ORIGINAL RESEARCH

Generalized Poverty-gap Orderings

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Abstract
This paper provides a characterization of a new class of ordinal poverty measures that are defined by means of the aggregate generalized poverty gap. To be precise, we propose to use the sum of the differences between the transformed fixed poverty line and the transformed level of income of each person below the line as our measure. If the transformation is strictly concave, the resulting measure is strictly inequality averse with respect to the incomes of the poor. In analogy to some existing results on inequality measurement, we show that the only relative (scale-invariant) members of our class are based on strictly concave power functions or the natural logarithm. Moreover, we show that our measures allow for a useful decomposition that is akin to those examined in some earlier contributions. In an empirical analysis, we compare the logarithmic variant of our index to two well-established alternative orderings. Unlike numerous indices that appear in the earlier literature, ours do not explicitly depend on the number of poor or on the total population size, thereby ruling out any direct influence of the head-count ratio on poverty comparisons.

Keywords Poverty measurement · Poverty gap · Focus axiom · Ethical inequality index

1 Introduction

Poverty continues to be a pressing (and, more to the point, depressing) issue throughout the world. This paper is intended to contribute to the literature by suggesting a new criterion for its measurement. Sen's (1976) seminal contribution paved the way for numerous subsequent studies, many of which are summarized in Zheng's (1997; 2000) comprehensive surveys. A collection of more recent essays on the topic is provided in the volume edited

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by Betti and Lemmi (2014). We endeavor to make a compelling case that our proposal presents an attractive alternative to existing measures and, moreover, that it constitutes a desirable and important tool to be used in applied policy choices.

One of the cornerstones of poverty measurement is the well-established focus condition (Sen, 1976). Basically, this property requires that poverty does not depend on the incomes of the non-poor—that is, the incomes of those above and at the poverty line. Focus plays an important role in our approach as well, and we broaden its scope to address some additional issues that may emerge. To illustrate, suppose that an individual above or at the poverty line in a society moves to another society where (s)he is also among the non-poor. All else unchanged, the effect of this move on the recipient country is that overall population size increases and the incidence of poverty remains unchanged. According to most existing poverty measures, poverty in the recipient country decreases since they measure the seriousness of poverty on a per-capita basis. Moreover, if the measure in question possesses some mild continuity property, the level of poverty can decrease even if the move is accompanied by a regressive transfer from a poor person to the rich migrant (through his or her economic activities, for example). By the same token, poverty becomes higher in the country from which this rich person moved away. That is, the majority of the standard poverty measures are sensitive to the existence of rich people in a society. It is natural that the migration of poor people affects local poverty levels. However, is the sensitivity to the existence of non-poor individuals as illustrated above plausible? If the answer is in the affirmative, a possible approach to poverty reduction is to increase the number of rich people while leaving the situation of the poor unchanged. We consider this an ethically unattractive position and, therefore, propose an alternative approach that does not exhibit this feature. We note that there are other ways of approaching this issue. For example, poverty measures employed within the European Union are typically based on poverty lines that are not fixed; instead, these lines are given by 60% of a country's median income. But this method comes with severe shortcomings: for instance, country-specific poverty lines can easily generate situations in which considerably wealthier societies end up being assigned higher levels of poverty, a situation that we consider undesirable. Our approach does not suffer from this drawback.

Even if one is primarily concerned with the evolution of world poverty over time rather than with cross-national comparisons, the above illustration translates easily into a setting in which migration is replaced by someone above the poverty line being brought into existence. Again, the question is whether such a population change decreases poverty even though none of the poor is affected by this change. To be clear, we agree that adding a person with a high level of well-being or income to an otherwise unchanged population is desirable from an overall welfare perspective. But, as most—if not all—contributors in this area would readily agree, an increase in social welfare is not identical to a decrease in poverty. Therefore, our objection to the dependence of poverty on the existence of non-poor people does not in any way conflict with the postulate that there may be a positive welfare effect associated with increasing the number of those who are well-off in a society.

Policy makers are frequently confronted with the dilemma of having to allocate scarce resources to candidate recipients so that trade-offs have to be solved. The position that we endorse is that world poverty should be reduced as much as possible. Because this means that we care about the number and the situation of those who are poor, it should not matter where these poor are located. Poverty is a phenomenon that has devastating effects on people's lives. It is therefore a phenomenon that is experienced by people and not by abstract entities such as a country. As a point in case, a person can feel hunger; a country cannot. To us, this is a strong argument in favor of an approach that reduces the poverty of people,

no matter where they are located—which is what our proposal does. To illustrate, consider an example in which an international agency has funds available that can be spent on poverty alleviation in either of two countries A and B. For the sake of simplicity, suppose that all poor have the same level of income. Country A is large with a total population of 100 million, country B is much smaller with a total population of ten million. There are five million poor in country A and five million poor in country B. Assume that the total funds can be more effectively used in the large country so that if A receives the funds, there will be two million fewer poor. On the other hand, if the funds go to country B, only one million can be lifted out of poverty. Thus, in absolute terms, the poverty reduction that can be achieved in A is twice as high as in B. In relative terms, the poverty rate would be reduced from 5% to 3% in country A, whereas the poverty rate in country B would be reduced from 50% to 40% if it were the beneficiary. To us, the answer is clear: the policy that reduces world poverty the most is preferable. The more poor can be lifted out of poverty, the better—we reject the idea that people should be discriminated against on the basis of the place where they live.

Of course, if there are secondary effects that are associated with these transfers, they must be properly taken into consideration. Once all consequences of allocating the resources are properly accounted for, the choice is clear in our mind: if the poverty of individuals with respect to their incomes is the social issue to be addressed, world poverty should be reduced as much as possible given the resource constraint. Because only a person can experience the sentiment of being poor, focusing on a reduction in poverty rates instead can prevent us from reaching this objective. As a referee pointed out, some types of secondary effects may affect different countries in different ways. For instance, if the poor in a small country attempt to initiate a riot, this is more likely to lead to social upheaval than if the same number of poor in a significantly larger country engage in this form of action. This is a valid point but it seems that it cannot be addressed within the confines of poverty measurement that is based on individual income distributions. If issues such as social cohesion are to be included as a criterion (or even become the major criterion), a poverty indicator can no longer serve as an adequate measure; rather, a much more comprehensive measure that also depends on determinants other than an income distribution is called for. Note that this observation applies not only to our contribution but to everything that has ever been written on the assessment of poverty as reflected by individual incomes. Thus, while considerations of this nature are highly important and relevant for many aspects of the situation a society finds itself in, they are well outside the realm of income-based poverty measurement. Although we fully acknowledge that not everyone may share our viewpoint, we are confident that this position is ethically very appealing and that our proposal is highly relevant for policy purposes.

Insensitivity with respect to the existence of the non-poor is closely associated with the spirit of Sen's (1976) focus property. Sen's original axiom requires that a change in the income level of a rich person does not affect the poverty level as long as the rich individual in question does not drop below the poverty line. This type of insensitivity is satisfied by almost all poverty measures. We impose an extended version of the focus axiom, which requires that the existence of people above and at the poverty line does not affect the poverty level. The existence-insensitivity variant is stronger than the income-insensitivity version represented by the focus axiom. See also the contributions of Subramanian (2002), Chakravarty et al. (2006), and Subramanian (2012) for discussions of variants of the focus axiom and the existence of people above and at the poverty line.

As a starting point, we assume the poverty line to be fixed in the sense that it does not depend on the income distribution under consideration. This assumption is widely

employed in the theoretical literature on poverty measurement. Moreover, it is consistent with the practice of determining the poverty line based on basic needs; for instance, the official poverty line of the United States, that of Canada since 2018, and the International Poverty Line set by the World Bank are determined according to this convention. We are well aware that some contributions in the existing literature (particularly in the empirical branch) employ endogenous poverty lines; perhaps the most prominent examples are those in which the poverty line is determined as a proportion of an average income (such as the median) within a country or within a group of countries. The use of distribution-dependent poverty lines may be suitable, for instance, if the objective is to gain insights into problems that involve social exclusion. In our view, however, approaches of this nature suffer from potentially quite serious shortcomings, especially if the poverty line is determined as a percentage of an aggregate income measure in a group of countries. This method can have quite counterintuitive effects. In particular, there is a distinct possibility that the relative poverty ranking of two countries depends on whether another country is a member of the group—a consideration that can be quite relevant indeed, for example in the context of Brexit. Ravallion and Chen (2011, 2019) perform a global comparison of poverty by introducing an endogenous poverty line; they use a relative-income approach to arrive at a poverty line that is increasing in the average income in each country. However, Ravallion and Chen (2011, 2019) assume that a poverty line is globally fixed in the utility dimension, although the line is endogenous in the income dimension. Thus, their approach is compatible with our framework if a utility function is explicitly incorporated.

One response to the issue of the presence of non-poor people alluded to earlier may be to modify an existing measure by multiplying its values by the population size (see, for instance, Ravallion 2020). However, we consider it preferable to address the problem by conducting an explicit axiomatic analysis. We show that our extended focus axiom leads to what we call the aggregate generalized poverty gap when combined with some normatively plausible axioms for a poverty ranking. The aggregate generalized poverty gap uses the sum of the differences between the transformed fixed poverty line and the transformed level of income of each person below the line as the criterion to assess poverty. The use of such a poverty ranking yields a quite different picture from that provided by traditional measures that do not possess the existence-independence feature of our extended focus axiom. This is illustrated in Sect. 5 by means of applications to the United States and 19 developing countries.

We stress that, although our main interest lies in the analysis of poverty as a world phenomenon, nothing prevents us from applying our measures to specific countries (or, more generally, to specific geographic regions). As a consequence of their additive structure, the members of our class are such that inter-regional comparisons can be performed without any concerns regarding possible inconsistencies.

Our approach to poverty measurement is ordinal in the sense of Sen (1976). That is, we do not attach any numerical significance to the values of a poverty index—only statements such as "poverty in A is higher than poverty in B" are considered meaningful. Therefore, we do not need numerical poverty values; it suffices to measure poverty by means of an ordering with the interpretation "is associated with at least as much poverty as." Thus, we do not address the issue of poverty measurement with ordinal variables as examined, for example, by Bennett and Hatzimasoura (2011) and by Seth and Yalonetzky (2021); the variable we use—income—is continuous in nature.

