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On the exponential diophantine equation $U_n^x + U_{n+1}^x = U_m$

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Abstract
Let $\{U_n\}_{n\geq 0}$ be the Lucas sequence. For integers x, n and m, we find all solutions to $U_n^x + U_{n+1}^x = U_m$. The equation was studied and claimed to be solved completely in Ddamulira and Luca (Ramanujan J 56(2):651–684, 2021) but there are some computational bugs in that publication because of the wrong statement of Mignotte's bound from Mignotte (A kit on linear forms in three logarithms. http://irma.math.unistra.fr/~bugeaud/travaux/kit.pdf, 2008). In this paper, the main result remains the same as in Ddamulira and Luca (Ramanujan J 56(2):651–684, 2021) but we focus on correcting the computational mistakes in Ddamulira and Luca (Ramanujan J 56(2):651–684, 2021), involving the application of Theorem 2.1 from Mignotte (A kit on linear forms in three logarithms. http://irma.math.unistra.fr/~bugeaud/travaux/kit.pdf, 2008).

Keyword Lucas sequences · Exponential Diophantine equations

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1 Introduction

1.1 Background

Let $r \geq 1$ be an integer, then $\mathbb{U} := \{U_n\}_{n \geq 0}$ is the Lucas sequence given by $U_0 = 0$, $U_1 = 1$, and

$$U_{n+2} = rU_{n+1} + U_n, \tag{1.1}$$

for all $n \geq 0$. Famous examples of Lucas sequences include the Fibonacci numbers, Mersenne numbers, Pell numbers, Jacobsthal numbers, among others. When $r = 1$, $\mathbb{U}$ becomes the Fibonacci sequence while when $r = 2$, $\mathbb{U}$ is the Pell sequence, with the same initial conditions. The first few terms of $\mathbb{U}$ are

$$\{0,\ 1,\ r,\ r^2 + 1,\ r^3 + 2r,\ r^4 + 3r^2 + 1,\ \ldots\}.$$

For the Lucas sequence $\mathbb{U}$, it is well known that

$$U_n^2 + U_{n+1}^2 = U_{2n+1}, \tag{1.2}$$

for all $n \geq 0$, which tells us that the sum of the squares of any two consecutive terms of $\mathbb{U}$ is also a term of $\mathbb{U}$. Consider the exponential Diophantine equation

$$U_n^x + U_{n+1}^x = U_m, \tag{1.3}$$

in nonnegative integers (n, m, x) which by (1.2), has the parametric solution $m = 2n + 1$ when $x = 2$ and for any $r > 1$, and the parametric solution $m = n + 2$ when $x = 1$ and $r = 1$. When $r = 1$, [3] proved that (1.3) has no integer solutions (n, m, x) with $n \geq 2$ and $x \geq 3$. Similarly, [6] proved for $r = 2$, that (1.3) has no positive integer solutions (n, m, x) with $x \neq 2$, while [7] solved (1.3) with $\mathbb{U} = \{F_n^{(k)}\}_{n \geq -(k-2)}$, the $k-$generalised Fibonacci sequence.

In [1], it was shown that there exists no positive integer solutions (r, n, m, x) of the Diophantine equation (1.3) with $r \geq 3$ and $x \neq 2$. However, this came with a big computational mistake in stating and applying Theorem 2.1 in [4]. In this paper, we restudy [1]. Still, we consider equation (1.3) in nonnegative integers (r, n, m, x) treating $r \geq 3$ as an integer parameter, since the cases $r \in \{1, 2\}$ were treated in [3] and [6] respectively. Our main result is the following.

1.2 Main Result

Theorem 1.1 *There is no positive integer solution (r, n, m, x) to the Diophantine equation (1.3) with $r \geq 3$ and $x \neq 2$.*

2 Methods

2.1 Preliminaries

Subsection 2.1 in [1] was correctly written. All results there-in are correct and therefore we also adopt this subsection as it appears from page 652 to page 654 of [1].

2.2 Linear forms in logarithms and continued fractions

This is Subsection 2.2 and 2.3 in [1]. All the results in these two Subsections 2.2 and 2.3 of [1] are well stated and explained. An exception was that the result of Mignotte from [4] was wrongly stated as Theorem 2 on page 655 of publication [1].

Here, we restate Theorem 2.1 on linear forms in three logarithms due to Mignotte from [4]. It will be used to express equations as linear forms, in three logarithms. The result from [4] is more general, but we quote it in the form we shall use. The following is Proposition 5.2 in [4]. The reader is also refereed to [5].

Theorem 2.1 (Mignotte, [4]) *Consider three algebraic numbers γ_1, γ_2 and γ_3, which are all real, greater than* 1 *and multiplicatively independent. Put*

$$\mathcal{D} := [\mathbb{Q}(\gamma_1, \gamma_2, \gamma_3) : \mathbb{Q}].$$

Let b_1, b_2, b_3 be coprime positive integers and consider

$$\Gamma := b_2 \log \gamma_2 - b_1 \log \gamma_1 - b_3 \log \gamma_3.$$

Put

$$d_1 := \gcd(b_1, b_2) = \frac{b_1}{b_1'} = \frac{b_2}{b_2'}, \quad d_3 := \gcd(b_2, b_3) = \frac{b_2}{b_2''} = \frac{b_3}{b_3''}.$$

Let A_1, A_2 and A_3 be real numbers such that

$$A_i \geq \max\{4, 4.296 \log \gamma_i + 2\mathcal{D} h(\gamma_i)\}, \quad i = 1, 2, 3 \;\; \textit{and} \;\; \Omega := A_1 A_2 A_3 \geq 100.$$

Put

$$b' := \left(\frac{b_1'}{A_2} + \frac{b_2'}{A_1}\right)\left(\frac{b_3''}{A_2} + \frac{b_2''}{A_3}\right) \;\; \textit{and} \;\; \log \mathcal{B} := \max\left\{0.882 + \log b', \frac{10}{\mathcal{D}}\right\}.$$

Then, either $\log |\Gamma| > -790.95 \Omega \mathcal{D}^2 (\log \mathcal{B})^2$, or one of the following conditions holds:

(i) *there exists two positive integers r_0 and s_0 such that $r_0 b_2 = s_0 b_1$ with $r_0 \leq 5.61 A_2 (\mathcal{D} \log \mathcal{B})^{\frac{1}{3}}$ and $s_0 \leq 5.61 A_1 (\mathcal{D} \log \mathcal{B})^{\frac{1}{3}}$.*

(ii) *there exists integers* r_1, s_1, t_1 *and* t_2, *with* $r_1 s_1 \neq 0$, *such that* $(t_1 b_1 + r_1 b_3)s_1 = r_1 b_2 t_2$, $\gcd(r_1, t_1) = \gcd(s_1, t_2) = 1$, *which also satisfy*

$$|r_1 s_1| \leq 5.61\delta A_3(\mathcal{D}\log\mathcal{B})^{\frac{1}{3}},$$
$$|s_1 t_1| \leq 5.61\delta A_1(\mathcal{D}\log\mathcal{B})^{\frac{1}{3}},$$
$$|r_1 t_2| \leq 5.61\delta A_2(\mathcal{D}\log\mathcal{B})^{\frac{1}{3}},$$

where $\delta = \gcd(r_1, s_1)$. *Moreover, when* $t_1 = 0$, *we can take* $r_1 = 1$, *and when* $t_2 = 0$, *we can take* $s_1 = 1$.

In the Mignotte bound given in [1], the $(\mathcal{D}\log\mathcal{B})^{\frac{1}{3}}$ term was wrongly written and applied with $(\mathcal{D}\log\mathcal{D})^{\frac{1}{3}}$ instead and since $\mathcal{D}$ is much smaller than $\mathcal{B}$, this resulted in upper bounds which are smaller than what they should actually have been. This was the main bug in that paper.
Finally, we present an analytic argument which is Lemma 7 from [2]. It is useful when obtaining upper bounds on some positive real variable involving powers of the logarithm of the variable itself.

Lemma 2.1 (Gúzman and Luca [2]) *If* $s \geq 1$, $T > (4s^2)^s$ *and* $T > \dfrac{x}{(\log x)^s}$, *then*

$$x < 2^s T(\log T)^s.$$

In addition to the above results, and all results stated in Section 2 of [1], we also perform computations with Mathematica.

3 Proof of Theorem 1.1

In this section, we consider equation (1.3) in nonnegative integers (r, n, m, x) treating $r \geq 3$ as an integer parameter, since the cases $r \in \{1, 2\}$ were treated in [3] and [6] respectively.

3.1 Trivial solutions

(a) Let $n = 0$, then (1.3) becomes

$$U_0^x + U_1^x = U_m,$$
$$0^x + 1^x = U_m,$$
$$U_m = 1,$$

so that $m = 1$. This solution $(r, n, m, x) = (r, 0, 1, x)$ for any $r \geq 3$ and $x \geq 1$, is trivial, so we omit it and assume that n is positive.

