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Drift estimation for a Lévy-driven Ornstein–Uhlenbeck process with heavy tails

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Abstract

We consider the problem of estimation of the drift parameter of an ergodic Ornstein–Uhlenbeck type process driven by a Lévy process with heavy tails. The process is observed continuously on a long time interval $[0, T]$, $T \to \infty$. We prove that the statistical model is locally asymptotic mixed normal and the maximum likelihood estimator is asymptotically efficient.

Keywords Lévy process · Ornstein–Uhlenbeck type process · Local asymptotic mixed normality · Heavy tails · Regular variation · Maximum likelihood estimator · Asymptotic observed information

Mathematics Subject Classification 62M05 · 60F05 · 60J75

1 Introduction, motivation, previous results

In this paper, we deal with an estimation of the drift parameter $\theta > 0$ of an ergodic one-dimensional Ornstein–Uhlenbeck process X driven by a Lévy process:

$$X_t = X_0 - \theta \int_0^t X_s \, ds + Z_t, \quad t \geq 0.$$

The process Z is a one-dimensional Lévy process with known characteristics and with infinite variance. The process X is observed continuously on a long time interval $[0, T]$, $T \to \infty$.

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The problem is to study asymptotic properties of the corresponding statistical model and to show that the maximum likelihood estimator of θ is asymptotically efficient in an appropriate sense. Although the continuous time observations are far from being realistic in applications, they are of theoretical importance since they can be considered as a limit of high frequency discrete models.

Since we deal with continuous observations, it is natural to assume that the Gaussian component of the Lévy process Z is not degenerate. In this case, the laws of observations corresponding to different values of θ are equivalent and the likelihood ratio has an explicit form.

There are a lot of papers devoted to inference for Lévy driven SDEs. Most of the literature treats the case of discrete time observations both in the high and low frequency setting. A general theory for the likelihood inference for continuously observed jump-diffusions can be found in Sørensen (1991).

A complete analysis of the drift estimation for continuously observed ergodic and non-ergodic Ornstein–Uhlenbeck process driven by a Brownian motion can be found in Höpfner (2014, Chapter 8.1).

For continuously observed square integrable Lévy driven Ornstein–Uhlenbeck processes, the local asymptotic normality (LAN) of the model and the asymptotic efficiency of the maximum likelihood estimator of the drift have been derived by Mai (2012, 2014) with the help of the theory of exponential families, see Küchler and Sørensen (1997).

High frequency estimation of a square integrable Lévy driven Ornstein–Uhlenbeck process with non-vanishing Gaussian component has been performed by Mai (2012, 2014). Kawai (2013) studied the asymptotics of the Fisher information for three characterizing parameters of Ornstein–Uhlenbeck processes with jumps under low frequency and high frequency discrete sampling. The existence of all moments of the Lévy process was assumed. Tran (2017) considered the ergodic Ornstein–Uhlenbeck process driven by a Brownian motion and a compensated Poisson process, whose drift and diffusion coefficients as well as its jump intensity depend on unknown parameters. He obtained the LAN property of the model in the high frequency setting.

We also mention the works by Hu and Long (2007, 2009a, b), Long (2009) and Zhang and Zhang (2013) devoted to the least-square estimation of parameters of the Ornstein–Uhlenbeck process driven by an α-stable Lévy process.

There is vast literature devoted to parametric inference for discretely observed Lévy processes (see, e.g. a survey by Masuda (2015)) and Lévy driven SDEs. More results on the latter topic can be found e.g. in Masuda (2013), Ivanenko and Kulik (2014), Kohatsu-Higa et al. (2017), Masuda (2019), Uehara (2019), Clément and Gloter (2015), Clément et al. (2019), Clément and Gloter (2019), Nguyen (2018) and Gloter et al. (2018) and the references therein.

In this paper, we fill the gap and analyse a continuously observed ergodic Ornstein–Uhlenbeck process driven by a Lévy process with heavy regularly varying tails of index $-\alpha$, $\alpha \in (0, 2)$, in the presence of a Gaussian component. It turns out that the log-likelihood in this model is quadratic, however the model is not asymptotically normal and we prove only the local asymptotic mixed normality (LAMN) property. We refer to Le Cam and Yang (2000) and Höpfner (2014) for the general theory of estimation for LAMN models.

The fact that the prelimiting log-likelihood is quadratic automatically implies that the maximum likelihood estimator is asymptotically efficient in the sense of Jeganathan's convolution theorem and attains the local asymptotic minimax bound. Another feature of our model is that the asymptotic observed information has spectrally positive $\alpha/2$-stable distribution. This implies that the limiting law of the maximum likelihood estimator has tails of the order $\exp(-x^{\alpha})$ and hence finite moments of all orders.

The paper is organized as follows. In the next section we formulate the assumptions of our model and the main results of the paper. Section 3 contains auxiliary results that will be used in the proof of the main Theorem 2.5. In particular, we calculate the tail of a product of two iid heavy-tail random variables (Lemma 3.2), a conditional law of inter-arrival times of a Poisson process, and prove a technically involved Lemma 3.7. Eventually in Sect. 4, the proofs of the main results are presented.

2 Setting and the main result

Consider a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$, $\mathbb{F}$ being right-continuous. Let Z be a Lévy process with the characteristic triplet (σ^2, b, ν) and the Lévy–Itô decomposition

$$Z_t = \sigma W_t + bt + \int_0^t \int_{|z|\leq 1} z\tilde{N}(\mathrm{d}z, \mathrm{d}s) + \int_0^t \int_{|z|>1} zN(\mathrm{d}z, \mathrm{d}s), \tag{2.1}$$

where W is a standard one-dimensional Brownian motion, N is a Poisson random measure on $\mathbb{R}\backslash\{0\}$ with the Lévy measure ν satisfying $\int_{\mathbb{R}}(z^2 \wedge 1)\,\nu(\mathrm{d}z) < \infty$, $\tilde{N}$ is the compensated Poisson random measure, and $b \in \mathbb{R}$.

For $\theta \in \mathbb{R}$, let X be an Ornstein–Uhlenbeck type process being a solution of the SDE

$$X_t = X_0 - \theta \int_0^t X_s\,\mathrm{d}s + Z_t, \quad t \geq 0, \tag{2.2}$$

where $\theta \in \mathbb{R}$ is an unknown parameter. The initial value $X_0 \in \mathscr{F}_0$ is a random variable whose distribution does not depend on θ. Note that X has an explicit representation

$$X_t = X_0 \mathrm{e}^{-\theta t} + \int_0^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}Z_s, \quad t \geq 0,$$

see, e.g. Applebaum (2009, Sections 4.3.5 and 6.3) and Sato (1999, Section 17).

Let $\mathbb{D} = D([0,\infty), \mathbb{R})$ be the space of real-valued càdlàg functions $\omega\colon [0,\infty) \to \mathbb{R}$ equipped with Skorokhod topology and Borel σ-algebra $\mathscr{B}(\mathbb{D})$. The space $(\mathbb{D}, \mathscr{B}(\mathbb{R}))$ is Polish, and $\mathscr{B}(\mathbb{D})$ coincides with the σ-algebra generated by the coordinate projections. We define a (right-continuous) filtration $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ consisting of σ-algebras

$$\mathscr{G}_t := \bigcap_{s>t} \sigma\Big(\omega_r : r \leq s, \omega \in \mathbb{D}\Big), \quad t \geq 0.$$

For each $\theta \in \mathbb{R}$, the process $X = (X_t)_{t\geq 0}$ induces a measure $\mathbf{P}^\theta$ on the path space $(\mathbb{D}, \mathscr{B}(\mathbb{D}))$. Let

$$\mathbf{P}^\theta_T = \mathbf{P}^\theta\Big|_{\mathscr{G}_T}$$

be a restriction of $\mathbf{P}^\theta$ to the σ-algebra $\mathscr{G}_T$.

