints fixed, so that no helicity is added to the tube as a whole – i.e. half a turn is subtracted from the section of the underlying tube that is in the upper right of Figure 10a, while half a turn is added to the section in the lower left. After reconnection, the result is that now fibril a_1 is joined to b_2 instead of b_1 , while a_2 is joined to b_1 . The result is that the upper flux sheet is untwisted and has zero helicity, while the lower sheet is twisted by one full turn (2π) and so its helicity is F^2 . We still have helicity conservation but no longer equipartition. Thus, as well as Wright's standard result, a range of other scenarios is possible, in which the central portion of a flux sheet is twisted by multiples of π before reconnection.

4.3. Local Reconnection of Flux Sheets Consisting of Three Fibrils

Next consider the case of sheets consisting of three fibrils (Figure 11a), each fibril having no twist and containing a flux $\frac{1}{3}F$. The initial mutual helicity is

$$H_m^i = 3\left(\frac{1}{3}F\right)^2 + 3\left(\frac{1}{3}F\right)^2 + 3\left(\frac{1}{3}F\right)^2 = F^2. \quad (23)$$

Suppose first that there are reconnections at the crossings marked by dots in Figure 11a to give the configuration shown in Figure 11b. A reconnection at each crossing converts mutual to self-helicity in the same way as in the previous subsection and the whole process is linear, so the total helicity is conserved as before.

But what happens if different crossings are reconnected, such as in Figure 12c. Here A_1B_1 reconnects with C_2D_2 instead of C_1D_1 , while A_2B_2 reconnects with C_3D_3 instead of C_2D_2 , and A_3B_3 reconnects with C_1D_1 instead of C_3D_3 . However, as can be seen, the problem is that now the two new flux bundles are not separated, since A_1C_2 and A_2C_3 still link with D_1B_3 . Of course, we could then allow extra reconnections at the joins of D_1B_3 with the other two fibrils, but the result is to give two flux ropes with exactly the same total helicity as before, even though the distribution between mutual and self-helicity may be different.

Is there any way to obtain a different amount of helicity in the resulting flux sheets? Yes indeed. Just as in the case described in Figure 10a, pairs of fibrils may have their central parts rotated before reconnection without changing the initial helicity, with the result that helicity equipartition during reconnection no longer applies, since the twists of the resulting reconnected sheets can differ by multiples of a turn.

5. Reconnection of Flux Shells

Do our results change when we model the flux tubes as shells rather than sheets? The way to address this is to regard the shells as consisting of a discrete number of fibrils (just as in our

Figure 11 Two of the different ways in which a pair of flux sheets intersecting at right angles and each consisting of 3 fibrils may reconnect. (a) The locations of the reconnections of fibrils A_1B_1 with C_1D_1 , A_2B_2 with C_2D_2 , and A_3B_3 with C_3D_3 , and (b) the result of the reconnections. (c) The locations of the reconnections of fibrils A_1B_1 with C_2D_2 , A_2B_2 with C_3D_3 , and A_3B_3 with C_1D_1 , and (d) the result of the reconnections.

Figure 12 (a) Two crossing flux tubes, each consisting of four fibrils. (b) The effect of reconnecting A_1B_1 to C_3D_3 , A_2B_2 to C_4D_4 , A_3B_3 to C_1D_1 , and A_4B_4 to C_2D_2 . (c) The resulting two flux tubes.

discussion of sheets) and then to carefully follow the effects of reconnecting each fibril. We first of all consider the local reconnection process and discover some differences between sheets and shells. Then we discuss how to recover Wright's result when regarding the structures as flux shells rather than sheets. Although Wright's result is the standard, most natural solution, it was derived for fluxes that have no twisting or writhing prior to reconnection. Naturally, there are, therefore, other scenarios that can produce different distributions of magnetic helicity in the reconnected sheaths by preconditioning the fluxes to have portions of non-zero twist. The self-helicity associated with sections of flux that have twist or writhe prior to reconnection needs to be accounted for in addition to the self-helicity associated with reconnection and footpoint location (see Wright, 2019).