(b) Let $n \geq 1$ and $x = 0$, then (1.3) becomes

$$\begin{aligned} U_n^0 + U_{n+1}^0 &= U_m, \\ U_m &= 2 \notin \mathbb{U}, \end{aligned}$$

for all $r \geq 3$. Therefore this case is not possible.

(c) Let $n \geq 1$ and $x = 1$, then (1.3) becomes

$$U_m = U_n + U_{n+1} < U_{n+2},$$

for all $r \geq 3$. It follows that the Diophantine equation (1.3) has no solution with $x = 1$.

(d) Let $n = 1$ and $x = 2$, then (1.3) becomes

$$\begin{aligned} U_n^2 + U_{n+1}^2 &= U_m, \\ 1^2 + r^2 &= U_m, \\ U_m &= 1 + r^2, \end{aligned}$$

so that $m = 2$. This solution $(r, n, m, x) = (r, 1, 2, 2)$ for any $r \geq 3$, is trivial, so we also omit it.

(e) Lastly, if $n = 1$, then

$$U_m = 1 + r^x. \tag{3.1}$$

From this point, we adopt the solution for this case, from Subsection 3.1, given on page 657, following equation (18) in [1]. After a simple computer search, we found no other solutions to equation (3.1) apart from the trivial ones given in (a) and (d).

3.2 Calculations when $n \in [2, 100]$ and $x \in [3, 100]$

This is Subsection 3.2 in [1]. We adopt this section as it appears on pages 658 and 659 in [1]. Subsection 3.2 in [1] was clearly written and Lemma 6 there-in was correctly stated and proved.

From now on, we assume $n \geq 2$, $x \geq 3$ and $\max\{n, x\} > 100$.

3.3 A small linear form in three logarithms

A small linear form in three logarithms was correctly deduced in [1] as equations (26) and (27). Lemma 7 in [1] summarizes these results, with proof. We therefore adopt this subsection from Subsection 3.3 in [1].

Next, we intend to apply Theorem 2.1 on equation (26) from [1] with the following data:

$$\gamma_1 := \alpha - \beta = \sqrt{r^2 + 4}, \quad \gamma_2 := \alpha, \quad \gamma_3 := U_{n+1}, \quad b_1 := 1, \quad b_2 := m, \quad b_3 := x.$$

Since $r \geq 3$ and $n \geq 2$, it is clear that $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}_{>1}$ and $\gcd(b_1, b_2) = \gcd(b_1, b_3) = \gcd(b_2, b_3) = 1$. It remains to show that $\gamma_1, \gamma_2, \gamma_3$ are multiplicatively independent.

3.4 Showing that $\gamma_1, \gamma_2, \gamma_3$ are multiplicatively independent

To show that $\gamma_1, \gamma_2, \gamma_3$ are multiplicatively independent, we follow the same explanation given in Subsection 3.4 of [1].

3.5 Applying Theorem 2.1

In Subsection 3.5 of [1], everything is fine and well proved from the start of this subsection up-to equation (34) in [1]. However, Theorem 2.1 was wrongly applied. We therefore start from equation (34) on page 662 in [1] and apply Theorem 2.1 correctly. We first restate this equation as

$$x < 1.38 \times 10^6 n\left(1 + \frac{1}{r\log r}\right)(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^2. \tag{3.2}$$

We now go back to possibilities (i) and (ii) in Theorem 2.1.

(i) In case (i), there are positive integers r_0, s_0, which may be assumed to be coprime, such that $r_0 b_2 = s_0 b_1$. So, we get $r_0 m = s_0$ and since r_0, s_0 are coprime, we take $r_0 = 1$, $s_0 = m$, and we get

$$m = s_0 < 5.61 A_1 (\mathcal{D}\log\mathcal{B})^{\frac{1}{3}}. \tag{3.3}$$

Since $x < (n-1)x + 1 < m$ by (19) in [1], and the fact that $x \geq 3$, $n \geq 2$, then (3.3) gives

$$x < 5.61 A_1 (\mathcal{D}\log\mathcal{B})^{\frac{1}{3}}. \tag{3.4}$$

Now, from (3.4), assuming first that $\log\mathcal{B} = 5$, then

$$\begin{aligned} x &< 5.61 \times 8.296\log(r+1)(2\times 5)^{\frac{1}{3}},\\ &< 101\log(r+1). \end{aligned} \tag{3.5}$$

Next, if $\log\mathcal{B} > 5$, then

$$\begin{aligned} x &< 5.61 \times 8.296\log(r+1)\left[2\log\left(\frac{0.3mx}{(\log(r+1))^2}\right)\right]^{\frac{1}{3}},\\ &< 59\log(r+1)\left[\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right]^{\frac{1}{3}}. \end{aligned} \tag{3.6}$$

Case (i) of Theorem 2.1 is done.

(ii) In case (ii), we have integers r_1, s_1, t_1 and t_2, with $r_1 s_1 \neq 0$, such that

$$(t_1 b_1 + r_1 b_3)s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1.$$

Thus, for us, we have

$$(t_1 + r_1 x)s_1 = r_1 m t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1. \tag{3.7}$$

Reducing equation (3.7) modulo r_1, we get $t_1 s_1 \equiv 0 \pmod{r_1}$ and since $\gcd(t_1, r_1) = 1$, we get that $r_1 | s_1$. So, we put $s_1 = r_1 s_1'$, and simplify both sides of (3.7) by r_1 to get

$$(t_1 + r_1 x)s_1' = m t_2.$$

Consequently, for us $\delta = \gcd(r_1, s_1) = r_1$. Hence,

$$\begin{aligned}
|r_1(r_1 s_1')| = |r_1 s_1| &\leq 5.61 \delta A_3 (\mathcal{D} \log \mathcal{B})^{\frac{1}{3}}, \\
&= 5.61 r_1 \times 8.296 n \log(r+1)(2 \log \mathcal{B})^{\frac{1}{3}}, \\
&< 59 n r_1 \log(r+1)(\log \mathcal{B})^{\frac{1}{3}}, \\
\text{so,} \quad |r_1 s_1'| &< 59 n \log(r+1)(\log \mathcal{B})^{\frac{1}{3}}, \\
|(r_1 s_1')t_1| = |s_1 t_1| &\leq 5.61 \delta A_1 (\mathcal{D} \log \mathcal{B})^{\frac{1}{3}}, \\
&= 5.61 r_1 \times 8.296 \log(r+1)(2 \log \mathcal{B})^{\frac{1}{3}}, \\
&< 59 r_1 \log(r+1)(\log \mathcal{B})^{\frac{1}{3}}, \\
\text{so,} \quad |s_1' t_1| &< 59 \log(r+1)(\log \mathcal{B})^{\frac{1}{3}}, \\
|r_1 t_2| &\leq 5.61 \delta A_2 (\mathcal{D} \log \mathcal{B})^{\frac{1}{3}}, \\
&= 5.61 r_1 \times 6.296 \log(r+1)(2 \log \mathcal{B})^{\frac{1}{3}}, \\
&< 45 r_1 \log(r+1)(\log \mathcal{B})^{\frac{1}{3}}, \\
\text{so,} \quad |t_2| &< 45 \log(r+1)(\log \mathcal{B})^{\frac{1}{3}}.
\end{aligned} \tag{3.8}$$

Assuming first that $\log \mathcal{B} = 5$, then

$$\begin{aligned}
&|r_1 s_1'| < 101 n \log(r+1), \\
&|s_1' t_1| < 101 \log(r+1), |t_2| < 77 \log(r+1).
\end{aligned} \tag{3.9}$$

So, if $t_2 = 0$, then by Theorem 2.1, $s_1 = 1 = r_1 = s_1'$ and

$$x = \frac{|t_1|}{|r_1|} \leq |s_1' t_1| < 101 \log(1+r),$$

which is back to the case in (3.5). If $t_2 \neq 0$, we return to equation (27) in [1], and multiply both sides by t_2 and get

$$|mt_2 \log \gamma_2 - t_2 \log \gamma_1 - xt_2 \log \gamma_3| < \frac{2.2|t_2|}{r^x}, \text{ but } (t_1 + r_1 x)s_1' = mt_2.$$

Thus,

$$\left| \log \left(\frac{\gamma_2^{t_1 s_1'}}{\gamma_1^{t_2}} \right) + x \log \left(\frac{\gamma_2^{r_1 s_1'}}{\gamma_3^{t_2}} \right) \right| < \frac{2.2|t_2|}{r^x}. \tag{3.10}$$

Inequality (3.10) is exactly the same inequality (37) in [1]. We skip the details of checking the multiplicative independence of the algebraic numbers η_1 and η_2 in inequality (37) of [1], since this was done correctly on page 664 of [1].

