



A simple proof of Slater's transformations for bilateral series ${}_r\psi_r$, ${}_{2r}\psi_{2r}$ and ${}_{2r-1}\psi_{2r-1}$

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Abstract

We give a simple proof of Slater's transformations for bilateral series ${}_r\psi_r$, ${}_{2r}\psi_{2r}$, and ${}_{2r-1}\psi_{2r-1}$, using the residue theorem only, without technical manipulation of the series.

Keywords Slater's transformations · ${}_r\psi_r$ series · Very-well-poised ${}_{2r}\psi_{2r}$ series · Bilateral basic hypergeometric series

Mathematics Subject Classification Primary 33D15; Secondary 05A30 · 33D60

1 Introduction

The bilateral basic hypergeometric series ${}_n\psi_n$ is defined by

$${}_n\psi_n \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_n \end{matrix}; q, x \right) := \sum_{-\infty < k < \infty} \frac{(a_1, \dots, a_n; q)_k}{(b_1, \dots, b_n; q)_k} x^k$$

for

$$\left| b_1 \cdots b_n a_1^{-1} \cdots a_n^{-1} \right| < |x| < 1, \quad |q| < 1, \quad a_i, b_i, x, q \in \mathbb{C}, \text{ and } a_i, qb_i^{-1} \notin q^{\mathbb{Z}_{>0}},$$

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where

$$(a_1, a_2, \dots, a_n; q)_k = (a_1; q)_k (a_2; q)_k \cdots (a_n; q)_k,$$

$$(a; q)_k = (a; q)_\infty / (aq^k; q)_\infty,$$

and

$$(a; q)_\infty = \prod_{i \geq 0} (1 - aq^i).$$

In [6, 7], Slater derived transformations for bilateral basic hypergeometric series. She applied the results by Sears [5] in [6], and she considered the basic analogue of the Barnes-type integrals in [7]. Exposition on the work by Slater is provided in Chapter 5 of the book by Gasper and Rahman [1]. On the other hand, Ito and Sanada gave a proof of the transformation for ${}_r\psi_r$ series (Theorem 1 below) and a proof of the transformation for very-well-poised-balanced ${}_{2r}\psi_{2r}$ series (Theorem 2 below) from the viewpoint of the connection problem associated with a Jackson integral [2].

The purpose of the present paper is to give a simple proof of these results by means of the residue theorem only, without technical manipulation of the series, and also to study the integrals which represent several functions in q -analysis. We refer the reader to our previous work [4] for the proof of Ramanujan's ${}_1\psi_1$ -sum and Bailey's ${}_6\psi_6$ -sum from the same point of view (See also [3]).

We also use the symbols

$$(a_1, a_2, \dots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty,$$

$$\theta(a) = (a, qa^{-1}; q)_\infty$$

and

$$\theta(a_1, \dots, a_n) = \theta(a_1) \cdots \theta(a_n).$$

In this paper, the base q is fixed to be a real number satisfying $0 < q < 1$ for simplicity.

2 ${}_r\psi_r$ series

Slater's transformation for ${}_r\psi_r$ series is given by the following, which is (5.4.3) of [1] ((4) of [6], (7.2.5) of [7], and (5.1) of [2]):

Theorem 1 *Suppose that*

$$c_i, c_i c_j^{-1} \notin q^{\mathbb{Z}}, \quad 1 \leq i \neq j \leq r,$$

for $r \geq 1$. Then we have

$$\begin{aligned} & \frac{\theta(Ax) \prod_{i=1}^r (b_i, qa_i^{-1}; q)_\infty}{\prod_{i=1}^r \theta(c_i)} {}_r\psi_r \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, x \right) \\ &= \sum_{s=1}^r \frac{q}{c_s} \frac{\theta(Axc_s q^{-1})}{\theta(c_s)} \frac{\prod_{i=1}^r (qb_i c_s^{-1}, c_s a_i^{-1}; q)_\infty}{\prod_{\substack{1 \leq i \leq r \\ i \neq s}} \theta(c_s c_i^{-1})} \\ & \times {}_r\psi_r \left(\begin{matrix} qa_1 c_s^{-1}, qa_2 c_s^{-1}, \dots, qa_r c_s^{-1} \\ qb_1 c_s^{-1}, qb_2 c_s^{-1}, \dots, qb_r c_s^{-1} \end{matrix}; q, x \right), \end{aligned} \tag{2.1}$$

where $A = \prod_{i=1}^r a_i c_i^{-1}$ and $|b_1 \cdots b_r a_1^{-1} \cdots a_r^{-1}| < |x| < 1$.