The poverty orderings that we advocate are motived by Blackorby and Donaldson's (1984) critical-level generalized-utilitarian population principles; see also Blackorby et al. (2005). Critical-level utilitarianism uses as a social criterion the sum of the individual

differences between lifetime well-being (utility) and a fixed critical level that represents a minimally acceptable standard of living. The interpretation of the critical level is straightforward: the ceteris-paribus addition of a person to an existing population is socially desirable if and only if the lifetime utility of the new individual is above the critical level. The generalized counterpart of critical-level utilitarianism employs transformed utilities, thereby allowing for the possibility of incorporating inequality aversion. This principle has also been applied in the context of sufficientarian theories of justice. A sufficientarian criterion pays special attention to those below a given threshold level of well-being but may allow the well-being of those who are at or above the threshold to matter to a lesser extent. In contrast, a poverty measure focuses exclusively on those who are below the poverty line. Sufficientarianism was first explored by Frankfurt (1987) who proposed to use the (relative) number of those below the threshold to define a sufficientarian ranking of income or utility distributions. Frankfurt's approach has been criticized in numerous subsequent contributions, including those of Crisp (2003), Brown (2005), Casal (2007), Huseby (2012), and Hirose (2016), to name but a few. Once the critical level is reinterpreted as the threshold, it appears quite natural to define a sufficientarian criterion as a principle that is based on the aggregate (transformed) shortfalls from the threshold in the first instance and use the aggregate surplus of those above the threshold as a tie-breaking device. This approach is explored by Bossert et al. (2021a, 2021b).

Returning to the context of poverty measurement, the notion of aggregate generalized gaps appears again as a natural candidate to assess the situation of the poor. Unlike a sufficientarian principle, however, poverty measurement restricts exclusive attention to the levels of well-being or income of those below the poverty line—there is no reason to consult the part of the distribution that applies to the non-poor. This viewpoint is unambiguously reflected in Sen's (1976) focus axiom which we, along with the majority of the subsequent literature, fully endorse (and even strengthen, as hinted at earlier). To be specific, the poverty orderings we propose and characterize are based on the aggregate difference between the transformed poverty line and the transformed incomes of the poor. An important subclass is obtained if the strict Pigou-Dalton transfer principle (Pigou, 1912 and Dalton, 1920) for the poor is added as a requirement. In this case, the transformation applied to the incomes and to the poverty line must be strictly concave.

It has been understood for some time now that coarse measures such as the head-count ratio are inadequate to truly capture the severity and the hardships still suffered by too many on this planet and, therefore, it is difficult to find advocates of the exclusive use of this ratio as an indicator of poverty. However, the head-count ratio continues to play a role in combination with other attributes of an income distribution, such as various notions of the aggregate income gap of the poor. A novel feature of this paper is that we deviate from this practice and propose a class of poverty measures that do not directly depend on the number of poor or the total population size. Of course, our measures still satisfy the standard monotonicity requirements usually imposed on an index of poverty. In particular, if there is a ceteris-paribus increase in the number of poor, this will lead to an increase in the value of our poverty measures because the aggregate generalized poverty gap increases as a consequence of such a change. The only impact that is absent in the case of our measures is an additional increase that results directly from the increase in the number of those below the poverty line—or from the associated increase in the head-count ratio. But this seems to be a defensible attribute of an index: as long as the requisite monotonicity properties are respected (which certainly is true in this case), there is nothing wrong with the absence of an additional (direct) effect through the head-count ratio. We do, of course, acknowledge that ours is not the only position that one can assume in this respect.

Because our measures do not depend on the number of poor per se, they can be interpreted as reverse indicators of well-being for the poor. This allows for a plausible and straightforward link to be established between our indicators and ethical measures of inequality for those below the poverty line. We show that a useful decomposition result can be obtained in this context. Ours is certainly not the first contribution that examines the normative connections between poverty and inequality (for example, Blackorby and Donaldson, 1980, and Vaughan, 1987, are precursors in that regard) but the actual relationship between poverty and inequality is different in our setting as a consequence of the observation that our measures do not depend on attributes such as the head-count ratio.

Our basic definitions are presented in Sect. 2. Section 3 is devoted to an axiomatic foundation of the poverty measures that we propose. We introduce and discuss several properties (the axioms) that we consider appealing, followed by our characterization results. In particular, we provide an axiomatization of our new class and the subclasses that result if we add strict inequality aversion and scale invariance to our list of axioms. In Sect. 4, we illustrate a decomposition property that the members of our class possess. Section 5 provides applications by offering a comparison between some existing poverty measures and the logarithmic ordering that belongs to our class. Section 6 concludes this paper. The proofs of our results appear in Appendix 1, and the independence of the axioms used in our characterizations is established in Appendix 2.

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers. The set of all (all non-negative, all positive) real numbers is denoted by $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$). For $n \in \mathbb{N}$, we use $\mathbf{1}^n$ to denote the vector that consists of n ones. We adopt the notational convention that the sum taken over an empty set is equal to zero. An income distribution for a population of $n \in \mathbb{N}$ individuals is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}_{++}^n$. In order to allow for poverty comparisons across different time periods (and across different geographical regions if so desired), we permit population size to vary and, thus, we consider the set of income distributions given by the union $\Omega = \cup_{n \in \mathbb{N}} \mathbb{R}_{++}^n$. As is standard practice in the literature, we focus on positive incomes. Note, however, that our first two theorems can easily be extended to cover zero and negative incomes as well. Our third theorem, however, cannot be generalized in this manner because the scale-invariance property that we employ in this result is not suitable if incomes are not restricted to be positive. For this reason, we assume incomes to be above zero throughout.

To identify the set of those who are poor, we employ a fixed level of income that we use as the poverty line $z \in \mathbb{R}_{++}$. For the class of poverty orderings characterized here, it does not matter whether we use the weak definition of the poor or the strong definition of the poor. According to the weak definition, everyone below the poverty line is considered poor but those whose incomes are equal to the poverty line are not; in contrast, the strong definition of the poor counts those located at the poverty line among the poor as well. As discussed by Donaldson and Weymark (1986), the strong definition of the poor may lead to impossibility results when it comes to the compatibility of various sets of properties in a fixed-population setting. We use the weak definition of the poor but reiterate that the strong definition is equivalent in the context of the orderings we characterize. Thus, for a poverty line $z \in \mathbb{R}_{++}$, the set of the poor in a distribution $x = (x_1, \ldots, x_n) \in \Omega$ is given by

$$\mathcal{P}(z;x) = \{i \in \{1, \ldots, n\} \mid x_i < z\}.$$

Given a poverty line and an income distribution such that the set of the poor is non-empty, we use $x^p = (x_i)_{i \in \mathcal{P}(z;x)}$ to denote the subvector of $x \in \Omega$ that corresponds to the incomes of the poor for the poverty line $z \in \mathbb{R}_{++}$. Letting x^{np} denote the subvector of $x \in \Omega$ that corresponds to the non-poor, a distribution $x \in \Omega$ can be partitioned into these two constituent parts so that $x = (x^p, x^{np})$.

Although we work with poverty lines that do not depend on the distribution under consideration, it is important to allow this line to vary. This is necessary in order to formulate properties such as the well-established requirement that an indicator of poverty be relative. Thus, we define a poverty ordering R to be a transitive and complete binary relation on the set of pairs $((z; x), (z; y))$ with $z \in \mathbb{R}_{++}$ and $x, y \in \Omega$. For a poverty line $z \in \mathbb{R}_{++}$ and two income distributions $x, y \in \Omega$, we interpret $(z; x)R(z; y)$ to mean that the poverty level of x is higher than or equal to that of y, given the poverty line z. The asymmetric and symmetric parts of R are denoted by P and I, that is, for a poverty line z, we write $(z; x)P(z; y)$ to say that there is more poverty in x than in y, and $(z; x)I(z; y)$ indicates that the poverty levels for the distributions x and y are equal.

A poverty ordering R is a generalized poverty-gap ordering if there exists a continuous and increasing function $g : \mathbb{R}_{++} \to \mathbb{R}$ such that, for all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$,

$$(z;x)R(z;y) \Leftrightarrow \sum_{i \in \mathcal{P}(z;x)} \big(g(z) - g(x_i) \big) \geq \sum_{i \in \mathcal{P}(z;y)} \big(g(z) - g(y_i) \big).$$

If the strong definition of the poor is employed (that is, if those whose incomes are equal to the poverty line are also considered poor), the generalized poverty-gap criterion remains unchanged because the contributions to overall poverty of these individuals is equal to $g(z) - g(z) = 0$. This equivalence of the weak and strong definitions of the poor is attributable to the fact that our orderings do not directly depend on the number of the poor or on total population size.

Recall that, according to our convention, the sum $\sum_{i \in \mathcal{P}(z;x)} \big(g(z) - g(x_i) \big)$ assumes the value zero if $\mathcal{P}(z;x) = \emptyset$. Thus, if there are no poor in the distribution x for the poverty line z, poverty is as low as possible because the value of this sum is positive whenever the set of poor is non-empty.

The generalized poverty-gap orderings are inspired by the critical-level generalized-utilitarian population principles introduced by Blackorby and Donaldson (1984); see also Blackorby et al. (2005) for a comprehensive treatment.

3 Characterizations

This section provides an axiomatic justification of the class of generalized poverty-gap orderings by showing that they are the only poverty orderings that possess some ethically appealing properties. We begin by introducing the properties (or axioms) used in this paper and then establish our main characterization result. In addition, we illustrate how this class can be narrowed down by adding further desirable axioms.

As is standard in the literature on poverty measurement (and on social index numbers in general), we assume that the identities of the income recipients are ethically irrelevant. The axiom of anonymity formalizes this assumption.

Anonymity. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, and for all $x, y \in \mathbb{R}_{++}^n$, if y is obtained from x by applying a permutation to the components of x, then $(z; x)I(z; y)$.

The focus axiom is a standard property in fixed-population poverty measurement; see Sen (1976). It requires that the incomes of the non-poor do not influence the poverty ranking. We use a strengthening that applies in a variable-population setting. This strengthening, which implies that not only the incomes of the non-poor but also their existence are irrelevant for poverty assessments, is a key property for our argument.

Extended focus. For all $z \in \mathbb{R}_{++}$, for all $x \in \Omega$, for all $n, m \in \mathbb{N}$, for all $y \in \mathbb{R}_{++}^n$, and for all $w \in \mathbb{R}_{++}^m$, if $y_i \geq z$ for all $i \in \{1, \ldots, n\}$ and $w_i \geq z$ for all $i \in \{1, \ldots, m\}$, then

$$(z;x)I(z;(x, y)) \text{ and } (z;(x, y))I(z;(x, w)).$$

This axiom first appeared in Subramanian (2002) under the name of strong focus. We note that, by restricting y and w to be of the same population size, this axiom implies the original focus axiom, which requires that $(z; (x, y))I(z; (x, w))$ for all $z \in \mathbb{R}_{++}$, for all $x \in \Omega$, for all $n \in \mathbb{N}$, and for all $y, w \in \mathbb{R}_{++}^n$ such that $y_i \geq z$ and $w_i \geq z$ for all $i \in \{1, \ldots, n\}$. Consider the special case of the extended focus axiom that is obtained for $n = 1$ and $y = (z)$. If a given population is augmented by an individual whose income is equal to the poverty line, the resulting distribution is associated with the same level of poverty as the original. This feature is related to the observation that the generalized poverty-gap orderings can equivalently be expressed in terms of the strong definition of the poor. As mentioned earlier, extended focus implies that not only the incomes of the non-poor but also their existence are irrelevant for the poverty ranking. This property is in stark contrast with the nonpoverty growth axiom proposed by Kundu and Smith (1983), which requires poverty to decrease if a non-poor person is brought into existence.