Figure 13 (a) An arcade of coronal loops, modelled as a pair of untwisted flux tubes AB and CD, which reconnect to create an overlying twisted flux rope AD together with underlying flux BC. (b) A possible sketch of the axes of the reconnected flux tubes as they writhe about the direct curves AD and BC

5.1. Local Effect of Reconnecting Flux Shells

Consider two intersecting flux tubes, each consisting of, for example, four fibrils that are located in a flux shell, as shown in Figure 12a. The first shell has fibrils A_1B_1 , A_2B_2 , A_3B_3 , and A_4B_4 , while the second shell has fibrils C_1D_1 , C_2D_2 , C_3D_3 , and C_4D_4 . Now reconnect A_1B_1 with C_3D_3 , A_2B_2 with C_4D_4 , A_3B_3 with C_1D_1 and A_4B_4 with C_2D_2 , in the manner shown in Figure 12b. This produces two flux tubes modelled as flux shells, each of twist π , as shown in Figure 12c. In the first tube, A_1 joins to C_3 , A_2 to C_4 , A_3 to C_1 , and A_4 to C_2 , while in the second tube B_1 joins to D_3 , B_2 to D_4 , B_3 to D_1 , and B_4 to D_2 . We therefore have equipartition of self-helicity, as in the case of flux sheets, so let us next include also the global effect of the orientation of the feet.

5.2. Deriving Wright's Result by Modelling the Flux Tubes as Flux Shells instead of Sheets

Consider a pair of flux tubes AB and CD that each consist of four fibrils located in a shell. The first possesses fibrils A_1B_1 , A_2B_2 , A_3B_3 , and A_4B_4 , while the second has fibrils C_1D_1 , C_2D_2 , C_3D_3 , and C_4D_4 . Suppose they reconnect to give two flux tubes AD and BC (Figure 13a), in which footpoint A_1 joins to D_3 , A_2 to D_4 , A_3 to D_1 and A_4 to D_2 , while B_1 joins to C_3 , B_2 to C_4 , B_3 to C_1 , and B_4 to C_2 .

Then, at first sight, it looks a simple matter to calculate the helicity, since the twist in AD appears to be π , but the problem is that it is not immediately obvious what is the effect of the orientation of the feet. Furthermore, it should be borne in mind that, the writhe and twist are not conserved, since one can be converted to another by an ideal motion while their total, namely, the magnetic self-helicity, remains the same. This means that the extra effect of the footpoint orientation may show up in a given example either as writhe or as extra twist or a combination of both.

One way to calculate the helicity is as follows. First of all, imagine the tube AD has its axis located in a vertical plane through AB, so that the centre of the footpoint D is located at D_0 on the line AB instead of at D (as shown in Figure 14a or, viewed from above in Figure 14b). Since this tube has no writhe, its helicity will come entirely from

Figure 14 (a) The reconnected flux rope AD aligned in a plane through AD_0 parallel to AB, and then moved to its final position by moving the right footpoint from D_0 to D. (b) Looking down on the flux tube AD_0 . The effects of (c) a rigid body rotation of AD_0 to AD and (d) a rotation of the feet of the fibrils to their final positions. (e) A perspective view of the final state when the tube is straightened out.

its twist, namely, in this case π . Next, suppose AD_0 is rigidly rotated through an angle $\theta_3 - \theta_1$ so that the footpoint D_0 moves to D. This means that the orientations of the fibril footpoints A_1, A_2, A_3, A_4 and D_1, D_2, D_3, D_4 have all rotated by an angle $\theta_3 - \theta_1$, as shown in Figure 14c. Thus, in order to bring the fibril footpoints to their correct locations as seen in Figure 13a, all of the fibril footpoints need to rotate by $(\theta_3 - \theta_1)$ when viewed from above, to give Figure 14d.