Proof Let $F(t)$ be a function defined by

$$F(t) = \frac{1}{t} \frac{\theta(Axt^{-1})}{\theta(t^{-1})} \prod_{i=1}^r \frac{(b_i t, qa_i^{-1} t^{-1}; q)_\infty}{\theta(c_i t)}. \tag{2.2}$$

First, for $k \in \mathbb{Z}$, we have

$$\begin{aligned} \text{Res}_{t=q^k} F(t) dt &= \lim_{t \rightarrow q^k} (t - q^k) F(t) \\ &= \frac{\theta(Ax) \prod_{i=1}^r \theta(a_i)}{(q; q)_\infty^2 \prod_{i=1}^r \theta(c_i)} \prod_{i=1}^r \frac{(b_i q^k; q)_\infty}{(a_i q^k; q)_\infty} x^k, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \text{Res}_{t=c_s^{-1} q^{1+k}} F(t) dt &= \lim_{t \rightarrow c_s^{-1} q^{1+k}} (t - c_s^{-1} q^{1+k}) F(t) \\ &= -\frac{q}{c_s} \frac{\theta(Axc_s q^{-1})}{(q; q)_\infty^2 \theta(c_s)} \frac{\prod_{i=1}^r \theta(c_s a_i^{-1})}{\prod_{\substack{1 \leq i \leq r \\ i \neq s}} \theta(c_s c_i^{-1})} \prod_{i=1}^r \frac{(qb_i c_s^{-1} q^k; q)_\infty}{(qa_i c_s^{-1} q^k; q)_\infty} x^k. \end{aligned} \tag{2.4}$$

Secondly, for real positive numbers R_1 and R_2 satisfying

$$R_j, c_i R_j \notin q^{\mathbb{Z}}, \quad 1 \leq i \leq r, \quad j = 1, 2,$$

we have

$$\begin{aligned} & F \left(R_1 e^{\sqrt{-1}\theta} q^{-l} \right) \\ &= \frac{\left(Ax R_1^{-1} e^{-\sqrt{-1}\theta} q^l, q A^{-1} x^{-1} R_1 e^{\sqrt{-1}\theta}; q \right)_\infty}{R_1 e^{\sqrt{-1}\theta} q^{-l} \left(R_1^{-1} e^{-\sqrt{-1}\theta} q^l, q R_1 e^{\sqrt{-1}\theta}; q \right)_\infty} \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{i=1}^r \frac{(b_i R_1 e^{\sqrt{-1}\theta}, q a_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l; q)_{\infty}}{(c_i R_1 e^{\sqrt{-1}\theta}, q c_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l; q)_{\infty}} \\
 & \times \frac{(A x R_1^{-1} e^{-\sqrt{-1}\theta}; q)_l}{(R_1^{-1} e^{-\sqrt{-1}\theta}; q)_l} \prod_{i=1}^r \frac{(q b_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}; q)_l}{(q c_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}; q)_l} \left(\frac{b_1 \cdots b_r}{a_1 \cdots a_r} \frac{1}{x}\right)^l, \\
 & F(R_2 e^{\sqrt{-1}\theta} q^l) \\
 & = \frac{(A x R_2^{-1} e^{-\sqrt{-1}\theta}, q A^{-1} x^{-1} R_2 e^{\sqrt{-1}\theta} q^l; q)_{\infty}}{R_2 e^{\sqrt{-1}\theta} q^l (R_2^{-1} e^{-\sqrt{-1}\theta}, q R_2 e^{\sqrt{-1}\theta} q^l; q)_{\infty}} \\
 & \times \prod_{i=1}^r \frac{(b_i R_2 e^{\sqrt{-1}\theta} q^l, q a_i^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}; q)_{\infty}}{(c_i R_2 e^{\sqrt{-1}\theta} q^l, q c_i^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}; q)_{\infty}} \\
 & \times \frac{(q A^{-1} x^{-1} R_2 e^{\sqrt{-1}\theta}; q)_l}{(R_2^{-1} e^{\sqrt{-1}\theta}; q)_l} \prod_{i=1}^r \frac{(a_i R_2 e^{\sqrt{-1}\theta}; q)_l}{(c_i R_2 e^{\sqrt{-1}\theta}; q)_l} x^l,
 \end{aligned}$$

and thus

$$\begin{aligned}
 & \left| \int_{C_1^{(l)}} F(t) dt \right| \\
 & = \left| \int_0^{2\pi} F(R_1 e^{\sqrt{-1}\theta} q^{-l}) R_1 \sqrt{-1} e^{\sqrt{-1}\theta} q^{-l} d\theta \right| \leq M_1 \left| \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \frac{1}{x} \right|^l, \tag{2.5}
 \end{aligned}$$

$$\left| \int_{C_2^{(l)}} F(t) dt \right| = \left| \int_0^{2\pi} F(R_2 e^{\sqrt{-1}\theta} q^l) R_2 \sqrt{-1} e^{\sqrt{-1}\theta} q^l d\theta \right| \leq M_2 |x|^l, \tag{2.6}$$

where $l \in \mathbb{Z}_{\geq 0}$, $C_1^{(l)} = \{R_1 e^{\sqrt{-1}\theta} q^{-l} \in \mathbb{C} \mid 0 \leq \theta \leq 2\pi\}$, $C_2^{(l)} = \{R_2 e^{\sqrt{-1}\theta} q^l \in \mathbb{C} \mid 0 \leq \theta \leq 2\pi\}$, and M_1, M_2 are positive numbers independent of l .