Another standard requirement demands poverty to be decreasing in the income levels of the poor.

Strict monotonicity below the poverty line. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, for all $x, y \in \mathbb{R}_{++}^n$, and for all $i \in \{1, \ldots, n\}$, if $z > x_i > y_i$ and $x_j = y_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$, then $(z; y)P(z; x)$.

The next axiom is a separability property that is familiar from the literature on population ethics. For our purposes, it is sufficient to restrict its scope to fixed population sizes but it can be extended to cover variable-population situations as well; see, for example, Blackorby and Donaldson (1984) or Blackorby et al. (2005, Chapter 5). Independence properties of this nature are ubiquitous in the design of social index numbers, even if they are often not expressed explicitly but are nevertheless present because the indexes in question satisfy them. This is the case because they are at the root of an additive structure that is common to most indexes that are used in practice. The property is intuitively appealing because it allows us to treat the contribution of each poor individual separately, thereby eliminating the influence of people who are not affected by the choice of favoring one income distribution over another with respect to their relative degrees of poverty. A thorough discussion of the role of independence properties can be found in Blackorby et al. (2005), for example.

Fixed-number independence. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x, y \in \mathbb{R}_{++}^n$, and for all $w, s \in \mathbb{R}_{++}^m$,

$$(z;(x,w))R(z;(y,w)) \iff (z;(x,s))R(z;(y,s)).$$

There are existing poverty measures that satisfy various decomposability properties—a prime example are some of the measures of Foster et al. (1984) that can be expressed as a function of a normalized income gap, the head-count ratio, and a measure of entropy; see also Blackburn (1989) for a related decomposition of the Watts measure (Watts, 1968). This type of decomposition is different in nature because it is a decomposition into aggregates rather than a decomposition into individual contributions to poverty. We note that an independence property of the type that we use is also satisfied by these measures because of their additive structure.

Continuity is a well-established robustness property. It ensures that 'small' changes in an income distribution do not lead to 'large' changes in the poverty ordering. We restrict its scope to incomes below and at the poverty line because, by the extended focus axiom, the changes in incomes of those at or above the poverty line are irrelevant. Because the incomes at the poverty line as well as the incomes of the poor are covered by extended focus, our continuity axiom must reflect this attribute by including incomes at the poverty line as well. The generalized poverty-gap orderings embody this requirement because, as alluded to earlier, these criteria remain unchanged if the strong definition of the poor rather than the weak definition is employed. We note that a weaker version of the continuity property that applies to incomes below the poverty line only is not sufficient for our characterization results because additional criteria satisfy it. See Appendix 2 for a concrete example.

Continuity below and at the poverty line. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, and for all $x \in (0, z]^n$, the sets $\{y \in (0, z]^n \mid (z;y)R(z;x)\}$ and $\{y \in (0, z]^n \mid (z;x)R(z;y)\}$ are closed in $(0, z]^n$.

The next axiom is an independence condition with respect to the poverty line. Suppose that, in two distributions x and y of the same population size, everyone is poor for a given poverty line z. Now suppose that the poverty line is increased to $z' > z$. Everyone remains poor under the new poverty line, and it seems reasonable to require that the relative ranking of x and y is unchanged if the new poverty line z' rather than the original z applies. The following property captures this requirement. Unlike the properties introduced so far, it imposes restrictions across different poverty lines.

Poverty-line independence. For all $z, z' \in \mathbb{R}_{++}$ such that $z < z'$, for all $n \in \mathbb{N}$, and for all $x, y \in \mathbb{R}_{++}^n$, if $x_i < z$ and $y_i < z$ for all $i \in \{1, \dots, n\}$, then

$$(z;x)R(z;y) \iff (z';x)R(z';y).$$

Combined with extended focus, poverty-line independence has the practically relevant implication that changes in the poverty line do not influence poverty assessments as long as the poor and the non-poor remain the same. This seems to be an eminently plausible feature of a poverty measure if the purpose of a poverty line is indeed, as we believe it to be the case, to separate those who are poor from those who are not.

We also impose an equity property that applies to the incomes of those below the poverty line. The strict transfer principle (see Pigou, 1912 and Dalton, 1920) restricted to the poor requires that a transfer from a poor person to one who is even poorer leads to a reduction in poverty, provided that the relative ranks of the two individuals are unchanged and no one else's income is changed.

Principle of transfers below the poverty line. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, for all $x, y \in \mathbb{R}_{++}^n$, for all $i, j \in \{1, \ldots, n\}$ such that $y_j < z$, and for all $\varepsilon \in \mathbb{R}_{++}$, if $x_i = y_i + \varepsilon \le y_j - \varepsilon = x_j$ and $x_k = y_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$, then $(z; y)P(z; x)$.

Finally, scale invariance is a common property that requires a poverty ordering to be relative. It demands that poverty comparisons are unchanged if both the poverty line and the fixed-population income distribution under consideration are multiplied by a common positive constant. Because we operate in an ordinal framework in the sense of Sen (1976), this is the natural way of phrasing such a requirement; note that we do not employ a numerically significant index the value of which can be required to be insensitive with respect to equal proportional changes. The standard currency independence that is implied by a scale-invariance condition also applies here. If the poverty line and all incomes are multiplied by the same constant, all poverty comparisons remain unchanged—and this is precisely the unit-independence condition that can be expected from an ordering.

Scale invariance. For all $z \in \mathbb{R}_{++}$, for all $n \in \mathbb{N}$, for all $x, y \in \mathbb{R}_{++}^n$, and for all $\lambda \in \mathbb{R}_{++}$,

$$(\lambda z; \lambda x)R(\lambda z; \lambda y) \iff (z; x)R(z; y).$$

As is the case for poverty-line independence, scale invariance involves different poverty lines. We note in passing that Zheng's (2007) unit-consistency property is conceptually weaker because some poverty measures that are not relative satisfy it. In an ordinal framework, the scope of unit consistency is restricted to poorer-than comparisons but it does not apply to situations in which poverty is equal.

With the exception of extended focus and poverty-line independence, all of the above axioms are well-established in the literature on poverty measurement. For example, the Watts ordering (Watts, 1968) and the Foster-Greer-Thorbecke orderings with a parameter value above one (Foster et al., 1984) satisfy them. Moreover, the Watts ordering also satisfies poverty-line independence.

The first observation of this section establishes that the generalized poverty-gap orderings are the only orderings that satisfy the axioms of anonymity, extended focus, strict monotonicity below the poverty line, fixed-number independence, continuity below and at the poverty line, and poverty-line independence. If these properties are considered appealing, the result provides a strong argument in favor of the use of these orderings. As our second result shows, adding the principle of transfer below the poverty line restricts the class of possible generalized poverty-gap orderings to those with a strictly concave function g. Our final theorem identifies the scale-invariant members of this class. It turns out that these are the ones associated with a function g that is either a power or a logarithmic function. The second of these three results follows immediately so we only prove the first and the third, and we do so in Appendix 1.

Our first theorem characterizes the class of generalized poverty-gap orderings. This constitutes the main result of the paper.

Theorem 1 *A poverty ordering R satisfies anonymity, extended focus, strict monotonicity below the poverty line, fixed-number independence, continuity below and at the poverty line, and poverty-line independence if and only if R is a generalized poverty-gap ordering.*

If the principle of transfers below the poverty line is added to the axioms of Theorem 1, the function g must be strictly concave. This is an immediate consequence of the

well-known result that the principle of transfers is equivalent to strict concavity in the context of an additive criterion.

Theorem 2 *A poverty ordering R satisfies anonymity, extended focus, strict monotonicity below the poverty line, fixed-number independence, continuity below and at the poverty line, poverty-line independence, and the principle of transfers below the poverty line if and only if R is a generalized poverty-gap ordering with a strictly concave function g.*

Our final theoretical result illustrates the consequences of adding scale invariance to the axioms of Theorem 2. The following theorem identifies the subclass of generalized poverty-gap orderings that are based on functions g given by a strictly concave power function or by the natural logarithm. Again, the proof is provided in Appendix 1.

Theorem 3 *A poverty ordering R satisfies anonymity, extended focus, strict monotonicity below the poverty line, fixed-number independence, continuity below and at the poverty line, poverty-line independence, the principle of transfers below the poverty line, and scale invariance if and only if R is a generalized poverty-gap ordering such that either there exists $r \in (0, 1)$ with $g(t) = t^r$ for all $t \in \mathbb{R}_{++}$, or $g(t) = \ln(t)$ for all $t \in \mathbb{R}_{++}$.*

If g is a power function with parameter $r \in (0, 1)$, the generalized poverty-gap ordering of Theorem 3 is given by

$$(z;x)R(z;y) \iff \sum_{i \in \mathcal{P}(z;x)} \left(z^r - x_i^r \right) \geq \sum_{i \in \mathcal{P}(z;y)} \left(z^r - y_i^r \right)$$

for all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$. Such a member of the class of generalized poverty-gap orderings can be seen as a counterpart of the poverty ordering induced by Chakravarty's (1983) poverty index that is based on total numbers. In the case of the natural logarithm, the resulting generalized poverty-gap ordering is defined as

$$(z;x)R(z;y) \iff \sum_{i \in \mathcal{P}(z;x)} \left(\ln(z) - \ln(x_i) \right) \geq \sum_{i \in \mathcal{P}(z;y)} \left(\ln(z) - \ln(y_i) \right)$$

for all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$. This poverty ordering is a total-numbers-based counterpart of the Watts ordering (Watts 1968).

4 A Decomposition

In analogy to the decomposability property of the Foster-Greer-Thorbecke index alluded to earlier, it is possible to express each member of the class of generalized poverty-gap orderings in terms of the income-gap ratio (a canonical poverty measure) and a prominent measure of inequality among the poor. Suppose now that R is viewed as a reverse welfare indicator for those below the poverty line; this is an interpretation that is well in line with our approach. Because neither the income-gap ratio nor the measure of inequality that we utilize are well-defined if the set of poor is empty, we only consider poverty lines $z \in \mathbb{R}_{++}$ and distributions $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing$ in this section.

