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ORIGINAL RESEARCH

Predicting expected idiosyncratic volatility: Empirical evidence from ARFIMA, HAR, and EGARCH models

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Abstract

We investigate the performances of the ARFIMA, HAR, and EGARCH models in capturing the time-varying property of idiosyncratic volatility (IVOL). We find that the expected IVOL predictions by HAR are superior. In diverse portfolio scenarios, a greater degree of judgment is required to assess the pricing ability of expected IVOLs. For the lowest value-weighted quintiles and the expected IVOL estimated by the HAR model, the IVOL-return relationship is negative. Conversely, the IVOL-return relationship is positive for the expected IVOL estimated by the EGARCH model. Further evidence suggests a complicated and mixed relationship between the expected IVOL estimated by the ARFIMA model and stock returns.

Keywords Asset Pricing · Idiosyncratic volatility · Time-varying · ARFIMA · HAR · EGARCH

JEL Classification C53 · G12 · G17

1 Introduction

Contradictory results stem from the time-varying nature of idiosyncratic volatility (IVOL), which inherently reflects firm-specific activities such as periodical disclosures and seasonal variations in operating activities. Ang et al. (2006) [AHXZ (2006) hereafter] propose one path of a negative relationship, known as the IVOL puzzle, using the IVOL from the current month as a proxy for the next month. Another route, following Fu (2009), supports a positive IVOL-return relationship by adopting the expected IVOL from Exponential Generalized Autoregressive Conditional Heteroskedasticity (EGARCH) models. Consequently, predicting the future value

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for a non-stationary random walk series, as described by AHXZ (2006), is deemed inappropriate, given the persistent impact of a random shock from a faraway time to the present (Fu 2009).

Recognizing the critical role of the time-series property of IVOL in examining its relationship with stock and market returns, as well as in selecting the approximate value for expected IVOL, we are motivated to compare the performances of other, unexplored dynamic models in catching the time-variation of IVOL and examine the relationship between the expected IVOL and stock returns. Beyond the EGARCH model, our paper also resorts to the Autoregressive Fractionally Integrated Moving Average (ARFIMA) model and calibrates the Heterogenous Autoregressive (HAR) model to capture the time-series property in IVOL. In particular, the ARFIMA model, enhancing the Autoregressive Moving Average (ARMA) model and ARIMA model, incorporates the conversion between factional integration and fractional difference, demonstrating clear performance in capturing long-run dependence in realized volatility prediction (Koop et al. 1997; Andersen et al. 2003; Bhardwaj and Swanson 2006). The HAR model reproduces the decay of autocorrelations over various horizons (Corsi et al. 2008; Corsi 2009). The HAR model treats the time series as immediately observable and is also straightforward to estimate (Bollerslev et al. 2016). Despite their frequent use in the realized volatility literature,[1] the ARFIMA and HAR models are surprisingly absent from IVOL literature.[2]

Meanwhile, we revisit the IVOL-return relationship on both the stock and portfolio levels using expected IVOLs from ARFIMA, HAR, and EGARCH models. The IVOL puzzle, stating that lower IVOL should be compensated by higher returns, has been contested under various model specifications. Notably, the one-month-lagged IVOL in AHXZ (2006) and Ang et al. (2009) [AHXZ (2009) hereafter] is questioned due to its unreliable nature and its autocorrelation of 0.33, contradicting their underlying random walk assumption. For instance, the IVOL predicted by the EGARCH model shows a positive relationship with future stock returns (Fu 2009). Chua et al. (2010) document a positive relationship conditioning on the unexpected aspect of IVOL, decomposing IVOL from AR(2) processes into expected and unexpected components. However, the positive IVOL-return relationship modeled by the ARIMA model is reversed due to short-term return reversals (Huang et al. 2010) or January effects (Peterson and Smedema 2011).

Previous studies have taken into account the time-series nature of financial variables, for instance, the time-varying expected business conditions (Campbell and Diebold 2009). However, there is no consensus in the relevant literature on the most appropriate model for capturing the time-series characteristics of IVOL. For instance, when estimating the conditional IVOL, Spiegel and Wang (2005), Fu (2009), and Guo et al. (2014) adopt an

[1] It has been demonstrated that the long-memory ARFIMA model outperforms over conventional models such as the AR, Moving Average (MA), ARMA, GARCH and Stochastic Volatility (SV) models, in the parsimonious way of volatility forecasts (Andersen et al. 2003; Bhardwaj and Swanson 2006). The ARFIMA model, however, is unable to effectively depict and reflect the true structure of data and lacks a clear economic meaning (Corsi et al. 2008; Jiang et al. 2017; Izzeldin et al. 2019). The multicomponent HAR model of Corsi (2009) and its augmented family (see Andersen et al. 2007; Andersen et al. 2011; Busch et al. 2011; Jou et al. 2013; Bollerslev et al. 2016 for more augmented HAR models) have subsequently been shown to fit and perform better towards the long memory characteristics (Patton and Sheppard 2009).

[2] Models like the ARFIMA, EGARCH and HAR models are commonly used in literatures on realized volatility to capture the long memory and nonlinearity in time series (e.g., Andersen et al. 2001; Corsi 2009). In contrast, rather than modelling IVOL itself, the research on IVOL focuses more on its causes and effects. The relationship between IVOL and various factors, such as firm characteristics, market conditions, and investor behaviour, is therefore frequently studied using simpler models (e.g., Pástor and Stambaugh 2003; AHXZ, 2006; AHXZ, 2009; Babenko et al. 2016).

EGARCH model in light of its relaxation of the symmetry requirement in the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model examined by Xu and Malkiel (2003). Fu's empirical findings demonstrate a positive relationship between expected stock returns and the EGARCH-estimated IVOL. Others, such as Diavatopoulos et al. (2008), Chua et al. (2010), and Bekaert et al. (2012), utilize the Autoregressive (AR) model, while some, including Huang et al. (2010) and Peterson and Smedema (2011), go further to test the Autoregressive Integrated Moving Average (ARIMA) model. Khovansky and Zhylyevskyy (2013) first apply the Gaussian Mixed Models (GMM).

Our sample includes stocks with share codes 10 and 11 traded on the NYSE, AMEX, and NASDAQ from CRSP during the period from July 1965 to December 2020. We calculate IVOL as the standard deviation of residuals with respect to the Fama–French three-factor model scaled by the number of trading days. Initially, we assess the long memory property of IVOL through unit root tests and graphical analyses (line graph, density graph, and autocorrelogram). Significant Augmented Dickey-Fuller (ADF) results are documented for both IVOL series and sorted portfolios, indicating that recent IVOLs are still impacted by past IVOLs. Subsequently, we select the best-fitting models. For the EGARCH model, we test nine permutations among its autoregressive parameter ($1 \leq p \leq 3$) and the moving average parameter ($1 \leq q \leq 3$). For the ARFIMA model, we test 16 permutations among its autoregressive parameter ($0 \leq p \leq 3$), and the moving average parameter ($0 \leq q \leq 3$), and estimate the long memory parameter d. For the HAR model, we identify the most significant autoregressive lags based on ACF (Autoregressive Function), PACF (Partial Autoregressive Function), and AIC (Akaike Information Criterion). All models are run recursively using IVOLs up to month t to predict IVOL in month $t+1$. We use notations such as Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR}, and Exp_IVOL^{EGARCH} to refer to IVOLs predicted by the ARFIMA, HAR and EGARCH models, respectively. The original IVOL is denoted as *Actual_IVOL*.

To assess each model's predictive ability, we compare all expected IVOLs to *Actual_IVOL*. Later, to examine the pricing ability in the IVOL-return relationship, we regress stock returns on each expected IVOL individually and collectively with controls. On the stock level, we begin by recursively estimating expected IVOLs using value-weighted and equal-weighted series. Autocorrelograms and ADF tests suggest long-term reliance in both series, where previous IVOLs continue to impact current IVOL. We find that the HAR model more accurately reproduces the time-variation of *Actual_IVOL* with 1-month, 3-month, and 9-month lags for value-weighted series and with 1-month, 2-month, 3-month, and 12-month lags for equal-weighted series. Notably, the IVOL puzzle is observed only for Exp_IVOL^{HAR} in value-weighted series, while the IVOL-return relationship for Exp_IVOL^{ARFIMA} remains positive, unaffected by the weighting scheme.

On the portfolio level, we employ the portfolio-sorting approach, frequently used in the relevant literature for examining relationships between variables and generally performing well for nonlinear relationships.[3] In the main section, we sort the data using three different

[3] For example, AHXZ (2006, 2009) form value-weighted quintile portfolios based on IVOL, size, book-to-market ratio, leverage, liquidity, volume, turnover, bid-ask spreads, and dispersion of analysts' forecasts. Fu (2009) sorts the sample on IVOL and one-month lagged return. Huang et al. (2010) explain the IVOL puzzle through return reversals and sort their sample on IVOL. Hur and Luma (2017) explain the dynamic of the negative relationship between aggregate IVOL and unrealized gains on capital gains overhang portfolios. Verousis and Voukelatos (2018) sort stocks into quintiles using their IVOL proxy, the cross-sectional dispersion of individual stock returns to the market return. Bi and Zhu (2020) study the variations in the relationship between value-at-risk and expected stock returns by using both the single sorting and double sorting methods, followed by the construction of value- and equal-weighted deciles.

methods: IVOL-sorting, size-sorting, and book-to-market ratio-sorting, respectively. Generally, we observe the existence of the IVOL puzzle in the lowest value-weighted portfolios, in contrast to the positive relationship in the lowest equal-weighted portfolios, across all sorting approaches. Furthermore, the IVOL puzzle exists when the HAR model outperforms. Positive IVOL-return relationships are found when the EGARCH model surpasses the other two models, but only in the two lowest quintiles. However, the precise direction of the IVOL-return relationship is inconclusive when the ARFIMA model outcompetes its rivals. Only the lowest two value-weighted quintiles show a concentration of the negative relationship; the other quintiles have a positive relationship. In the robustness check, we discover a similar pattern for quintile portfolios sorted by *BETA* as well as for various exclusion schemes of the IVOL building under 10-day and 11-day. We discover that the IVOL explanation for the *BETA* anomaly put forth by Liu et al. (2018) depends on the model applied for the IVOL prediction. We further include analysis for portfolios sorted by the sentimentalized IVOL. The previously described pattern becomes confused when the joint factor is included, and a positive Exp_IVOL^{HAR}-return relationship is first noticed.