The residue theorem combined with inequalities (2.5) and (2.6) leads to

$$\begin{aligned}
 & \sum_{-\infty < k < \infty} \operatorname{Res}_{t=q^k} F(t) dt + \sum_{s=1}^r \sum_{-\infty < k < \infty} \operatorname{Res}_{t=c_s^{-1} q^{1+k}} F(t) dt \\
 & = \lim_{l \rightarrow \infty} \frac{1}{2\pi \sqrt{-1}} \left(\int_{C_1^{(l)}} + \int_{C_2^{(l)}} \right) F(t) dt = 0, \tag{2.7}
 \end{aligned}$$

if $\left| b_1 \cdots b_r a_1^{-1} \cdots a_r^{-1} \right| < |x| < 1$, where $C_1^{(l)}$ is in the counterclockwise direction and $C_2^{(l)}$ is in the clockwise direction.

Consequently, combining (2.3), (2.4) with (2.7), we obtain

$$\begin{aligned}
 0 &= \frac{\theta(Ax) \prod_{i=1}^r \theta(a_i)}{(q; q)_\infty^2 \prod_{i=1}^r \theta(c_i)} \sum_{-\infty < k < \infty} \prod_{i=1}^r \frac{(b_i q^k; q)_\infty}{(a_i q^k; q)_\infty} x^k \\
 &\quad - \sum_{s=1}^r \frac{q}{c_s} \frac{\theta(Ax c_s q^{-1}) \prod_{i=1}^r \theta(c_s a_i^{-1})}{(q; q)_\infty^2 \theta(c_s) \prod_{\substack{1 \leq i \leq r \\ i \neq s}} \theta(c_s c_i^{-1})} \sum_{-\infty < k < \infty} \prod_{i=1}^r \frac{(q b_i c_s^{-1} q^k; q)_\infty}{(q a_i c_s^{-1} q^k; q)_\infty} x^k \\
 &= \frac{\theta(Ax) \prod_{i=1}^r (q a_i^{-1}, b_i; q)_\infty}{(q; q)_\infty^2 \prod_{i=1}^r \theta(c_i)} \sum_{-\infty < k < \infty} \prod_{i=1}^r \frac{(a_i; q)_k}{(b_i; q)_k} x^k \\
 &\quad - \sum_{s=1}^r \frac{q}{c_s} \frac{\theta(Ax c_s q^{-1}) \prod_{i=1}^r (c_s a_i^{-1}, q b_i c_s^{-1}; q)_\infty}{(q; q)_\infty^2 \theta(c_s) \prod_{\substack{1 \leq i \leq r \\ i \neq s}} \theta(c_s c_i^{-1})} \sum_{-\infty < k < \infty} \prod_{i=1}^r \frac{(q a_i c_s^{-1}; q)_k}{(q b_i c_s^{-1}; q)_k} x^k,
 \end{aligned}$$

if $|b_1 \cdots b_r a_1^{-1} \cdots a_r^{-1}| < |x| < 1$. This completes the proof of (2.1). □

3 Very-well-poised-balanced ${}_2r\psi_{2r}$ series

Slater's transformation for very-well-poised-balanced ${}_2r\psi_{2r}$ series is given by the following, which is (5.5.2) of [1] ((1.1) of [2]):

Theorem 2 *Suppose that*

$$a, a_i, a_i a^{-1}, a_i a_j^{-1}, a_i^2 a^{-1}, a_i a_j a^{-1} \notin q^{\mathbb{Z}}, \quad 1 \leq i \neq j \leq r - 2,$$

for $r \geq 3$, and $|a^{r-1} q^{r-2}| < \left| \prod_{i=1}^{2(r-1)} b_i \right|$. Then we have

$$\begin{aligned}
 &\frac{\prod_{i=1}^{2(r-1)} (q b_i^{-1}, q a b_i^{-1}; q)_\infty}{(q a, q a^{-1}; q)_\infty \prod_{i=1}^{r-2} \theta(a_i a^{-1}, a_i)} \\
 &\quad \times {}_{2r}\psi_{2r} \left(\begin{matrix} q a^{\frac{1}{2}}, -q a^{\frac{1}{2}}, b_1, b_2, \dots, b_{2(r-1)} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, q a b_1^{-1}, q a b_2^{-1}, \dots, q a b_{2(r-1)}^{-1} \end{matrix}; q, \frac{q^{r-2} a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right) \\
 &= \sum_{s=1}^{r-2} \frac{\prod_{i=1}^{2(r-1)} (q a a_s^{-1} b_i^{-1}, q a_s b_i^{-1}; q)_\infty}{(q a_s^2 a^{-1}, q a a_s^{-2}; q)_\infty \theta(a_s a^{-1}, a_s) \prod_{\substack{1 \leq i \leq r-2 \\ i \neq s}} \theta(a_i a_s^{-1}, a_s a_i a^{-1})} \\
 &\quad \times {}_{2r}\psi_{2r} \left(\begin{matrix} q a_s a^{-\frac{1}{2}}, -q a_s a^{-\frac{1}{2}}, b_1 a_s a^{-1}, b_2 a_s a^{-1}, \dots, b_{2(r-1)} a_s a^{-1} \\ a_s a^{-\frac{1}{2}}, -a_s a^{-\frac{1}{2}}, q a_s b_1^{-1}, q a_s b_2^{-1}, \dots, q a_s b_{2(r-1)}^{-1} \end{matrix}; q, \frac{q^{r-2} a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right). \tag{3.1}
 \end{aligned}$$