In order to establish the equivalence of the laws $\mathbf{P}^\theta_T$ and $\mathbf{P}^{\theta_0}_T$, $\theta, \theta_0 \in \mathbb{R}$, we have to make the following assumption.

$\mathbf{A}_\sigma$: The Brownian component of Z is non-degenerate, i.e. $\sigma > 0$.

Proposition 2.1 *Let $\mathbf{A}_\sigma$ hold true. Then for each $T > 0$, any $\theta, \theta_0 \in \mathbb{R}$*

$$\mathbf{P}^\theta_T \sim \mathbf{P}^{\theta_0}_T,$$

and the likelihood ratio is given by

$$L_T(\theta_0,\theta)=\frac{\mathrm{d}\mathbf{P}_T^{\theta}}{\mathrm{d}\mathbf{P}_T^{\theta_0}}=\exp\Big(-\frac{\theta-\theta_0}{\sigma^2}\int_0^T\omega_s\,\mathrm{d}m_s^{(\theta_0)}-\frac{(\theta-\theta_0)^2}{2\sigma^2}\int_0^T\omega_s^2\,\mathrm{d}s\Big), \tag{2.3}$$

where

$$\begin{aligned}m_t^{(\theta_0)}&=\omega_t-\omega_0+\theta_0\int_0^t\omega_s\,\mathrm{d}s-bt-\sum_{s\le t}\Delta\omega_s\mathbb{I}(|\Delta\omega_s|>1)\\&\quad-\int_0^t\int_{|x|\le 1}x\Big(\mu(\mathrm{d}x,\mathrm{d}s)-\nu(\mathrm{d}x)\mathrm{d}s\Big)\end{aligned}$$

is the continuous local martingale component of ω under the measure $\mathbf{P}_T^{\theta_0}$, and the random measure

$$\mu(\mathrm{d}x,\mathrm{d}s)=\sum_s\mathbb{I}(\Delta\omega_s\neq 0)\delta_{(\Delta\omega_s,s)}(\mathrm{d}x,\mathrm{d}s)$$

is defined by the jumps of ω.

Proof See Jacod and Shiryaev (2003, Theorem III-5-34). □

Consider a family of statistical experiments

$$\Big(\mathbb{D},\mathscr{G}_T,\{\mathbf{P}_T^{\theta}\}_{\theta>0}\Big)_{T>0}. \tag{2.4}$$

Our goal is to establish local asymptotic mixed normality (LAMN) of these experiments under the assumption that the process Z has heavy tails. We make the following assumption.
$\mathbf{A}_\nu$: The Lévy measure ν has a regularly varying heavy tail of the order $\alpha\in(0,2)$, i.e.

$$H(R):=\int_{|z|>R}\nu(\mathrm{d}z)\in\mathrm{RV}_{-\alpha},\quad R>0.$$

In other words, $H\colon(0,\infty)\to(0,\infty)$ and there is a positive function $l=l(R)$ slowly varying at infinity such that

$$H(R)=\frac{l(R)}{R^\alpha},\quad R>0.$$

Let us consider the function

$$\tilde H(R)=\alpha\int_R^\infty\frac{H(z)}{z}\,\mathrm{d}z,\quad R>0.$$

Since $H(z)>0$, $z>0$, the function $\tilde H$ is *absolutely continuous* and *strictly decreasing*. Moreover, by Karamata's theorem, see e.g. Resnick (2007, Theorem 2.1 (a)), applied to the function $\frac{H(z)}{z}\in\mathrm{RV}_{-\alpha-1}$ we obtain the equivalence

$$\lim_{R\to\infty}\frac{\tilde H(R)}{H(R)}=\lim_{R\to\infty}\frac{\alpha\displaystyle\int_R^\infty\frac{H(z)}{z}\,\mathrm{d}z}{H(R)}=1.$$

We use the function $\tilde H$ to introduce the *continuous* and *monotone increasing* scaling $\{\phi_T\}_{T>0}$ defined by the relation

$$\frac{1}{\phi_T}:=\tilde H^{-1}\Big(\frac{1}{T}\Big), \tag{2.5}$$

where $\tilde H^{-1}(R):=\inf\{u>0\colon\tilde H(u)=R\}$ is the (continuous) inverse of $\tilde H$. It is easy to see that $\phi_T\in\mathrm{RV}_{-1/\alpha}$.

Remark 2.2 We make use of the absolutely continuous and strictly decreasing function $\tilde{H}$ just for convenience in order to avoid technicalities connected with the inversion of càdlàg functions. For instance, one can equivalently define $\phi_T := \big(H^{\leftarrow}(1/T)\big)^{-1}$ for the generalized inverse $H^{\leftarrow}(R) := \inf\{u > 0\colon H(u) > R\}$, see Bingham et al. (1987, Chapter 1.5.7).

Remark 2.3 By Theorem 17.5 in Sato (1999), for each $\theta > 0$ the Ornstein–Uhlenbeck process X has an invariant distribution if $\int_{|z|>1} \ln|z|\,\nu(\mathrm{d}z) < \infty$. The latter inequality easily follows from Assumption $\mathbf{A}_\nu$ and Potter's bounds (3.2).

Example 2.4 Let the jump part of the process Z be an α-stable Lévy process, i.e. for $\alpha \in (0, 2)$ and $c_-, c_+ \geq 0$, $c_- + c_+ > 0$, let

$$\nu(\mathrm{d}z) = \Big(\frac{c_-}{|z|^{1+\alpha}}\mathbb{I}(z<0) + \frac{c_+}{z^{1+\alpha}}\mathbb{I}(z>0)\Big)\,\mathrm{d}z.$$

Then

$$H(R) = \tilde{H}(R) = \frac{c_- + c_+}{\alpha R^\alpha},$$
$$\tilde{H}^{-1}(T) = \Big(\frac{c_- + c_+}{\alpha}\Big)^{1/\alpha}\frac{1}{T^{1/\alpha}},$$

and

$$\phi_T = \Big(\frac{\alpha}{c_- + c_+}\Big)^{1/\alpha}\frac{1}{T^{1/\alpha}}.$$

The main result is the LAMN property of our model.

Theorem 2.5 *Let $\mathbf{A}_\sigma$ and $\mathbf{A}_\nu$ hold true. Then the family of statistical experiments (2.4) is locally asymptotically mixed normal at each $\theta_0 > 0$, namely for each $u \in \mathbb{R}$*

$$\mathrm{Law}\Big(\ln L_T(\theta_0, \theta_0 + \phi_T u)\Big|\mathbf{P}_T^{\theta_0}\Big) \to \mathcal{N}\sqrt{\frac{\mathcal{S}^{(\alpha/2)}}{2\sigma^2\theta_0}}u - \frac{1}{2}\frac{\mathcal{S}^{(\alpha/2)}}{2\sigma^2\theta_0}u^2, \quad T \to \infty,$$

where $\mathcal{N}$ is a standard Gaussian random variable and $\mathcal{S}^{(\alpha/2)}$ is an independent spectrally positive $\alpha/2$-stable random variable with the Laplace transform

$$\mathbf{E}\mathrm{e}^{-\lambda\mathcal{S}^{(\alpha/2)}} = \mathrm{e}^{-\Gamma(1-\frac{\alpha}{2})\lambda^{\alpha/2}}, \quad \lambda \geq 0. \tag{2.6}$$

Theorem 2.5 is based on the following key result.