Overall, our study contributes to the comprehensive investigation of the time-varying property of IVOL and addresses the IVOL puzzle under various model specifications as well as stock compositions within the portfolio. We expand the toolkit with the ARFIMA and HAR models to the discussion on the IVOL puzzle.[4] Our empirical results unfold attention on lower portfolios across various sorted contexts., underscoring the importance of carefully assessing the risk and return dynamics when making investment decisions. However, it is inconclusive which model outperforms in predicting IVOL and capturing its time-varying feature. By merely using the EGARCH model for the prediction, we are unable to refute the existence of the IVOL puzzle. The inclusion of the ARFIMA and HAR models emphasizes the need for careful consideration of the IVOL-return relationship.

The remainder of this paper is organized as follows. Section 2 reviews related literature on the IVOL puzzle and the time-series properties of IVOL. Section 3 describes data and introduces the calculation of IVOL and the mechanisms of EGARCH, ARFIMA, and HAR models. The empirical findings for the IVOL series and sorted portfolios are presented in Section 4. We discuss robustness in Section 5 and include a further analysis in Section 6. Section 7 concludes.

2 Related literature

2.1 The IVOL puzzle

AHXZ (2006) point out a considerable difference in the monthly average returns between the quintiles with the highest idiosyncratic volatility and the lowest idiosyncratic volatility. As indicated by this anomalous negative correlation, the difference of 1.06% is substantial

[4] Our paper focuses on the time-series property of IVOL and introduces novel methodologies (i.e., the ARFIMA and HAR models) in IVOL estimation and the IVOL-return relationship, which is to fill the important research gap from previous studies. For instance, Fink et al. (2012) and Guo et al. (2014) provide comprehensive examinations of the look-ahead bias in the IVOL estimation process. Their expected IVOLs are estimated based on three sets of EGARCH(p,q) models, namely, using the full sample, using Fu (2009)'s method, and correcting the look-ahead bias. However, the incorporation of the ARFIMA and HAR models in analyzing the time-variation property of IVOL is absent in the previous IVOL literature.

at the portfolio level. At the firm level, the negative coefficient stands. The subsequent work by AHXZ (2009), investigates further by extending the IVOL puzzle findings made by AHXZ (2006) in the U.S. market to international stock markets such as those in G7 countries, even while contending with the effects of size, value, momentum, volume, liquidity, market frictions, and information asymmetry (see Jiang et al. 2009; Babenko et al. 2016; Chen and Strebulaev 2019, for more negative relationship). However, Fu (2009) illustrates a positive relationship between future stock returns and their IVOL measure using an EGARCH model and emphasizes the IVOL measure in their ability to capture the time-variant characteristic. Bergbrant and Kassa (2021) suggest the use of various out-of-sample EGARCH models dynamically and find the IVOL-return relationship to be positive after ruling out the noise brought by using just one out-of-sample EGARCH model. From Brockman et al. (2022), the IVOL premium is present in 57 different countries worldwide (see Malkiel and Xu 2002; Spiegel and Wang 2005; Boehme et al. 2009; Chua et al. 2010; Brockman et al. 2022, for more positive relationship). The positive relationship appears to be more inclined to the under-diversification argument offered by classical theories (Levy 1978; Merton 1987), while the negative relationship is explained within the context of mispricing (Shleifer and Vishny 1997; Brav et al. 2010).

Mixed relationships between IVOL and stock returns have been documented in other studies. For instance, Guo and Savickas (2006) find that when combined with the aggregate market realized volatility, the negligible forecasting power from value-weighted IVOL becomes significantly negative. Duan et al. (2010) discover a significant monthly return difference between the stocks with the highest and lowest IVOL quintiles while the negative IVOL-return relationship clusters among stocks with a high short interest. Huang et al. (2010) ascribe the negative relationship to return reversals and demonstrate that the magnitude of this bias depends on the IVOL estimation approach used. As shown in Rachwalski and Wen (2016), stocks with high idiosyncratic volatility earn lower returns initially for a few temporary months (during the previous six months), but subsequently see continuously higher returns. The lower return over a short period reflects the delay in incorporating risk news. The IVOL puzzle is more substantial when receiving a lower attention from sophisticated investors and is concentrated in the first half of the month following portfolio formation (see, for instance, Bali and Cakici 2008; Khovansky and Zhylyevskyy 2013; Cao and Han 2016 for more details).

2.2 The time-series properties of IVOL

The current body of research employs various techniques in an effort to dynamically capture the time variation exhibited in the IVOL series. According to earlier investigations, IVOL decays slowly and has a strong first-order autocorrelation (Amihud and Hurvich 2004; Lewellen 2004; Campbell and Yogo 2006). As a result, the approach of regressing the lagged IVOL on stock returns to gauge their relationship may prove unproductive (Jiang and Lee 2006). The recent research most frequently uses the EGARCH model, which is selected as a natural estimator for IVOL (Spiegel and Wang 2005; Fu 2009; Huang et al. 2010; Peterson and Smedema 2011; Fink et al. 2012; Guo et al. 2014; Cao and Han 2016; Bergbrant and Kassa 2021; Brockman et al. 2022). Fu (2009) explicitly elaborates on the adoption of the EGARCH model and finds a positive IVOL-return relationship with the constraint that stocks have a minimum of 15 trading days in a month in the pooled sample. Because the autocorrelation of 0.33 during Fu (2009)'s sample period deviates from the underlying implication of random walk in AHXZ (2006, 2009), which is reaffirmed by the Dickey-Fuller test result, Fu (2009) contends that the one-month lagged IVOL in AHXZ (2006,2009) is not necessarily a proper proxy for the expected IVOL. The advantage of the EGARCH model in relaxing the non-negative parameter restrictions makes it able to reflect the asymmetry of volatilities. Fu (2009) evaluates nine permutations of the EGARCH model with the auto-regressive parameter p to be between 1 to 3 and the moving average parameter q to also be between 1 to 3 and then chooses the best-fitting model based on the lowest AIC with an expanding window of the previous 30 months.

Due to its reflection of the time-series features and its reduction of the majority of serial autocorrelation, Diavatopoulos et al. (2008) and Chua et al. (2010) decide to employ the AR(2) model based on AIC. Bekaert et al. (2012) describe IVOL at the aggregate level by AR(1) and AR(3) processes. In accordance with the time-series characteristics of IVOL, Huang et al. (2010) and Peterson and Smedema (2011) implement the ARIMA model over a 24-month rolling window. The direction of the IVOL-return relationship is dependent on the estimation windows, following Khovansky and Zhylyevskyy (2013), who initially implement the GMM procedure to estimate IVOL. The GMM approach frees the previous two-pass method in that it immediately makes an estimation without being affected by the duration of available stock returns and does not require estimating IVOL as the first step based on the Fama–French model. Nevertheless, this GMM approach is parametrically constrained and has distributional restrictions. Aslanidis et al. (2019) extend Boyer et al. (2010)'s methodology for calculating expected idiosyncratic skewness, to obtain the expected IVOL throughout regressions on the one-period lagged IVOL with a 240-month rolling window.

Overall, the existence of the IVOL puzzle is currently being researched and the best-fitting model for IVOL estimation is inclusive. By utilizing the novel ARFIMA model and HAR model to capture the time-series evolution of IVOL and to explore the IVOL puzzle with a comparison to the EGARCH model, our paper adds to the field.

3 Data and methodology

3.1 Data and IVOL calculation

The sample period is from July 1965 to December 2020, with a total of 666 months. Daily stock returns including all the ordinary common equities (share code 10 or 11) on the NYSE, AMEX, and NASDAQ (exchange code 1, 2, or 3) are collected from CRSP. There are overall 28,523 stocks identified by the unique PERMNO during the sample period without any exclusion. We include common time-series control variables in the regressions (AHXZ, 2006; Peterson and Smedema 2011). Monthly excess market return (*MKT*), *HML*, *SMB*, *MOM*, *LMW,* and risk-free rate are from Professor Kenneth French's website.[5] The risk-free rate is the monthly T-bill return compounded from the simple daily rate from Ibbotson and Associates Inc. Excess market return *MKT* is calculated by subtracting the risk-free rate. *HML* stands for high book-to-market ratio minus low book-to-market ratio while *SMB* stands for small market capitalization minus big market capitalization. *MOM* stands for the momentum factor, which is the difference between the average return on the two high prior (2–12 month) return portfolios and the two low prior (2–12 month) return

[5] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

portfolios. *ST_Rev* is the short-term reversal factor, which is the difference between the average return on the two high prior (1 month) return portfolios and the two low prior (1 month) return portfolios. We also include the liquidity factor (*PS*) of Pástor and Stambaugh (2003), which is available from Professor Stambaugh's website.[6]

To be consistent with previous studies and for a better comparison, we adopt the prevalent measurement of computing the standard deviation of the residuals from the Fama and French (1993) three-factor model[7] as follows:

$$XRET_{i,n} = \alpha_i + \beta_i RET_{M,n} + h_i HML_n + s_i SMB_n + \epsilon_{i,n}, \tag{1}$$

where $XRET_{i,n}$ is the excess return for stock i in day n and $RET_{M,n}$, $HML_{n,}$ and SMB_n are the three Fama–French factors in day n. The standard deviation of the residual series on stock i and day n will be computed each month for stock i following the estimate in order to indicate the stock i's monthly IVOL. In particular, IVOL is scaled by the square root of the number of trading days inside the corresponding month and is estimated monthly using daily data. The exclusion applies to stocks with less than 5 trading days in a month (different exclusions of days will be carried out in the robust tests).

3.2 Unit root test and long memory

To determine whether the time series of IVOL has a long memory, the Augmented Dickey-Fuller test (ADF test) and autocorrelogram are used. The null hypothesis of the ADF test is that there exists a unit root in the time-series sample. The null hypothesis is rejected when the *P*-value is below the predetermined significance level, proving that the time series sample does not have a unit root and is therefore stationary. In this case, we can state that this time series sample has a long memory if the autocorrelogram demonstrates that lagged terms continue to lay influence on present terms. A forecast of the future IVOL is possible, in a sense, for instance, if the time series of IVOL has a long memory, and historical IVOLs are still influencing future IVOLs. As stated by Fu (2009), by capturing the time-varying characteristic of IVOL, we can predict expected IVOL using the EGARCH model. We can then examine the relationship between the predicted expected IVOL and the stock return. Similarly, we may also predict the expected IVOL using the ARFIMA and HAR models, both of which are established on the principle of long memory.