Proof Let $F(t)$ be a function defined by

$$\begin{aligned} F(t) &= \frac{\prod_{i=1}^{2(r-1)} (qb_i^{-1}t^{-1}, qab_i^{-1}t; q)_\infty}{\theta(t^{-1}, at) \prod_{i=1}^{r-2} \theta(a_i a^{-1}t^{-1}, a_i t)} \\ &= \frac{\prod_{i=1}^{2(r-1)} (qb_i^{-1}t^{-1}, qab_i^{-1}t; q)_\infty}{(t^{-1}, qt, at, qa^{-1}t^{-1}; q)_\infty \prod_{i=1}^{r-2} (a_i a^{-1}t^{-1}, qa_i^{-1}at, a_i t, qa_i^{-1}t^{-1}; q)_\infty}, \end{aligned} \quad (3.2)$$

which satisfies $F(t) = F(a^{-1}t^{-1})$.

First, for $k \in \mathbb{Z}$, we have

$$\begin{aligned} \operatorname{Res}_{t=q^k} F(t) dt &= \lim_{t \rightarrow q^k} (t - q^k) F(t) \\ &= \frac{q^k \times \prod_{i=1}^{2(r-1)} \theta(b_i)}{(q; q)_\infty^2 \theta(a) \prod_{i=1}^{r-2} \theta(a_i a^{-1}, a_i)_\infty} \prod_{i=1}^{2(r-1)} \frac{(qab_i^{-1}q^k; q)_\infty}{(b_i q^k; q)_\infty} \left(\frac{(aq)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \operatorname{Res}_{t=a_s a^{-1} q^k} F(t) dt &= \lim_{t \rightarrow a_s a^{-1} q^k} (t - a_s a^{-1} q^k) F(t) \\ &= \frac{a_s a^{-1} q^k \times \prod_{i=1}^{2(r-1)} \theta(qa a_s^{-1} b_i^{-1})}{(q; q)_\infty^2 \theta(aa_s^{-1}, a_s, a_s^2 a^{-1}) \prod_{\substack{1 \leq i \leq r-2 \\ i \neq s}} \theta(a_i a_s^{-1}, a_s a_i a^{-1})} \\ &\quad \times \prod_{i=1}^{2(r-1)} \frac{(qa_s b_i^{-1} q^k; q)_\infty}{(a_s b_i a^{-1} q^k; q)_\infty} \left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k. \end{aligned} \quad (3.4)$$

Secondly, for $k \in \mathbb{Z}$, we have

$$\begin{aligned} \operatorname{Res}_{t=a^{-1} q^{-k}} F(t) dt &= -a^{-1} \operatorname{Res}_{u=q^k} F(a^{-1} u^{-1}) u^{-2} du \\ &= -a^{-1} \operatorname{Res}_{u=q^k} F(u) u^{-2} du = -a^{-1} q^{-2k} \operatorname{Res}_{u=q^k} F(u) du \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \operatorname{Res}_{t=a_s^{-1} q^{-k}} F(t) dt &= -a^{-1} \operatorname{Res}_{u=a^{-1} a_s q^k} F(a^{-1} u^{-1}) u^{-2} du \\ &= -a^{-1} \operatorname{Res}_{u=a^{-1} a_s q^k} F(u) u^{-2} du = -a a_s^{-2} q^{-2k} \operatorname{Res}_{u=a^{-1} a_s q^k} F(u) du \end{aligned} \quad (3.6)$$

by the change of integration variable from t to u by $t = a^{-1} u^{-1}$ with the equality $F(t) = F(a^{-1} t^{-1})$.

Thirdly, for real positive numbers R_1 and R_2 satisfying

$$R_j, aR_j, a_i R_j, aa_i^{-1} R_j \notin q^{\mathbb{Z}}, \quad 1 \leq i \leq r - 2, \quad j = 1, 2,$$