The income-gap ratio for a poverty line $z \in \mathbb{R}_{++}$ and an income distribution $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing$ is defined by

$$IGR(z;x) = \frac{\sum_{i \in \mathcal{P}(z;x)}(z - x_i)}{|\mathcal{P}(z;x)|z}.$$

In addition to the income-gap ratio, the Atkinson-Kolm-Sen measure of inequality appears in our decomposition analysis. To define this measure, we begin by introducing the notion of an equally-distributed-equivalent income for a poverty line $z \in \mathbb{R}_{++}$ and an income distribution $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \emptyset$. An income level $\xi^p \in \mathbb{R}_{++}$ is an equally-distributed-equivalent income of the poor for z and x if

$$\left(z; \xi^p \mathbf{1}^{|\mathcal{P}(z;x)|}\right) I(z;x^p). \tag{1}$$

That is, if everyone among the poor has the same income ξ^p, the resulting equal distribution among the poor is associated with the same level of poverty as the restriction of the original distribution x to those who are poor. Because $\mathcal{P}(z;x) \neq \emptyset$, the existence of a unique equally-distributed-equivalent income level for the poor is guaranteed whenever the ordering R satisfies extended focus, strict monotonicity below the poverty line, and continuity below and at the poverty line. This uniqueness allows us to express ξ^p as a function $\Xi^p(z;x)$ that depends on the poverty line z and the distribution under consideration x.

If R is a generalized poverty-gap ordering with a function g, (1) is equivalent to

$$|\mathcal{P}(z;x)|(g(z) - g(\xi^p)) = \sum_{i \in \mathcal{P}(z;x)} \left(g(z) - g(x_i)\right)$$

or, solving for ξ^p and using the uniqueness of the equally-distributed-equivalent income,

$$\xi^p = \Xi^p(z;x) = g^{-1}\left(\frac{1}{|\mathcal{P}(z;x)|} \sum_{i \in \mathcal{P}(z;x)} g(x_i)\right) \tag{2}$$

for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \emptyset$.

Letting, for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$,

$$\mu(x^p) = \frac{1}{|\mathcal{P}(z;x)|} \sum_{i \in \mathcal{P}(z;x)} x_i$$

denote the arithmetic mean of the incomes of the poor, the Atkinson-Kolm-Sen ethical index of inequality for the poor is defined as

$$I_{AKS}^p(z;x) = \frac{\mu(x^p) - \Xi^p(z;x)}{\mu(x^p)} = 1 - \frac{\Xi^p(z;x)}{\mu(x^p)} \tag{3}$$

for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \emptyset$; see Kolm (1969), Atkinson (1970), and Sen (1973). This index measures the percentage income shortfall of the poor that can be attributed to inequality among the poor. Because percentage shortfalls are employed in its definition, the Atkinson-Kolm-Sen index is based on relative considerations. An alternative relative ethical index of inequality is explored in Blackorby and Donaldson (1978).

We can now express a generalized poverty-gap ordering in terms of the income-gap ratio and the Atkinson-Kolm-Sen index of inequality for the poor. First, observe that, for all $z \in \mathbb{R}_{++}$ and for all $x \in \Omega$ such that $\mathcal{P}(z;x) \neq \emptyset$,

$$\mu(x^p) = z(1 - IGR(z;x))$$

and, thus, by (2) and (3),

$$\Xi^p(z;x) = \mu(x^p)\big(1 - I_{AKS}^p(z;x)\big) = z(1 - IGR(z;x))(1 - I_{AKS}^p(z;x)).$$

Using (1), we obtain

$$(z;x)R(z;y) \Leftrightarrow |\mathcal{P}(z;x)|\big[g(z) - g\big(z(1 - IGR(z;x))(1 - I_{AKS}^p(z;x))\big)\big]$$
$$\geq |\mathcal{P}(z;y)|\big[g(z) - g\big(z(1 - IGR(z;y))(1 - I_{AKS}^p(z;y))\big)\big]$$

for all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$ such that $\mathcal{P}(z;x) \neq \varnothing \neq \mathcal{P}(z;y)$. This expression demonstrates that any generalized poverty-gap ordering allows for a decomposition that employs the income-gap ratio and the Atkinson-Kolm-Sen index (provided, of course, that the set of poor is non-empty).

The approach outlined in this section differs from Blackorby and Donaldson's (1980) examination of the possible links between welfare, poverty, and inequality among the poor. In their contribution, Blackorby and Donaldson define a relative poverty index as a function of the head-count ratio and the Atkinson-Kolm-Sen index for the poor. Our relationship between poverty and inequality is similar but deviates from Blackorby and Donaldson's (1980) formulation because our poverty measures do not depend on the head-count ratio and, therefore, it is possible to interpret them as inverse welfare indicators for the poor. See also Vaughan (1987) for some links between welfare and poverty. Bossert (1990b) explores the relationships between Blackorby and Donaldson's (1980) classes of poverty measures and the replication-invariance principle.

5 Applications

We apply one of the scale-invariant generalized poverty-gap orderings identified in Theorem 3 to statistical data on income or consumption-expenditure distributions. Our main purpose is to compare the poverty evaluation according to our ordering with those obtained when using some standard relative ordinal indexes of poverty. Because one of the comparison measures is given by the Watts (1968) ordering that is based on a logarithmic criterion, it is natural to choose the generalized poverty-gap ordering that corresponds the function defined by $g(t) = \ln(t)$ for all $t \in \mathbb{R}_{++}$. We denote this ordering by $R_{\ln}$. The logarithmic measure is also a member of the class characterized by Chakravarty et al. (2006). Because all comparison measures are per-capita-based poverty orderings, an important difference between the features of those orderings and ours is how total population size is taken into account in poverty measurement. To highlight this aspect of their difference, we provide two applications in different practical contexts.

The first application is an intertemporal comparison of national poverty conducted for the United States. Using Waves III to XI of the Luxembourg Income Study (LIS) datasets, obtained from the Current Population Survey—Annual Social and Economic Supplement, referred to as the March Supplement until 2002, we compare the trend in poverty in the United States between 1991 and 2019 according to our logarithmic ordering with the trends that are obtained when using the comparison measures; see, for instance, Rodgers (1991) for an earlier and quite detailed comparison of poverty measures applied to US data. Recall that a fixed poverty line is employed in the USA and, thus, this constitutes a natural practical application of the approach to poverty measurement that we advocate here. In addition, in view of the population growth in the USA from 255 million in 1991 to 329 million in

2019 at annual rates between 0.5 and 1.4 per cent during this period, it is worth examining how this significant population growth affects intertemporal poverty comparisons.

The second application consists of a cross-national comparison of poverty that involves 19 developing countries. Using the World Bank's PovcalNet, which provides standard poverty indexes that are calculated from nationally representative household surveys, we compare national poverty in 19 developing countries in 2010 by employing the International Poverty Line. Those 19 countries are chosen so as to include both small-population and large-population countries since our purpose is to highlight the difference between our logarithmic ordering and the comparison measures in their treatment of total population size.

The poverty orderings with which we compare our logarithmic ordering are two members of the class of Foster-Greer-Thorbecke orderings (Foster et al., 1984) and the Watts ordering (Watts, 1968). The Foster-Greer-Thorbecke ordering R_{FGT}^{α} is defined by letting, for all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}_{++}^{n}$, and for all $y \in \mathbb{R}_{++}^{m}$,

$$(z;x)R_{FGT}^{\alpha}(z;y) \iff \frac{1}{n}\sum_{i\in\mathcal{P}(z;x)}\left(\frac{z-x_i}{z}\right)^{\alpha} \geq \frac{1}{m}\sum_{i\in\mathcal{P}(z;y)}\left(\frac{z-y_i}{z}\right)^{\alpha}$$

where $\alpha \in \mathbb{R}_+$ is a parameter. For $\alpha = 0$ and $\alpha = 1$, we obtain the head-count ratio and the poverty-gap ordering, respectively. If $\alpha = 2$, the squared poverty-gap ordering results. Unlike the head-count ratio and the poverty-gap ordering, the latter is distributionally sensitive. In general, a Foster-Greer-Thorbecke ordering satisfies the principle of transfers below the poverty line if and only if $\alpha > 1$. For our application, we employ the head-count ratio ordering R_{FGT}^{0} and the squared poverty-gap ordering R_{FGT}^{2}.

The Watts ordering R_W is closely related to the logarithmic generalized poverty-gap ordering $R_{\ln}$. It is defined by letting, for all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}_{++}^{n}$, and for all $y \in \mathbb{R}_{++}^{m}$,

$$(z;x)R_W(z;y) \iff \frac{1}{n}\sum_{i\in\mathcal{P}(z;x)}\left(\ln(z) - \ln(x_i)\right) \geq \frac{1}{m}\sum_{i\in\mathcal{P}(z;y)}\left(\ln(z) - \ln(y_i)\right).$$

That is, the Watts ordering is the per-capita counterpart of the logarithmic generalized poverty-gap ordering. Because of this analogy, the logarithmic generalized poverty-gap ordering seems to be a suitable choice for the comparisons carried out in this section. See Zheng (1993) for an axiomatization of a representation of the Watts index.

Although we only consider ordinal comparisons, we perform our computations using the specific representations of $R_{\ln}$, R_{FGT}^{0}, R_{FGT}^{2}, and R_W given by

$$\sum_{i\in\mathcal{P}(z;x)}\left(\ln(z) - \ln(x_i)\right),$$

$$\frac{|\mathcal{P}(z;x)|}{n},$$

$$\frac{1}{n}\sum_{i\in\mathcal{P}(z;x)}\left(\frac{z-x_i}{z}\right)^{2},$$

and

$$\frac{1}{n}\sum_{i\in\mathcal{P}(z;x)}\left(\ln(z) - \ln(x_i)\right).$$

We begin with the comparison of the four poverty orderings, $R_{\ln}$, R_{FGT}^0, R_{FGT}^2, and R_W, in light of the trends in poverty in the United States over the period from 1991 to 2019 that are obtained by these measures. As already noted, we use Waves III to XI of the LIS datasets, obtained from the Current Population Survey. These datasets are used by the United States Bureau of Census for estimating the official poverty rate, which is given by the head-count ratio R_{FGT}^0. Although the official poverty rate of the United States is calculated using gross incomes, we employ disposable household cash incomes and augment them by including in-kind food benefits (such as food stamps) to obtain measures that better reflect actual poverty. See Notten and de Neubourg (2011) and Smeeding (2006) for the use of disposable cash income in measuring poverty in the United States. These household incomes are equivalized by means of the square root equivalence scale; therefore, the unit of analysis is the individual. When calculating the poverty measures, missing and non-positive incomes are excluded.

The official poverty line of the United States is updated annually, using the Consumer Price Index for all Urban Consumers. Strictly speaking, there is a set of 48 poverty lines, each of which defines the poverty threshold for a specific type of family categorized by size and composition. Meanwhile, the weighted average poverty thresholds are also published by the United States Bureau of Census for all family sizes. These weighted average thresholds take into account how many (primary) families each threshold applies to. Since our primary purpose is the comparison of the four poverty measures, we use, for simplicity, the weighted average poverty threshold for a family of four. This figure is equivalized by means of the square root equivalence scale. As pointed out by numerous authors, the official poverty line of the United States is rather low compared to median income. Between 1991 and 2019, the ratio of the equivalized weighted average threshold to the median equivalized disposable cash income varies between 30 and 42 per cent.