3.3 The EGARCH model

The EGARCH model, which Nelson (1991) extended based on Engel (1982)'s ARCH model and Bollerslev (1986)'s GARCH model, accommodates the asymmetry in volatility, known as the leverage effect, where the return volatility increases more after stock price declines due to the increase of leverage ratio, and relaxes the parameter restriction of non-negative variance in the earlier two models. Assuming that the IVOL estimation process's residuals from the Fama–French three-factor model (Eq. (1)) follow a normal distribution and are serially independent,

[6] https://finance.wharton.upenn.edu/~stambaug/.

[7] Our results are robust to the choice of alternative factor models for IVOL calculation, namely, the Carhart (1997) model and the Fama and French (2015) five-factor model. Furthermore, our results are also aligning with the previous studies (Khasawneh et al. 2023) that report similar IVOL outcomes.

$$\epsilon_{i,n} \sim N\left(0, \sigma_{i,n}^2\right), \tag{2}$$

and the conditional variance $\sigma_{i,n}^2$ follows the EGARCH (*p,q*) process,

$$\ln\sigma_{i,n}^2 = \alpha_i + \sum_{l=1}^{p} b_{i,l} ln\sigma_{i,n-1}^2 + \sum_{k=1}^{q} \gamma_{i,k} \left\{ \theta\left(\frac{\varepsilon_{i,n-k}}{\sigma_{i,n-k}}\right) + \gamma\left[\left|\frac{\varepsilon_{i,n-k}}{\sigma_{i,n-k}}\right| - \left(\frac{2}{\pi^{\frac{1}{2}}}\right)\right]\right\}, \tag{3}$$

where the conditional variance $\sigma_{i,n}^2$ has the past *p* periods and return shocks have the past *q* periods.

The EGARCH model has already been applied in the IVOL prediction (for further information on how to use EGARCH models, see Spiegel and Wang 2005; Fu 2009). It has been proven that the expected IVOL, as calculated by the EGARCH model, is robustly and positively correlated with stock returns.

3.4 The ARFIMA model

The fractal market hypothesis, which takes into account the nonlinear causal relationship between irrational investor expectations and the market's response to information, presenting the market structure and characteristics under normal circumstances, serves as the rationale of the ARFIMA model. The long memory is represented by parameter *d* in the ARFIMA model, and the short-term first-order property of time series is represented by parameters *p* and *q*. Following Granger and Joyeux (1980), the ARFIMA model can be expressed as in the following Eq. (4),

$$\emptyset(L)(1 - L)^d X_t = \theta(L)\epsilon_t, \tag{4}$$

where $\phi(L)$ and $\theta(L)$ are lag polynomials of finite orders. ϵ_t, which is only defined for the integer value of *d*, is a stationary noise series. *L* is the lag operator and is generalized to fractional differences using binomial expansion. The long memory parameter *d* is a real number in the range [-0.5,0.5] within which the time series will be stable, and more weight is given to older data. The ARFIMA model is a development of the ARIMA model to non-integer values of *d*. As the ARFIMA(*p,d,q*) model degrades into the ARMA(*p,q*) model, the time series will specifically have long memory in the region [0,0.5], medium memory in the region [-0.5,0] and short memory when $d = 0$.

It is widely acknowledged that the ARFIMA model can demonstrate the first-order long-term and short-term correlation of time series and can depict the fractional feature via parameter *d*, making it superior to other models that only take into consideration short or long memories as well as the autocorrelogram, which relies on intuition and subjective judgment (Granger and Joyeux 1980).

3.5 The HAR model

The fat-tailed, leptokurtic, and scaling characteristics of time series cannot be replicated by the standard GARCH model and other stochastic models. Corsi (2009) also disputes factional integrated models such as the ARFIMA model for their complexity and information loss. Then the HAR model is raised with the addictive autoregressive cascade with the goal of easily and parsimoniously capturing and forecasting the long-memory feature of time series. Based on Müller et al. (1993)'s Heterogeneous Market Hypothesis, the HAR model incorporates information from diverse time scales of different market participants as shown in Eq. (5),

$$V_t = \alpha + \beta_1 V_{t-1} + \beta_2 V_{t-5} + \beta_3 V_{t-22} + \varepsilon_t, \tag{5}$$

where the time series of volatility in time t is estimated by the daily (V_{t-1}), weekly (V_{t-5}), and monthly (V_{t-22}) volatility.

In terms of the IVOL study, Chua et al. (2010) applied AR models to the IVOL time series and discovered that the AR(2) model fits the data best under the criterion of AIC. Their sample period is from July 1963 to December 2003. In this study, we calibrate Corsi's HAR model and choose the autoregressive lags for the value-weighted and equal-weighted IVOL series, as well as the sorted portfolios based on AIC. We examine heterogeneous time scales and choose the significant lags rather than being constrained by the precise AR lags. Therefore, our calibrated HAR model does not require lags from short-, near- and long-term strictly but is flexible for the lags included to consider the time-variation in IVOL and at the same time absorb varied impact from different time scales.

4 Empirical results

This section first describes the basic statistical features of the time-series of value-weighted and equal-weighted *Actual_IVOL* for the entire sample and portfolios through the line graph, density distribution graph, autocorrelogram, and unit root test. The series will be evaluated to see if it possesses the long memory characteristic. Later, it will address how well the ARFIMA, HAR, and EGARCH models can capture time-varying traits, such as the long memory feature. Finally, we examine the IVOL-return relationship for the expected IVOLs at both stock and portfolio levels.

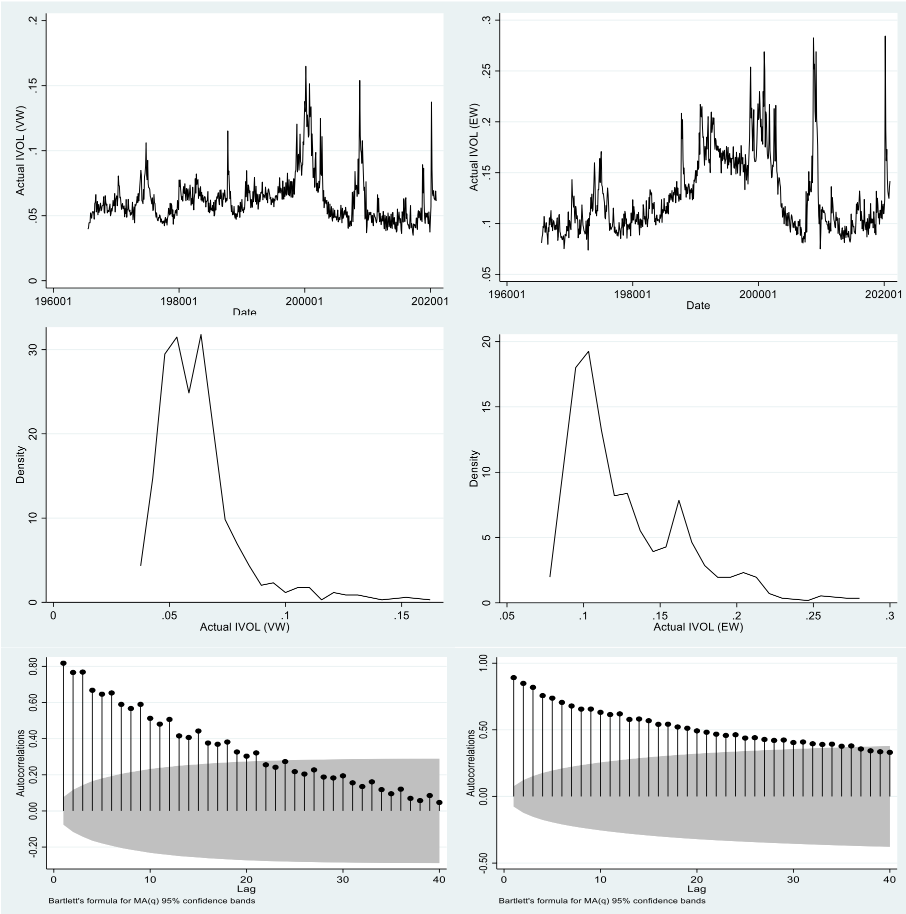
4.1 Time-variation in actual IVOL

4.1.1 Actual IVOL series

Figure 1 displays line graphs, density distribution graphs, and autocorrelograms (from top to bottom) for the monthly value-weighted *Actual_IVOL* (left) and equal-weighted *Actual_IVOL* (right), respectively. The statistical findings with the ADF test for unit root are listed in Panel A of Table 1. Both the value-weighted and equal-weighted *Actual_IVOL* are leptokurtic, right-skewed and have fat tails. The equal-weighted *Actual_IVOL* series has a higher standard deviation of 0.0374, making it more volatile. The value-weighted *Actual_IVOL*, however, is more right-skewed and has a higher kurtosis. Based on the ADF test, the null hypothesis that a unit root exists in the sample series is rejected at a 1% significance level by both value-weighted and equal-weighted *Actual_IVOL*s. The monthly time-series of value-weighted and equal-weighted *Actual_IVOL*s are stationary in some degree across the sample period between July 1965 to December 2020. Moreover, the autocorrelograms reveal that even 20 lags later, the long memory persists for both value-weighted and equal-weighted *Actual_IVOL*s, indicating that prior *Actual_IVOL*s continue to have a strong influence on recent *Actual_IVOL*s.