we have

$$\begin{aligned} & F\left(R_1 e^{\sqrt{-1}\theta} q^{-l}\right) \\ &= \frac{\prod_{i=1}^{2(r-1)}(qb_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l, qab_i^{-1} R_1 e^{\sqrt{-1}\theta}; q)_\infty}{(R_1^{-1} e^{-\sqrt{-1}\theta} q^l, qR_1 e^{\sqrt{-1}\theta}, aR_1 e^{\sqrt{-1}\theta}, qa^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l; q)_\infty} \\ &\quad \times \frac{1}{\prod_{i=1}^{r-2}(a_i a^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l, qa a_i^{-1} R_1 e^{\sqrt{-1}\theta}, a_i R_1 e^{\sqrt{-1}\theta}, qa_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l; q)_\infty} \\ &\quad \times \frac{\prod_{i=1}^{2(r-1)}(b_i a^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}; q)_l}{(R_1^{-1} e^{-\sqrt{-1}\theta} q^l, a^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}; q)_l \prod_{i=1}^{r-2}(a_i a^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}, a_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}; q)_l} \\ &\quad \times \left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^l, \\ & F(R_2 e^{\sqrt{-1}\theta} q^l) \\ &= \frac{\prod_{i=1}^{2(r-1)}(qb_i^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}, qab_i^{-1} R_2 e^{\sqrt{-1}\theta} q^l; q)_\infty}{(R_2^{-1} e^{-\sqrt{-1}\theta}, qR_2 e^{\sqrt{-1}\theta} q^l, aR_2 e^{\sqrt{-1}\theta} q^l, qa^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}; q)_\infty} \\ &\quad \times \frac{1}{\prod_{i=1}^{r-2}(a_i a^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}, qa a_i^{-1} R_2 e^{\sqrt{-1}\theta} q^l, a_i R_2 e^{\sqrt{-1}\theta} q^l, qa_i^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}; q)_\infty} \\ &\quad \times \frac{\prod_{i=1}^{2(r-1)}(b_i R_2 e^{\sqrt{-1}\theta}; q)_l}{(qR_2 e^{\sqrt{-1}\theta}, aR_2 e^{\sqrt{-1}\theta}; q)_l \prod_{i=1}^{r-2}(qa a_i^{-1} R_2 e^{\sqrt{-1}\theta}, a_i R_2 e^{\sqrt{-1}\theta}, ; q)_l} \left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^l, \end{aligned}$$

and thus

$$\left| \int_{C_1^{(l)}} F(t) dt \right| = \left| \int_0^{2\pi} F(R_1 e^{\sqrt{-1}\theta} q^{-l}) R_1 \sqrt{-1} e^{\sqrt{-1}\theta} q^{-l} d\theta \right| \leq M_1 \left| \frac{a^{r-1} q^{r-2}}{\prod_{i=1}^{2(r-1)} b_i} \right|^l, \tag{3.7}$$

$$\left| \int_{C_2^{(l)}} F(t) dt \right| = \left| \int_0^{2\pi} F(R_2 e^{\sqrt{-1}\theta} q^l) R_2 \sqrt{-1} e^{\sqrt{-1}\theta} q^l d\theta \right| \leq M_2 \left| \frac{a^{r-1} q^r}{\prod_{i=1}^{2(r-1)} b_i} \right|^l, \tag{3.8}$$

where $l \in \mathbb{Z}_{\geq 0}$, $C_1^{(l)} = \{R_1 e^{\sqrt{-1}\theta} q^{-l} \in \mathbb{C} \mid 0 \leq \theta \leq 2\pi\}$, $C_2^{(l)} = \{R_2 e^{\sqrt{-1}\theta} q^l \in \mathbb{C} \mid 0 \leq \theta \leq 2\pi\}$, and M_1, M_2 are positive numbers independent of l .

The residue theorem combined with inequalities (3.7) and (3.8) leads to

$$\begin{aligned} & \sum_{-\infty < k < \infty} \left\{ \operatorname{Res}_{t=q^k} F(t) dt + \operatorname{Res}_{t=a^{-1}q^{-k}} F(t) dt \right\} \\ & + \sum_{s=1}^{r-2} \sum_{-\infty < k < \infty} \left\{ \operatorname{Res}_{t=a_s a^{-1} q^k} F(t) dt + \operatorname{Res}_{t=a_s^{-1} q^{-k}} F(t) dt \right\} \\ & = \lim_{l \rightarrow \infty} \frac{1}{2\pi\sqrt{-1}} \left(\int_{C_1^{(l)}} + \int_{C_2^{(l)}} \right) F(t) dt = 0, \end{aligned} \tag{3.9}$$

if $|a^{r-1}q^{r-2}| < \left| \prod_{i=1}^{2(r-1)} b_i \right|$, where $C_1^{(l)}$ is in the counterclockwise direction and $C_2^{(l)}$ is in the clockwise direction.

On the other hand, for $k \in \mathbb{Z}$, we have

$$\begin{aligned} & \operatorname{Res}_{t=q^k} F(t) dt + \operatorname{Res}_{t=a^{-1}q^{-k}} F(t) dt \\ & = (q^k - a^{-1}q^{-k}) \times \frac{\prod_{i=1}^{2(r-1)} \theta(b_i)}{(q; q)_\infty^2 \theta(a) \prod_{i=1}^{r-2} \theta(a_i a^{-1}, a_i)} \\ & \quad \times \prod_{i=1}^{2(r-1)} \frac{(qab_i^{-1}q^k; q)_\infty}{(b_i q^k; q)_\infty} \left(\frac{(aq)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k \\ & = -a^{-1} \times \frac{\prod_{i=1}^{2(r-1)} \theta(b_i)}{(q; q)_\infty^2 (qa, qa^{-1}; q)_\infty \prod_{i=1}^{r-2} \theta(a_i a^{-1}, a_i)} \\ & \quad \times \frac{1 - aq^{2k}}{1 - a} \prod_{i=1}^{2(r-1)} \frac{(qab_i^{-1}q^k; q)_\infty}{(b_i q^k; q)_\infty} \left(\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k \end{aligned} \tag{3.10}$$