Theorem 2.6 *Let $\mathbf{A}_\sigma$ and $\mathbf{A}_\nu$ hold true. Then for each $\theta_0 > 0$*

$$\mathrm{Law}\Big(\phi_T^2\int_0^T X_s^2\,\mathrm{d}s\Big|\mathbf{P}_T^{\theta_0}\Big) \to \frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}, \quad T \to \infty,$$

where $\mathcal{S}^{(\alpha/2)}$ is a random variable with the Laplace transform (2.6).

Corollary 2.7 *Let $\mathbf{A}_\sigma$ and $\mathbf{A}_\nu$ hold true. Then for each $\theta_0 > 0$*

$$\mathrm{Law}\Big(\phi_T\int_0^T X_s\,\mathrm{d}W_s, \phi_T^2\int_0^T X_s^2\,\mathrm{d}s\Big|\mathbf{P}_T^{\theta_0}\Big) \to \Big(\mathcal{N}\sqrt{\frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}}, \frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}\Big), \quad T \to \infty. \tag{2.7}$$

Proposition 2.1 and Theorem 2.5 allow us to establish asymptotic distribution of the maximum likelihood estimator $\hat{\theta}_T$ of θ. Moreover, the special form of the likelihood ratio guarantees that $\hat{\theta}_T$ is asymptotically efficient.

Corollary 2.8 1. *Let $\mathbf{A}_\sigma$ hold true. Then the maximum likelihood estimator $\hat\theta_T$ of θ satisfies*

$$\hat\theta_T = \theta_0 - \frac{\int_0^T \omega_s \,\mathrm{d}m_s^{(\theta_0)}}{\int_0^T \omega_s^2 \,\mathrm{d}s}. \tag{2.8}$$

2. *Let $\mathbf{A}_\sigma$ and $\mathbf{A}_\nu$ hold true. Then*

$$\operatorname{Law}\Big(\frac{\hat\theta_T-\theta_0}{\phi_T}\Big|\mathbf{P}_T^{\theta_0}\Big) \to \sigma\sqrt{2\theta_0}\cdot\frac{\mathcal{N}}{\sqrt{\mathcal{S}^{(\alpha/2)}}},\quad T\to\infty. \tag{2.9}$$

The maximum likelihood estimator $\hat\theta_T$ is asymptotically efficient in the sense of the convolution theorem and the local asymptotic minimax theorem for LAMN models, see Höpfner (2014, Theorems 7.10 and 7.12).

Remark 2.9 It is instructive to determine the tails of the random variable $\mathcal{N}/\sqrt{\mathcal{S}^{(\alpha/2)}}$: for each $\alpha\in(0,2)$

$$\limsup_{x\to+\infty} x^{-\alpha}\ln\mathbf{P}\Big(\frac{|\mathcal{N}|}{\sqrt{\mathcal{S}^{(\alpha/2)}}}>x\Big)<0, \tag{2.10}$$

and in particular all moments of the r.h.s. of (2.9) are finite.

3 Auxiliary results

We decompose the Lévy process Z into a compound Poisson process with heavy jumps, and the rest. Consider the non-decreasing function $R_T = T^\rho\colon [1,\infty)\to[1,\infty)$, where $\rho\geq 0$ will be chosen later.

Denote

$$\begin{aligned}
\eta_t^T &= \int_0^t\int_{|z|>R_T} zN(\mathrm{d}z,\mathrm{d}s),\\
\xi_t^T &= \sigma W_t + \int_0^t\int_{|z|\leq R_T} z\tilde N(\mathrm{d}z,\mathrm{d}s),\\
b_T &= b + \int_{1<|z|\leq R_T} z\nu(\mathrm{d}z),\\
Z_t^T &= Z_t - \eta_t^T = \xi_t^T + b_T t.
\end{aligned}$$

For each $T\geq 1$, the process η^T is a compound Poisson process with intensity $H(R_T)$, the iid jumps $\{J_k^T\}_{k\geq1}$ occurring at arrival times $\{\tau_k^T\}_{k\geq1}$, such that

$$\begin{aligned}
&\mathbf{P}(|J_k^T|\geq z) = \frac{H(z)}{H(R_T)},\quad z\geq R_T,\\
&\mathbf{P}(\tau_{k+1}^T-\tau_k^T>u) = \mathrm{e}^{-H(R_T)u},\quad u\geq 0.
\end{aligned}$$

Denote also by N^T the Poisson counting process of η^T; it is a Poisson process with intensity $H(R_T)$.

We decompose the Ornstein–Uhlenbeck process X into a sum

$$\begin{aligned} X_t &= X_t^T + X_t^{\eta^T}, \\ X_t^T &:= X_0 \mathrm{e}^{-\theta t} + \int_0^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}Z_s^T, \\ X_t^{\eta^T} &:= \int_0^t \mathrm{e}^{-\theta(t-s)}\,\mathrm{d}\eta_s^T. \end{aligned} \tag{3.1}$$

Since $H(\cdot) \in \mathrm{RV}_{-\alpha}$ and $\phi_\cdot \in \mathrm{RV}_{-1/\alpha}$, $\alpha \in (0,2)$, by Potter's bounds (see, e.g. Resnick (2007, Proposition 2.6 (ii)) for each $\varepsilon > 0$ there are constants $0 < c_\varepsilon \le C_\varepsilon < \infty$ such that for $u \ge 1$

$$\begin{aligned} \frac{c_\varepsilon}{u^{\alpha+\varepsilon}} &\le H(u) \le \frac{C_\varepsilon}{u^{\alpha-\varepsilon}}, \\ \frac{c_\varepsilon}{u^{\frac{1}{\alpha}+\varepsilon}} &\le \phi_u \le \frac{C_\varepsilon}{u^{\frac{1}{\alpha}-\varepsilon}}. \end{aligned} \tag{3.2}$$

The following Lemma gives useful asymptotics of the truncated moments of the Lévy measure ν.

Lemma 3.1 1. *For $\alpha \in (0,1]$ and any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that*

$$\int_{1<|z|\le R} |z|\nu(\mathrm{d}z) \le C(\varepsilon) R^{1-\alpha+\varepsilon}. \tag{3.3}$$

2. *For $\alpha \in (1,2)$ there is $C > 0$ such that*

$$\int_{1<|z|\le R} |z|\nu(\mathrm{d}z) \le C. \tag{3.4}$$

3. *For $\alpha \in (0,2)$ and any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that*

$$\int_{1<|z|\le R} z^2\nu(\mathrm{d}z) \le C(\varepsilon) R^{2-\alpha+\varepsilon}. \tag{3.5}$$

Proof To prove the first inequality we integrate by parts and note that for any $\varepsilon > 0$

$$\begin{aligned} \int_{1<|z|\le R} |z|\nu(\mathrm{d}z) = -\int_{(1,R]} z\,\mathrm{d}H(z) = -zH(z)\Big|_1^R + \int_{(1,R]} H(z)\,\mathrm{d}z \\ \le H(1) + C_\varepsilon \int_1^R \frac{\mathrm{d}z}{z^{\alpha-\varepsilon}}. \end{aligned}$$

Hence (3.3) follows for any $\varepsilon > 0$ and (3.4) is obtained if we choose $\varepsilon \in (0, \alpha - 1)$. The estimate (3.5) is obtained analogously to (3.3). □

The next Lemma will be used to determine the tail behaviour of the product of any two independent normalized jumps $|J_k^T||J_l^T|/R_T^2$, $k \ne l$.