We use *Actual_IVOL*s from the previous t months to predict expected IVOLs through the ARFIMA, HAR, and EGARCH models in month $t+1$ recursively to avoid the look-ahead

Time-Series Properties of *Actual_IVOL*



Value-weighted *Actual_IVOL* **Equal-weighted *Actual_IVOL***

Fig. 1 Time-Series Properties of *Actual_IVOL*. This set of figures shows the line graph, the density distribution graph, and the autocorrelogram (from top to bottom) of value-weighted (left) and equal-weighted (right) *Actual_IVOLs*. The monthly *Actual_IVOL* is computed as the standard deviation of the residuals with respect to the Fama–French three-factor model of daily stock returns and is scaled by the square root of the number of trading days in each month. The value-weighted *Actual_IVOL* is taken monthly according to the market capitalization of each stock. The sample period is from July 1965 to December 2020. The shadow in the autocorrelogram (the bottom) is Bartlett's formula for the MA(q) 95% confidence band

bias (Guo et al. 2014; Fink et al. 2012). Fu (2009, 2010)'s approach that requiring the first 30 months to generate the initial forecast results in 636 expected IVOLs remaining in the sample.[8] When it comes to the EGARCH model, we test nine different models following Fu (2009): EGARCH(1,1), EGARCH(1,2), EGARCH(1,3), EGARCH(2,1), EGARCH(2,2),

[8] Following Spiegel and Wang (2005) and Guo et al. (2014), we additionally require the first 60 months to generate the initial forecast. Our results are robust, which are in line with Fu (2010) and Fink et al. (2012).

Table 1 Descriptive statistics for *Actual_IVOL* series and sorted portfolios

Panel A Actual_IVOL Series											
Value-Weighted	Mean	Std. Dev	Skewness	Kurtosis	ADF Test	Equal-Weighted	Mean	Std. Dev	Skewness	Kurtosis	ADF Test
	0.0626	0.0182	2.0740	9.1623	-4.8490***		0.1283	0.0376	1.2297	4.3216	-4.3350***
Panel B Value-Weighted Actual_IVOL Portfolios						*Panel C Equal-Weighted Actual_IVOL Portfolios*					
	Mean	Std. Dev	Skewness	Kurtosis	ADF Test		Mean	Std. Dev	Skewness	Kurtosis	ADF Test
Sorted on Actual_IVOL						*Sorted on Actual_IVOL*					
Low	0.0408	0.0117	1.7008	7.5985	-5.236***	Low	0.0412	0.0112	1.6791	8.8116	-5.587***
2	0.0684	0.0185	1.4554	6.5191	-5.089***	2	0.0703	0.0193	1.3852	6.0402	-4.957***
3	0.0980	0.0276	1.2906	4.9233	-4.531***	3	0.1003	0.0283	1.2773	4.8626	-4.553***
4	0.1399	0.0404	1.2598	4.4727	-4.353***	4	0.1440	0.0418	1.2535	4.4581	-4.366***
High	0.2427	0.0716	1.0785	3.7964	-4.404***	High	0.2860	0.0921	1.0701	3.7941	-4.011***
Sorted on Size						*Sorted on Size*					
Low	0.0890	0.0264	1.0887	5.696	-4.050***	Low	0.1449	0.0415	1.2562	4.3621	-4.381***
2	0.0900	0.0214	2.5486	15.4767	-6.211***	2	0.0912	0.0219	2.6589	16.8509	-6.217***
3	0.0781	0.0188	2.3964	14.7389	-6.124***	3	0.0788	0.0188	2.3736	14.4815	-6.135***
4	0.0691	0.0176	2.0245	10.2771	-5.656***	4	0.0697	0.0177	2.0422	10.609	-5.694***
High	0.0525	0.016	2.0471	8.9981	-4.993***	High	0.0591	0.0167	2.1341	10.1121	-5.372***
Sorted on BM						*Sorted on BM*					
Low	0.0773	0.0336	1.8153	8.1283	-4.316***	Low	0.1558	0.0611	1.3329	5.2118	-4.693***
2	0.0633	0.0223	2.0941	8.7577	-5.034***	2	0.1137	0.0363	1.2645	4.6772	-4.570***
3	0.0601	0.0176	1.8355	7.9299	-5.125***	3	0.1082	0.029	1.3471	5.8118	-4.767***
4	0.0601	0.0166	1.7753	7.809	-5.168***	4	0.1134	0.0307	1.1359	4.4557	-4.234***
High	0.0669	0.0193	1.7885	8.4247	-4.844***	High	0.1408	0.0394	1.2246	4.3526	-4.064***

This table represents the descriptive statistics for the monthly time series and sorted portfolios during the sample period from July 1965 to December 2020. We show the mean, standard deviation (Std. Dev.), skewness, kurtosis, and augmented Dickey-Fuller test result (ADF test) to check for the presence of the unit root. The monthly *Actual_IVOL* is computed as the standard deviation of residuals with respect to the Fama–French three-factor model of daily stock returns and is scaled by the square root of the number of trading days in each month. Panel A reports results for the value-weighted and equal-weighted *Actual_IVOL* series where the monthly value-weighted *Actual_IVOL* is computed based on the corresponding market capitalization of each stock. Panel B (Panel C) displays the basic statistical results of value-weighted (equal-weighted) portfolios. Following Fama and French (1992), stocks are sorted into quintiles monthly based on their *Actual_IVOL*, size (based on NYSE breakpoints), and BM ratio, respectively, where BM ratio is the book value of equity from the previous fiscal year divided by the market capitalization from the previous calendar year. Then each quintile is value-weighted according to the corresponding market capitalization of each stock or simply equal-weighted. MacKinnon approximate *p*-value: *** $p<0.01$, ** $p<0.05$, * $p<0.1$

EGARCH(2,3), EGARCH(3,1), EGARCH(3,2), and EGARCH(3,3). In other words, these nine models are permutations of the autoregressive parameter, $1 \leq p \leq 3$, and the moving average parameter, $1 \leq q \leq 3$. The converged model with the lowest AIC will be selected following the estimation. Each model estimating approach employs the selection procedure. The parameters of its best-fit model are used to predict expected IVOLs, denoted as Exp_IVOL^{EGARCH}. For example, in Panel C of Table 2, 96.23% of all estimations adopt the EGARCH(1,1) model, and the remaining 3.77% are generated by the EGARCH(2,1) model for the value-weighted *Actual_IVOL* series. As for the equal-weighted *Actual_IVOL* series, the EGARCH(1,1) model is the best-fitting for 99.06% of estimations.

The ARFIMA model is suitable to describe the time-varying characteristics in the monthly time-series of value-weighted and equal-weighted *Actual_IVOL*, according to the analysis of Fig. 1 and Panel A of Table 1. As indicated, the ARFIMA model depicts a long memory via the parameter *d* and describes the short-term first-order characteristic via parameters *p* and *q*. We test 16 permutations: ARFIMA(0,*d*,0), ARFIMA(0,*d*,1), ARFIMA(0,*d*,2), ARFIMA(0,*d*,3), ARFIMA(1,*d*,0), ARFIMA(1,*d*,1), ARFIMA(1,*d*,2), ARFIMA(1,*d*,3), ARFIMA(2,*d*,0), ARFIMA(2,*d*,1), ARFIMA(2,*d*,2), ARFIMA(2,*d*,3), ARFIMA(3,*d*,0), ARFIMA(3,*d*,1), ARFIMA(3,*d*,2), ARFIMA(3,*d*,3), with the autoregressive parameter, $0 \leq p \leq 3$ and the moving average parameter, $0 \leq q \leq 3$. As a result, the long memory parameter *d* is where the ARFIMA model estimation's crucial point is located. We shall state that the *Actual_IVOL* series is stationary and has a long memory if *d* falls within the region [0,0.5]. Hence, by selecting the model with the lowest AIC, we will also choose the ARFIMA model that fits the data the best. The best-fitting ARFIMA model was found in Panel A of Table 2, where $p=0$ and $q=0$ were set for the value-weighted *Actual_IVOL* series (34.91% across all estimations) and $p=1$ and $q=0$ for the equal-weighted *Actual_IVOL* series (28.3% across all estimations). The long memory parameter *d* is estimated in this circumstance to be 0.4961 and 0.4940, respectively. Both values are inside the region [0,0.5] and significant at the 1% level. This result reconfirms that using ARFIMA models to an *Actual_IVOL* time series is appropriate.

We also require the first 30 months to generate the first ARFIMA prediction. Then, using the *Actual_IVOL* from the previous *t* months to forecast the *Actual_IVOL* in month $t+1$, we estimate the parameters recursively and denote the predicted values as Exp_IVOL^{ARFIMA}. The best-fitted findings for the ARFIMA model differ from the relatively consistent best-fitted results for the EGARCH model, depending on the observation. As compared to the EGARCH model, we would anticipate that the more adaptable ARFIMA model will be better able to capture the time-variation in the *Actual_IVOL* series and provide a more precise expected IVOL forecast for the subsequent period.

For the HAR model, we examine various AR lags using partial autocorrelations and AIC statistics, and we arrive at the best-fitting model for the *Actual_IVOL* series. One, three and nine-month lags of the *Actual_IVOL* are given in Panel B of Table 2 for the value-weighted *Actual_IVOL* series, which is affected by the previous short-, near- and long-term periods with $Actual\_IVOL_{t-1}$ contributing the most from a coefficient of 0.5362. Recent *Actual_IVOL*s with one, two, and three-month lags have a greater impact on the equal-weighted *Actual_IVOL* series. The preceding long-term $Actual\_IVOL_{t-12}$ even shares a coefficient of 0.0769 with the present $Actual\_IVOL_t$. In order to predict the expected IVOL in month $t+1$, we first record the coefficients from the recursive OLS regressions of the observations up through month *t* and denote the predictions as Exp_IVOL^{HAR}. To generate the initial forecast, we also need the first 30 months' worth of observations.