The trends in poverty in the United States according to the four poverty measures are summarized in Fig. 1, where the poverty line employed is the official poverty threshold of the United States. We normalize the values of the poverty measures calculated for each year between 1991 and 2019 by dividing them by the corresponding 1991 values. As a remark aside, the trends obtained for the line that is given by 125 per cent of the equivalized weighted average poverty threshold (corresponding to 38 to 52 per cent of the median equivalized disposable cash income during the period) are very similar and we do not exhibit them here.

As Fig. 1 shows, poverty in the United States is increasing, in particular from 2000 until the late 2010s, if evaluated by $R_{\ln}$. By definition, the trends in poverty as measured by $R_{\ln}$ and R_W fluctuate synchronously. The fluctuations according to R_W are weaker in magnitude reflecting the population growth in the United States. While the poverty comparisons by $R_{\ln}$ and R_W are the same when comparing adjacent periods, their comparisons for more distant periods may differ. For instance, $R_{\ln}$ concludes that poverty in 2019 is higher than in 1991, whereas R_W reaches the opposite conclusion. In addition, R_W concludes that poverty in the United States from the late 1990s onward, except for a few years (such as 2014), is lower than in the early 1990s. An increasing trend in poverty in the United States is less obvious if R_{FGT}^2 is employed. In particular, R_{FGT}^2 concludes, more clearly than $R_{\ln}$, that poverty in the United States from the late 1990s onward (with the exception of 2014) is lower than in the early 1990s. Furthermore, the trend in poverty as measured by the head-count ratio is decreasing over the sampling period; the corresponding official poverty rates do not exhibit a clear decreasing trend (Semega et al., 2020). The decreasing trend in the head-count ratio suggests that the number of the poor may have decreased over the sampling
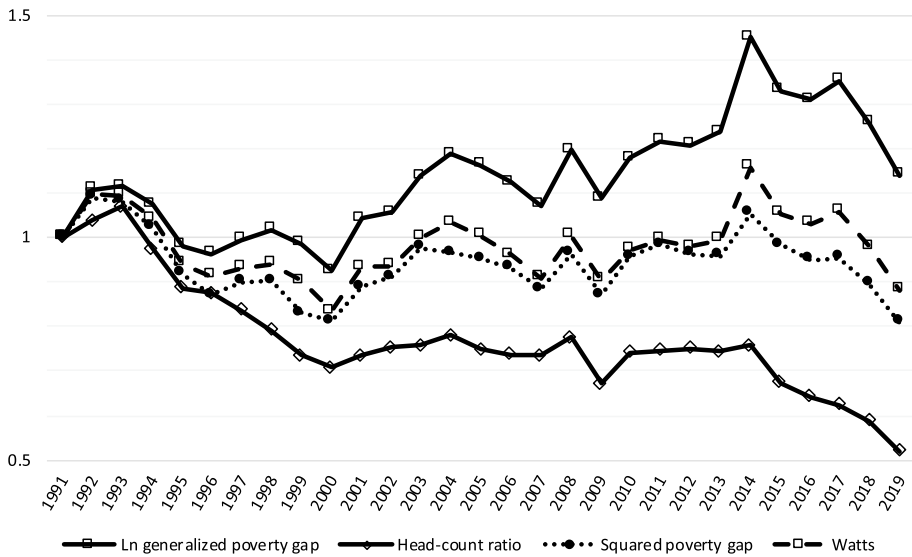
Fig. 1 Trends in poverty in the United States over the period 1991–2019

period. Indeed, the number of the poor has decreased from 29 million in 1991 to 19.5 million in 2019. Meanwhile, $R_{\ln}$ concludes that poverty in 2019 is higher than in 1991. In view of the reduction in the number of the poor, this means that the depth of poverty has increased. As seen from Fig. 1, this increase in the depth of poverty may be overlooked if either or both of R_W and R^2_{FGT} are employed, even together with the head-count ratio R^0_{FGT} and the number of the poor, because these per-capita-based measures discount the depth of poverty by population growth.

We note that, because they discount by population growth, the per-capita-based measures may conclude a reduction in poverty even though the number of the poor increases. Indeed, the population in the United States increased by 8.4 per cent from 290 million in 2003 to 314 million in 2012 and the number of the poor has also increased by 7.6 per cent from 25 million in 2003 to 27 million in 2012, so that the head-count ratio R^0_{FGT} has decreased accordingly over this period. Moreover, if evaluated by R_W and R^2_{FGT}, poverty in the United States in 2012 is lower than in 2003 by 2.0 per cent and 1.6 per cent, respectively. Meanwhile, as seen from Fig. 1, our logarithmic ordering $R_{\ln}$ concludes that poverty in the United States in 2012 is higher than in 2003. These observations suggest that our logarithmic ordering is useful in assessing long-term trends in poverty.

Next, we perform some cross-national comparisons of poverty in 19 developing countries in 2010. The 19 countries are chosen so as to include both small-population and large population countries in each of the three categories classified by the World Bank, namely, the low-income, lower-middle-income, and upper-middle-income countries (as of 2010); see Table 1. Using the World Bank's PovcalNet online calculator, we obtain the head-count ratio, the squared poverty gap, and the Watts measure calculated from the national household surveys of those 19 countries in terms of per capita household consumption expenditure; PovcalNet employs the per-capita equivalence scale to adjust for household size. Since consumption expenditure is used

instead of income, the minimum equivalized consumption expenditure is positive for each country. We set the poverty line at 1.90 USD per person per day; PovcalNet converts this line into local currencies through the 2011 purchasing power parity conversion factors, and those lines in local currency units are converted to 2010 values with intertemporal price deflators in each country. Our logarithmic measure is calculated by using the population data obtained from the Penn World Table version 10.0 (Feenstra et al. 2015).

The results are reported in Table 1. The poverty rankings of the 19 developing countries, ordered from the greatest poverty to the least, are presented in parentheses. The difference in the poverty rankings corresponding to $R_{\ln}$ and R_W is, by definition, to be attributed to whether we measure national poverty on a total or a per-capita basis. Note that the only difference between R_W and R^2_{FGT} in their poverty ranking of the 19 countries is the switch of adjacent ranks of Bangladesh and Nepal. Thus, their difference in the ways of measuring the depth of poverty does not make much of a difference in their rankings. Therefore, for the 19 countries we compared, the difference in the poverty rankings corresponding to $R_{\ln}$ and R^2_{FGT} can also be attributed to whether we measure national poverty on a total or a per-capita basis.

Table 1 Poverty indexes measured at \$1.9 (2011 PPP) per person per day in 2010

	Population	$R_{\ln}$	R^0_{FGT}	R^2_{FGT}	R_W
Low-income countries					
Bangladesh (BDG)	147.5754	6.0683 (6)	19.18 (8)	0.95 (11)	4.11 (10)
Gambia (GMB)	1.7932	0.1851 (14)	25.35 (6)	3.27 (5)	10.32 (5)
Madagascar (MDG)	21.1516	12.6569 (3)	78.18 (1)	22.03 (1)	59.84 (1)
Mali (MLI)	15.0494	3.1961 (8)	50.29 (4)	6.72 (4)	21.24 (4)
Nepal (NPL)	27.0132	1.0151 (9)	14.99 (10)	0.99 (10)	3.76 (11)
Lower-middle-income countries					
Egypt (EGY)	82.7612	0.2523 (11)	2.15 (15)	0.06 (16)	0.30 (16)
Georgia (GEO)	4.0991	0.2109 (12)	11.96 (12)	1.64 (9)	5.14 (9)
India (IND)	1234.2812	114.1710 (1)	32.81 (5)	2.49 (7)	9.25 (7)
Indonesia (IDN)	241.8342	6.4570 (5)	13.29 (11)	0.60 (12)	2.67 (12)
Nigeria (NGA)	158.5032	56.2918 (2)	56.40 (3)	12.62 (3)	35.51 (3)
Sri Lanka (LKA)	20.2617	0.1136 (15)	2.83 (14)	0.13 (14)	0.56 (14)
Vietnam (VNM)	87.9677	0.9031 (10)	4.03 (13)	0.27 (13)	1.03 (13)
Zambia (ZMB)	13.6060	7.2698 (4)	65.82 (2)	19.87 (2)	53.43 (2)
Upper-middle-income countries					
Namibia (NAM)	2.1189	0.2004 (13)	23.47 (7)	2.99 (6)	9.46 (6)
Russian Federation (RUS)	143.4793	0.0355 (18)	0.09 (19)	0.007 (18)	0.02 (18)
South Africa (ZAF)	51.2170	3.3547 (7)	16.17 (9)	2.05 (8)	6.55 (8)
Thailand (THA)	67.1950	0.0097 (19)	0.13 (18)	0.002 (19)	0.01 (19)
Tunisia (TUN)	10.6352	0.0524 (17)	2.00 (16)	0.12 (15)	0.49 (15)
Turkey (TUR)	72.3270	0.0673 (16)	0.77 (17)	0.01 (17)	0.09 (17)

For each of the logarithmic generalized poverty-gap ordering, the head-count ratio ordering, the squared povertygap ordering, and the Watts ordering, the corresponding poverty ranking of countries is presented in parentheses. The population size and the value of specific representation of the logarithmic generalized poverty-gap ordering are in units of million. The reported values of specific representations of the head-count ratio ordering, the squared poverty-gap ordering, and the Watts ordering are multiplied by 100