So we are now in position to apply Theorem 3 (in [1]) to the left-hand side of inequality (38) in [1]. We first compute $\log B_1$ and $\log B_2$. By properties (15) on page 654 of [1],

$$\begin{aligned}
h(\eta_1) &\leq |s_1' t_1| h(\gamma_2) + |t_2| h(\gamma_1), \\
&\leq 101 \log(r+1) \cdot \frac{1}{2} \log(r+1) + 77 \log(r+1) \cdot \log(r+1) \\
&< 128 (\log(r+1))^2, \\
h(\eta_2) &\leq |r_1 s_1'| h(\gamma_2) + |t_2| h(\gamma_3), \\
&\leq 101 n \log(r+1) \cdot \frac{1}{2} \log(r+1) + 77 \log(r+1) \cdot n \log(r+1) \\
&< 128 n (\log(r+1))^2.
\end{aligned}$$

By the same argument on page 664 of [1], we can take

$$\log B_1 := 128(\log(r+1))^2, \quad \log B_2 := 128n(\log(r+1))^2.$$

Next, we bound

$$\frac{1}{2 \log B_2} + \frac{x}{2 \log B_1} = \frac{1}{256(\log(r+1))^2} \left(\frac{1}{n} + x \right) < \frac{x+1}{256(\log(r+1))^2},$$

and so,

$$b' := \frac{x+1}{256(\log(r+1))^2}.$$

Now, Theorem 3 in [1] gives

$$\begin{aligned}\log|\Gamma_1| &> -24.34\times 2^4(\max\{\log b'+0.14, 10.5, 0.5\})^2\\ &\quad\times(128(\log(r+1))^2)(128n(\log(r+1))^2)\\ &> -6.4\times 10^6 n(\log(r+1))^4 M^2,\end{aligned} \tag{3.11}$$

where $M := \max\{\log b'+0.14, 10.5, 0.5\}$. In case $M=10.5$, then

$$\log b'+0.14<10.5, \quad \text{or} \quad b'<31572,$$

which gives

$$b' := \frac{x+1}{256(\log(r+1))^2} < 31572.$$

Thus,

$$\begin{aligned} x+1 &< 31572\times 256(\log(r+1))^2;\\ x &< 8.1\times 10^6(\log(r+1))^2. \end{aligned} \tag{3.12}$$

Next, suppose

$$M=\log b'+0.14=\log(e^{0.14}b')<\log(1.5b')=\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right).$$

Comparing (3.11) and equation (38) in [1], we get

$$x\log r-\log(2.2|t_2|)<6.4\times 10^6 n(\log(r+1))^4\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2.$$

Since $|t_2|<77\log(r+1)$, then

$$\begin{aligned} x &< \frac{\log(170\log(r+1))}{\log r}\\ &\quad+6.4\times 10^6 n\left(\frac{\log(r+1)}{\log r}\right)(\log(r+1))^3\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2. \end{aligned} \tag{3.13}$$

The summand in the right-hand side of (3.13) is less than 5 for all $r\geq 3$. Using inequality (32) in [1], we get

$$x<6.41\times 10^6 n\left(1+\frac{1}{r\log r}\right)(\log(r+1))^3\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2. \tag{3.14}$$

Now, we are done with the case that $\log \mathcal{B} = 5$ in (3.8). Next, we go back to (3.8) and assume that $\log \mathcal{B} > 5$. Then

$$
\begin{aligned}
|r_1 s_1'| &< 59n \log(r+1)\left(\log\left(\frac{0.3mx}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}},\\
|s_1' t_1| &< 59 \log(r+1)\left(\log\left(\frac{0.3mx}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}},\\
|t_2| &< 45 \log(r+1)\left(\log\left(\frac{0.3mx}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}},
\end{aligned}
\tag{3.15}
$$

by the inequality (31) in [1]. So, if $t_2 = 0$, then by Theorem 2.1, $s_1 = 1 = r_1 = s_1'$ and

$$
\begin{aligned}
x = \frac{|t_1|}{|r_1|} &\leq |s_1' t_1| < 59 \log(r+1)\left(\log\left(\frac{0.3mx}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\\
&< 59 \log(r+1)\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}},
\end{aligned}
$$

where we have used the inequality $m < (n+1)x + 2 < (n+1)(x+1)$ (because $n \geq 2$), and we are back to the case in (3.6). If $t_2 \neq 0$, we return to inequality (27) in [1] and multiply both sides by t_2 and get

$$
|mt_2 \log \gamma_2 - t_2 \log \gamma_1 - xt_2 \log \gamma_3| < \frac{2.2|t_2|}{r^x}, \quad \text{but} \quad (t_1 + r_1 x)s_1' = mt_2.
$$

Therefore,

$$
\left|\log\left(\frac{\gamma_2^{t_1 s_1'}}{\gamma_1^{t_2}}\right) + x \log\left(\frac{\gamma_2^{r_1 s_1'}}{\gamma_3^{t_2}}\right)\right| < \frac{2.2|t_2|}{r^x}. \tag{3.16}
$$

Inequality (3.16) is again of the form

$$
|\Gamma_2| < \frac{2.2|t_2|}{r^x}, \quad \text{where} \quad |\Gamma_2| := \log \eta_1 + x \log \eta_2. \tag{3.17}
$$

It was already checked that η_1 and η_2 are multiplicatively independent. See page 664 in [1], after inequality (38).

So, we are now in position to apply Theorem 3 (in [1]) to the left-hand side of (3.17). We first compute $\log B_1$ and $\log B_2$. By properties (15) on page 654 of [1],

$$\begin{aligned} h(\eta_1) &\leq |s_1' t_1| h(\gamma_2) + |t_2| h(\gamma_1) \\ &\leq 59 \log(r+1) \left(\log\left(\frac{0.3mx}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}} \cdot \frac{1}{2} \log(r+1) \\ &\quad + 45 \log(r+1) \left(\log\left(\frac{0.3mx}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}} \cdot \log(r+1) \\ &< 75 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} h(\eta_2) &\leq |r_1 s_1'| h(\gamma_2) + |t_2| h(\gamma_3) \\ &\leq 59n \log(r+1) \left(\log\left(\frac{0.3mx}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}} \cdot \frac{1}{2} \log(r+1) \\ &\quad + 45 \log(r+1) \left(\log\left(\frac{0.3mx}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}} \cdot n \log(r+1) \\ &< 75n (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}. \end{aligned}$$

Since $\frac{|\log \gamma_i|}{2} \leq h(\gamma_i)$, for all $i = 1, 2, 3$, it follows, by the absolute value inequality, that the same inequalities are satisfied by the numbers $\frac{|\log \eta_i|}{2}$ for $i = 1, 2$. Thus, since $\mathcal{D} = 2$, we can take

$$\begin{aligned} \log B_1 &:= 75 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}, \\ \log B_2 &:= 75n (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}. \end{aligned}$$

Next, we bound

$$\begin{aligned} \frac{1}{2\log B_2} + \frac{x}{2\log B_1} &= \frac{1}{150 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}} \left(\frac{1}{n} + x \right) \\ &< \frac{x+1}{150 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}}, \end{aligned}$$

and so,

$$b' := \frac{x+1}{150(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}}.$$

Now, Theorem 3 in [1] gives

$$\begin{aligned}\log|\Gamma_2| &> -24.34 \times 2^4(\max\{\log b' + 0.14, 10.5, 0.5\})^2\\ &\quad \times 75(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\\ &\quad \times 75n(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\\ &> -2.2 \times 10^6 n(\log(r+1))^4\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\Bigg)^{\frac{1}{3}} M^2, \end{aligned} \tag{3.19}$$

where $M := \max\{\log b' + 0.14, 10.5, 0.5\}$. In case $M = 10.5$, then

$$\log b' + 0.14 < 10.5, \quad \text{or} \;\; b' < 31572,$$

which gives

$$b' := \frac{x+1}{150(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}} < 31572.$$

Therefore,

$$\begin{aligned} x+1 &< 31572 \times 150(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}},\\ x &< 4.8 \times 10^6(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}. \end{aligned} \tag{3.20}$$

Next, suppose

$$\begin{aligned} M &= \log b' + 0.14 = \log(e^{0.14}b') < \log(1.5b')\\ &= \log\left[\frac{x+1}{100(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}}\right]. \end{aligned} \tag{3.21}$$

Comparing (3.19) and (3.17), we get

$$x\log r-\log(2.2|t_2|)<2.2\times10^6 n(\log(r+1))^4 \times\left(\log\left[\frac{x+1}{100(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2. \tag{3.22}$$

Since

$$|t_2|<45\log(r+1)\left(\log\left(\frac{0.3mx}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}<45\log(r+1)\times\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}},$$

then

$$\begin{aligned} x<{}&\frac{\log\left[99\log(r+1)\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\right]}{\log r}\\ &+2.2\times10^6 n\left(\frac{\log(r+1)}{\log r}\right)(\log(r+1))^3\\ &\times\left(\log\left[\frac{x+1}{100(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2. \end{aligned} \tag{3.23}$$

Using inequality (32) on page 662 of [1], we get

$$\begin{aligned} x<{}&\frac{\log\left[99\log(r+1)\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\right]}{\log r}\\ &+2.2\times10^6 n\left(1+\frac{1}{r\log r}\right)(\log(r+1))^3\\ &\times\left(\log\left[\frac{x+1}{100(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2. \end{aligned} \tag{3.24}$$

Now, we are done with the case that $\log\mathcal{B}>5$ in (3.8).