After that, we make a comparison between the Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR},, and Exp_IVOL^{EGARCH} and the *Actual_IVOL* for the relevant month. To determine how well expected

Table 2 The model selection for *Actual_IVOL* series

Panel A The ARFIMA Model						*Panel C The EGARCH Model*					
Value-Weighted			Equal-Weighted			Value-Weighted			Equal-Weighted		
Best-Fitted	Freq	Percent (%)	Best-Fitted	Freq	Percent (%)	Best-Fitted	Freq	Percent (%)	Best-Fitted	Freq	Percent (%)
(0,*d*,0)	222	34.91	(0,*d*,0)	100	15.72	(1,1)	612	96.23	(1,1)	630	99.06
(1,*d*,0)	7	1.1	(1,*d*,0)	180	28.3	(2,1)	24	3.77	(2,1)	4	0.63
(3,*d*,0)	26	4.09	(2,*d*,0)	2	0.31				(3,1)	2	0.31
(0,*d*,1)	107	16.82	(0,*d*,1)	49	7.7						
(2,*d*,1)	12	1.89	(1,*d*,1)	2	0.31						
(3,*d*,1)	36	5.66	(2,*d*,1)	11	1.73						
(0,*d*,2)	92	14.47	(0,*d*,2)	141	22.17						
(1,*d*,2)	7	1.1	(2,*d*,2)	2	0.31						
(2,*d*,2)	14	2.2	(0,*d*,3)	149	23.43						
(3,*d*,2)	23	3.62									
(0,*d*,3)	22	6.46									
(1,*d*,3)	15	2.36									
(2,*d*,3)	5	0.79									
(3,*d*,3)	48	7.55									

Panel B The HAR Model

Value-Weighted

$$Actual\_IVOL_t = 0.0047 + 0.5362 Actual\_IVOL_{t-1} + 0.298 Actual\_IVOL_{t-3} + 0.0915 Actual\_IVOL_{t-9} + \epsilon_t$$

Equal-Weighted

$$Actual\_IVOL_t = 0.0061 + 0.6053 Actual\_IVOL_{t-1} + 0.1607 Actual\_IVOL_{t-2} + 0.1106 Actual\_IVOL_{t-3} + 0.0769 Actual\_IVOL_{t-12} + \epsilon_t$$

This table lists all feasible possibilities of the best-fit ARFIMA, HAR, and EGARCH models for monthly time series of value-weighted (in Panel A) and equal-weighted (in Panel B) *Actual_IVOL* during the sample period from July 1965 to December 2020. The monthly *Actual_ IVOL* is computed as the standard deviation of the residuals with respect to the Fama–French three-factor model of daily stock returns and is scaled by the square root of the number of trading days in each month. Following Fu (2009), the initial expected IVOL is predicted using the first 30 observations. Based on observations up through month *t*-1, the following estimates are generated recursively to forecast the expected IVOL in month *t*. ARFIMA models with permutations among the autoregressive parameter $0 \le p \le 3$ and the moving average parameter $0 \le q \le 3$ are evaluated. The converged model with the lowest Akaike Information Criterion (AIC) For each estimation is chosen. Autoregressive (AR) lags for the HAR model are determined according to partial autocorrelations and AIC statistics. For each selection process, EGARCH models with permutations among the autoregressive parameter $1 \le p \le 3$ and the moving average parameter $1 \le q \le 3$ are tested. The converged permutation with the lowest AIC statistics is then selected

Table 3 The comparison of expected IVOLs for *Actual_IVOL* series

	Mean	(1)	(2)	(3)	(4)
Panel A Value-Weighted					
$Exp\_IVOL^{ARFIMA}_t$	0.0594	0.1344***			-0.1516
$Exp\_IVOL^{HAR}_t$	0.0626		0.9625***		-0.0082
$Exp\_IVOL^{EGARCH}_t$	0.0042			5.7575***	0.9892***
Panel B Equal-Weighted					
$Exp\_IVOL^{ARFIMA}_t$	0.1196	0.6871***			-0.0182
$Exp\_IVOL^{HAR}_t$	0.128		0.9692***		1.1462***
$Exp\_IVOL^{EGARCH}_t$	0.0177			3.0582***	-0.5404*

This table reports the mean of Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR}, and Exp_IVOL^{EGARCH} and their coefficients when regressed on the *Actual_IVOL* individually and collectively during the sample period from July 1965 to December 2020. In Panel A, expected IVOLs are predicted based on the value-weighted *Actual_IVOL* series while in Panel B, expected IVOLs are predicted based on the equal-weighted *Actual_IVOL* series. Columns (1), (2), and (3) contain the coefficient β when regressing the *Exp_IVOL* predicted by each model on the *Actual_IVOL* as in the following Eq. (8):

$$Actual\_IVOL_t = \alpha_t + \beta_t Exp\_IVOL_t + \varepsilon_t. \quad (8)$$

The last column (4) contains the coefficient β^{ARFIMA},β^{HAR}, andβ^{EGARCH} when regressing the expected IVOLs collectively on the *Actual_IVOL* as in the following Eq. (9):

$$Actual\_IVOL_t = \alpha_t + \beta_t^{ARFIMA} Exp\_IVOL_t^{ARFIMA} + \beta_t^{HAR} Exp\_IVOL_t^{HAR} + \beta_t^{EGARCH} Exp\_IVOL_t^{EGARCH} + \varepsilon_t. \quad (9)$$

t-statistics are reported in parentheses: *** $p<0.01$, ** $p<0.05$, * $p<0.1$

IVOLs catch the time-varying trait inside the *Actual_IVOL*, we regress the *Actual_IVOL* on expected IVOLs individually and collectively and anticipate their coefficients to be close to 1. In Table 3, whether value-weighted or equal-weighted, the mean of Exp_IVOL^{HAR} is closest to the mean of the *Actual_IVOL*. When regressed independently, all the expected IVOLs are significantly related to the *Actual_IVOL*. However, when regressed jointly, the direction of Exp_IVOL^{EGARCH} and Exp_IVOL^{ARFIMA} switches from positive to negative. Only the direction and significance of the Exp_IVOL^{HAR} coefficient remain consistent and close to 1. As it stands, the HAR model outperforms the ARFIMA and EGARCH models in terms of reflecting the time-variation inside both value-weighted and equal-weighted *Actual_IVOL* series.

4.1.2 Actual IVOL portfolios

We then describe the pattern for *Actual_IVOL* in terms of sorted portfolios. First, we sort all stocks into quintiles each month based on their *Actual_IVOL*. In Panel B and C of Table 1, we record the basic statistics for each value-weighted and equal-weighted *Actual_IVOL* quintile. The lowest *Actual_IVOL* quintile has the largest right-skewness and leptokurtosis but the least variation for both value-weighted and equal-weighted portfolios, while the highest *Actual_IVOL* quintile shows the reverse trend. Overall, value-weighted *Actual_IVOL* portfolios are more right-skewed and leptokurtic, aligning with the outcomes for the full sample. However, only the lowest equal-weighted *Actual_IVOL* quintile, which stands out from all other quintiles, has a high kurtosis of 8.8116. The stationary for *Actual-IVOL*-sorted portfolios is indicated by the fact that all the ADF statistics are significant at the 1% level.

The results of model selection and model comparison for each *Actual_IVOL* quintile are included in the Online Appendix Tables A.1 and A.2, following the same sorting and selection procedure. For both value-weighted and equal-weighted *Actual_IVOL* portfolios, the best-fitted EGARCH and ARFIMA models vary over a wider range, especially for the ARFIMA model. The choice for the EGARCH(1,1) model still predominates, nevertheless. For the first three quintiles, we choose the ARFIMA(0,*d*,0) model, and for the top two quintiles, we select the ARFIMA(0,*d*,3) model. Value-weighted and equal-weighted *Actual_IVOL*s almost always have the same AR lags for the HAR model for every quintile. The main influence is always accounted for by $Actual\_IVOL_{t-1}$. $Actual\_IVOL_{t-4}$ lays a negative impact on the current *Actual_IVOL* if it is included as an AR lag. Exp_IVOL^{HAR} outperforms the competition in that its coefficient is closer to 1.

The portfolio-sorting process is also done in accordance with some commonly used firm characteristics such as size and book-to-market ratio (BM ratio). Here, we sort all stocks into quintiles according to NYSE breakpoints to reduce the noise brought by small-sized stocks (Fama and French 1992; Bali and Cakici 2008). The sorting process is balanced monthly. For size-sorted quintiles, the basic statistics in Panel B and C of Table 1 demonstrate a different pattern. The three middle quintiles are more right-skewed, have higher kurtosis, and have higher ADF test statistics. In the Online Appendix A.1 and A.3, we report the selection and comparison among models in detail. First of all, the EGARCH(1,1) model remains dominant across all size-sorted quintiles, regardless of value-weighted or equal-weighted. Except for the ARFIMA(1,*d*,0) model of the second-lowest value-weighted size quintile and the two-lowest equal-weighted size quintiles, the ARFIMA(0,*d*,0) model predominates. With respect to the HAR model, almost all quintiles have lags of one, two, and three months. The lowest value-weighted quintile has lags of one, two, three, four, nine, and ten months, whereas the lowest equal-weighted quintile has lags of one, two, three, and four months. Nevertheless, the coefficient of the one-month lag of *Actual_IVOL* has the greatest impact on the present *Actual_IVOL* for all portfolios, followed by the three-month lag of *Actual_IVOL*. If *Actual_IVOL* with a four-month latency is chosen, the effect is always negative. In each quintile, when comparing expected IVOLs to the *Actual_IVOL*, the Exp_IVOL^{HAR} has the closest mean and stands out as the most prominent regressor with a close value of 1 with a consistent direction.

Next, we sort all stocks into quintiles according to their BM ratio, which is calculated following Fama and French (1992) by using the book value of equity from the previous fiscal year upon the market capitalization from the previous calendar year. This procedure is also monthly balanced. We report the basic statistics for each value-weighted and equal-weighted BM quintile in Panel B and C of Table 1. All BM-sorted portfolios are right-skewed and leptokurtic. The results of the ADF test show that previous *Actual_IVOL*s still have an impact on the current *Actual_IVOL*. The EGARCH(1,1) model is still suitable for all value-weighted and equal-weighted BM-quintiles, as described in the Online Appendix Table A.1. Here, different ARFIMA models, including the ARFIMA(0,*d*,0), ARFIMA(1,*d*,0) and ARFIMA(0,*d*,3) models are chosen. Except for the lowest and the highest equal-weighted BM-quintiles, the HAR model selection is dominated by one-, two-, and three-month lags. The characteristics of expected IVOLs remain the same for BM-sorted portfolios (see the Online Appendix Table A.4).

4.2 The expected IVOL-return relationship

In this section, we revisit the IVOL puzzle by using expected IVOLs from the ARFIMA, HAR, and EGARCH models at both the stock and portfolio levels.