from (3.3) and (3.5), and

$$\begin{aligned} & \operatorname{Res}_{t=a_s a^{-1} q^k} F(t) dt + \operatorname{Res}_{t=a_s^{-1} q^{-k}} F(t) dt \\ & = (a_s a^{-1} q^k - a_s^{-1} q^{-k}) \times \frac{\prod_{i=1}^{2(r-1)} \theta(a_s b_i a^{-1})}{(q; q)_\infty^2 \theta(aa_s^{-1}, a_s, a_s^2 a^{-1}) \prod_{\substack{1 \leq i \leq r-2 \\ i \neq s}} \theta(a_i a_s^{-1}, a_i a_s a^{-1})} \\ & \quad \times \prod_{i=1}^{2(r-1)} \frac{(qa_s b_i^{-1} q^k; q)_\infty}{(a_s b_i a^{-1} q^k; q)_\infty} \left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k \\ & = -a_s^{-1} \times \frac{\prod_{i=1}^{2(r-1)} \theta(a_s b_i a^{-1})}{(q; q)_\infty^2 (qa_s^2 a^{-1}, qa a_s^{-2}; q)_\infty \theta(aa_s^{-1}, a_s) \prod_{\substack{1 \leq i \leq r-2 \\ i \neq s}} \theta(a_i a_s^{-1}, a_i a_s a^{-1})} \\ & \quad \times \frac{1 - a_s^2 a^{-1} q^{2k}}{1 - a_s^2 a^{-1}} \prod_{i=1}^{2(r-1)} \frac{(qa_s b_i^{-1} q^k; q)_\infty}{(a_s b_i a^{-1} q^k; q)_\infty} \left(\frac{q^{r-2} a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k \end{aligned} \tag{3.11}$$

from (3.4) and (3.6).

Consequently, combining (3.10), (3.11) with (3.9), we obtain

$$\begin{aligned}
 0 &= -a^{-1} \times \frac{\prod_{i=1}^{2(r-1)} \theta(b_i)}{(q; q)_{\infty}^2 (qa, qa^{-1}; q)_{\infty} \prod_{i=1}^{r-2} \theta(a_i a^{-1}, a_i)} \\
 &\times \sum_{-\infty < k < \infty} \frac{1 - aq^{2k}}{1 - a} \prod_{i=1}^{2(r-1)} \frac{(qab_i^{-1} q^k; q)_{\infty}}{(b_i q^k; q)_{\infty}} \left(\frac{q^{r-2} a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k \\
 &- \sum_{s=1}^{r-2} a_s^{-1} \times \frac{\prod_{i=1}^{2(r-1)} \theta(a_s b_i a^{-1})}{(q; q)_{\infty} (qa_s^2 a^{-1}, qa a_s^{-2}; q)_{\infty} \theta(aa_s^{-1}, a_s) \prod_{1 \leq i \leq r-2, i \neq s} \theta(a_i a_s^{-1}, a_i a_s a^{-1})} \\
 &\times \sum_{-\infty < k < \infty} \frac{1 - a_s^2 a^{-1} q^{2k}}{1 - a_s^2 a^{-1}} \prod_{i=1}^{2(r-1)} \frac{(qa_s b_i^{-1} q^k; q)_{\infty}}{(a_s b_i a^{-1} q^k; q)_{\infty}} \left(\frac{q^{r-2} a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k, \tag{3.12}
 \end{aligned}$$

which is equivalent to

$$\begin{aligned}
 &\frac{\prod_{i=1}^{2(r-1)} (qb_i^{-1}, qab_i^{-1}; q)_{\infty}}{(qa, qa^{-1}; q)_{\infty} \prod_{i=1}^{r-2} \theta(a_i a^{-1}, a_i)} \\
 &\times \sum_{-\infty < k < \infty} \frac{1 - aq^{2k}}{1 - a} \prod_{i=1}^{2(r-1)} \frac{(b_i; q)_k}{(qab_i^{-1}; q)_k} \left(\frac{q^{r-2} a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k \\
 &= \sum_{s=1}^{r-2} \frac{\prod_{i=1}^{2(r-1)} (qa a_s^{-1} b_i^{-1}, qa_s b_i^{-1}; q)_{\infty}}{(qa_s^2 a^{-1}, qa a_s^{-2}; q)_{\infty} \theta(a_s a^{-1}, a_s) \prod_{1 \leq i \leq r-2, i \neq s} \theta(a_i a_s^{-1}, a_i a_s a^{-1})} \\
 &\times \sum_{-\infty < k < \infty} \frac{1 - a_s^2 a^{-1} q^{2k}}{1 - a_s^2 a^{-1}} \prod_{i=1}^{2(r-1)} \frac{(a_s b_i a^{-1}; q)_k}{(qa_s b_i^{-1}; q)_k} \left(\frac{q^{r-2} a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right)^k. \tag{3.13}
 \end{aligned}$$