However, when the flux tube is straightened out and viewed in perspective (Figure 14e), rather than being curved and viewed from above (Figures 14b–d), it can be seen that the two ends of the flux tube have been rotated in opposite directions when going from Figure 14c to Figure 14d. This provides an extra twist of $2(\theta_3 - \theta_1)$ to the ends of the tube in addition to the twist of π provided by the local effect of reconnection, and it may show up either as extra flux rope twist or as writhe. Thus, in total, the self-helicity of the reconnected flux rope AD is

$$H_s = \frac{F^2}{2\pi} [\pi + 2(\theta_3 - \theta_1)], \quad (24)$$

and, if there is no writhe, its twist is therefore

$$\Phi_R = \pi + 2(\theta_3 - \theta_1), \quad (25)$$

in agreement with Wright's result (Equation 9).

Another way to calculate the extra helicity above the local reconnection value ($\frac{1}{2} F^2$) due to the footpoint orientations is by supposing this extra helicity is all in the form of writhe. Suppose, as in Figure 13b, that the axis of the flux tube starts out near A pointing upwards

Figure 15 (a) The writhed axis AMD of the flux tube AD, whose projection onto the xy -plane makes a curve that starts out at A inclined an angle $\theta_3 - \theta_1$ to the x -axis, passes through P and then ends up at D also inclined at $\theta_3 - \theta_1$. (b) The tantrix sphere containing points that map from the ends of the unit vectors that are parallel to the flux tube axis.

and in a vertical plane through AB. Then it rotates to be horizontal over the midpoint between A and D, and, during the second half of its path, it ends up approaching D along a curve lying in a vertical plane through D.

Writhe can be calculated in terms of the so-called “tantrix curve” (Berger and Prior, 2006). The unit vector that is tangent to the axis of the writhing flux tube AD maps points on the axis of the tube to points on a unit sphere, called the tantrix sphere. The tantrix curve is then the path that the tip of the tangent vector traces out on the surface of the sphere. The writhe of a closed curve is given in terms of the spherical area $\mathcal{A}$ enclosed by the tantrix curve as

$$\mathcal{W} = \frac{\mathcal{A}}{2\pi} - 1. \quad (26)$$

For a flux tube such as AD, the tantrix curve is completed in two steps: first the tube is extended by adding sections below the xy -plane that change the tangent direction to the vertical and so add sections to the tantrix curve that extend it to the north and south poles along geodesics; secondly, the curve is closed by adding a semicircle.

Thus, in Figure 15a the points A, M and D, map to A', M', and D' on the tantrix curve (Figure 15a), with M' lying on the curve $\phi = 0$ of longitude. Then the footpoints A and D are continued down to N and S below the xy -plane, which map to N' and S' on the tantrix. Finally, the curve is completed by a semicircle which maps onto the curve $\phi = \pi$ of longitude on the tantrix. Now the area between the curves $\phi = 0$ and $\phi = \pi$ is 2π and the area we require is that area minus the area between the curves $\phi = 0$ and the curve N'A'D'S' (namely, $\phi = \theta_3 - \theta_1$), which gives an area of $4\pi(\theta_3 - \theta_1)$. Thus,

$$\mathcal{A} = 2\pi - 4\pi(\theta_3 - \theta_1), \quad (27)$$

and so

$$\mathcal{W} = 2(\theta_3 - \theta_1), \quad (28)$$

as required.

5.3. Generalisations of Wright’s Standard Result

We have verified Wright’s result for the standard natural reconnection of flux sheaths when there is initial twisting or writhing present. However, it is possible to reconnect the field in other ways, as follows, depending on the direction in which flux sheaths approach one

Figure 16 Reconnection of a pair of untwisted flux tubes AB and CD, viewed in perspective (top) or from above (middle) or from the side, in which the first fibrils to reconnect are (a) A_2B_2 with C_4D_4 , or (b) A_3B_3 with C_4D_4 or (c) A_3B_3 with C_1D_1 .

another before reconnecting and also on a possible twisting of part of the sheaths. As mentioned in Wright's (2019) discussion of sheets, any initial twisting or writhing needs to be accounted for in addition to the self-helicities expected from reconnecting fluxes with no twist or writhe.