4.2.1 Stock-level analysis

For stock-level analysis, we form both the value-weighted excess stock returns monthly according to the market capitalization as well as the equal-weighted excess stock returns. To examine whether the IVOL puzzle exists, we regress the excess stock returns on the expected IVOLs respectively and collectively in combination with additional time-series control variables. Our comparison is justified by the fact that we require the significance for each expected IVOL when regressing both individually and collectively. We also need the coefficients for each expected IVOL to point in the same direction. The model with the highest adjusted-R^2 and the lowest Root Mean Squared Error (RMSE) is then picked.

From Panel A of Table 4, we could observe that when regressing each expected IVOL with control variables separately (in regressions (1), (2), and (3)), the Exp_IVOL^{ARFIMA} and Exp_IVOL^{HAR} exhibit the capacity to price the value-weighted returns. The direction of Exp_IVOL^{EGARCH} alters from negative to positive while the Exp_IVOL^{ARFIMA} and Exp_IVOL^{HAR} remain consistent and significant when all expected IVOLs and control variables are taken into account in regression (5). The Exp_IVOL^{ARFIMA} has an adjusted-R^2 of 4.52% in regression (1) which is marginally greater than the Exp_IVOL^{HAR}'s in regression (2) as well as an RMSE of 0.2696 is slightly lower than the Exp_IVOL^{HAR}'s. All expected IVOLs in Panel B of Table 4 for the equal-weighted returns are significant whether they are regressed separately or jointly, except the direction of Exp_IVOL^{HAR}, which is inconsistently changing from positive to negative. Once more, Exp_IVOL^{ARFIMA} outperforms Exp_IVOL^{EGARCH} with a higher adjusted-R^2 of 5.77% and a lower RMSE of 0.2680. The results are not affected after comparing with the results of *Actual_IVOL* (in regressions (4) and (5)) as a benchmark. Given that the IVOL-return relationship depends on the model being utilized, we are unable to draw the conclusion that the IVOL puzzle exists. Specifically, the IVOL puzzle only arises for the Exp_IVOL^{HAR} in the value-weighted series but not for other circumstances.

Overall, Exp_IVOL^{ARFIMA} exhibits consistency, maintains positive significance, and holds the lowest RMSE and the highest adjusted-R^2 for pricing both value-weighted and equal-weighted *Actual_IVOL* series. The Exp_IVOL^{EGARCH} is an additional option with a positive pricing ability for the equal-weighted *Actual_IVOL* series. Only the value-weighted *Actual_IVOL* series employing the HAR model with one-, three-, and nine-month lags could reveal the IVOL puzzle.

4.2.2 Sorted portfolios analysis

We further expand our results to value-weighted and equal-weighted *Actual_ IVOL*-sorted, size-sorted, and BM-sorted portfolios. First, the Exp_IVOL^{ARFIMA} and Exp_IVOL^{EGARCH} have better pricing ability than Exp_IVOL^{HAR} across all *Actual_IVOL* quintiles. Table 5 shows that, although Exp_IVOL^{HAR} more closely resembles and fits the *Actual_IVOL* variation observed in earlier analyses, the Exp_IVOL^{HAR} is either inconsequential or inconsistent when it comes to pricing stock returns in *Actual_IVOL*-sorted portfolios. For example, the EGARCH model for the *Actual_IVOL* quintile 2 and the ARFIMA model for the *Actual_IVOL* quintile 3, 4, and 5 are both consistent choices within the same quintile whether weighted equally or by value. The value-weighted *Actual_IVOL* quintile 1 chooses the ARFIMA model in contrast to its equal-weighted equivalent, which selects the EGARCH model. It is possible that the underlying pricing feature residing in *Actual_IVOL*-sorted

Table 4 The IVOL-return relationship for *Actual_IVOL* series

	Panel A Value-Weighted					*Panel B Equal-Weighted*				
	(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
$Exp\_IVOL^{ARFIMA}_t$	1.2561***				1.4103***	1.1807***				1.8966***
$Exp\_IVOL^{HAR}_t$		-2.1180***			-14.2621***		0.6125**			-2.4154*
$Exp\_IVOL^{EGARCH}_t$			-5.6069		90.0862***			2.6111***		9.8895***
$Actual\_IVOL_{t-1}$				-2.4121***	-2.6405**				0.2007	-1.6352**
MKT_t	0.0029	0.0025	0.0026	0.0023	0.0021	0.0025	0.0025	0.0025	0.0027	0.0026
SMB_t	0.0015	0.0019	0.0013	0.0007	0.0017	0.0010	0.0009	0.0008	0.0013	0.0006
HML_t	-0.0094**	-0.0092**	-0.0096**	-0.0102**	-0.0099**	-0.0096**	-0.0097**	-0.0097**	-0.0096**	-0.0102**
MOM_t	-0.0064**	-0.0076***	-0.0073***	-0.0076***	-0.0073***	-0.0066**	-0.0069***	-0.0066**	-0.0071***	-0.0062**
ST_Rev_t	-0.0065*	-0.0060	-0.0064	-0.0064	-0.0042	-0.0050	-0.0064*	-0.0064*	-0.0064	-0.0049
PS_t	0.4790***	0.4010**	0.4589***	0.2761	0.2111	0.4618**	0.5244***	0.5330***	0.5225***	0.2818*
Constant	-0.4292***	-0.2237***	-0.3308***	-0.2080***	0.2328**	-0.4963***	-0.4308***	-0.3985***	-0.3784***	-0.2427***
RMSE	0.2696	0.2697	0.2713	0.2692	0.2566	0.2680	0.2709	0.2704	0.2715	0.2624
Adj-R^2 (%)	4.52	4.51	3.33	5.35	12.28	5.77	3.70	4.11	3.17	7.83

This table compares the pricing ability to value-weighted and equal-weighted stock returns for the Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR}, and Exp_IVOL^{EGARCH}, predicted from the best-fitting ARFIMA, HAR, and EGARCH models respectively. The sample period is from July 1965 to December 2020. The excess value-weighted return in Panel A and the excess equal-weighted return in Panel B serve as the dependent variables *XRET* for all regressions. *MKT, SMB,* and *HML* are Fama–French factors from Professor Kenneth French's website. *MOM* is the momentum factor and *ST_Rev* is the short-term reversal factor. *PS* is the liquidity factor of Pástor and Stambaugh (2003). In columns (1), (2), (3), and (4), we simply incorporate the Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR}, Exp_IVOL^{EGARCH}, and the one-month lag of *Actual_IVOL*, separately with control variables as illustrated in the following Eq. (10):

$$XRET_t = \alpha_t + \beta_t Exp\_IVOL_t\left(Actual\_IVOL_{t-1}\right) + \beta_t^{Controls} Controls_t + \varepsilon_t. \quad (10)$$

In column (5), we further regress all expected IVOLs with the one-month lag of *Actual_IVOL* jointly on the excess stock returns with control variables (in Eq. (11)):

$$XRET_t = \alpha_t + \beta_t^{ARFIMA} Exp\_IVOL_t^{ARFIMA} + \beta_t^{HAR} Exp\_IVOL_t^{HAR} + \beta_t^{EGARCH} Exp\_IVOL_t^{EGARCH} + \beta_t^{Actual\_IVOL} Actual\_IVOL_{t-1} + \beta_t^{Controls} Controls_t + \varepsilon_t. \quad (11)$$

The Newey-West (1987) robust *t*-statistics are reported in parentheses: *** $p<0.01$, ** $p<0.05$, * $p<0.1$

Table 5 The IVOL-return relationship for *Actual_IVOL*-sorted portfolios

		Panel A Value-Weighted					*Panel B Equal-Weighted*				
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
Low	$Exp\_IVOL^{ARFIMA}_t$	-1.6110*				-2.0679**	0.1641				-0.8756
	$Exp\_IVOL^{HAR}_t$		-0.8345			-17.2809**		2.4888**			0.1766
	$Exp\_IVOL^{EGARCH}_t$			8.3369		193.0869***			27.4018***		37.7651**
	$Actual\_IVOL_{t-1}$				-1.6023*	-2.5512				1.1601	-1.4497
	RMSE	0.2709	0.2717	0.2717	0.2715	0.2626	0.2719	0.2709	0.2700	0.2713	0.2693
	Adj-R^2 (%)	3.62	3.09	3.10	3.43	8.07	2.99	3.68	4.37	3.19	4.21
2	$Exp\_IVOL^{ARFIMA}_t$	0.9440*				0.8099	-0.0664				-0.1326
	$Exp\_IVOL^{HAR}_t$		-0.0085			-5.5352*		0.0073			-5.1087
	$Exp\_IVOL^{EGARCH}_t$			5.8234*		42.1496**			5.3907*		41.7454**
	$Actual\_IVOL_{t-1}$				-0.6380	-1.9691				-0.5933	-1.9673
	RMSE	0.2711	0.2717	0.2712	0.2717	0.2639	0.2719	0.2719	0.2714	0.2719	0.2661
	Adj-R^2 (%)	3.55	3.07	3.42	3.24	6.14	3.04	3.04	3.39	3.20	5.83
3	$Exp\_IVOL^{ARFIMA}_t$	0.6153**				0.6901	0.5883**				0.8139*
	$Exp\_IVOL^{HAR}_t$		0.0910			-6.2079***		0.0737			-6.0168***
	$Exp\_IVOL^{EGARCH}_t$			2.5043		32.7110***			2.3374		30.8054***
	$Actual\_IVOL_{t-1}$				-0.3556	-1.8027**				-0.3752	-1.8456**
	RMSE	0.2711	0.2716	0.2712	0.2716	0.2638	0.2714	0.2718	0.2715	0.2718	0.2637
	Adj-R^2 (%)	3.52	3.15	3.42	3.26	6.94	3.43	3.09	3.36	3.23	7.01
4	$Exp\_IVOL^{ARFIMA}_t$	0.8634***				1.3089***	1.0653***				1.4927***
	$Exp\_IVOL^{HAR}_t$		0.2432			-4.9353***		0.3054			-3.2804**
	$Exp\_IVOL^{EGARCH}_t$			1.4186*		16.4852***			1.5438**		11.3566***
	$Actual\_IVOL_{t-1}$				-0.1001	-1.3978**				-0.0509	-1.5119**
	RMSE	0.2690	0.2713	0.2709	0.2715	0.2628	0.2673	0.2714	0.2709	0.2717	0.2619
	Adj-R^2 (%)	5.02	3.34	3.64	3.26	8.34	6.25	3.33	3.69	3.15	9.28