To summarize this subsection, we see that inequality (3.5) is contained in inequality (30) of [1], which is also contained in inequality (3.12). Moreover, inequality (3.6) is also contained in inequality (3.20), so we remain to summarize inequalities (3.2), (3.12), (3.14), (3.20) and (3.24) in the following Lemma 3.1 which we have just proved.

Lemma 3.1 *Let $n \geq 2$, $x \geq 3$ and $r \geq 3$ satisfy (1.3), then one of the following inequalities hold.*

$$x < 8.1 \times 10^6 (\log(r+1))^2, \tag{3.25}$$

$$x < 4.8 \times 10^6 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}, \tag{3.26}$$

$$x < 1.38 \times 10^6 n \left(1 + \frac{1}{r \log r}\right) (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^2, \tag{3.27}$$

$$x < 6.41 \times 10^6 n \left(1 + \frac{1}{r \log r}\right) (\log(r+1))^3 \left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2} \right) \right)^2, \tag{3.28}$$

$$\begin{aligned} x <& \frac{\log\left[99 \log(r+1)\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}\right]}{\log r} \\ &+ 2.2 \times 10^6 n \left(1 + \frac{1}{r \log r}\right) (\log(r+1))^3 \\ &\times \left(\log\left[\frac{x+1}{100(\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}} \right] \right)^2. \end{aligned} \tag{3.29}$$

3.6 More inequalities in terms of *n* and *x*

Here, we adopt the results from Subsection 3.6 of [1] as they are. All results there are correctly stated and proved.

3.7 Another inequality among *r*, *n*, *m* and *x*

In this subsection, we adopt Lemma 10 and Lemma 11 from Subsection 3.7 of [1].

3.8 The case $n \leq 100$

We first seek bounds on r. Having the bounds in r and n, we get bounds on x using Lemma 3.1. Finally, for a fixed r, we use Baker-Davenport on estimate (27) in [1] to lower x. The hope is that in all cases, we get $x \leq 100$, a case which has already been treated.

Note that Lemma 3.2 is analogous to Lemma 12 in [1]. However, Lemma 3.2 has a larger bound on r. We reprove it below.

Lemma 3.2 *When $n \le 100$, we have $r \le 10^{12}$.*

Proof Assume $r > 10^{12}$. Then

$$x > \frac{\kappa r^2 \log r + 1}{1 + \frac{5}{r}} > \frac{\kappa r^2 \log r}{1 + \frac{5}{r}} > \frac{\kappa r^2 \log r}{1.01} > \frac{n r^2 \log r}{2.02},$$

by Lemmas 11 and 9 in [1]. We now go through the possibilities in Lemma 3.1.

(i) We start with case (ii), that is,

$$\begin{aligned} \frac{n r^2 \log r}{2.02} &< x < 8.1 \times 10^6 (\log(r+1))^2, \\ r^2 &< \frac{2.02}{2 \log 10^{12}} \times 8.1 \times 10^6 (\log(r+1))^2, \quad \text{since } n \ge 2, \quad r > 10^{12}, \\ r &< 545 \log(r+1), \quad \text{or equivalently,} \\ \frac{r}{\log r} &\le 545. \end{aligned}$$

We now apply Lemma 2.1 with the data: $x = r$, $s = 1$ and $T = 545 > (4s^2)^s = 4$. We get $r < 2 \times 545 \log 545 = 6868$, a contradiction.

(ii) Assume we are in case (ii), then

$$x + 1 < 4.8 \times 10^6 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}. \tag{3.30}$$

Now,

$$\begin{aligned} x + 1 > x > \frac{n r^2 \log r}{2.02} &= n r^2 \log(r+1) \left(\frac{\log r}{2.02 \log(r+1)} \right) \\ &> \frac{n r^2 \log(r+1)}{2.03}, \quad \text{since} \quad r > 10^{12}. \end{aligned}$$

Put $y := \dfrac{x+1}{n \log(r+1)}$. The above inequality becomes $y > \dfrac{r^2}{2.03}$. Inequality (3.30) can be rewritten in terms of y as

$$\begin{aligned} y = \frac{x+1}{n \log(r+1)} &< x + 1 < 4.8 \times 10^6 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}} \\ y &< 4.8 \times 10^6 (\log(r+1))^2 \left(\log\left(0.3 n^2 (n+1) y^2 \right) \right)^{\frac{1}{3}} \\ &< 4.8 \times 10^6 (\log(r+1))^2 \left(\log\left(0.3 \cdot 100^2 (100+1) y^2 \right) \right)^{\frac{1}{3}} \end{aligned}$$

$$< 4.8 \times 10^6 (\log(r+1))^2 \big(13 + 2\log y\big)^{\frac{1}{3}}.$$

We look at the function

$$f(y) := \frac{y}{\big(13 + 2\log y\big)^{\frac{1}{3}}}.$$

Its derivative is

$$f'(y) = \frac{6\log y + 37}{3\big(13 + 2\log y\big)^{\frac{4}{3}}} > 0,$$

so our function is increasing. Since $f(y) < 4.8 \times 10^6 (\log(r+1))^2$, and $y > \dfrac{r^2}{2.03}$, then

$$f\left(\frac{r^2}{2.03}\right) < 4.8 \times 10^6 (\log(r+1))^2,$$

$$\frac{\left(\dfrac{r^2}{2.03}\right)}{\left[(13 + 2\log\left(\dfrac{r^2}{2.03}\right)\right]^{\frac{1}{3}}} < 4.8 \times 10^6 (\log(r+1))^2.$$

This gives

$$\frac{r}{(\log r)^{\frac{3}{2}}} < 5916.$$

We now apply Lemma 2.1 with the data: $x = r, s = \frac{3}{2}$ and $T = 5916 > (4s^2)^s = 27$. We get $r < 2^{\frac{3}{2}} \times 5916 (\log 5916)^{\frac{3}{2}} < 428311$, a contradiction.

(iii) Assume we are in case (iii), then we use the same substitution $y := \dfrac{x+1}{n\log(r+1)}$ to get

$$\begin{aligned}
y &= \frac{x+1}{n\log(r+1)} < \frac{x+1}{n} \\
&< 1.38 \times 10^6 \left(1 + \frac{1}{r\log r}\right)(\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^2 \\
&< 1.38 \times 10^6 \times 1.001 (\log(r+1))^2 \big(\log\big(0.3 n^2 (n+1) y^2\big)\big)^2 \\
&= 1.38 \times 10^6 \times 1.001 (\log(r+1))^2 \big(\log\big(0.3 \times 100^2 (100+1) y^2\big)\big)^2 \\
&< 1.39 \times 10^6 (\log(r+1))^2 \big(13 + 2\log y\big)^2.
\end{aligned}$$

By similar arguments in (ii), the function

$$f(y) := \frac{y}{\big(13 + 2\log y\big)^2},$$

is also increasing, so we deduce that

$$\frac{\left(\frac{r^2}{2.03}\right)}{\left[13 + 2\log\left(\frac{r^2}{2.03}\right)\right]^2} < 1.39 \times 10^6 (\log(r+1))^2,$$
$$\text{or} \quad \frac{r}{(\log r)^2} < 77835.$$

We again apply Lemma 2.1 with the data: $x = r$, $s = 2$ and $T = 77835 > (4s^2)^s = 256$. We get $r < 2^2 \times 77835(\log 77835)^2 < 3.95 \times 10^7$, a contradiction.

(iv) We go to case (iv) of Lemma 3.1, and still use the same substitution $y := \dfrac{x+1}{n\log(r+1)}$ to get

$$\begin{aligned}
y &= \frac{x+1}{n\log(r+1)} < \frac{x+1}{n} \\
&< 6.41 \times 10^6 \left(1 + \frac{1}{r\log r}\right)(\log(r+1))^3 \left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2 \\
&< 6.41 \times 10^6 \times 1.001(\log(r+1))^3 \big(\log\big(0.006 n^2 y^2\big)\big)^2 \\
&= 6.41 \times 10^6 \times 1.001(\log(r+1))^3 \big(\log\big(0.006 \times 100^2 y^2\big)\big)^2 \\
&< 6.42 \times 10^6 (\log(r+1))^3 \big(4.1 + 2\log y\big)^2.
\end{aligned}$$

By similar arguments as in (ii), the function

$$f(y) := \frac{y}{\big(4.1 + 2\log y\big)^2},$$

is also increasing, so we deduce that

$$\frac{\left(\frac{r^2}{2.03}\right)}{\left[4.1 + 2\log\left(\frac{r^2}{2.03}\right)\right]^2} < 6.42 \times 10^6 (\log(r+1))^3$$
$$\text{or} \quad \frac{r}{(\log r)^{\frac{5}{2}}} < 38757.$$

We again apply Lemma 2.1 with the data: $x = r$, $s = \frac{5}{2}$ and $T = 38757 > (4s^2)^s = 3125$. We get $r < 2^{\frac{5}{2}} \times 38757(\log 38757)^{\frac{5}{2}} < 7.96 \times 10^7$, a contradiction.