The differences in the poverty rankings generated by $R_{\ln}$ and each of the comparison measures, R_W, R^2_{FGT}, and R^0_{FGT}, are illustrated in Figs. 2, 3, and 4, respectively. For each point in one of these figures, the vertical coordinate is the rank according to $R_{\ln}$ and the horizontal coordinate is the rank according to each of the comparison measures. The size of each circle plotted represents the population size of the corresponding country. Note that, in Figs. 2, 3, and 4, for any two countries plotted in such a way that one is to the southeast of the other, the relative ranking of poverty for those two countries is reversed, depending on which of the respective two orderings is used for poverty measurement. For instance, Figs. 2 and 3 illustrate that, while R_W and R^2_{FGT} conclude that poverty in South Africa is higher than in Bangladesh, $R_{\ln}$ provides the opposite evaluation. Figures 2, 3, and 4 show that, while all of R_W, R^2_{FGT}, and R^0_{FGT} conclude that poverty in Georgia is higher than in Vietnam, $R_{\ln}$ indicates that poverty in Vietnam is worse. Not surprisingly, our logarithmic ordering tends to assess the poverty of a country with a larger population as being higher than that of a less populated country. Indeed, it concludes that poverty in India is the
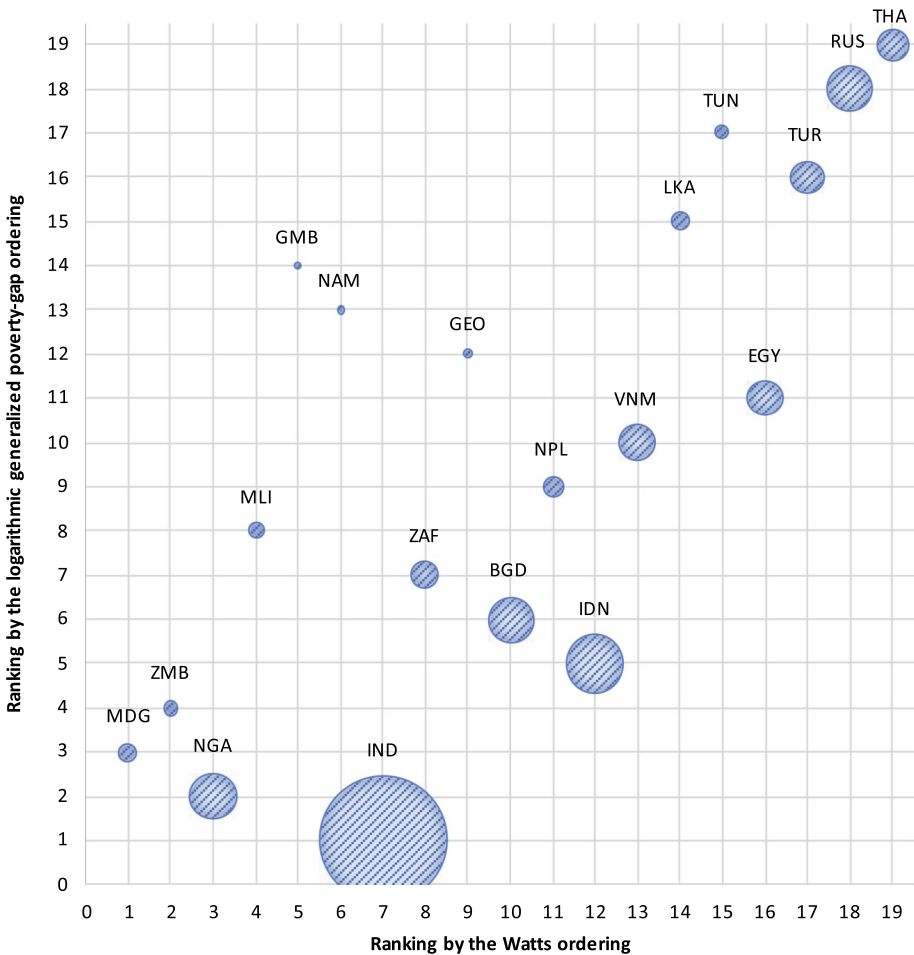
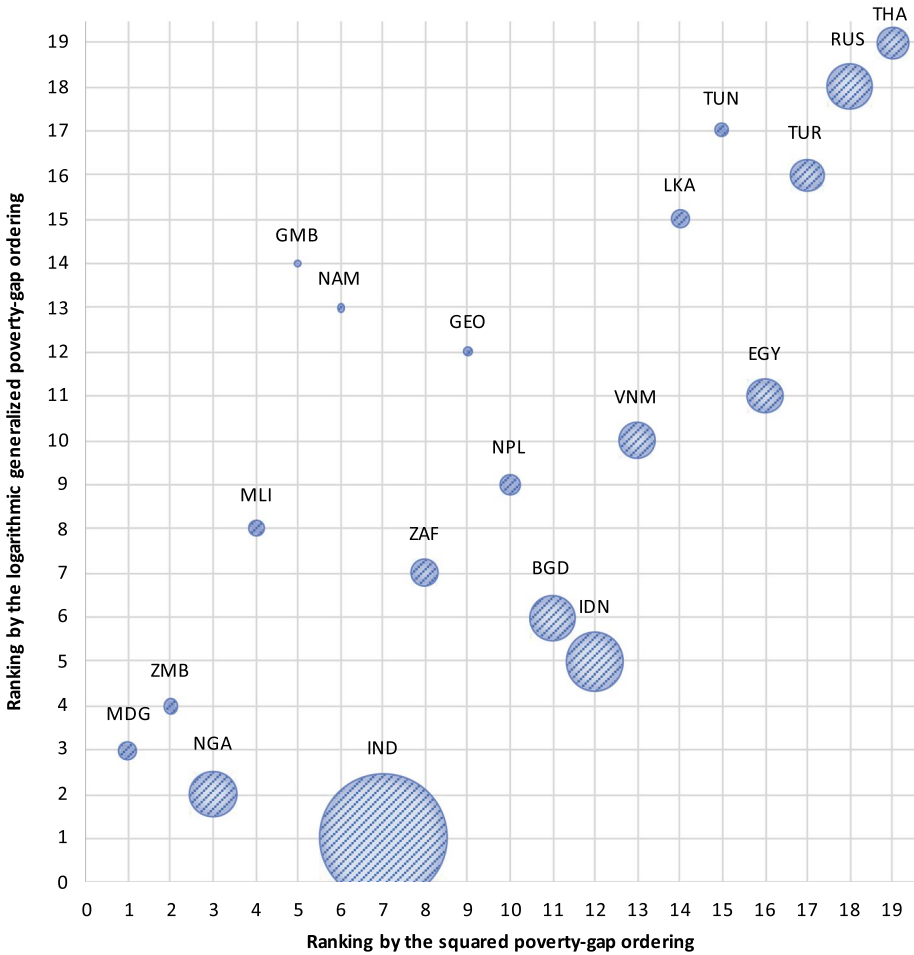**Fig. 2** Comparison of the poverty rankings by $R_{\ln}$ and R_W

Fig. 3 Comparison of the poverty rankings by $R_{\ln}$ and R^2_{FGT}

highest among the 19 countries. However, it should be noted that there are still instances in which our ordering ranks less populated countries as poorer than countries with a larger population; see, for example, the relative ranks of (i) Madagascar, Zambia, Mali, South Africa, and Nepal vis-à-vis Vietnam and Egypt; (ii) Madagascar and Zambia vis-à-vis Bangladesh and Indonesia; and (iii) Gambia, Namibia, and Georgia vis-à-vis Sri Lanka.

6 Conclusion

This paper provides a theoretical justification of and some empirical observations on generalized poverty-gap orderings. An important feature of these measures is that poverty levels are not affected by any changes in the number of non-poor people. In practical contexts, this insensitivity with respect to the existence of the non-poor implies that poverty should be measured without discounting by population growth or total population size. As we
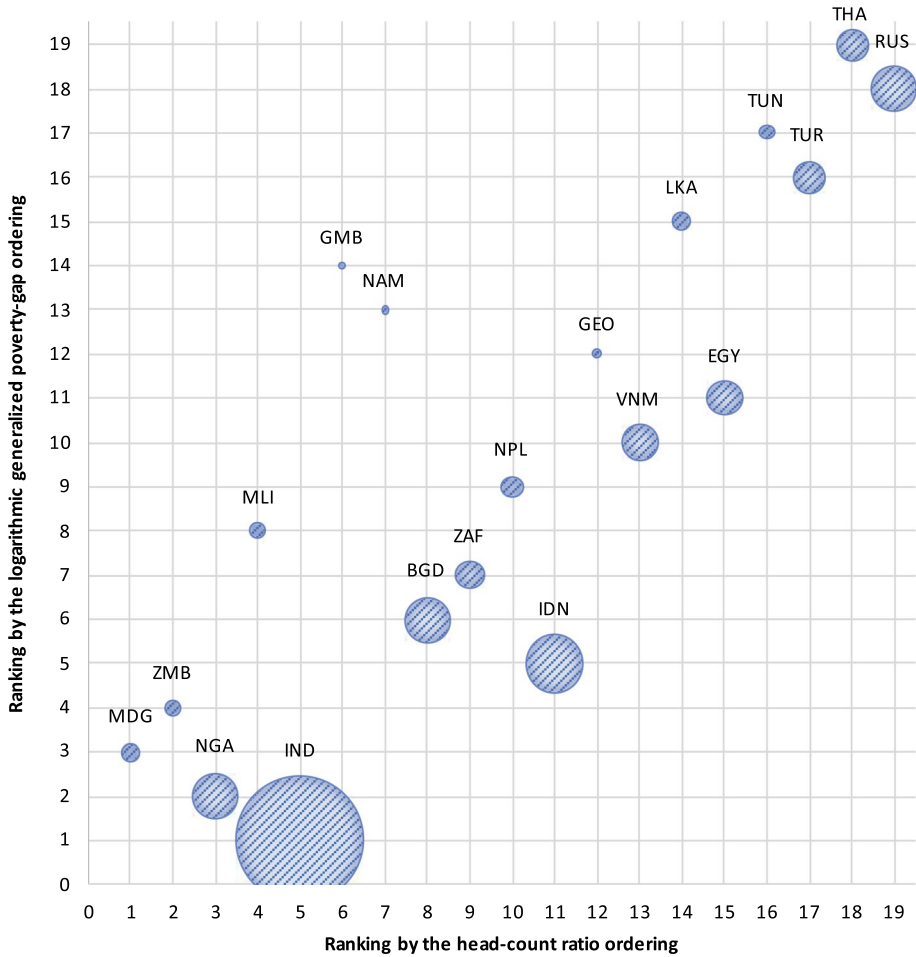
Fig. 4 Comparison of the poverty rankings by $R_{\ln}$ and R^0_{FGT}

demonstrated by applying the logarithmic generalized poverty-gap ordering to statistical data on income distributions in the United States, a long-term trend in poverty evaluated by our measure will be in stark contrast to those obtained by standard per-capita-based poverty measures if there is significant population growth. Analogously, cross-national poverty comparisons made by our measure are different from those obtained by per-capita-based measures because ours considers geographical or national aggregates of poverty. Global poverty can be assessed in a straightforward manner by adding these aggregates.

Appendix 1: Proofs

This appendix provides the proofs of Theorems 1 and 3; as mentioned in the main text, the result of Theorem 2 is immediate so that no formal proof is required.

Theorem 1 is proven by employing some observations from the literature on population ethics and adapting them to the poverty framework considered here; see also Huseby (2012) and Bossert et al. (2021b) for some links between population ethics and sufficientarian theories of justice. As a first step, we focus on income distributions such that everyone's income is below or at the poverty line. To this end, for all $z \in \mathbb{R}_{++}$ and for all $x = (x_1, \ldots, x_n) \in \Omega$, let

$$\mathcal{P}^s(z;x) = \{i \in \{1, \ldots, n\} \mid x_i \leq z\}$$

be the set of the poor and people whose incomes are at the poverty line, that is, the set of the poor according to the strong definition of the poor. Analogously to our notation, x^p and x^{np}, of the incomes of the poor and of the non-poor according to the weak definition of the poor, we use x^{ps} and x^{nps} to denote the subvectors of $x \in \Omega$ that correspond to the incomes of the poor and of the non-poor for a given poverty line z according to the strong definition of the poor. The following auxiliary result shows that the generalized poverty-gap criterion must apply in these situations.