First, consider an arcade of coronal loops, modelled as a pair of untwisted flux sheaths AB and CD, which reconnect to create an overlying twisted flux rope AD together with underlying flux BC (Figure 16). Suppose the loop AB consists of four fibrils A_1B_1 , A_2B_2 , A_3B_3 , A_4B_4 , while CD consists of fibrils C_1D_1 , C_2D_2 , C_3D_3 , C_4D_4 . The simplest and most obvious way for the flux tubes AB and CD to touch and reconnect is in a symmetric manner, which first leads to A_2B_2 reconnecting with C_4D_4 , as shown in Figure 16a. However, an alternative is for them to approach asymmetrically, with, for instance, tube CD lying closer to the solar surface than AB and approaching at differing angles. For instance, in Figure 16b the first field lines to reconnect are A_3B_3 and C_4D_4 , whereas in Figure 16c the first ones are A_3B_3 and C_1D_1 .

Secondly, consider the nature of the local reconnection regardless of the orientation of the feet. We found for flux sheets that rotation of the central part of a sheet before reconnection could lead to a different distribution of twist, and the same is true for flux sheaths, but with an important difference. Consider, for example, the case shown in Figure 17 of two flux sheaths, each consisting of four fibrils, approaching one another and reconnecting. Suppose sheath AB is rotated by $\pi/4$, so that A_1B_1 reconnects with C_2D_2 instead of C_3D_3 , while A_2B_2 reconnects with C_3D_3 , A_3B_3 reconnects with C_4D_4 and A_4B_4 reconnects with C_1D_1 . Thus, after reconnection A_1 , A_2 , A_3 , and A_4 connect to C_2 , C_3 , C_4 , and C_1 , respectively, while B_1 , B_2 , B_3 , and B_4 connect to D_2 , D_3 , D_4 , and D_1 , respectively. The result is that

Figure 17 Local reconnection of a pair of flux tubes AB and CD in which flux tube AB is first rotated about its centre by $\pi/4$ before reconnection, leading to a twist of $\pi/4$ in BD and $3\pi/4$ in AC.

instead of both sheaths being twisted by $\pi/2$, sheath AC is twisted by $3\pi/4$, while sheath BD is twisted by $\pi/4$.

This may be generalised to allow the central part of sheath AB to twist by any multiple of $\pi/4$ before reconnection, so that after reconnection AC will be twisted by any corresponding multiple of $\pi/4$, while BD is twisted by an amount such that magnetic helicity is conserved. A further generalisation is to allow the sheaths to consist of, say, n fibrils, in which case pre-twisting by any multiple of π/n is possible, and so the final twist of AC could be any multiple of π/n . In the limit as $n \rightarrow \infty$, rotation by any angle before reconnection is possible and so AC could be twisted by any corresponding amount after reconnection, with again BD twisted by the amount that conserves magnetic helicity. This differs from our previous consideration of flux sheets, when only rotation by discrete multiples of π was allowed.

6. Reconnection of Flux Tubes

Flux tubes, which consist of a continuum of field lines spread through a volume, rather than along a line or in a shell, are potentially much more complex than flux sheets or flux shells. For example, they may be initially twisted ropes, when Linton, Dahlburg, and Antiochos (2001) and Linton and Priest (2003) showed that one can obtain tunnelling with double reconnection rather than slingshot reconnection. Another complex example is the case of double separator reconnection between two flux tubes that was considered by Priest and Longcope (2017).

However, here we focus on much simpler scenarios, by generalising from the above results for flux sheets and sheaths. So, the question that arises is: do the results for simple two-dimensional sheets or sheaths apply to three-dimensional flux tubes? We would like to suggest that yes they do, on the following grounds.