(v) Lastly, we are in case (v) of Lemma 3.1, and still use the same substitution $y := \dfrac{x+1}{n\log(r+1)}$ to get

$$\begin{aligned}
y &= \frac{x+1}{n\log(r+1)} < \frac{x+1}{n}\\
&< \frac{\log\left[99\log(r+1)\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\right]}{n\log r}\\
&\quad + 2.2\times 10^6\left(1+\frac{1}{r\log r}\right)(\log(r+1))^3\\
&\quad \times\left(\log\left[\frac{x+1}{100(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2\\
&< \frac{\frac{1}{3}\log 99\log(r+1)\log(0.3n^2(n+1)y^2)}{2\log 10^{12}} + 2.2\times 10^6\times 1.01(\log(r+1))^3\\
&\quad \times\left(\log\left[\frac{0.01n^2y^2}{(\log(0.3n^2(n+1)y^2))^{\frac{1}{3}}}\right]\right)^2.
\end{aligned}$$

Since $2 \le n \le 100$ for this subsection, we have

$$\begin{aligned}
y &< \frac{\frac{1}{3}\log 99\log(r+1)\log(0.3\times 100^2(100+1)y^2)}{2\log 10^{12}} + 2.2\\
&\quad \times 10^6\times 1.01(\log(r+1))^3\\
&\quad \times\left(\log\left[\frac{0.01\times 100^2y^2}{(\log(0.3\times 2^2(2+1)y^2))^{\frac{1}{3}}}\right]\right)^2\\
&< (13+2\log y)\log(r+1) + 2.3\times 10^6(\log(r+1))^3\left(\log\left[\frac{100y^2}{(\log(3.6y^2))^{\frac{1}{3}}}\right]\right)^2\\
&< (13+2\log y)(\log(r+1))^3 + 2.3\times 10^6(\log(r+1))^3\\
&\quad \times(10\log y - \log\log 3.6y^2)^2\\
&< 26\log y\log(r+1)\cdot 2.3\times 10^6(\log(r+1))^3\times(10\log y)^2\\
&< 6\times 10^9(\log(r+1))^4(\log y)^2.
\end{aligned}$$

The function

$$f(y) := \frac{y}{(\log y)^2},$$

has derivative

$$f'(y) := \frac{\log y - 2}{(\log y)^3} > 0,$$

for all $y > e^2$, which is our case. Thus $f(y)$ is also increasing, so we deduce that

$$\frac{\left(\frac{r^2}{2.03}\right)}{\left[\log\left(\frac{r^2}{2.03}\right)\right]^2} < 6 \times 10^9 (\log(r+1))^4,$$
$$\text{or} \quad \frac{r}{(\log r)^3} < 220727.$$

We again apply Lemma 2.1 with the data: $x = r$, $s = 3$ and $T = 220727 > (4s^2)^s = 46656$. We get $r < 2^3 \times 220727(\log 220727)^3 < 3.3 \times 10^9$, a contradiction.

This completes the proof of Lemma 3.2. $\square$

Now, having bounds $2 \le n \le 100$ and $3 \le r \le 10^{12}$, the inequalities in Lemma 3.1 become

(i)
$$\begin{aligned} x &< 8.1 \times 10^6 (\log(r+1))^2 \\ &\le 8.1 \times 10^6 (\log(10^{12}+1))^2 \\ &< 6.2 \times 10^9. \end{aligned}$$

(ii)
$$\begin{aligned} x &< 4.8 \times 10^6 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}} \\ &\le 4.8 \times 10^6 (\log(10^{12}+1))^2 \left(\log\left(\frac{0.3(100+1)(x+1)^2}{(\log(3+1))^2}\right)\right)^{\frac{1}{3}}. \end{aligned}$$
$$\frac{x}{\log x} < 7.4 \times 10^9.$$

We apply Lemma 2.1 with the data: $s = 1$ and $T = 7.4 \times 10^9 > (4s^2)^s$. We get $x < 2 \times 7.4 \times 10^9 \log 7.4 \times 10^9 < 3.4 \times 10^{11}$.

(iii)
$$\begin{aligned} x &< 1.38 \times 10^6 n \left(1 + \frac{1}{r \log r}\right) (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^2 \\ &\le 1.38 \times 10^6 \times 100 \times 1.01 (\log(10^{12}+1))^2 \left(\log\left(\frac{0.3(100+1)(x+1)^2}{(\log(3+1))^2}\right)\right)^2. \end{aligned}$$

$$\frac{x}{(\log x)^2} < 3.3 \times 10^{12}.$$

We apply Lemma 2.1 with the data: $s = 2$ and $T = 3.3 \times 10^{12} > (4s^2)^s$. We get $x < 2^2 \times 3.3 \times 10^{12}(\log 3.3 \times 10^{12})^2 < 1.1 \times 10^{16}$.

(iv)

$$\begin{aligned} x &< 6.41 \times 10^6 n\left(1 + \frac{1}{r\log r}\right)(\log(r+1))^3\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2 \\ &\le 6.41 \times 10^6 \times 100 \times 1.01(\log(10^{12}+1))^3\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(3+1))^2}\right)\right)^2. \end{aligned}$$

$$\frac{x}{(\log x)^2} < 1.4 \times 10^{13}.$$

We apply Lemma 2.1 with the data: $s = 2$ and $T = 1.4 \times 10^{13} > (4s^2)^s$. We get $x < 2^2 \times 1.4 \times 10^{13}(\log 1.4 \times 10^{13})^2 < 5.2 \times 10^{16}$.

(v)

$$\begin{aligned} x &< \frac{\log\left[99\log(10^{12}+1)\left(\log\left(\frac{0.3(100+1)(x+1)^2}{(\log(3+1))^2}\right)\right)^{\frac{1}{3}}\right]}{\log 3} \\ &\quad + 2.2 \times 10^6 \times 100 \times 1.01(\log(10^{12}+1))^3 \\ &\quad \times \left(\log\left[\frac{x+1}{100(\log(3+1))^2\left(\log\left(\frac{0.3(2+1)(x+1)^2}{(\log(10^{12}+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2 \\ &< 4400(\log(x+1))^{\frac{1}{3}} + 4.7 \times 10^{12}(\log(x+1) - (-1)\log 458(\log(x+1))^{\frac{1}{3}})^2. \end{aligned}$$

$$\frac{x}{(\log x)^3} < 2.9 \times 10^{13}.$$

We apply Lemma 2.1 with the data: $s = 3$ and $T = 2.9 \times 10^{13} > (4s^2)^s$. We get $x < 2^3 \times 2.9 \times 10^{13}(\log 2.9 \times 10^{13})^3 < 7 \times 10^{18}$.

From all the above inequalities, we have $x < 7 \times 10^{18}$. Now we perform the Baker-Davenport reduction on relation (27) in [1], for $2 \le n \le 100$, $3 \le r \le 10^{12}$ and $x < 7 \times 10^{18}$. This also gives $m < 7.1 \times 10^{20}$ via inequality (19) in [1]. We return to (27) in [1] and rewrite it as

$$\left| x\frac{\log U_{n+1}}{\log\alpha} - m + \frac{\log\sqrt{r^2+4}}{\log\alpha} \right| < \frac{2.2}{r^x\log\alpha}. \tag{3.31}$$

We now apply Lemma 5 from [1], with the following data:

$$M := 7.1 \times 10^{20}, \quad \tau := \frac{\log U_{n+1}}{\log\alpha}, \quad \mu := \frac{\log\sqrt{r^2+4}}{\log\alpha}, \quad A := \frac{2.2}{\log\alpha} \text{ and } B := r.$$

A computer search in Mathematica reveals that $x \le 89$, which is a contradiction (an already treated case in Subsection 3.2). This computation lasted for 8 days on an 8GB RAM laptop.

3.9 The case $n > 100$

In this subsection, we adopt relations (54) and (55) on page 674 of [1]. Using bounds obtained in Lemma 3.1, the statement in Lemma 13 of [1] is correct. Its proof follows exactly as given in [1], but using relations given here in Lemma 3.1. For this reason, we adopt the statement of Lemma 13 in [1]. We also adopt relations (56) to (58) in [1], since they are well derived. However, for us here, we use the fact that for $r \geq 10^{12}$, then we still maintain $\dfrac{\log(r+1)}{\log r} < 1.0001$.
Next, we state and prove Lemma 3.3 below. This is analogous to Lemma 14 in [1], though it fell short out of the actual ranges.