Lemma 1 *If a poverty ordering R satisfies anonymity, extended focus, strict monotonicity below the poverty line, fixed-number independence, and continuity below and at the poverty line, then, for all $z \in \mathbb{R}_{++}$, there exists a continuous and increasing function $g^z : (0, z] \to \mathbb{R}$ such that, for all $x, y \in \Omega$,*

$$(z;x)R(z;y) \Leftrightarrow \sum_{i \in \mathcal{P}^s(z;x)} \left(g^z(z) - g^z(x_i) \right) \geq \sum_{i \in \mathcal{P}^s(z;y)} \left(g^z(z) - g^z(y_i) \right). \tag{4}$$

Proof Because none of the axioms in the lemma statement involve any changes in the poverty line, we can, without loss of generality, assume that $z \in \mathbb{R}_{++}$ is fixed. Setting $y = (z) \in \mathbb{R}_{++}$ in the first part of extended focus, it follows that $(z; x)I(z; (x, z))$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}_{++}^n$ (and, thus, for all $x \in (0, z]^n$). This means that z is a fixed critical level for R (see Blackorby and Donaldson, 1984). Combined with the axioms of anonymity, strict monotonicity below the poverty line, fixed-number independence, and continuity below and at the poverty line, this implies that there exists a continuous and increasing function $g^z : (0, z] \to \mathbb{R}$ such that

$$(z;x)R(z;y) \Leftrightarrow \sum_{i \in \mathcal{P}^s(z;x)} \left(g^z(x_i) - g^z(z) \right) \leq \sum_{i \in \mathcal{P}^s(z;y)} \left(g^z(y_i) - g^z(z) \right) \tag{5}$$

for all $x, y \in \Omega$. This observation is an immediate consequence of adapting the requisite results of Blackorby and Donaldson (1984) and Blackorby et al. (2005, Chapters 4 and 6) to our domain; the requisite strong monotonicity property employed in their results is implied by the conjunction of strict monotonicity below the poverty line and continuity below and at the poverty line. These earlier observations make use of a fundamental theorem of Gorman (1968) regarding the additive structure that results from separability properties such as that imposed by fixed-number independence; see also Blackorby et al. (1978, Theorem 4.7) for a detailed proof of Gorman's theorem. Note that a less-than-or-equal-to inequality is obtained in (5) because the requisite monotonicity property requires decreasingness rather than increasingness. Clearly, (5) is equivalent to (4). □

We now use this lemma to prove Theorem 1.

Proof of Theorem 1 That the generalized poverty-gap orderings satisfy anonymity, extended focus, strict monotonicity below the poverty line, fixed-number independence, continuity below and at the poverty line, and poverty-line independence is straightforward to verify.

Conversely, suppose that R is a poverty ordering that satisfies the axioms. Let $z \in \mathbb{R}_{++}$, $n, m \in \mathbb{N}$, $x \in \mathbb{R}_{++}^n$, and $y \in \mathbb{R}_{++}^m$. We can, without loss of generality, assume that both $\mathcal{P}^s(z;x)$ and $\mathcal{P}^s(z;y)$ are non-empty; this is the case because $(z; x)I(z; (x, z))$ by extended focus and, thus, $(z; x)R(z; y)$ if and only if $(z; (x, z))R(z; y)$ by transitivity. Of course, the same argument applies to y. Using our notational convention, we write x and y as $x = (x^{ps}, x^{nps})$ and $y = (y^{ps}, y^{nps})$. Note that x^{nps} or y^{nps} may be the empty vector, which is the case if everyone is poor in the requisite distribution. By extended focus and transitivity, $(z; x)R(z; y)$ if and only if $(z; x^{ps})R(z; y^{ps})$ and, by Lemma 1, it follows that (4) is true for all $z \in \mathbb{R}_{++}$ and for all $x, y \in \Omega$.

We now use poverty-line independence to show that the functions g^z can be chosen to be the same for all poverty lines $z \in \mathbb{R}_{++}$. To this end, let $z, z' \in \mathbb{R}_{++}$ be two poverty lines such that $z < z'$. Let $x \in (0, z)^2$ and define $a, b \in \mathbb{R}_{++}$ by

$$a = (g^z)^{-1}\left(\frac{g^z(x_1) + g^z(x_2)}{2}\right) \quad \text{and} \quad b = (g^{z'})^{-1}\left(\frac{g^{z'}(x_1) + g^{z'}(x_2)}{2}\right),$$

that is, a and b are the quasilinear means of x generated by g^z and $g^{z'}$, respectively. These are well-defined since g^z and $g^{z'}$ are continuous and increasing on $(0, z)$. Furthermore, a and b are located in the open interval $(0, z)$. Using (4), we obtain

$$(z;(a, a))I(z;x)$$

and

$$(z';(b, b))I(z';x).$$

By poverty-line independence, $(z; (a, a))I(z; x)$ implies that

$$(z';(a, a))I(z';x).$$

By the transitivity of R, we obtain

$$(z';(a, a))I(z';(b, b)).$$

This implies $a = b$ since R satisfies strict monotonicity below the poverty line. Thus, we obtain that, for all $x \in (0, z)^2$,

$$(g^z)^{-1}\left(\frac{g^z(x_1) + g^z(x_2)}{2}\right) = (g^{z'})^{-1}\left(\frac{g^{z'}(x_1) + g^{z'}(x_2)}{2}\right).$$

From Theorem 83 of Hardy, Littlewood, and Pólya (1934; 1952, p. 66), it follows that g^z and the restriction of $g^{z'}$ to $(0, z)$ must be increasing affine transformations of each other. Because the function g^z in (4) is unique up to increasing affine transformations, g^z and $g^{z'}$ can, without loss of generality, be chosen to be the same on $(0, z)$. Since this applies to all $z, z' \in \mathbb{R}_{++}$ with $z < z'$, there exists a continuous and increasing function $g : \mathbb{R}_{++} \to \mathbb{R}$ such that, for all $z, t \in \mathbb{R}_{++}$, $g^z(t) = g(t)$. Therefore, R must be a generalized poverty-gap ordering since the generalized poverty-gap orderings remain unchanged even if the strong definition of the poor is employed. □

We conclude this appendix by providing a proof of Theorem 3.

Proof of Theorem 3 As established in Theorem 2, all generalized poverty-gap orderings with a strictly concave function g (and, thus, the subclass identified in the statement of Theorem 3) satisfy the axioms of anonymity, extended focus, strict monotonicity below the poverty line, fixed-number independence, continuity below and at the poverty line, poverty-line independence, and the principle of transfers below the poverty line. It is straightforward to verify that the orderings based on the functions g of the statement also satisfy scale invariance.

To prove the converse implication, note first that R must be a generalized poverty-gap ordering with a strictly concave function $g : \mathbb{R}_{++} \to \mathbb{R}$ by virtue of Theorem 2. Let $z \in \mathbb{R}_{++}$ and $x \in (0, z)^2$. Let $a \in (0, z)$ denote the quasilinear mean of x generated by g, that is,

$$a = g^{-1} \left(\frac{g(x_1) + g(x_2)}{2} \right).$$

Since R is the generalized poverty-gap ordering associated with g, we obtain

$$(z;x)I(z;(a,a)).$$

From scale invariance, it follows that, for all $\lambda \in \mathbb{R}_{++}$,

$$(\lambda z; \lambda x)I(\lambda z; \lambda(a, a)).$$

Thus, we obtain that

$$g^{-1} \left(\frac{g(\lambda x_1) + g(\lambda x_2)}{2} \right) = \lambda a = g^{-1} \left(\frac{g(x_1) + g(x_2)}{2} \right) \lambda$$

for all $x \in (0, z)^2$ and for all $\lambda \in \mathbb{R}_{++}$. Since $z \in \mathbb{R}_{++}$ was chosen arbitrarily, we obtain that, for all $x \in \mathbb{R}_{++}^2$ and for all $\lambda \in \mathbb{R}_{++}$,

$$g^{-1} \left(\frac{g(\lambda x_1) + g(\lambda x_2)}{2} \right) = g^{-1} \left(\frac{g(x_1) + g(x_2)}{2} \right) \lambda.$$

Applying the second part of Theorem 2 in Section 3.1.3 of Aczél (1966, p. 153), the solution to this functional equation is given by

$$g(t) = \alpha \ln(t) + \beta \quad \text{or} \quad g(t) = \alpha t^r + \beta$$

for some $\alpha, r \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$. Since g is strictly concave, r must be less than one. Furthermore, by the definition of a generalized poverty-gap ordering, we can, without loss of generality, assume that $\alpha = 1$ and $\beta = 0$ so that the functions of the theorem statement result. □

Appendix 2: Independence of the axioms

The generalized poverty-gap ordering associated with the transformation $g(t) = t$ for all $t \in \mathbb{R}_{++}$ satisfies all of our axioms of Theorems 2 and 3 except the principle of transfers below the poverty line. If a generalized poverty-gap ordering is associated with the transformation $g(t) = -\exp(-t)$ for all $t \in \mathbb{R}_{++}$, it satisfies all of our axioms of Theorem 3 except scale invariance. Furthermore, the Watts ordering R_W satisfies all of our axioms of Theorems 1, 2, and 3 except extended focus. We show that each of the other five axioms of Theorems 1, 2, and 3 is independent.

We define the poverty ordering R^1 as follows. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}_{++}^n$, and for all $y \in \mathbb{R}_{++}^m$,

$$(z;x)R^1(z;y) \iff \sum_{i \in \mathcal{P}(z;x)} (\ln(\alpha_i) - \ln(x_i)) \geq \sum_{i \in \mathcal{P}(z;y)} (\ln(\alpha_i) - \ln(y_i)),$$

where $\alpha_1 = 2z$ and $\alpha_j = z$ for all $j \in \mathbb{N} \setminus \{1\}$. The poverty ordering R^1 satisfies all axioms of Theorems 1, 2, and 3 except anonymity. Letting $z \in \mathbb{R}_{++}$, consider $x = (2z, z/2)$ and $y = (z/2, 2z)$. Then, yP^1x follows since $\mathcal{P}(z;x) = \{2\}$, $\mathcal{P}(z;y) = \{1\}$, and $\ln(\alpha_2) - \ln(x_2) = \ln(z) - \ln(z/2) < \ln(2z) - \ln(z/2) = \ln(\alpha_1) - \ln(y_1)$.