First of all, let us regard the flux tube as being composed of a series of flux sheaths (Figure 18). Then reconnection of the flux tube cannot simply be modelled as the sequential reconnection, as in Figure 8, of the set of flux sheaths for the following reason. Suppose we start with two flux sheaths, an inner one and an outer one, as in Figure 18a, with the outer sheaths each consisting of four fibrils, namely, A_1B_1 , A_2B_2 , A_3B_3 , A_4B_4 in the first tube and C_1D_1 , C_2D_2 , C_3D_3 , A_4B_4 in the second tube. Then, at first sight, there seems to be no problem in reconnecting these fibrils in the outer sheaths to give the situation shown in Figure 18b. However, a difficulty arises since the fibrils in Figure 18b do not form smooth

Figure 18 (a) Regarding flux tubes as a collection of sheaths (in this case two), each consisting of a collection of fibrils (in this case four). (b) The effect of reconnecting first the outer sheath and then the inner sheath.

Figure 19 Regarding a flux tube as a collection of sheets.

sheaths, because, if the number of fibrils is increased, there will eventually be two fibrils between C_1D_1 and C_4D_4 in the one sheath and between A_2B_2 and A_3B_3 in the other sheath which cannot reconnect, since their paths are blocked by the inner sheaths. That is to say, the first fibril lies above the inner sheath of the upper flux tube and the second fibril lies below the lower flux tube. Reconnection of these two fibrils can, however, still take place after the inner sheaths have reconnected.

As a better alternative, a flux tube may be sliced into a series of flux sheets (Figure 19), so that reconnection of the flux tubes may be modelled as the sequential reconnection, as in Figure 8, of the set of flux sheets. Figure 19 illustrates the case with seven sheets, such that field line A_3B_3 reconnects with field line C_1D_1 , and then sheet a_1 reconnects with sheet c_1 , followed by a_2 with c_2 , a_3 with c_3 , a_4 with c_4 , a_5 with c_5 , a_6 with c_6 , and finally field line A_1B_1 with field line C_3D_3 . There seems therefore to be no conceptual problem to regarding the reconnection of a three-dimensional flux tube as the reconnection of an infinite sequence of flux sheets.

Figure 20 The variation of (a) Φ_R and (b) Φ_U with s/w for various values of L/w .

7. Application of Wright's Method to the Eruption of a Simple Arcade in an Eruptive Solar Flare

In previous publications, we have considered the influence of magnetic helicity conservation on the creation of twist in an eruptive solar flare (Priest, Longcope, and Janvier, 2016; Priest and Longcope, 2017). We included as a hypothesis magnetic helicity equipartition during reconnection, but Wright (2019) has suggested that there will not generally be such equipartition, so here we reconsider the role of helicity conservation and its role in creating twist in eruptive flares and coronal mass ejections by adopting Wright's method.

7.1. Φ_R and Φ_U as functions of s/w and L/w

Following Priest, Longcope, and Janvier (2016), suppose that a preflare sheared arcade is modelled simply as a pair of flux tubes AB and CD with significant mutual helicity, and suppose that the tubes reconnect to give an eruptive flux rope AD together with underlying flux BC (Figure 2). The aim is to calculate the self-helicity in the eruptive flux rope. Wright (2019) has proposed that, for this situation the twist in the eruptive flux tube AD if it has no writhe is given by

$$\Phi_R = \pi + 2(\theta_3 - \theta_1), \quad (29)$$

while the twist in the underlying flux tube U is

$$\Phi_U = 2(\theta_3 + \theta_2) - \pi, \quad (30)$$

as proved in Section 3, and so we may calculate its variation with various parameters as follows. Φ_R and Φ_U are functions of θ_1 , θ_2 and θ_3 (indicated in Figure 2), and therefore of the arcade shear (s/w) and length (L/w) normalised with respect to its width (w).