Lemma 3.2 *Let $U_R \ge 1$ and $V_R \ge 1$ be two independent random variables with probability distribution function*

$$\mathbf{P}(U_R > x) = \mathbf{P}(V_R > x) = \bar{F}_R(x) = \frac{H(xR)}{H(R)}, \quad R \ge 1, \quad x \ge 1.$$

Then for each $\varepsilon \in (0,\alpha)$ there is $C(\varepsilon) > 0$ such that for all $R \ge 1$ and all $x \ge 1$

$$\mathbf{P}(U_R V_R > x) \le \frac{C(\varepsilon)}{x^{\alpha-\varepsilon}}.$$

Proof Recall that Potter's bounds Resnick (2007, Proposition 2.6 (ii)) imply that for each $\varepsilon > 0$ there is $C_0(\varepsilon) > 0$ such that for each $x \geq 1$ and $R \geq 1$

$$\bar{F}_R(x) = \frac{H(xR)}{H(R)} \leq \frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}}.$$

Moreover,

$$\bar{F}_R(x) \equiv 1, \quad x \in [0, 1].$$

For $x > 1$ we write

$$\begin{aligned}\mathbf{P}(U_R V_R > x) &= \int_1^\infty \int_{x/u}^\infty \mathrm{d}F_R(v)\, \mathrm{d}F_R(u) \\ &= \Big(\int_1^x + \int_x^\infty\Big) \bar{F}_R(x/u)\, \mathrm{d}F_R(u) = I_R^{(1)}(x) + I_R^{(2)}(x).\end{aligned}$$

Then

$$I_R^{(2)}(x) = \int_x^\infty \bar{F}_R(x/u)\, \mathrm{d}F_R(u) \leq \int_x^\infty \mathrm{d}F_R(u) \leq \bar{F}_R(x) \leq \frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}}.$$

Eventually,

$$\begin{aligned}I_R^{(1)}(x) &\leq \frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}} \int_1^x u^{\alpha-\varepsilon}\, \mathrm{d}F_R(u) = -\frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}} \int_1^x u^{\alpha-\varepsilon}\, \mathrm{d}\bar{F}_R(u) \\ &= -\frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}} u^{\alpha-\varepsilon} \bar{F}_R(u)\Big|_1^x + (\alpha-\varepsilon)\frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}} \int_1^x u^{\alpha-1-\varepsilon} \bar{F}_R(u)\, \mathrm{d}u \\ &\leq \frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}} + (\alpha-\varepsilon)\frac{C_0(\varepsilon)^2}{x^{\alpha-\varepsilon}} \int_1^x \frac{u^{\alpha-1-\varepsilon}}{u^{\alpha-\varepsilon}}\, \mathrm{d}u \\ &\leq \frac{C_0(\varepsilon)}{x^{\alpha-\varepsilon}} + (\alpha-\varepsilon)\frac{C_0(\varepsilon)^2}{x^{\alpha-\varepsilon}} \ln x \\ &\leq \frac{C(\varepsilon)}{x^{\alpha-2\varepsilon}}\end{aligned}$$

for some $C(\varepsilon) > 0$. □

Remark 3.3 A finer tail asymptotics of products of iid non-negative Pareto type random variables can be found in Rosiński and Woyczyński (1987, Theorem 2.1) and Jessen and Mikosch (2006, Lemma 4.1 (4)). In Lemma 3.2, however, we establish rather rough estimates which are valid for the families of iid random variables $\{U_R, V_R\}_{R\geq 1}$.

The following useful Lemma will be used to determine the conditional distribution of the interarrival times of the compound Poisson process η^T.

Lemma 3.4 *Let $T > 0$ and let $N = (N_t)_{t\in[0,T]}$ be a Poisson process, $\{\tau_k\}_{k\geq 1}$ be its arrival times, $\tau_0 = 0$. Then for each $m \geq 1$, and $1 \leq j < j + k \leq m$*

$$\mathbf{P}(\tau_{j+k} - \tau_j \leq s | N_T = m) = \mathbf{P}\Big(\sigma_k \leq \frac{s}{T}\Big), \quad s \in [0, 1], \tag{3.6}$$

where σ_k is a Beta$(m, k-1)$*-distributed random variable with density*

$$f_{\sigma_k}^{(m)}(u) = \frac{m!}{(k-1)!(m-k)!} u^{k-1}(1-u)^{m-k}, \quad u \in [0, 1], \quad m \geq 1,\ 1 \leq k \leq m. \tag{3.7}$$

Proof It is well known that the conditional distribution of the arrival times $\tau_1, \dots, \tau_m$, given that $N_T = m$, coincides with the distribution of the order statistics obtained from m samples from the population with uniform distribution on $[0, T]$, see Sato (1999, Proposition 3.4).

Let, for brevity, $T = 1$. The joint density of (τ_j, τ_{j+k}), $1 \le j < j+k \le m$ is well known, see e.g. Balakrishnan and Nevzorov (2003, Chapter 11.10):

$$f^{(m)}_{\tau_j,\tau_{j+k}}(u, v) = c_{j,k,m} \cdot u^{j-1}(v-u)^{k-1}(1-v)^{m-j-k}\mathbb{I}(0 \le u < v \le 1),$$
$$c_{j,k,m} = \frac{m!}{(j-1)!(k-1)!(m-j-k)!},$$

and consequently

$$f^{(m)}_{\tau_{j+k}-\tau_j,\tau_j}(u, v) = c_{j,k,m} \cdot v^{j-1}u^{k-1}(1-u-v)^{m-j-k}, \quad u, v, u+v \in [0, 1].$$

Hence, the probability density of the difference $\tau_{j+k} - \tau_j$ is obtained by integration w.r.t. $v \in [0, 1]$,

$$f^{(m)}_{\tau_{j+k}-\tau_j}(u) = c_{k,j,m} \cdot u^{k-1} \int_0^{1-u} v^{j-1}(1-u-v)^{m-j-k}\,\mathrm{d}v$$
$$\overset{v=(1-u)z}{=} c_{j,k,m} \cdot u^{k-1} \cdot (1-u)^{m-k} \int_0^1 z^{j-1}(1-z)^{m-j-k}\,\mathrm{d}z.$$

Recalling the definition of the Beta-function, we get

$$\int_0^1 z^{j-1}(1-z)^{m-j-k}\,\mathrm{d}z = \frac{(j-1)!(m-j-k)!}{(m-k)!},$$

which yields the desired result. □

Lemma 3.5 *Let $\mathbf{A}_\nu$ hold true and $\{\phi_T\}$ be the scaling defined in (2.5). Then for any $\rho \in [0, \frac{1}{\alpha})$*

$$\phi_T^2[\eta^T]_T \xrightarrow{\mathrm{d}} \mathcal{S}^{(\alpha/2)}, \quad T \to \infty,$$

where $\mathcal{S}^{(\alpha/2)}$ is a spectrally positive $\alpha/2$-stable random variable with Laplace transform (2.6).