This completes the proof of (3.1). □

Remark 1 If we change r to $r + 2$ and substitute $b_{2r+1} = a^{\frac{1}{2}}$, $b_{2r+2} = -a^{\frac{1}{2}}$ in (3.1), we obtain (5.5.1) of [1] ((7) of [6], and (7.2.1.1) of [7]), since

$$\begin{aligned}
 \frac{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}; q)_{\infty}}{\theta(a)} &= \frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_{\infty}}{(qa, qa^{-1}; q)_{\infty}}, a_s \frac{(a^{\frac{1}{2}} a_s^{-1}, -a^{\frac{1}{2}} a_s^{-1}; q)_{\infty}}{\theta(aa_s^{-1}, a_s^2 a^{-1})} \\
 &= \frac{(qa^{\frac{1}{2}} a_s^{-1}, -qa^{\frac{1}{2}} a_s^{-1}; q)_{\infty}}{\theta(a_s a^{-1}) (qa_s^2 a^{-1}, qa a_s^{-2}; q)_{\infty}}.
 \end{aligned}$$

Remark 2 To obtain (5.5.1) of [1] by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t) = \frac{\prod_{i=1}^{2r} (qb_i^{-1}t^{-1}, qab_i^{-1}t; q)_\infty}{(t^{-1}, qt, at, qa^{-1}t^{-1}; q)_\infty \prod_{i=1}^r (a_i a^{-1}t^{-1}, qa_i^{-1}at, a_i t, qa_i^{-1}t^{-1}; q)_\infty}.$$

4 Very-well-poised ${}_2r\psi_{2r}$ and ${}_{2r-1}\psi_{2r-1}$ series

If we change r to $r + 1$ and substitute $b_{2r-1} = q^{\frac{1}{2}}a^{\frac{1}{2}}$, $b_{2r} = -q^{\frac{1}{2}}a^{\frac{1}{2}}$ in (3.1), we obtain the following transformation for very-well-poised ${}_{2r}\psi_{2r}$ series:

Theorem 3 *Suppose that*

$$a, a_i, a_i a^{-1}, a_i a_j^{-1}, a_i^2 a^{-1}, a_i a_j a^{-1} \notin q^{\mathbb{Z}}, \quad 1 \leq i \neq j \leq r - 1,$$

for $r \geq 2$, and $|a^{r-2}q^{r-1}| < \left| \prod_{i=1}^{2(r-1)} b_i \right|$. Then we have

$$\begin{aligned} & \frac{\prod_{i=1}^{2(r-1)} (qb_i^{-1}, qab_i^{-1}; q)_\infty}{(q^2a, q^2a^{-1}; q^2)_\infty \prod_{i=1}^{r-1} \theta(a_i a^{-1}, a_i)} \\ & \times {}_{2r}\psi_{2r} \left(\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b_1, b_2, \dots, b_{2(r-1)} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, qab_1^{-1}, qab_2^{-1}, \dots, qab_{2(r-1)}^{-1} \end{matrix}; q, -\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right) \\ & = \sum_{s=1}^{r-1} \frac{\prod_{i=1}^{2(r-1)} (qa a_s^{-1} b_i^{-1}, qa_s b_i^{-1}; q)_\infty}{(q^2 a_s^2 a^{-1}, q^2 a a_s^{-2}; q^2)_\infty \theta(a_s a^{-1}, a_s) \prod_{\substack{1 \leq i \leq r-1 \\ i \neq s}} \theta(a_i a_s^{-1}, a_s a_i a^{-1})} \\ & \times {}_{2r}\psi_{2r} \left(\begin{matrix} qa a_s^{-\frac{1}{2}}, -qa_s a^{-\frac{1}{2}}, b_1 a_s a^{-1}, b_2 a_s a^{-1}, \dots, b_{2(r-1)} a_s a^{-1} \\ a_s a^{-\frac{1}{2}}, -a_s a^{-\frac{1}{2}}, qa_s b_1^{-1}, qa_s b_2^{-1}, \dots, qa_s b_{2(r-1)}^{-1} \end{matrix}; q, -\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i} \right). \end{aligned} \tag{4.1}$$

Remark 3 To obtain (4.1) by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t) = \frac{(q^{\frac{1}{2}}a^{-\frac{1}{2}}t^{-1}, q^{\frac{1}{2}}a^{\frac{1}{2}}t, -q^{\frac{1}{2}}a^{-\frac{1}{2}}t^{-1}, -q^{\frac{1}{2}}a^{\frac{1}{2}}t; q)_\infty \prod_{i=1}^{2(r-1)} (qb_i^{-1}t^{-1}, qab_i^{-1}t; q)_\infty}{(t^{-1}, qt, at, qa^{-1}t^{-1}; q)_\infty \prod_{i=1}^{r-1} (a_i a^{-1}t^{-1}, qa_i^{-1}at, a_i t, qa_i^{-1}t^{-1}; q)_\infty}.$$