Lemma 3.3 *If* $r \geq 4$, *then* $r \leq 3.3 \times 10^{14}$.

Proof We now use the bounds on x given in Lemma 3.1.

(i) Here,

$$\begin{aligned} 50r^2 \log r &< x < 8.1 \times 10^6 (\log(r+1))^2 \\ r^2 &< \left(\frac{8.1 \times 10^6}{50}\right)\left(\frac{\log(r+1)}{\log r}\right)\log(r+1). \\ \frac{r}{\log r} &< 403. \end{aligned}$$

We apply Lemma 2.1 with the data: $s = 1$ and $T = 403 > (4s^2)^s = 4$. We get $r < 2 \times 403 \log 403 < 4836$, a contradiction.

(ii) In this situation,

$$x + 1 < 4.8 \times 10^6 (\log(r+1))^2 \left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}.$$

In case $\dfrac{x+1}{\log(r+1)} \leq n$, we have

$$\begin{aligned} \frac{x+1}{\log(r+1)} \leq n &< 196 \log(r+1)\left(\log\left(\frac{2.5x}{\log(r+1)}\right)\right)^2. \\ x + 1 &< 196(\log(r+1))^2 \left(\log\left(\frac{2.5x}{\log(r+1)}\right)\right)^2. \end{aligned}$$

Putting $y := \dfrac{x}{\log(r+1)}$, we get $y < 196 \log(r+1)(\log 2.5y)^2$. Note that

$$y = \frac{x}{\log(r+1)} > \frac{nr^2\log r}{2.01\log(r+1)} \geq \frac{101r^2}{2.01\left(\frac{\log(r+1)}{\log r}\right)}$$
$$> \frac{101r^2}{2.02} = 50r^2.$$

In the above inequalities, we have used the fact that $r \geq 10^{12}$. The function $f(y) = \frac{y}{(\log 2.5y)^2}$ is increasing for all $y > \frac{e^2}{2.5}$, which is our case. Hence, the inequality $y < 196\log(r+1)(\log 2.5y)^2$ should hold when y is replaced with $50r^2$. This gives

$$50r^2 < 196\log(r+1)(\log 2.5 \times 50r^2)^2.$$
$$\frac{r}{(\log r)^{\frac{3}{2}}} < 91.$$

Applying Lemma 2.1 with the data: $s = \frac{3}{2}$ and $T = 91 > (4s^2)^s = 27$. We get $r < 2^{\frac{3}{2}} \times 91(\log 91)^{\frac{3}{2}} < 2466$, a contradiction.
Thus $n < \frac{x+1}{\log(r+1)}$. Since $0.3(n+1) < n < \frac{x+1}{\log(r+1)}$, we can conclude that

$$x+1 < 4.8 \times 10^6(\log(r+1))^2\left(\log\left(\frac{(x+1)^3}{(\log(r+1))^3}\right)\right)^{\frac{1}{3}}$$
$$< 7 \times 10^6(\log(r+1))^2\left(\log\left(\frac{x+1}{\log(r+1)}\right)\right)^{\frac{1}{3}}.$$

Putting $y := \frac{x+1}{\log(r+1)}$, we get $y < 7 \times 10^6\log(r+1)(\log y)^{\frac{1}{3}}$. The function $f(y) = \frac{y}{(\log y)^{\frac{1}{3}}}$ is increasing for all $y > 50r^2 > 50(10^{12})^2 = 5 \times 10^{24}$, so we get that the above inequality should hold when y is replaced with $50r^2$. Thus,

$$50r^2 < 7 \times 10^6\log(r+1)(\log(50r^2))^{\frac{1}{3}}.$$
$$\frac{r}{(\log r)^2} < 528.$$

We again apply Lemma 2.1 with the data: $s = 2$ and $T = 528 > (4s^2)^s = 256$. We get $r < 2^2 \times 528(\log 528)^2 < 8.4 \times 10^4$, a contradiction.

(iii) In the third case, we have

$$x+1 < 1.38 \times 10^6 n\left(1 + \frac{1}{r\log r}\right)(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^2$$

$$< 1.39 \times 10^6 n(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^2.$$

We have already shown in (ii) that the case $\frac{x+1}{\log(r+1)} \leq n$ is impossible. Thus, $n < \frac{x+1}{\log(r+1)}$ and still we use the fact that $0.3(n+1) < n < \frac{x+1}{\log(r+1)}$. We can now write

$$x+1 < 1.39 \times 10^6 \left[196 \log(r+1)\left(\log\left(\frac{2.5x}{\log(r+1)}\right)\right)^2\right](\log(r+1))^2 \\ \times \left(\log\left(\frac{(x+1)^3}{(\log(r+1))^3}\right)\right)^2.$$

Putting $y := \frac{x+1}{\log(r+1)}$, we get

$$y < 4.72 \times 10^8 (\log(r+1))^2 (\log 2.5y)^2 (\log y)^2 \\ < 4.72 \times 10^8 (\log(r+1))^2 (\log 2.5y)^4.$$

The function $f(y) = \frac{y}{(\log 2.5y)^4}$ is increasing for all $y > 50r^2 > 50(10^{12})^2 = 5 \times 10^{25}$, so we get that the above inequality should hold when y is replaced with $50r^2$. Thus,

$$50r^2 < 4.72 \times 10^8 (\log(r+1))^2 (\log(2.5 \times 50r^2))^4.$$
$$\frac{r}{(\log r)^3} < 2.9 \times 10^5.$$

We again apply Lemma 2.1 with the data: $s = 3$ and $T = 2.9 \times 10^5 > (4s^2)^s = 46656$. We get $r < 2^3 \times 2.9 \times 10^5 (\log 2.9 \times 10^5)^3 < 4.7 \times 10^9$, a contradiction.

(iv) In case (iv) of Lemma 3.1, we have

$$x+1 < 6.41 \times 10^6 n\left(1+\frac{1}{r\log r}\right)(\log(r+1))^3\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2 \\ < 6.5 \times 10^6 n(\log(r+1))^3\left(\log\left(\frac{x+1}{(\log(r+1))^2}\right)\right)^2.$$

We have already shown in (ii) that the case $\frac{x+1}{\log(r+1)} \le n$ is impossible. Thus, $n < \frac{x+1}{\log(r+1)}$ and we can now write

$$x+1 < 6.5 \times 10^6 \left[196\log(r+1)\left(\log\left(\frac{2.5x}{\log(r+1)}\right)\right)^2\right](\log(r+1))^2$$
$$\times \left(\log\left(\frac{(x+1)^2}{(\log(r+1))^2}\right)\right)^2.$$

Putting $y := \frac{x+1}{\log(r+1)}$, we get

$$\begin{aligned} y &< 1.3 \times 10^9 (\log(r+1))^2 (\log 2.5y)^2 (\log y^2)^2 \\ &< 5.2 \times 10^9 (\log(r+1))^2 (\log 2.5y)^4. \end{aligned}$$

The function $f(y) = \frac{y}{(\log 2.5y)^4}$ is increasing for all $y > 50r^2 > 50(10^{12})^2 = 5 \times 10^{24}$, so we get that the above inequality should hold when y is replaced with $50r^2$. Thus,

$$\begin{aligned} 50r^2 &< 5.2 \times 10^9 (\log(r+1))^2 (\log(2.5 \times 50r^2))^4. \\ \frac{r}{(\log r)^3} &< 9.6 \times 10^5. \end{aligned}$$

We again apply Lemma 2.1 with the data: $s = 3$ and $T = 9.6 \times 10^5 > (4s^2)^s = 46656$. We get $r < 2^3 \times 9.6 \times 10^5 (\log 9.6 \times 10^5)^3 < 2.1 \times 10^{10}$, a contradiction.

(v) In the last case of Lemma 3.1, we have

$$x+1 < \frac{\log\left[99\log(r+1)\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\right]}{\log r}$$
$$+ 2.2 \times 10^6 n \left(1 + \frac{1}{r\log r}\right)(\log(r+1))^3$$
$$\times \left(\log\left[\frac{x+1}{100(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2.$$

We have already shown in the previous cases that $\frac{x+1}{\log(r+1)} \le n$ is impossible. Thus, $n < \frac{x+1}{\log(r+1)}$ and still we use the fact that $0.3(n+1) < n < \frac{x+1}{\log(r+1)}$.