Proof The process $t \mapsto \phi_T^2[\eta^T]_t$ is a compound Poisson process with Lévy measure ν_T with the tail

$$H_T(u) = \int_u^\infty \nu_T(\mathrm{d}z) = H\Big(\frac{\sqrt{u}}{\phi_T} \vee R_T\Big), \quad u > 0.$$

The Laplace transform of $\phi_T^2[\eta^T]_T$ has the cumulant

$$K_T(\lambda) := \ln \mathbf{E}\mathrm{e}^{-\lambda\phi_T^2[\eta^T]_T} = -T \int_0^\infty \Big(\mathrm{e}^{-\lambda u} - 1\Big)\,\mathrm{d}H_T(u), \quad \lambda \ge 0.$$

Integrating by parts yields

$$K_T(\lambda) = -T\big(\mathrm{e}^{-\lambda u} - 1\big)H_T(u)\Big|_0^\infty - \lambda T \int_0^\infty \mathrm{e}^{-\lambda u} H_T(u)\,\mathrm{d}u. \tag{3.8}$$

Since the first summands on the r.h.s. of (3.8) vanish, it is left to evaluate the integral term. Taking into account (2.5), namely that $\frac{1}{T} = \tilde{H}(\frac{1}{\phi_T})$, we write for any $u_0 > 0$

$$\begin{aligned} K_T(\lambda) &= -\lambda T \int_0^\infty \mathrm{e}^{-\lambda u} H_T(u)\,\mathrm{d}u \\ &= -\lambda T \int_0^{u_0} \mathrm{e}^{-\lambda u} H\Big(\frac{\sqrt{u}}{\phi_T} \vee R_T\Big)\,\mathrm{d}u \\ &\quad -\lambda \frac{H(1/\phi_T)}{\tilde{H}(1/\phi_T)} \frac{1}{H(1/\phi_T)} \int_{u_0}^\infty \mathrm{e}^{-\lambda u} H\Big(\frac{\sqrt{u}}{\phi_T} \vee R_T\Big)\,\mathrm{d}u \\ &= -I_T^{(1)}(\lambda) - I_T^{(2)}(\lambda). \end{aligned}$$

It is evident that $\lim_{T\to\infty} \frac{H(1/\phi_T)}{\tilde{H}(1/\phi_T)} = 1$. Moreover for $\rho \in [0, 1/\alpha)$ due to Resnick (2007, Proposition 2.4), the convergence

$$\lim_{T\to\infty} \frac{H\big(\frac{\sqrt{u}}{\phi_T} \vee R_T\big)}{H\big(\frac{1}{\phi_T}\big)} = \frac{1}{u^{\alpha/2}}$$

holds uniformly on each half-line $[u_0, \infty)$, $u_0 > 0$, and thus for each $u_0 > 0$

$$\lim_{T\to\infty} I_T^{(2)}(\lambda) = \lambda \int_{u_0}^\infty \frac{\mathrm{e}^{-\lambda u}}{u^{\alpha/2}}\,\mathrm{d}u.$$

Further we estimate

$$I_T^{(1)}(\lambda) \le 2\lambda T \phi_T^2 \int_0^{\sqrt{u_0}/\phi_T} y H(y)\,\mathrm{d}y.$$

Note that $y \mapsto yH(y)$ is integrable at 0 by the definition of the Lévy measure, $0 \le -\int_0^1 y^2 \mathrm{d}H(y) < \infty$, and the integration by parts. Eventually by Karamata's theorem (Resnick 2007, Theorem 2.1 (a))

$$\begin{aligned} I_T^{(1)} &\le 2\lambda \frac{H(1/\phi_T)}{\tilde{H}(1/\phi_T)} \cdot \frac{\phi_T^2}{H(1/\phi_T)} \cdot \frac{\int_0^{\sqrt{u_0}/\phi_T} yH(y)\,\mathrm{d}y}{\frac{u_0}{\phi_T^2} H(\frac{\sqrt{u_0}}{\phi_T})} \cdot \frac{u_0}{\phi_T^2} \cdot H\Big(\frac{\sqrt{u_0}}{\phi_T}\Big) \\ &\to \frac{2\lambda}{2-\alpha} u_0^{1-\frac{\alpha}{2}}, \quad T \to \infty. \end{aligned}$$

Hence choosing $u_0 > 0$ sufficiently small and letting $T \to \infty$ we obtain the convergence of K_T to the cumulant of a spectrally positive stable random variable

$$\lim_{T\to\infty} K_T(\lambda) = -\lambda \int_0^\infty \frac{\mathrm{e}^{-\lambda u}}{u^{\alpha/2}}\,\mathrm{d}u = -\Gamma\Big(1 - \frac{\alpha}{2}\Big)\lambda^{\alpha/2}.$$

□

Lemma 3.6 *For any* $\rho \in [0, 1/\alpha)$ *and any* $\theta > 0$

$$\begin{aligned} &\phi_T^2 |X_T^T|^2 \xrightarrow{\mathrm{d}} 0, \quad T \to \infty, \\ &\phi_T^2 \textstyle\int_0^T |X_s^T|^2\,\mathrm{d}s \xrightarrow{\mathrm{d}} 0, \quad T \to \infty. \end{aligned} \tag{3.9}$$

Proof

$$|X_s^T|^2 \le 3|X_0|^2 \mathrm{e}^{-2\theta s} + 3\mathrm{e}^{-2\theta s}\Big|\int_0^s \mathrm{e}^{\theta r}\,\mathrm{d}\xi_r^T\Big|^2 + 3b_T^2 \mathrm{e}^{-2\theta s}\Big|\int_0^s \mathrm{e}^{\theta r}\,\mathrm{d}r\Big|^2$$
$$= a_1(s) + a_2(s) + a_3(s).$$

By the Itô isometry and Lemma 3.1, for any $\varepsilon > 0$ we estimate for each $s \ge 0$

$$\phi_T^2 \mathbf{E} a_2(s) = \phi_T^2 \cdot \frac{3}{2\theta}\Big(\sigma^2 + \int_{|z|\le R_T} z^2 \nu(\mathrm{d}z)\Big)\cdot \mathrm{e}^{-2\theta s}(\mathrm{e}^{2\theta s} - 1)$$
$$\le C_1 \cdot T^{-\frac{2}{\alpha}+\varepsilon+\rho(2-\alpha+\varepsilon)}. \tag{3.10}$$

Analogously, Lemma 3.1 yields

$$b_T^2 \le \begin{cases} C_2 T^{2\rho(1-\alpha+\varepsilon)}, & \alpha \in (0,1],\\ C_2, & \alpha \in (1,2).\end{cases}$$

and hence for each $s \ge 0$

$$\phi_T^2 \cdot a_3(s) \le C_3 \max\{1, T^{2\rho(1-\alpha+\varepsilon)}\}\cdot T^{-\frac{2}{\alpha}+\varepsilon} \to 0. \tag{3.11}$$

Finally, for $s \ge 0$

$$\phi_T^2 a_1(s) \le C_4 |X_0|^2 \cdot T^{-\frac{2}{\alpha}+\varepsilon} \to 0 \quad \text{a.s. as } T \to +\infty. \tag{3.12}$$