Table 5 (continued)

		Panel A Value-Weighted					*Panel B Equal-Weighted*				
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
High	$Exp\_IVOL^{ARFIMA}_t$	1.0546***				1.5936***	1.0653***				1.4927***
	$Exp\_IVOL^{HAR}_t$		0.3871***			0.3465		0.3054			-3.2804**
	$Exp\_IVOL^{EGARCH}_t$			0.8069***		-0.5374			1.5438**		11.3566***
	$Actual\_IVOL_{t-1}$				0.1492	-1.0839***				-0.0509	-1.5119**
	RMSE	0.2558	0.2700	0.2695	0.2707	0.2499	0.2578	0.2695	0.2693	0.2703	0.2494
	Adj-R^2 (%)	14.06	4.24	4.56	3.52	18.05	12.77	4.72	4.85	3.81	17.60

This table compares the pricing ability to value-weighted and equal-weighted stock returns for portfolios sorted on the *Actual_IVOL*. The sorting procedure is balanced monthly. Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR}, and Exp_IVOL^{EGARCH} are predicted from the best-fitting ARFIMA, HAR, and EGARCH models within each *Actual_IVOL* quintile respectively. The sample period is from July 1965 to December 2020. The excess value-weighted return in Panel A and the excess equal-weighted return in Panel B serve as the dependent variables *XRET* for all regressions. *MKT*, *SMB,* and *HML* are Fama–French factors from Professor Kenneth French's website. *MOM* is the momentum factor and *ST_Rev* is the short-term reversal factor. *PS* is the liquidity factor of Pástor and Stambaugh (2003). In columns (1), (2), (3), and (4), we simply incorporate each expected IVOL and the one-month lag of *Actual_IVOL* separately with control variables as illustrated in Eq. (10). In column (5), we further regress all expected IVOLs with the one-month lag of *Actual_IVOL* jointly on the excess stock returns with control variables (in Eq. (11)). For more information on regressions, see the table notes in Table 4. The Newey-West (1987) robust *t*-statistics are reported in parentheses: *** $p<0.01$, ** $p<0.05$, * $p<0.1$

portfolios can be better captured by the ARFIMA and EGARCH models. Only the lowest value-weighted *Actual_IVOL* quintile for the Exp_IVOL^{ARFIMA} has the IVOL puzzle.

We do observe the IVOL puzzle for all value-weighted portfolios as well as for equal-weighted portfolios 4 and 5 concerning size quintiles. In Table 6, all of these IVOL puzzles, except for the value-weighted portfolio 2 by the Exp_IVOL^{ARFIMA}, are discovered to be predicted by the Exp_IVOL^{HAR}. The lowest three equal-weighted quintiles include only two positive Exp_IVOL^{EGARCH}-return relationships. The two smallest size quintiles, like *Actual_IVOL* quintiles, are appropriate for various models and have coefficients that point in opposite directions.

When it comes to quintiles that are sorted using the BM ratio, the pattern is distinct from previously sorted portfolios. As reported in Table 7, excluding quintile 2, value-weighted BM-quintiles generally exhibit an IVOL puzzle, whereas all equal-weighted BM-quintiles present a positive IVOL-return relationship. Nevertheless, only quintile 2, which is either value-weighted or equal-weighted, chooses the EGARCH model with a positive coefficient.

In conclusion, we could notice that the IVOL puzzle always exists in the lowest value-weighted portfolios from the aforementioned *Actual-IVOL*-, size-, and BM-sorted approaches. In contrast, the two lowest equal-weighted quintiles always show a positive IVOL-return relationship. Furthermore, focusing on the individual model, we also observe the negative IVOL-return relationship across all HAR-predicted quintiles, regardless of how the data are sorted (*Actual_IVOL*, size, or BM), whether they are value- or equal-weighted. Although the EGARCH model only outperforms in the bottom two quintiles, we also confirm that the Exp_IVOL^{EGARCH}-return relationship is positive. The IVOL puzzle brought by the Exp_IVOL^{ARFIMA}, on the other hand, is only present in the lowest two value-weighted quintiles and is more circumstantial. It is difficult to pinpoint the precise choice of models for predicting expected IVOLs as it depends on the stock composition within the portfolio.

5 Robustness checks

5.1 Results for BETA-sorted portfolios

We estimate *BETA* following Liu et al. (2018)'s explanation of the beta anomaly, which states that stocks with low beta have higher earnings than stocks with high beta. They argue that the beta anomaly is caused by the interaction between the positive beta-IVOL correlation and the negative IVOL-return relationship among overpriced stocks. In our sample, we expect that the coefficient of the best expected IVOL in explaining stock returns will also be negative.

To reconcile non-synchronous trading effects, we regress the monthly excess stock return of i in month t on the monthly market excess return in both month t and month t-1 with a 60-month rolling window. We require stocks to have at least 36 months of returns and the excess return is computed by subtracting the one-month US treasury bill. We denote the sum of coefficients of the current market excess return and the one-month lagged market excess return as β_i (Dimson 1979). Then following Vasicek (1973), we compute *BETA* by shrinking β_i with weight ω_i as in Eq. (6),

Table 6 The IVOL-return relationship for size-sorted portfolios

		Panel A Value-Weighted					*Panel B Equal-Weighted*				
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
Low	$Exp\_IVOL^{ARFIMA}_t$	-3.8403***				-3.4783***	0.7547***				0.0923
	$Exp\_IVOL^{HAR}_t$		-3.1565***			-9.9382***		0.8617***			-0.3771
	$Exp\_IVOL^{EGARCH}_t$			-10.9284***		53.1377***			2.7397***		6.8826**
	$Actual\_IVOL_{t-1}$				-3.0117***	-1.7744**				0.4753**	-1.1872*
	RMSE	0.2448	0.2612	0.2664	0.2617	0.2302	0.2701	0.2698	0.2694	0.2706	0.2659
	Adj-R^2 (%)	21.32	10.43	6.83	11.27	29.06	4.30	4.55	4.82	3.62	5.19
2	$Exp\_IVOL^{ARFIMA}_t$	-0.6274**				-0.9041***	-0.6149**				-0.8209***
	$Exp\_IVOL^{HAR}_t$		0.2541			-3.4880**		0.4125			-2.8793*
	$Exp\_IVOL^{EGARCH}_t$			4.1266***		16.3184***			4.0896***		13.0410***
	$Actual\_IVOL_{t-1}$				-0.2139	-0.8903				-0.0764	-0.6360
	RMSE	0.2709	0.2717	0.2710	0.2717	0.2683	0.2709	0.2716	0.2709	0.2717	0.2688
	Adj-R^2 (%)	3.67	3.10	3.61	3.10	5.36	3.67	3.14	3.70	3.08	4.94
3	$Exp\_IVOL^{ARFIMA}_t$	0.6996*				0.3864	1.0790***				0.7529**
	$Exp\_IVOL^{HAR}_t$		-1.6499**			-5.0038**		-1.5701**			-4.6304**
	$Exp\_IVOL^{EGARCH}_t$			1.0097		21.1722**			1.0882		21.1512**
	$Actual\_IVOL_{t-1}$				-1.6807**	-1.3232				-1.6213**	-1.3425
	RMSE	0.2712	0.2707	0.2717	0.2708	0.2668	0.2697	0.2708	0.2717	0.2709	0.2658
	Adj-R^2 (%)	3.41	3.83	3.10	4.18	6.40	4.53	3.76	3.09	4.11	6.95
4	$Exp\_IVOL^{ARFIMA}_t$	0.2971				0.1253	0.7222				0.0632
	$Exp\_IVOL^{HAR}_t$		-2.1861***			-12.6354***		-2.3601***			-11.3612***
	$Exp\_IVOL^{EGARCH}_t$			-2.4425		68.2458***			-2.7000		59.0002***
	$Actual\_IVOL_{t-1}$				-2.2607***	-1.6689				-2.3718***	-1.6713
	RMSE	0.2717	0.2700	0.2717	0.2701	0.2613	0.2712	0.2697	0.2716	0.2699	0.2617
	Adj-R^2 (%)	3.10	4.28	3.11	4.84	9.55	3.41	4.48	3.12	5.04	9.40

Table 6 (continued)

		Panel A Value-Weighted					Panel B Equal-Weighted				
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
High	$Exp\_IVOL^{ARFIMA}_t$	0.5273				0.6435	-0.4784				-0.6718
	$Exp\_IVOL^{HAR}_t$		-2.3231***			-3.0634		-2.2573***			-14.1775***
	$Exp\_IVOL^{EGARCH}_t$			-2.3985		25.4870***			-2.9401		89.2304***
	$Actual\_IVOL_{t-1}$				-2.5519***	-2.5695*				-2.4120***	-1.8488
	RMSE	0.2713	0.2699	0.2716	0.2696	0.2680	0.2715	0.2699	0.2716	0.2698	0.2589
	Adj-R^2 (%)	3.29	4.32	3.06	5.03	6.31	3.18	4.30	3.10	4.92	10.44

This table compares the pricing ability to value-weighted and equal-weighted stock returns for portfolios sorted on size. The sorting procedure is based on NYSE breakpoints and is balanced monthly. Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR}, and Exp_IVOL^{EGARCH} are predicted from the best-fitting ARFIMA, HAR, and EGARCH models within each Size quintile respectively. The sample period is from July 1965 to December 2020. The excess value-weighted return in Panel A and the excess equal-weighted return in Panel B serve as the dependent variables *XRET* for all regressions. *MKT*, *SMB,* and *HML* are Fama–French factors from Professor Kenneth French's website. *MOM* is the momentum factor and *ST_Rev* is the short-term reversal factor. *PS* is the liquidity factor of Pástor and Stambaugh (2003). In columns (1), (2), (3), and (4), we simply incorporate each expected IVOL and the one-month lag of *Actual_IVOL* separately with control variables as illustrated in Eq. (10). In column (5), we further regress all expected IVOLs with the one-month lag of *Actual_IVOL* jointly on the excess stock returns with control variables (in Eq. (11)). For more information on regressions, see the table notes in Table 4. The Newey-West (1987) robust *t*-statistics are reported in parentheses: *** $p<0.01$, ** $p<0.05$, * $p<0.1$