We can now write

$$
\begin{aligned}
x+1 &< 3\log(r+1)\left(\log\left(\frac{x+1}{\log(r+1)}\right)\right)^{\frac{1}{3}} \\
&\quad + 2.3\times 10^{6}\left[196\log(r+1)\left(\log\left(\frac{2.5x}{\log(r+1)}\right)\right)^{2}\right](\log(r+1))^{3} \\
&\quad \times\left(\log\left(\frac{x+1}{\log(r+1)}\right)-\left(100\log(r+1)\log\left(\frac{0.3(n+1)(x+1)^{2}}{(\log(r+1))^{2}}\right)\right)^{\frac{1}{3}}\right)^{2}.
\end{aligned}
$$

Putting $y := \dfrac{x+1}{\log(r+1)}$, we get

$$
\begin{aligned}
y &< 3(\log y)^{\frac{1}{3}} + 4.51\times 10^{8}\,(\log 2.5y)^{2}\,(\log(r+1))^{3}\left(\log y-(\log y^{3})^{\frac{1}{3}}\right)^{2} \\
&< 3(\log y)^{\frac{1}{3}} + 4.51\times 10^{8}\,(\log 2.5y)^{2}\,(\log(r+1))^{3}\,(\log y)^{2} \\
&< 1.4\times 10^{9}\,(\log 2.5y)^{5}\,(\log(r+1))^{3}.
\end{aligned}
$$

The function $f(y) = \dfrac{y}{(\log 2.5y)^5}$ is increasing for all $y > 50r^2 > 50(10^{12})^2 = 5\times 10^{24}$, so we get that the above inequality should hold when y is replaced with $50r^2$. Thus

$$
\begin{aligned}
50r^{2} &< 1.4\times 10^{9}(\log(r+1))^{3}(\log(2.5\times 50r^{2}))^{5}. \\
\frac{r}{(\log r)^{4}} &< 1.6\times 10^{8}.
\end{aligned}
$$

We again apply Lemma 2.1 with the data: $s = 4$ and $T = 1.6\times 10^8 > (4s^2)^s$. We get $r < 2^4\times 1.6\times 10^8(\log 1.6\times 10^8)^4 < 3.3\times 10^{14}$. □

At this point, having bounds on r makes it easy to find bounds on x. For example,

$$
\begin{aligned}
n &< 196\log(r+1)\left(\log\left(\frac{2.5x}{\log(r+1)}\right)\right)^{2} \\
&< 196\log(3.3\times 10^{14}+1)\left(\log\left(\frac{2.5x}{\log(4+1)}\right)\right)^{2} \\
&< 6826(\log 2.5x)^{2}.
\end{aligned}
$$

Next, we go through the five cases of Lemma 3.1.

(i) If x is in case (i), then we have

$$
\begin{aligned}
x &< 8.1\times 10^{6}(\log(r+1))^{2} \\
&< 8.1\times 10^{6}(\log(3.3\times 10^{14}+1))^{2} \\
&< 9.1\times 10^{9}.
\end{aligned}
$$

(ii) In the second case,

$$
\begin{aligned}
x &< 4.8\times 10^6(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}\\
&< 4.8\times 10^6(\log(3.3\times 10^{14}+1))^2\left(\log\left(\frac{0.3(6827(\log 2.5x)^2)(x+1)^2}{(\log(4+1))^2}\right)\right)^{\frac{1}{3}}\\
&< 1.68\times 10^{10}\log x.
\end{aligned}
$$

By Lemma 2.1 with the data: $s=1$ and $T=1.68\times 10^{10}>(4s^2)^s$, we get $x<2\times 1.68\times 10^{10}(\log 1.68\times 10^{10})<8\times 10^{11}$.

(iii) In case (iii),

$$
\begin{aligned}
x &< 1.38\times 10^6 n\left(1+\frac{1}{r\log r}\right)(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^2\\
&< 1.39\times 10^6(6826(\log 2.5x)^2)(\log(3.3\times 10^{14}+1))^2\\
&\quad\times\left(\log\left(\frac{0.3(6827(\log 2.5x)^2)(x+1)^2}{(\log(4+1))^2}\right)\right)^2\\
&< 8.3\times 10^{15}(\log x)^2.
\end{aligned}
$$

By Lemma 2.1 with the data: $s=2$ and $T=8.3\times 10^{15}>(4s^2)^s$, we get $x<2^2\times 8.3\times 10^{15}(\log 8.3\times 10^{15})^2<4.5\times 10^{19}$.

(iv) Here,

$$
\begin{aligned}
x &< 6.41\times 10^6 n\left(1+\frac{1}{r\log r}\right)(\log(r+1))^3\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2\\
&< 6.42\times 10^6(6826(\log 2.5x)^2)(\log(3.3\times 10^{14}+1))^3(\log(x+1))^2\\
&< 1.4\times 10^{15}(\log x)^4.
\end{aligned}
$$

By Lemma 2.1 with the data: $s=4$ and $T=1.4\times 10^{15}>(4s^2)^s$, we get $x<2^4\times 1.4\times 10^{15}(\log 1.4\times 10^{15})^4<3.32\times 10^{22}$.

(v) Lastly, we have

$$
\begin{aligned}
x &< \frac{\log\left[99\log(3.3\times 10^{14}+1)\left(\log\left(\frac{0.3(6827(\log 2.5x)^2)(x+1)^2}{(\log(4+1))^2}\right)\right)^{\frac{1}{3}}\right]}{\log 4}\\
&\quad+2.3\times 10^6(6826(\log 2.5x)^2)(\log(3.3\times 10^{14}+1))^3\\
&\quad\times\left(\log\left[\frac{x+1}{100(\log(4+1))^2\left(\log\left(\frac{0.3(2+1)(3+1)^2}{(\log(3.3\times 10^{14}+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2
\end{aligned}
$$

$$< 10\log x + 5.9\times 10^{14}(\log 2.5x)^2(\log(x+1))^2$$
$$< 6\times 10^{14}(\log x)^5.$$

By Lemma 2.1 with the data: $s = 5$ and $T = 6\times 10^{14} > (4s^2)^s$, we get $x < 2^5\times 6\times 10^{14}(\log 6\times 10^{14})^5 < 8.8\times 10^{23}$.

Therefore $r < 3.3\times 10^{14}$ and $x < 8.8\times 10^{23}$ and $n < 6826(\log 2.5x)^2 < 2.2\times 10^7$. Further, by relation (19) in [1], one gets that $m < 2\times 10^{31}$. Inequality (56) in [1] gives that

$$\left|\frac{\log\sqrt{r^2+4}}{\log\alpha} - \frac{x(n+1)-m}{x-1}\right| < \frac{1}{r^n(x-1)\log\alpha} < \frac{1}{16(x-1)^2}, \tag{3.32}$$

where in the last inequality, we used the fact the $r^n = r^2r^{n-2} \geq 16x > 16(x-1)$. In particular, the ratio $\dfrac{x(n+1)-m}{x-1}$ is a convergent of $\dfrac{\log\sqrt{r^2+4}}{\log\alpha}$. Since $x < 8.8\times 10^{23} < F_{120}$, it follows that

$$\frac{x(n+1)-m}{x-1} = \frac{p_k}{q_k},$$

for some $k \in [0, 119]$. So, we apply Lemma 4 in [1] on inequality (3.32) with the data:

$$M := 2\times 10^{31}, \quad \tau := \frac{\log\sqrt{r^2+4}}{\log\alpha}, \quad u := x(n+1)-1, \quad \text{and} \quad v := x-1.$$

With the help of a computer search in Mathematica, we checked all these possibilities over all the values for $4 \leq r \leq 3.3\times 10^{14}$ and found that $n \leq 52$, which is a contradiction. This computation lasted 3.5 days on an 8GB RAM laptop.

3.10 The case $r = 3$

The case $r = 3$ is special since we do not know that $\kappa > 0$, so some of the inequalities used for the case $r \geq 4$ do not apply.
If $n \leq 100$, then Lemma 3.1 gives five possibilities on bounds of x. We go through the possibilities.

(i) If x is in case (i), then we have

$$x < 8.1\times 10^6(\log(r+1))^2 = 8.1\times 10^6(\log(3+1))^2 < 1.6\times 10^7.$$

(ii) In the second case,

$$x < 4.8\times 10^6(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^{\frac{1}{3}}$$

$$\leq 4.8\times 10^6(\log(3+1))^2\left(\log\left(\frac{0.3(101)(x+1)^2}{(\log(3+1))^2}\right)\right)^{\frac{1}{3}}$$
$$< 1.63\times 10^7\log x.$$

By Lemma 2.1 with the data: $s = 1$ and $T = 1.63\times 10^7 > (4s^2)^s$, we get $x < 2\times 1.63\times 10^7(\log 1.63\times 10^7) < 5.42\times 10^8$.

(iii) In case (iii),

$$x < 1.38\times 10^6 n\left(1+\frac{1}{r\log r}\right)(\log(r+1))^2\left(\log\left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2}\right)\right)^2$$
$$< 1.8\times 10^6(100)(\log(3+1))^2\left(\log\left(\frac{0.3(101)(x+1)^2}{(\log(3+1))^2}\right)\right)^2$$
$$< 1.1\times 10^{10}(\log x)^2.$$

By Lemma 2.1 with the data: $s = 2$ and $T = 1.1\times 10^{10} > (4s^2)^s$, we get $x < 2^2\times 1.1\times 10^{10}(\log 1.1\times 10^{10})^2 < 2.4\times 10^{13}$.