For any $\rho \in [0, 1/\alpha)$ we can choose $\varepsilon > 0$ sufficiently small such that the bounds in (3.10) and (3.11) and (3.12) converge to 0 as $T \to \infty$ which gives (3.9). Integrating these inequalities w.r.t. $s \in [0, T]$ results in an additional factor T on the r.h.s. of these estimates, and convergence to 0 still holds true for $\varepsilon > 0$ sufficiently small. □

Lemma 3.7 *For any $\rho > \frac{1}{2\alpha}$ and any $\theta > 0$*

$$\phi_T^2 \int_0^T X_{s-}^{\eta^T}\,\mathrm{d}\eta_s^T \xrightarrow{\mathrm{d}} 0, \quad T \to \infty.$$

Proof The Ornstein–Uhlenbeck process X^{η^T} as well as its integral w.r.t. η^T can be written explicitly in the form of sums:

$$X_t^{\eta^T} = \sum_{j=1}^{\infty} J_j^T \mathrm{e}^{-\theta(t-\tau_j^T)} \mathbb{I}_{[\tau_j^T,\infty)}(t),$$
$$X_{\tau_k^T -}^{\eta^T} = \sum_{j=1}^{k-1} J_j^T \mathrm{e}^{-\theta(\tau_k^T - \tau_j^T)}, \quad k \ge 1,$$
$$\int_0^T X_{s-}^{\eta^T}\,\mathrm{d}\eta_s^T = \sum_{k=1}^{N_T^T} X_{\tau_k^T -}^{\eta^T} J_k^T = \sum_{k=1}^{N_T^T} J_k^T \sum_{j=1}^{k-1} J_j^T \mathrm{e}^{-\theta(\tau_k^T - \tau_j^T)}.$$

As always, we agree that $\sum_{j=k}^{m} = 0$ for $m < k$.

Note that for $N_T^T = 0$ and $N_T^T = 1$, $\int_0^T X_{s-}^{\eta^T}\,\mathrm{d}\eta_s^T = 0$. For $m \ge 2$, on the event $N_T^T = m$ we get the estimate

$$\Big|\int_0^T X_{s-}^{\eta^T}\,\mathrm{d}\eta_s^T\Big| \le \sum_{k=2}^{m} |J_k^T| \sum_{j=1}^{k-1} |J_j^T| \mathrm{e}^{-\theta(\tau_k^T - \tau_j^T)} = \sum_{j=1}^{m-1}\sum_{k=1}^{m-j} |J_{j+k}^T||J_j^T| \mathrm{e}^{-\theta(\tau_{j+k}^T - \tau_j^T)}. \tag{3.13}$$

We also take into account that for all $m \geq 2$ and $1 \leq j < j+k \leq m$

$$\mathrm{Law}\left(|J^T_{j+k}||J^T_j|\mathrm{e}^{-\theta(\tau^T_{j+k}-\tau^T_j)}\Big|N^T_T = m\right) \overset{\mathrm{d}}{=} \mathrm{Law}\left(R^2_T \cdot U_T \cdot V_T \cdot \mathrm{e}^{-\theta T\sigma_k}\right),$$

where U_T, V_T are iid random variables with probability law

$$\mathbf{P}(U_T \geq x) = \frac{H(xR_T)}{H(R_T)}, \quad x \geq 1,$$

and σ_k, $k = 1, \ldots, m-1$, is a Beta$(k, m-1+k)$-distributed random variable independent of U_T and V_T with probability density (3.7). For each $m \geq 0$ denote by $\mathbf{P}^{(m)}_T$ the conditional law $\mathbf{P}(\,\cdot\,|N^T_T = m)$.

For some $\varepsilon \in (0, \frac{2-\alpha}{\alpha})$ which will be chosen sufficiently small later, and for each $m \geq 2$ define the family of positive weights

$$w_{k,m} = \left(C(\alpha,\varepsilon)\cdot(m-1)\cdot k^{\frac{2}{\alpha}-\varepsilon}\right)^{-1}, \quad k = 1, \ldots, m-1,$$

where

$$C(\alpha,\varepsilon) = \sum_{k=1}^{\infty} k^{-\frac{2}{\alpha}+\varepsilon} < \infty$$

is the normalizing constant. With this construction for each $m \geq 2$

$$\sum_{k=1}^{m-1}\sum_{j=1}^{m-k} w_{k,m} = \sum_{k=1}^{m-1}(m-k)w_{k,m} = \frac{1}{C(\alpha,\varepsilon)}\sum_{k=1}^{m-1}\frac{m-k}{m-1}\cdot k^{-\frac{2}{\alpha}+\varepsilon} \leq 1. \tag{3.14}$$

Let $\gamma > 0$. In order to show that the sum (3.13) multiplied by ϕ^2_T converges to zero, we take into account (3.14) and write

$$\begin{aligned}
&\mathbf{P}^{(m)}_T\left(\phi^2_T\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}|J^T_{j+k}||J^T_j|\mathrm{e}^{-\theta(\tau_{j+k}-\tau_j)} > \gamma\right)\\
&\leq \mathbf{P}^{(m)}_T\left(\phi^2_T\sum_{k=1}^{m-1}\sum_{j=1}^{m-k}|J^T_{j+k}||J^T_j|\mathrm{e}^{-\theta(\tau_{j+k}-\tau_j)} > \gamma\sum_{k=1}^{m-1}\sum_{j=1}^{m-k}w_{k,m}\right)\\
&\leq \sum_{k=1}^{m-1}\sum_{j=1}^{m-k}\mathbf{P}\left(\phi^2_T R^2_T U_T V_T \mathrm{e}^{-\theta T\sigma_k} > \gamma w_{k,m}\right)\\
&= \sum_{k=1}^{m-1}(m-k)\mathbf{P}\left(\phi^2_T R^2_T \cdot U_T V_T \mathrm{e}^{-\theta T\sigma_k} > \gamma w_{k,m}\right).
\end{aligned}$$

Applying Lemma 3.2 and the independence of $U_T V_T$ and σ_k we obtain for some $\varepsilon > 0$

$$\begin{aligned}
p_{k,m}(T) &= \mathbf{P}\Big(\phi_T^2 R_T^2 \cdot U_T V_T \mathrm{e}^{-\theta T \sigma_k} > \gamma w_{k,m}\Big) \\
&= \frac{m!}{(k-1)!(m-k)!} \int_0^1 \mathbf{P}\Big(U_T V_T > \gamma \frac{w_{k,m}}{\phi_T^2 R_T^2} \cdot \mathrm{e}^{\theta T u}\Big)(1-u)^{m-k} u^{k-1}\,\mathrm{d}u \\
&\le C(\varepsilon) \frac{m!}{(k-1)!(m-k)!} \Big(\gamma \frac{w_{k,m}}{\phi_T^2 R_T^2}\Big)^{-\alpha+\varepsilon} \int_0^1 \mathrm{e}^{-\theta T u(\alpha-\varepsilon)}(1-u)^{m-k} u^{k-1}\,\mathrm{d}u \\
&\le C(\varepsilon) \frac{m!}{(k-1)!(m-k)!} \Big(\gamma \frac{w_{k,m}}{\phi_T^2 R_T^2}\Big)^{-\alpha+\varepsilon} \int_0^\infty \mathrm{e}^{-\theta T u(\alpha-\varepsilon)} u^{k-1}\,\mathrm{d}u \\
&\le C(\varepsilon,\alpha,\gamma) \frac{m!}{(k-1)!(m-k)!} \Big(k^{\frac{2}{\alpha}-\varepsilon}(m-1)T^{-\frac{2}{\alpha}+2\rho+\varepsilon}\Big)^{\alpha-\varepsilon} \frac{(k-1)!}{(\theta T(\alpha-\varepsilon))^k} \\
&\le C(\varepsilon,\alpha,\gamma) \cdot T^{(-\frac{2}{\alpha}+2\rho+\varepsilon)(\alpha-\varepsilon)} \cdot \frac{m! m^2 k^2}{(m-k)!} \cdot \frac{1}{(\theta T(\alpha-\varepsilon))^k},
\end{aligned}$$