For $z \in \mathbb{R}_{++}$, $n \in \mathbb{N}$, and $x \in \mathbb{R}_{++}^n$, let $x^* = (x_1^*, \dots, x_n^*)$ denote the censored income distribution corresponding to x defined by

$$x_i^* = \min\{x_i, z\}$$

for all $i \in \{1, \dots, n\}$. Define the poverty ordering R^2 by letting, for all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}_{++}^n$, and for all $y \in \mathbb{R}_{++}^m$,

$$(z;x)R^2(z;y) \iff \sum_{i=1}^{n} (z^2 - (x_i^*)^2) \leq \sum_{i=1}^{m} (z^2 - (y_i^*)^2).$$

The poverty ordering R^2 satisfies all axioms of Theorems 1, 2, and 3 except strict monotonicity below the poverty line.

For $n \in \mathbb{N}$ and for $x \in \mathbb{R}_{++}^n$, let $(x_{[1]}, \dots, x_{[n]})$ denote a permutation of x such that $x_{[1]} \leq \cdots \leq x_{[n]}$; that is, the incomes in such a permutation are ranked from lowest to highest, with ties being broken arbitrarily. We define the poverty ordering R^3 as follows. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}_{++}^n$, and for all $y \in \mathbb{R}_{++}^m$,

$$(z;x)R^3(z;y) \iff \sum_{i=1}^{n} \frac{1}{i}\left(z - x_{[i]}^*\right) \geq \sum_{i=1}^{m} \frac{1}{i}\left(z - y_{[i]}^*\right).$$

That is, the poverty ordering R^3 applies an absolute variant of a generalized Gini ordering (Gini 1912) with the weights $1/i$ to the shortfalls of censured incomes from a poverty line; see Mehran (1976), Donaldson and Weymark (1980), Weymark (1981), and Bossert (1990a) for various generalizations of the Gini ordering. The poverty ordering R^3 satisfies all axioms of Theorems 1, 2, and 3 except fixed-number independence.

We define the poverty ordering R^4 as follows. For all $z \in \mathbb{R}_{++}$, for all $n, m \in \mathbb{N}$, for all $x \in \mathbb{R}_{++}^n$, and for all $y \in \mathbb{R}_{++}^m$, $(z;x)R^4(z;y)$ if and only if

$$|\mathcal{P}(z;x)| > |\mathcal{P}(z;y)|$$

or

$$|\mathcal{P}(z;x)| = |\mathcal{P}(z;y)| \text{ and } \sum_{i\in\mathcal{P}(z;x)} (\ln(z) - \ln(x_i)) \geq \sum_{i\in\mathcal{P}(z;y)} (\ln(z) - \ln(y_i)).$$

That is, R^4 is the lexical composition of the comparison of the number of the poor and the logarithmic generalized poverty-gap ordering. The poverty ordering R^4 satisfies all axioms of Theorems 1, 2, and 3 except continuity below and at the poverty line. Note that R^4 also satisfies the weakening of continuity below and at the poverty line that applies to incomes below (but not at) the poverty line only. Thus, this example also serves to illustrate that the stronger version is indeed needed for our characterization results.

Finally, define the poverty ordering R^5 by letting, for all $z \in \mathbb{R}$ and for all $x, y \in \Omega$,

$$(z;x)R^5(z;y) \iff \sum_{i\in\mathcal{P}(z;x)} \left(\frac{z - x_i}{z}\right)^2 \geq \sum_{i\in\mathcal{P}(z;y)} \left(\frac{z - y_i}{z}\right)^2.$$

This is a variant of the squared poverty-gap ordering that measures poverty on a total basis. The poverty ordering R^5 satisfies all axioms of Theorems 1, 2, and 3 except poverty-line independence; for example, for $x = (9, 1)$ and $y = (4, 4)$, we obtain $(10;x)P^5(10;y)$ if $z = 10$, whereas $(15;y)P^5(15;x)$ follows if $z = 15$.

References

Aczél, J. (1966). *Lectures on functional equations and their applications*. London: Academic Press.

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory, 2,* 244–263.

Bennett, C. J., & Hatzimasoura, C. (2011). Poverty measurement with ordinal data. Working Paper 2011-14, George Washington University, Institute for International Economic Policy.

Betti, G., & Lemmi, A. (Eds.). (2014). *Poverty and social exclusion*. London: Routledge.

Blackburn, M. L. (1989). Poverty measurement: An index related to a Theil measure of inequality. *Journal of Business & Economic Statistics, 7,* 475–481.

Blackorby, C., Bossert, W., & Donaldson, D. (2005). *Population issues in social choice theory, welfare economics, and ethics*. New York: Cambridge University Press.

Blackorby, C., & Donaldson, D. (1978). Measures of relative equality and their meaning in terms if social welfare. *Journal of Economic Theory, 18,* 59–80.

Blackorby, C., & Donaldson, D. (1980). Ethical indices for the measurement of poverty. *Econometrica, 48,* 1053–1060.

Blackorby, C., & Donaldson, D. (1984). Social criteria for evaluating population change. *Journal of Public Economics, 25,* 13–33.

Blackorby, C., Primont, D., & Russell, R. R. (1978). *Duality, separability, and functional structure: theory and economic applications*. New York: North-Holland.

Bossert, W. (1990). An axiomatization of the single-series Ginis. *Journal of Economic Theory, 50,* 82–92.

Bossert, W. (1990). Population replications and ethical poverty measurement. *Mathematical Social Sciences, 20,* 227–238.

Bossert, W., Cato, S., & Kamaga, K. (2021a). Critical-level sufficientarianism. *Journal of Political Philosophy, forthcoming.* https://doi.org/10.1111/jopp.12267.

Bossert, W., Cato, S., & Kamaga, K. (2021b). Thresholds, critical levels, and generalized sufficientarian principles. *Economic Theory*, *forthcoming*.

Brown, C. (2005). Priority or sufficiency...or both? *Economics & Philosophy, 21,* 199–220.

Casal, P. (2007). Why sufficiency is not enough. *Ethics, 117,* 296–326.

Chakravarty, S. R. (1983). A new index of poverty. *Mathematical Social Sciences, 6,* 307–313.

Chakravarty, S. R., Kanbur, R., & Mukherjee, D. (2006). Population growth and poverty measurement. *Social Choice and Welfare, 26,* 471–483.

Crisp, R. (2003). Equality, priority, and compassion. *Ethics, 113,* 745–763.

Dalton, H. (1920). The measurement of the inequality of incomes. *Economic Journal, 30,* 348–361.

Donaldson, D., & Weymark, J. A. (1980). A single-parameter generalization of the Gini indices of inequality. *Journal of Economic Theory, 22,* 67–86.

Donaldson, D., & Weymark, J. A. (1986). Properties of fixed-population poverty indices. *International Economic Review, 27,* 667–688.

Feenstra, R. C., Inklaar, R., & Timmer, M. P. (2015). The next generation of the Penn World Table. *American Economic Review, 105,* 3150–3182.

Foster, J. E., Greer, J., & Thorbecke, E. (1984). A class of decomposable poverty measures. *Econometrica, 52,* 761–766.

Frankfurt, H. (1987). Equality as a moral ideal. *Ethics, 98,* 21–43.

Gini, C. (1912). *Variabilità e mutabilità. Contributo allo Studio delle Distribuzioni e delle Relazioni Statistiche*. Bologna: C. Cuppini.

Gorman, W. M. (1968). The structure of utility functions. *Review of Economic Studies, 35,* 367–390.

Hardy, G., Littlewood, J. E., & Pólya, G. (Eds.). (1952). *Inequalities* (2nd ed.). Cambridge: Cambridge University Press.

Hirose, I. (2016). Axiological sufficientarianism. In C. Fourie & A. Rid (Eds.), *What is enough? Sufficiency, justice, and health* (pp. 51–68). Oxford University Press.

Huseby, R. (2012). Sufficiency and population ethics. *Ethical Perspectives, 19,* 187–206.

Kolm, S.-C. (1969). The optimal production of social justice. In J. Margolis & H. Guitton (Eds.), *Public economics* (pp. 145–200). London: Macmillan.

Kundu, A., & Smith, T. E. (1983). An impossibility theorem on poverty indices. *International Economic Review, 24,* 423–434.

Mehran, F. (1976). Linear measures of income inequality. *Econometrica, 44,* 805–809.

Notten, G., & de Neubourg, C. (2011). Monitoring absolute and relative poverty: "Not enough'' is not the same as "much less''. *Review of Income and Wealth, 57,* 247–269.

Pigou, A. (1912). *Wealth and welfare*. Macmillan.

Ravallion, M. (2020). On measuring global poverty. *Annual Review of Economics, 12,* 167–188.

Ravallion, M., & Chen, S. (2011). Weakly relative poverty. *Review of Economics and Statistics, 93,* 1251–1261.

Ravallion, M., & Chen, S. (2019). Global poverty measurement when relative income matters. *Journal of Public Economics, 177,* 104046.

Rodgers, J. R. (1991). Does the choice of poverty index matter in practice? *Social Indicators Research, 24,* 233–252.

Semega, J., Kollar, M., Shrider, E. A., & Creamer, J. (2020). Income and poverty in the United States: 2019. Current Population Report P60–270, U.S. Census Bureau, Washington D.C. Available at https://www.census.gov/library/publications/2020/demo/p60-270.html.

Sen, A. (1973). *On economic inequality*. Oxford: Oxford University Press.

Sen, A. (1976). Poverty: An ordinal approach to measurement. *Econometrica, 44,* 219–231.

Seth, S., & Yalonetzky, G. (2021). Assessing deprivation with an ordinal variable: Theory and application to sanitation deprivation in Bangladesh. *World Bank Economic Review, 35,* 793–811.

Smeeding, T. (2006). Poor people in rich nations: The Unites States in comparative perspective. *Journal of Economic Perspectives, 20,* 69–90.

Subramanian, S. (2002). Counting the poor: An elementary difficulty in the measurement of poverty. *Economics and Philosophy, 18,* 277–285.

Subramanian, S. (2012). The focus axiom and poverty: On the co-existence of precise language and ambiguous meaning in economic measurement. *Economics: The Open-Access, Open-Assessment E-Journal, 6,* 1–21.

Vaughan, R. N. (1987). Welfare approaches to the measurement of poverty. *Economic Journal, 97,* 160–170.

Watts, H. (1968). An economic definition of poverty. In D. P. Moynihan (Ed.), *On understanding poverty* (pp. 316–329). New York: Basic Books.

Weymark, J. A. (1981). Generalized gini inequality indices. *Mathematical Social Sciences, 1,* 409–430.

Zheng, B. (1993). An axiomatic characterization of the Watts poverty index. *Economics Letters, 42,* 81–86.

Zheng, B. (1997). Aggregate poverty measures. *Journal of Economic Surveys, 11,* 123–162.

Zheng, B. (2000). Poverty orderings. *Journal of Economic Surveys, 14,* 427–466.

Zheng, B. (2007). Unit-consistent poverty indices. *Economic Theory, 31,* 113–142.