First of all, consider Φ_R , where $\tan \theta_1 = w/(L + s)$ and $\tan \theta_3 = w/s$, so that

$$\tan(\theta_3 - \theta_1) = \frac{\tan \theta_3 - \tan \theta_1}{1 + \tan \theta_3 \tan \theta_1} = \frac{w/s - w/(L + s)}{1 + w^2/[s(L + s)]} = \frac{Lw}{s^2 + sL + w^2}. \quad (31)$$

As s increases from 0 to ∞ , the expression $2(\theta_3 - \theta_1)$ in Equation 29 decreases from $2 \arctan(L/w)$ to 0, while, as L/w decreases from ∞ to 0, $2 \arctan(L/w)$ decreases from π to 0. The resulting behaviour of Φ_R with s/w and L/w is plotted in Figure 20a.

Next, consider Φ_U given by Equation 30. Since $\tan \theta_2 = w/(L - s)$ and $\tan \theta_3 = w/s$,

$$\tan(\theta_3 + \theta_2) = \frac{\tan \theta_3 + \tan \theta_2}{1 - \tan \theta_3 \tan \theta_2} = \frac{w/s + w/(L - s)}{1 - w^2/[s(L - s)]} = \frac{Lw}{-s^2 + sL - w^2}. \quad (32)$$

Thus, as $s \rightarrow 0$, $2(\theta_3 + \theta_2)$ tends to $\arctan(-L/w)$, which lies between π and 2π , so that Φ_U lies between 0 and π . As $s \rightarrow \infty$, $2(\theta_3 + \theta_2) \rightarrow 2\pi$ and $\Phi_U \rightarrow \pi$. Also, when $s = -\frac{1}{2}L + \sqrt{(w^2 + L^2/4)}$, $\theta_3 + \theta_2 = \pi/2$ and $\Phi_U = \pi$. The resulting behaviour of Φ_U with s/w and L/w is sketched in Figure 20b.

Finally, consider

$$\Phi_R + \Phi_U = 2(2\theta_3 + \theta_2 - \theta_1), \quad (33)$$

for which

$$\tan \frac{1}{2}(\Phi_R + \Phi_U) = -\frac{2sL^2w}{(w^2 + s^2)^2 - (s^2 - w^2)L^2}. \quad (34)$$

Thus, as $s \rightarrow 0$ or ∞ , $\Phi_R + \Phi_U \rightarrow 2\pi$, whereas, when $L \gg w$, $\Phi_R + \Phi_U \rightarrow 0$. In general, as found by Priest, Longcope, and Janvier (2016), $\Phi_R + \Phi_U$ lies between 0 and 2π .

The results in Figure 20 can also be compared with some of the properties given in Wright (2019), who found the overlying tube would always have a twist greater than a half turn (i.e., $\Phi_R/\pi > 1$) and the underlying tube would always have a twist less than a half turn (i.e. $\Phi_U/\pi < 1$).

For the specific situation in which the lines AB and CD in Figure 2 are parallel and the fluxes lie adjacent to one another (so are touching and ready to reconnect), we recover the situation in Figure 8b of Wright (2019). In terms of our notation, this corresponds to $L/w \rightarrow \infty$, $s/w \rightarrow \infty$ and $s < L$. Figure 20 shows that the overlying and underlying tubes both have a self-helicity of π , equivalent to a twist of a half turn, in agreement with Equation 23 of Wright (2019) when his $\theta_A \rightarrow 0$ and $\theta_C \rightarrow 0$.

8. Conclusion

A huge puzzle in the basic theory of reconnection has been to understand the nature of the conversion of, say, mutual magnetic helicity into self-helicity during the reconnection of two magnetic flux tubes, which in turn is important in determining the twist that can be added to a flux tube (such as an erupting solar flux rope on a large scale or, say, spicules on a small scale). The main unanswered question is: when two flux tubes reconnect, how is the resulting self-helicity (which can be either twist or writhe) partitioned between the two flux tubes? One indisputable constraint is total magnetic helicity conservation, so that the fall, say, in mutual helicity during reconnection must equal the rise in the sum of the self-helicities of the two flux tubes. But there must be a second condition that determines the distribution of the total self-helicity between the two reconnected flux tubes.