where we have used the well known relation $\int_0^\infty a u^n \mathrm{e}^{-au}\,\mathrm{d}u = n!/a^n$, $a > 0$, $n \ge 0$, as well as the elementary estimates $(m-1)^{\alpha-\varepsilon} \le m^2$ and $k^{(\frac{2}{\alpha}-\varepsilon)(\alpha-\varepsilon)} \le k^2$ which are valid for $\varepsilon > 0$ and $\alpha \in (0,2)$.

Hence

$$\begin{aligned}
p(T) :&= \sum_{m=2}^\infty \Big[\mathbf{P}(N_T^T = m) \sum_{k=1}^{m-1} (m-k) p_{k,m}(T)\Big] \\
&= C(\varepsilon,\alpha,\gamma) \cdot T^{(-\frac{2}{\alpha}+2\rho+\varepsilon)(\alpha-\varepsilon)} \\
&\quad \times \sum_{m=2}^\infty \Big[\mathrm{e}^{-T\cdot H(R_T)} \frac{(T\cdot H(R_T))^m}{m!} \sum_{k=1}^{m-1} (m-k) \frac{m! m^2 k^2}{(m-k)!} \cdot \frac{1}{(\theta T(\alpha-\varepsilon))^k}\Big] \\
&= C(\varepsilon,\alpha,\gamma) \cdot \mathrm{e}^{-T\cdot H(R_T)} \cdot T^{(-\frac{2}{\alpha}+2\rho+\varepsilon)(\alpha-\varepsilon)} \\
&\quad \times \sum_{k=1}^\infty \frac{k^2}{(\theta T(\alpha-\varepsilon))^k} \sum_{m=k+1}^\infty m^2 \frac{(T\cdot H(R_T))^m}{(m-k-1)!}.
\end{aligned} \tag{3.15}$$

To evaluate the inner sum we use the formula $\sum_{j=0}^\infty (j+k)^2 a^j/j! = \mathrm{e}^a(a^2+2ak+a+k^2)$ to obtain

$$\begin{aligned}
&\sum_{m=k+1}^\infty m^2 \frac{(T\cdot H(R_T))^m}{(m-k-1)!} \\
&\quad = (T\cdot H(R_T))^{k+1} \sum_{j=0}^\infty (j+k+1)^2 \frac{(T\cdot H(R_T))^j}{j!} \\
&\quad \le 3\Big((T\cdot H(R_T))^{k+3} + (k+1)^2 (T\cdot H(R_T))^{k+1}\Big)\mathrm{e}^{T\cdot H(R_T)}.
\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), it is left to estimate two summands. For the first one, we use the formula $\sum_{k=1}^\infty k^2 q^k = q(q+1)/(1-q)^3$, $|q| < 1$, to get

$$S_1 = \sum_{k=1}^\infty \frac{k^2}{(\theta T(\alpha-\varepsilon))^k}(T\cdot H(R_T))^{k+3} \le C_1 \cdot T^3 \cdot H(R_T)^4.$$

For the second summand, we use the formula $\sum_{k=1}^{\infty} k^2(k+1)^2 q^k = 4q(q^2+4q+1)/(1-q)^5$, $|q| < 1$, to get

$$S_2 = \sum_{k=1}^{\infty} \frac{k^2(k+1)^2}{(\theta T(\alpha-\varepsilon))^k}(T \cdot H(R_T))^{k+1} \leq C_2 \cdot T \cdot H(R_T)^2.$$

Combining (3.15) with the bounds for S_1 and S_2 we obtain

$$p(T) \leq C \cdot T^{(-\frac{2}{\alpha}+2\rho+\varepsilon)(\alpha-\varepsilon)} \cdot \left(T^{3-4\rho(\alpha-\varepsilon)} + T^{1-2\alpha\rho+\varepsilon}\right).$$

Since $\rho > \frac{1}{2\alpha}$, one can choose $\varepsilon > 0$ sufficiently small to obtain the limit $p(T) \to 0$, $T \to \infty$. □

4 Proofs of the main results

Proof of Theorem 2.6 Let $\rho \in (\frac{1}{2\alpha}, \frac{1}{\alpha})$ be fixed. With the help of the decomposition (3.1) we may write

$$\int_0^T X_s^2 \,\mathrm{d}s = \int_0^T (X_s^{\eta^T})^2 \,\mathrm{d}s + \int_0^T (X_s^T)^2 \,\mathrm{d}s + 2\int_0^T X_s^T \cdot X_s^{\eta^T} \,\mathrm{d}s. \tag{4.1}$$

Then by Lemma 3.6, $\phi_T^2 \int_0^T (X_s^T)^2 \,\mathrm{d}s \xrightarrow{\mathrm{d}} 0$. Recall that X^{η^T} satisfies the SDE

$$\mathrm{d}X_t^{\eta^T} = -\theta X_t^{\eta^T} \mathrm{d}t + \mathrm{d}\eta_t^T, \quad X_0^{\eta^T} = 0.$$

The Itô formula applied to the process X^{η^T} yields

$$\left(X_T^{\eta^T}\right)^2 = -2\theta \int_0^T \left(X_s^{\eta^T}\right)^2 \mathrm{d}s + 2\int_0^T X_{s-}^{\eta^T} \,\mathrm{d}\eta_s^T + [\eta^T]_T. \tag{4.2}$$

The decomposition (3.1) implies that $(X_T^{\eta^T})^2 \leq 2X_T^2 + 2(X_T^T)^2$. Since for $\theta > 0$ the process X has an invariant distribution (see, e.g. Sato 1999, Theorem 17.5 and Remark 2.3), we get that $\phi_T^2 X_T^2 \to 0$ in law. On the other hand, $\phi_T^2 (X_T^T)^2 \to 0$ in law by Lemma 3.6. Therefore, Lemmas 3.5, 3.7 and (4.2) yield

$$\phi_T^2 \int_0^T \left(X_s^{\eta^T}\right)^2 \mathrm{d}s \xrightarrow{\mathrm{d}} \frac{\mathcal{S}^{(\alpha/2)}}{2\theta}, \quad T \to \infty.$$

Eventually, the last integral in (4.1) multiplied by ϕ_T^2 converges to 0 by the Cauchy–Schwarz inequality. □