(iv) Here,

$$x < 6.41\times 10^6 n\left(1+\frac{1}{r\log r}\right)(\log(r+1))^3\left(\log\left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2}\right)\right)^2$$
$$< 8.4\times 10^6(100)(\log(3+1))^3(\log(x+1))^2$$
$$< 2.3\times 10^9(\log x)^2.$$

By Lemma 2.1 with the data: $s = 2$ and $T = 2.3\times 10^9 > (4s^2)^s$, we get $x < 2^2\times 2.3\times 10^9(\log 2.3\times 10^9)^2 < 4.3\times 10^{12}$.

(v) Lastly, we have

$$x < \frac{\log\left[99\log(3+1)\left(\log\left(\frac{0.3(101)(x+1)^2}{(\log(3+1))^2}\right)\right)^{\frac{1}{3}}\right]}{\log 3} + 2.9\times 10^6(100)(\log(3+1))^3$$
$$\times\left(\log\left[\frac{x+1}{100(\log(3+1))^2\left(\log\left(\frac{0.3(2+1)(3+1)^2}{(\log(3+1))^2}\right)\right)^{\frac{1}{3}}}\right]\right)^2$$
$$< 4\log x + 7.8\times 10^8(\log(x+1))^2$$
$$< 8\times 10^8(\log x)^3.$$

By Lemma 2.1 with the data: $s = 3$ and $T = 8\times 10^8 > (4s^2)^s$, we get $x < 2^3\times 8\times 10^8(\log 8\times 10^8)^3 < 5.6\times 10^{13}$.

In all cases, $x < 5.6\times 10^{13}$. Now we perform the Baker-Davenport reduction on relation (27) in [1] for $2 \leq n \leq 100$, $r = 3$ and $x < 5.6\times 10^{13}$. This also gives

$m < 5.7 \times 10^{15}$ via inequality (19) in [1]. We return to (27) in [1] and rewrite it as

$$\left| x \frac{\log U_{n+1}}{\log \alpha} - m + \frac{\log \sqrt{r^2+4}}{\log \alpha} \right| < \frac{2.2}{r^x \log \alpha}. \tag{3.33}$$

We now apply Lemma 5 in [1] with the following data:

$$M := 5.7 \times 10^{15}, \quad \tau := \frac{\log U_{n+1}}{\log \alpha}, \quad \mu := \frac{\log \sqrt{r^2+4}}{\log \alpha}, \quad A := \frac{2.2}{\log \alpha} \quad \text{and} \quad B := r.$$

A Mathematica code performed Baker-Davenport reduction and revealed that $x \le 222$. This gives a better bound on x.

If $n > 100$, then we have relation (60) on page 680 of [1]. We keep the notation r and α although this Subsection applies to $r = 3$ for which $\alpha = \dfrac{3+\sqrt{13}}{2}$. Put $\ell := \min\{n-1, x-1\}$. The lower bound in inequality (58) of [1] still applies and gives

$$\ell \log 3 < -\log |\Gamma_3| < 195 \left(\max \left\{ \log \left(\frac{2.5x}{\log 4} \right),\ 10.5 \right\} \right)^2 (\log 4)^2.$$

If $\ell = n - 1$, then

$$(n-1) \log 3 < 375 \left(\max \left\{ \log \left(\frac{2.5x}{\log 4} \right),\ 10.5 \right\} \right)^2.$$

In case the maximum above is 10.5, then $\log \left(\dfrac{2.5x}{\log 4} \right) < 10.5$ and so $x < 20140$. This further implies that $(n-1)\log 3 < 4.2 \times 10^4$ so that $n < 3.9 \times 10^4$.

In case the maximum is $\log \left(\dfrac{2.5x}{\log 4} \right)$, then

$$(n-1) \log 3 < 375 \left(\log \left(\frac{2.5x}{\log 4} \right) \right)^2.$$

This gives $n < 120(\log x)^2$. We now go through the five possibilities of Lemma 3.1.

(i) If x is in case (i), then we have

$$x < 8.1 \times 10^6 (\log(r+1))^2 = 8.1 \times 10^6 (\log(3+1))^2 < 1.6 \times 10^7.$$

(ii) In the second case,

$$x < 4.8 \times 10^6 (\log(r+1))^2 \left(\log \left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^{\frac{1}{3}}$$

$$\leq 4.8 \times 10^6 (\log(3+1))^2 \left(\log \left(\frac{0.3(121(\log x)^2)(x+1)^2}{(\log(3+1))^2} \right) \right)^{\frac{1}{3}}$$
$$< 2.1 \times 10^7 (\log x)^2.$$

By Lemma 2.1 with the data: $s = 2$ and $T = 2.1 \times 10^7 > (4s^2)^s$, we get $x < 2^2 \times 2.1 \times 10^7 (\log 2.1 \times 10^7)^2 < 2.4 \times 10^{10}$.

(iii) In case (iii),

$$x < 1.38 \times 10^6 n \left(1 + \frac{1}{r \log r}\right) (\log(r+1))^2 \left(\log \left(\frac{0.3(n+1)(x+1)^2}{(\log(r+1))^2} \right) \right)^2$$
$$< 1.8 \times 10^6 (120(\log x)^2)(\log(3+1))^2 \left(\log \left(\frac{0.3(121(\log x)^2)(x+1)^2}{(\log(3+1))^2} \right) \right)^2$$
$$< 5.8 \times 10^{10} (\log x)^4.$$

By Lemma 2.1 with the data: $s = 4$ and $T = 5.8 \times 10^{10} > (4s^2)^s$, we get $x < 2^4 \times 5.8 \times 10^{10} (\log 5.8 \times 10^{10})^4 < 3.51 \times 10^{17}$.

(iv) Here,

$$x < 6.41 \times 10^6 n \left(1 + \frac{1}{r \log r}\right) (\log(r+1))^3 \left(\log \left(\frac{x+1}{\frac{512}{3}(\log(r+1))^2} \right) \right)^2$$
$$< 8.4 \times 10^6 (120(\log x)^2)(\log(3+1))^3 (\log(x+1))^2$$
$$< 2.7 \times 10^9 (\log x)^4.$$

By Lemma 2.1 with the data: $s = 4$ and $T = 2.7 \times 10^9 > (4s^2)^s$, we get $x < 2^4 \times 2.7 \times 10^9 (\log 2.7 \times 10^9)^4 < 9.61 \times 10^{15}$.

(v) Lastly, we have

$$x < \frac{\log \left[99 \log(3+1) \left(\log \left(\frac{0.3(120(\log x)^2)(x+1)^2}{(\log(3+1))^2} \right) \right)^{\frac{1}{3}} \right]}{\log 3}$$
$$+ 2.9 \times 10^6 (120(\log x)^2)(\log(3+1))^3$$
$$\times \left(\log \left[\frac{x+1}{100(\log(3+1))^2 \left(\log \left(\frac{0.3(101+1)(3+1)^2}{(\log(3+1))^2} \right) \right)^{\frac{1}{3}}} \right] \right)^2$$
$$< 4 \log x + 9.3 \times 10^8 (\log(x+1))^4$$
$$< 3.8 \times 10^9 (\log x)^4.$$

By Lemma 2.1 with the data: $s = 4$ and $T = 3.8 \times 10^9 > (4s^2)^s$, we get $x < 2^4 \times 3.8 \times 10^9 (\log 3.8 \times 10^9)^4 < 1.44 \times 10^{16}$.

So, in all instances, $x < 1.44 \times 10^{16}$, and now $n < 120(\log x)^2 < 1.67 \times 10^5$. Further, by relation (19) in [1], one gets that $m < 2.41 \times 10^{21}$.
Since Lemma 13 in [1] still applies, then it follows that inequality (60) in [1] gives

$$\left| \frac{\log \sqrt{r^2+4}}{\log \alpha} - \frac{x(n+1)-m}{x-1} \right| < \frac{1}{3^{n-1}(x-1)\log \alpha} < \frac{1}{3(x-1)^2}, \tag{3.34}$$

In particular, the ratio $\frac{x(n+1)-m}{x-1}$ is a convergent of $\frac{\log \sqrt{r^2+4}}{\log \alpha}$. Since $x < 1.44 \times 10^{16} < F_{80}$, it follows that

$$\frac{x(n+1)-m}{x-1} = \frac{p_k}{q_k},$$

for some $k \in [0, 79]$. So, we apply Lemma 4 in [1] on inequality (3.34) with the data:

$$M := 2.41 \times 10^{21}, \quad \tau := \frac{\log \sqrt{r^2+4}}{\log \alpha}, \quad u := x(n+1)-1, \quad \text{and} \quad v := x-1.$$

With the help of a computer search in Mathematica, we checked for $r = 3$ and found that $n \leq 37$, which is a contradiction.
Next, if $\ell = x - 1$, then we adopt the remaining results given on pages $681 - 683$ of [1]. □

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Declarations

Competing interest The authors declare no competing interests.

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