Priest, Longcope, and Janvier (2016) and Priest and Longcope (2017), in their consideration of eruptive solar flares, suggested helicity equipartition, so that the same amount of self-helicity is added to the two tubes. More recently, Wright (2019) has proposed a clever technique for determining the second constraint and so deducing the resulting partition of self-helicity. The purpose of this paper has been to investigate and develop his proposal in detail and, in particular, to consider explicit calculations for flux sheets, flux sheaths, and flux tubes.

Our analysis and that of Wright (2019) imply that there is both a local and a global aspect to the effect of reconnection, and so our treatment is split into three parts. The first is to analyse the local effect of reconnection, which is found to produce an equipartition of the self-helicity (and therefore the twist) added to the structures, regardless of whether they are sheets, sheaths or tubes. The second is to consider the additional global effect of the orientation of the feet of the sheet, shell or tube. In general, it is found to add different amounts of magnetic self-helicity to the two structures. The third part is to allow for twisting of the central region of the sheet, sheath or tube before reconnection.

The local effect of reconnection is demonstrated both for sheets, sheaths, and tubes to be one of equipartition. This is achieved by splitting the structure into a series of fibrils and then letting the fibrils reconnect, one after the other.

However, for the global effect, namely, the effect of the orientation of the feet of the structures after reconnection, the extra amounts of self-helicity added to the tubes differ and depend on their orientations. This is because what matters is the orientation of the feet relative to the new orientation of the flux tube after reconnection, which differ from the initial orientation before reconnection. For flux sheets, the orientation before reconnection needs to be chosen carefully. For flux sheaths, Wright's results are also verified, by either regarding the self-helicity solely in terms of twist and considering carefully the locations of the feet of the fibrils or by regarding it as writhe and using the so-called tantrix method.

If the central part of a flux sheet, sheath or tube is twisted before reconnection while the feet are held fixed, then there is no longer local helicity equipartition and the untwisted calculation needs to be modified to include additional contributions. However, sheets behave very differently from sheaths or tubes, since sheets can have their central parts rotated only by a multiple of π , whereas sheaths and tubes can be rotated by arbitrary angles. The result is that after reconnection an extra amount of twist has been added to one flux tube and subtracted from the other.

In the future, it will be interesting to conduct numerical experiments to test the applicability of the result, and in particular to produce much more realistic analysis of the twist acquired by an erupting flux rope during an eruptive solar flare or coronal mass ejection. In addition, the energetics of the configurations needs to be investigated, so as to decide which transitions are favourable and therefore which are likely to lead to reconnection, as discussed in Priest, Longcope, and Janvier (2016) and Priest and Longcope (2017). Furthermore, the results can be applied to build an improved model for zipper reconnection (Priest and Longcope, 2017) in order to explain the early stages of a flare or coronal mass ejection, where reconnection starts at one location and spreads along the polarity inversion line to create flare ribbons and a rising arcade of reconnected coronal loops.

Acknowledgements ERP is grateful to friends in Bozeman, where this research was initiated, and we also are grateful to Andrew Wright for helpful comments.