Proof of Corollary 2.7 The decomposition

$$X_t = X_t^T + X_t^{\eta^T} = X_0 \mathrm{e}^{-\theta t} + b_T \frac{1-\mathrm{e}^{-\theta s}}{2\theta} + \int_0^t \mathrm{e}^{-\theta(t-s)} \,\mathrm{d}\xi_s^T + X_t^{\eta^T}$$

allows us to write

$$\int_0^T X_s \,\mathrm{d}W_s = \int_0^T X_s^T \,\mathrm{d}W_s + \int_0^T X_s^{\eta^T} \,\mathrm{d}W_s$$

as well as (4.1). It is easy to check that $\phi_T \int_0^T X_s^T \, \mathrm{d}W_s \xrightarrow{\mathrm{d}} 0$. Indeed, due to the independence of X_0 and W

$$\phi_T \int_0^T X_0 \mathrm{e}^{-\theta s} \, \mathrm{d}W_s = \phi_T \cdot X_0 \cdot \int_0^T \mathrm{e}^{-\theta s} \, \mathrm{d}W_s \to 0 \quad \text{a.s.}$$

and obviously by Lemma 3.7

$$\phi_T b_T \int_0^T (1 - \mathrm{e}^{-\theta s}) \, \mathrm{d}W_s \xrightarrow{\mathrm{d}} 0.$$

Finally by the estimate (3.10) of Lemma 3.6

$$\begin{aligned}
&\mathbf{E}\Big[\phi_T \int_0^T \Big(X_s^T - X_0 \mathrm{e}^{-\theta s} - b_T \frac{1 - \mathrm{e}^{-\theta s}}{2\theta}\Big) \, \mathrm{d}W_s\Big]^2 \\
&= \phi_T^2 \mathbf{E}\Big[\int_0^T \int_0^s \mathrm{e}^{-\theta(s-r)} \, \mathrm{d}\xi_r^T \, \mathrm{d}W_s\Big]^2 \\
&= \phi_T^2 \int_0^T \mathbf{E}\Big[\int_0^s \mathrm{e}^{-\theta(s-r)} \, \mathrm{d}\xi_r^T\Big]^2 \mathrm{d}s \\
&= \phi_T^2 \int_0^T \int_0^s \mathrm{e}^{-2\theta(s-r)} \, \mathrm{d}r \, \mathrm{d}s \cdot \Big(\sigma^2 + \int_{|z| \le R_T} z^2 \nu(\mathrm{d}z)\Big) \to 0, \quad T \to \infty.
\end{aligned}$$

Taking into account the argument in the proof of Theorem 2.6, we conclude that it is sufficient to consider the limiting behaviour of the pair $\big(\phi_T \int_0^T X_s^{\eta^T} \, \mathrm{d}W_s, \phi_T^2 \int_0^T (X_s^{\eta^T})^2 \, \mathrm{d}s\big)$.

The processes η^T and W are independent and

$$M_t^T = \int_0^t X_s^{\eta^T} \, \mathrm{d}W_s, \quad t \ge 0,$$

is a continuous local martingale with the angle bracket

$$\langle M^T \rangle_t = \int_0^t \big(X_s^{\eta^T}\big)^2 \, \mathrm{d}s,$$

which is independent of W. Then for $u, v \in \mathbb{R}$ we get

$$\begin{aligned}
\mathbf{E} \exp\Big(iu\phi_T M_T^T + iv\phi_T^2 \langle M^T \rangle_T\Big) &= \mathbf{E}\Big[\mathbf{E}\Big[\exp\Big(iu\phi_T M_T^T + iv\phi_T^2 \langle M^T \rangle_T\Big)\Big|\mathscr{F}_T^{\eta^T}\Big]\Big] \\
&= \mathbf{E}\Big[\exp\Big(iv\phi_T^2 \langle M^T \rangle_T\Big)\mathbf{E}\Big[\exp\Big(iu\phi_T M_T^T\Big)\Big|\mathscr{F}_T^{\eta^T}\Big]\Big] \\
&= \mathbf{E}\Big[\exp\Big(iv\phi_T^2 \langle M^T \rangle_T\Big) \exp\Big(-\frac{u^2}{2}\phi_T^2 \langle M^T \rangle_T\Big)\Big] \\
&= \mathbf{E} \exp\Big(\Big(iv - \frac{u^2}{2}\Big)\phi_T^2 \langle M^T \rangle_T\Big) \\
&\to \mathbf{E} \exp\Big(\Big(iv - \frac{u^2}{2}\Big)\frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}\Big) \\
&= \mathbf{E} \exp\Big(iu\mathcal{N}\sqrt{\frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}} + iv\frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}\Big), \quad T \to \infty.
\end{aligned}$$

□

Proof of Theorem 2.5 The statement of the theorem follows immediately from Proposition 2.1 and Corollary 2.7. Indeed, for each $\theta_0 > 0$ and $u \in \mathbb{R}$ we use the formula (2.3) for the likelihood ratio as well as semimartingale decompositions (2.1) and (2.2) to conclude that

$$\begin{aligned} \ln L_T(\theta_0, \theta_0 + \phi_T u) &= -\frac{\phi_T u}{\sigma^2} \int_0^T X_s \, \mathrm{d}(\sigma W_s) - \frac{(\phi_T u)^2}{2\sigma^2} \int_0^T X_s^2 \, \mathrm{d}s \\ &= -\frac{u}{\sigma} \cdot \phi_T \int_0^T X_s \, \mathrm{d}W_s - \frac{u^2}{2\sigma^2} \cdot \phi_T^2 \int_0^T X_s^2 \, \mathrm{d}s \\ &\xrightarrow{\mathrm{d}} -\frac{u}{\sigma} \cdot \mathcal{N} \sqrt{\frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}} - \frac{u^2}{2\sigma^2} \cdot \frac{\mathcal{S}^{(\alpha/2)}}{2\theta_0}, \quad T \to \infty. \end{aligned}$$

□

Proof of Corollary 2.8 . The relation (2.8) follows from Proposition 2.1. Due to the linear-quadratic form of the likelihood ratio, the maximum likelihood estimator coincides with the so-called central sequence. This implies the asymptotic efficiency in the aforementioned sense. The limit (2.9) follows from Corollary 2.7. □

Proof of Remark 2.9 For $x > 0$,

$$\mathbf{P}\Big(\frac{|\mathcal{N}|}{\sqrt{\mathcal{S}^{(\alpha/2)}}} > x\Big) \leq \mathbf{P}\Big(\mathcal{S}^{(\alpha/2)} \leq x^{\alpha-2}\Big) + \mathbf{P}(|\mathcal{N}| > x^{\alpha/2}) = p_1(x) + p_2(x).$$

By the well known property of the Gaussian distribution

$$p_2(x) \leq \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-x^\alpha/2}}{x^{\alpha/2}}.$$

To estimate $p_1(x)$, we apply the exponential Chebyshev inequality to get

$$\begin{aligned} p_1(x) = \mathbf{P}\Big(\mathcal{S}^{(\alpha/2)} \leq x^{\alpha-2}\Big) &\leq \inf_{\lambda>0} \mathrm{e}^{\lambda x^{\alpha-2}} \mathbf{E} \mathrm{e}^{-\lambda \mathcal{S}^{(\alpha/2)}} \\ &= \inf_{\lambda>0} \mathrm{e}^{\lambda x^{\alpha-2} - \Gamma(1-\frac{\alpha}{2})\lambda^{\alpha/2}} \leq \exp\Big(-C(\alpha) x^\alpha\Big) \end{aligned}$$

for some $C(\alpha) > 0$. Hence the estimate (2.10) follows. □

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