Table 7 The IVOL-return relationship for BM-sorted portfolios

		Panel A Value-Weighted					*Panel B Equal-Weighted*				
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
Low	$Exp\_IVOL^{ARFIMA}_t$	-0.4533***				-0.5912***	1.2764***				1.2837***
	$Exp\_IVOL^{HAR}_t$		-0.167			-6.0908***		1.0216***			1.4020
	$Exp\_IVOL^{EGARCH}_t$			3.0206***		32.6090***			2.2529***		-1.4653
	$Actual\_IVOL_{t-1}$				-0.5169	-1.336				0.7804***	-0.8333**
	RMSE	0.2706	0.2717	0.2707	0.2716	0.2553	0.2575	0.2656	0.2665	0.2666	0.2550
	Adj-R^2 (%)	3.91	3.09	3.71	3.42	12.95	13.05	7.52	6.88	5.95	13.25
2	$Exp\_IVOL^{ARFIMA}_t$	0.0372				-0.1504	0.9523***				0.5621*
	$Exp\_IVOL^{HAR}_t$		0.9762**			-5.0078*		0.9537***			-0.7914
	$Exp\_IVOL^{EGARCH}_t$			9.4187***		40.2319***			3.9872***		8.7426***
	$Actual\_IVOL_{t-1}$				0.3367	-1.0743				0.5648**	-1.1170
	RMSE	0.2719	0.2712	0.2698	0.2715	0.2647	0.2698	0.2701	0.2693	0.2707	0.2655
	Adj-R^2 (%)	3.04	3.50	4.51	3.10	6.31	4.57	4.37	4.94	3.59	5.48
3	$Exp\_IVOL^{ARFIMA}_t$	0.6382				0.2573	0.9120***				0.9699**
	$Exp\_IVOL^{HAR}_t$		-1.4877**			-16.6139***		0.5313			-3.6488*
	$Exp\_IVOL^{EGARCH}_t$			-0.7566		113.3592***			3.4151***		18.6851***
	$Actual\_IVOL_{t-1}$				-1.8470***	-2.1256				0.0043	-1.5409*
	RMSE	0.2714	0.2711	0.2720	0.2708	0.2572	0.2696	0.2716	0.2710	0.2719	0.2642
	Adj-R^2 (%)	3.40	3.64	3.03	4.24	10.35	4.71	3.31	3.77	3.07	6.55
4	$Exp\_IVOL^{ARFIMA}_t$	0.6762				1.4119**	0.7607***				1.3573*
	$Exp\_IVOL^{HAR}_t$		-3.1723***			-32.0834***		-0.4489			-8.0951***
	$Exp\_IVOL^{EGARCH}_t$			-11.1591**		207.2866***			-0.3175		32.6232***
	$Actual\_IVOL_{t-1}$				-3.1959***	-2.2329*				-0.8063***	-2.1373**
	RMSE	0.2716	0.2686	0.2710	0.2682	0.2507	0.2703	0.2717	0.2719	0.2712	0.2612
	Adj-R^2 (%)	3.25	5.40	3.69	6.42	17.21	4.31	3.30	3.11	3.87	9.74

Table 7 (continued)

		Panel A Value-Weighted					*Panel B Equal-Weighted*				
		(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
High	$Exp\_IVOL^{ARFIMA}_t$	-0.6433				-0.4888	1.0266***				1.3532**
	$Exp\_IVOL^{HAR}_t$		-4.8925***			-24.2398***		0.3625			-1.0705
	$Exp\_IVOL^{EGARCH}_t$			-21.8272***		129.6375***			1.3651*		6.2325
	$Actual\_IVOL_{t-1}$				-4.5781***	-2.3005*				-0.0759	-1.8824***
	RMSE	0.2713	0.2592	0.2646	0.2592	0.2448	0.2661	0.2716	0.2715	0.2720	0.2598
	Adj-R^2 (%)	3.52	11.92	8.23	12.90	21.21	7.28	3.36	3.49	3.14	9.38

This table compares the pricing ability to value-weighted and equal-weighted stock returns for portfolios sorted on the BM ratio, which is the book value of equity from the previous fiscal year divided by the market capitalization from the previous calendar year. The sorting procedure is balanced monthly. Exp_IVOL^{ARFIMA}, Exp_IVOL^{HAR}, and Exp_IVOL^{EGARCH} are predicted from the best-fitting ARFIMA, HAR, and EGARCH models within each BM quintile respectively. The sample period is from July 1965 to December 2020. The excess value-weighted return in Panel A and the excess equal-weighted return in Panel B serve as the dependent variables *XRET* for all regressions. *MKT*, *SMB,* and *HML* are Fama–French factors from Professor Kenneth French's website. *MOM* is the momentum factor and *ST_Rev* is the short-term reversal factor. *PS* is the liquidity factor of Pástor and Stambaugh (2003). In columns (1), (2), (3), and (4), we simply incorporate each expected IVOL and the one-month lag of *Actual_IVOL* separately with control variables as illustrated in Eq. (10). In column (5), we further regress all expected IVOLs with the one-month lag of *Actual_IVOL* jointly on the excess stock returns with control variables (in Eq. (11)). For more information on regressions, see the table notes in Table 4. The Newey-West (1987) robust *t*-statistics are reported in parentheses: *** $p<0.01$, ** $p<0.05$, * $p<0$

$$BETA_i = \omega_i * \beta_i + \left(1 - \omega_i\right) * 1, \tag{6}$$

where weight ω_i is defined as,

$$\omega_i = \frac{\frac{1}{\sigma^2(\beta_i)}}{\frac{1}{\sigma^2(\beta_i)} + \frac{1}{\hat{\sigma}^2(\beta)}}. \tag{7}$$

Specifically, in the above Eq. (7), $\sigma^2(\beta_i)$ is the variance of β_i; $\hat{\sigma}^2(\beta)$ is the estimate of the cross-sectional variance of true betas, computed by taking the difference between the cross-sectional variance of β_i and the cross-sectional mean of $\sigma^2(\beta_i)$.

We include the results for *BETA*-quintiles in the Online Appendix (Tables A.5, A.6, and A.7). Similar to the general characteristics observed in the previous sorted approaches, the lowest two equal-weighted quintiles still reveal a positive IVOL-return relationship. However, the relationship is positive rather than negative for the lowest value-weighted quintile. Regarding the beta anomaly, we could only find negative relationships for the Exp_IVOL^{HAR} in value-weighted portfolios, whereas positive relationships are presented from the Exp_IVOL^{ARFIMA} and Exp_IVOL^{EGARCH}.

5.2 Different exclusion schemes for IVOL

In the main section above, we have excluded stocks having less than 5 trading days in a month. As noted in Bali and Cakici (2008), however, there is no consensus in the extant literature about the exclusion of stocks. Different ways of applying the data collection criteria such as selecting from monthly, daily, or even intra-day data, weighting equally or by value, or excluding small-sized, low-priced, or illiquid stocks, lead to positive, negative, or compromised conclusions for the investigation of the IVOL-return relationship. By implementing two additional exclusion strategies during the IVOL construction process, we corroborate our findings. Since the exact number of stocks excluded between the 10-day scheme and 11-day scheme is significant, we choose to exclude both stocks with less than 10 trading days in a month and stocks with less than 11 trading days in a month. Although unreported, it is noteworthy that our main findings are not affected by the exclusion scheme.

6 Further analysis

We further take into account portfolios that are sorted on the sentimentalized idiosyncratic volatility [hereafter sentimentalized IVOL], which is the product between the investor sentiment index aligned by Huang et al. (2015) and IVOL. We find a significant pricing effect, shifting from negative to positive with the growth of the sentimentalized IVOL itself, on the cross-section of stock returns.

Both value-weighted and equal-weighted portfolios show novel patterns within sentimentalized IVOL quintiles when the IVOL-return relationship is examined (see the Online Appendix Table A.8). Previously, all Exp_IVOL^{HAR} always displayed the IVOL puzzle. In quintiles sorted on the sentimentalized IVOL, the lowest value-weighted quintile and the two bottom equal-weighted quintiles have a positive coefficient with Exp_IVOL^{HAR}. Value-weighted quintiles 4 even have no discernible IVOL-return association. We might locate the IVOL puzzle for the Exp_IVOL^{HAR} in quintiles 4 and 5 when equally weighted.

7 Conclusion

The time-varying property of IVOL is crucial in examining the relationship between IVOL and stock returns and selecting the appropriate model for estimating IVOL. In this paper, we evaluate the three expected IVOLs from the ARFIMA, HAR, and EGARCH models for their capacity to replicate and capture the time-variation property within the *Actual_IVOL* series and portfolios. We also look at how the three expected IVOLs and stock returns are related. Empirically, we find that the Exp_IVOL^{HAR} beats its two counterparts by a wide margin when it comes to simulating the variation in both the *Actual_IVOL* series and portfolios. This benefit cannot, however, be extended to the pricing ability. There is no all-encompassing model that can estimate expected IVOLs and reproduce its time-variation simultaneously. As a result, the performance of expected IVOLs in the IVOL-return relationship varies depending on the model used.

Our findings add to the ongoing debate over the IVOL puzzle. Under the EGARCH model, in which the best-fitted model is selected recursively, the IVOL-return relationship is consistently positive. The findings of Fu (2009) are likewise consistent with this positive relationship. The advantage of the HAR model in capturing the time-varying property suggests that different models might yield varying insights into the IVOL-return relationship. We discover the presence of the IVOL puzzle for Exp_IVOL^{HAR} in our sample period of July 1965 to December 2020, where the best-fitted HAR model is calibrated under the particular value-weighted and equal-weighted series or portfolios. The direction of the relationship between the Exp_IVOL^{ARFIMA} and stock returns is undetermined, so each case must be separately examined. Overall, there is not a universally recognized and standardized model that can effectively capture the dynamic time-dependency of IVOL and duplicate it. Therefore, if we solely rely on the expected IVOL derived from a single model, we would be unable to challenge the IVOL puzzle of AHXZ (2006). Continuous scrutiny of model assumptions and methodologies is still essential. While the inconclusive IVOL puzzle is still under investigation and depends on the sample composition, our findings imply some unique portfolios for the puzzle and call for more cautionary portfolio construction and decision-making. By recognizing the limitations of relying on one certain model, practitioners need to refine their risk management practices, with higher awareness of lower portfolios.

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