Disclosure of Potential Conflicts of Interest The authors declare that they have no conflicts of interest.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Arnol'd, V.I.: 1974, The asymptotic Hopf invariant and its applications. *Sel. Math. Sov.* **5**, 327.
- Berger, M.A.: 1999, Introduction to magnetic helicity. *Plasma Phys. Control. Fusion* **41**(12B), B167. [DOI](#). [ADS](#).
- Berger, M.A., Field, G.B.: 1984, The topological properties of magnetic helicity. *J. Fluid Mech.* **147**, 133.
- Berger, M.A., Prior, C.: 2006, The writhe of open and closed curves. *J. Phys. A, Math. Gen.* **39**(26), 8321. [DOI](#). [ADS](#).
- Canfield, R.C., Pevtsov, A.A.: 1998, Helicity of solar active-region magnetic fields. In: Balasubramaniam, K.S., Harvey, J.W., Rabin, M. (eds.) *Synoptic Solar Physics, ASP Conf. Ser.* **140**, 131.
- Close, R., Parnell, C.E., Priest, E.R., Longcope, D.W.: 2004, Recycling of the solar corona's magnetic field. *Astrophys. J.* **612**, L81. [DOI](#).
- Galsgaard, K., Nordlund, Å.: 1996, The heating and activity of the solar corona: I boundary shearing of an initially homogeneous magnetic field. *J. Geophys. Res.* **101**, 13445. [DOI](#).
- Heyvaerts, J., Priest, E.R.: 1984, Coronal heating by reconnection in DC current systems – a theory based on Taylor's hypothesis. *Astron. Astrophys.* **137**, 63.
- Linton, M.G., Priest, E.R.: 2003, Three-dimensional reconnection of untwisted magnetic flux tubes. *Astrophys. J.* **595**, 1259. [DOI](#).
- Linton, M.G., Dahlburg, R.B., Antiochos, S.K.: 2001, Reconnection of twisted flux tubes as a function of contact angle. *Astrophys. J.* **553**, 905. [DOI](#).
- Moffatt, H.K.: 1969, The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117.
- Moffatt, H.K.: 1978, *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press, Cambridge.
- Parker, E.N.: 1972, Topological dissipation and the small-scale fields in turbulent gases. *Astrophys. J.* **174**, 499. [DOI](#).
- Parnell, C.E., Priest, E.R.: 1995, A converging flux model for the formation of an X-ray bright point above a supergranule cell. *Geophys. Astrophys. Fluid Dyn.* **80**, 255. [DOI](#).
- Parnell, C.E., Priest, E.R., Titov, V.S.: 1994, A model for x-ray bright points due to unequal cancelling flux sources. *Solar Phys.* **153**, 217. [DOI](#).
- Pevtsov, A.A., Canfield, R.C., Metcalf, T.R.: 1995, Latitudinal variation of helicity of photospheric magnetic fields. *Astrophys. J.* **440**, L109. [DOI](#).
- Pevtsov, A.A., Canfield, R.C., Sakurai, T., Hagino, M.: 2008, On the solar-cycle variation of the hemispheric helicity rule. *Astrophys. J.* **677**, 719. [DOI](#).
- Priest, E.R., Longcope, D.W.: 2017, Flux-rope twist in eruptive flares and CMEs: due to zipper and main-phase reconnection. *Solar Phys.* **292**, 25. [DOI](#). [ADS](#).
- Priest, E.R., Heyvaerts, J., Title, A.: 2002, A Flux Tube Tectonics model for solar coronal heating driven by the magnetic carpet. *Astrophys. J.* **576**, 533. [DOI](#).
- Priest, E.R., Longcope, D.W., Janvier, M.: 2016, Evolution of magnetic helicity during eruptive flares and coronal mass ejections. *Solar Phys.* **291**, 2017. [DOI](#).
- Priest, E.R., Parnell, C.E., Martin, S.F.: 1994, A converging flux model of an X-ray bright point and an associated canceling magnetic feature. *Astrophys. J.* **427**, 459. [DOI](#).
- Taylor, J.B.: 1974, Relaxation of toroidal plasma and generation of reverse magnetic fields. *Phys. Rev. Lett.* **33**, 1139.
- Wilmot-Smith, A.L., Hornig, G., Pontin, D.I.: 2009, Magnetic braiding and quasi-separatrix layers. *Astrophys. J.* **704**, 1288. [DOI](#). [ADS](#).
- Woltjer, L.: 1958, A theorem on force-free fields. *Proc. Natl. Acad. Sci. USA* **44**, 489.
- Wright, A.N.: 2019, Partitioning of magnetic helicity in reconnected flux tubes. *Astrophys. J.* **878**(2), 102. [DOI](#). [ADS](#).
- Wright, A.N., Berger, M.A.: 1989, The effect of reconnection upon the linkage and interior structure of magnetic flux tubes. *J. Geophys. Res.* **94**, 1295. [DOI](#).