Remark 4 If we change r to $r + 2$ and substitute $b_{2r+1} = a^{\frac{1}{2}}$, $b_{2r+2} = -a^{\frac{1}{2}}$ in (4.1), we obtain (5.5.4) of [1] ((8) of [6], and (7.2.1.3) of [7]), since

$$\begin{aligned} & (qa^{-\frac{1}{2}}, qa^{\frac{1}{2}}, -qa^{-\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_\infty (q^{\frac{1}{2}}a^{-\frac{1}{2}}, q^{\frac{1}{2}}a^{\frac{1}{2}}, -q^{\frac{1}{2}}a^{-\frac{1}{2}}, -q^{\frac{1}{2}}a^{\frac{1}{2}}; q)_\infty \\ & = (qa, qa^{-1}; q)_\infty, \end{aligned}$$

and

$$\begin{aligned} & (qa^{\frac{1}{2}}a_s^{-1}, qa_s a^{-\frac{1}{2}}, -qa^{\frac{1}{2}}a_s^{-1}, -qa_s a^{-\frac{1}{2}}; q)_\infty \\ & \times (q^{\frac{1}{2}}a^{\frac{1}{2}}a_s^{-1}, q^{\frac{1}{2}}a^{-\frac{1}{2}}a_s, -q^{\frac{1}{2}}a^{\frac{1}{2}}a_s^{-1}, -q^{\frac{1}{2}}a^{-\frac{1}{2}}a_s; q)_\infty = (qa_s^2 a^{-1}, qa a_s^{-2}; q)_\infty. \end{aligned}$$

Remark 5 To obtain (5.5.4) of [1] by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t) = \frac{(q^{\frac{1}{2}}a^{-\frac{1}{2}}t^{-1}, q^{\frac{1}{2}}a^{\frac{1}{2}}t, -q^{\frac{1}{2}}a^{-\frac{1}{2}}t^{-1}, -q^{\frac{1}{2}}a^{\frac{1}{2}}t; q)_{\infty} (qa^{-\frac{1}{2}}t^{-1}, qa^{\frac{1}{2}}t, -qa^{-\frac{1}{2}}t^{-1}, -qa^{\frac{1}{2}}t; q)_{\infty}}{(t^{-1}, qt, at, qa^{-1}t^{-1}; q)_{\infty}} \times \frac{\prod_{i=1}^{2r} (qb_i^{-1}t^{-1}, qab_i^{-1}t; q)_{\infty}}{\prod_{i=1}^{r+1} (a_i a^{-1}t^{-1}, qa_i^{-1}at, a_i t, qa_i^{-1}t^{-1}; q)_{\infty}}.$$

The substitution $b_{2(r-1)} = \pm q^{\frac{1}{2}}a^{\frac{1}{2}}$ in (4.1) implies the following transformation for very-well-poised ${}_{2r-1}\psi_{2r-1}$ series:

Theorem 4 Suppose that

$$a, a_i, a_i a^{-1}, a_i a_j^{-1}, a_i^2 a^{-1}, a_i a_j a^{-1} \notin q^{\mathbb{Z}}, \quad 1 \leq i \neq j \leq r - 2,$$

for $r \geq 3$, and $\left| a^{r-\frac{5}{2}} q^{r-\frac{3}{2}} \right| < \left| \prod_{i=1}^{2r-3} b_i \right|$. Then we have

$$\begin{aligned} & \frac{(\pm q^{\frac{1}{2}}a^{-\frac{1}{2}}, \pm q^{\frac{1}{2}}a^{\frac{1}{2}}; q)_{\infty} \prod_{i=1}^{2r-3} (qb_i^{-1}, qab_i^{-1}; q)_{\infty}}{(qa, qa^{-1}; q)_{\infty} \prod_{i=1}^{r-2} \theta(a_i a^{-1}, a_i)} \\ & \times {}_{2r-1}\psi_{2r-1} \left(\begin{matrix} qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b_1, b_2, \dots, b_{2r-3} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, qab_1^{-1}, qab_2^{-1}, \dots, qab_{2r-3}^{-1} \end{matrix}; q, \pm \frac{q^{r-\frac{5}{2}} a^{r-\frac{3}{2}}}{\prod_{i=1}^{2r-3} b_i} \right) \\ & = \sum_{s=1}^{r-2} \frac{(\pm q^{\frac{1}{2}}a^{\frac{1}{2}}a_s^{-1}, \pm q^{\frac{1}{2}}a^{-\frac{1}{2}}a_s; q)_{\infty} \prod_{i=1}^{2r-3} (qaa_s^{-1}b_i^{-1}, qa_s b_i^{-1}; q)_{\infty}}{(qa_s^2 a^{-1}, qaa_s^{-2}; q)_{\infty} \theta(a_s a^{-1}, a_s) \prod_{\substack{1 \leq i \leq r-2 \\ i \neq s}} \theta(a_i a_s^{-1}, a_s a_i a^{-1})} \\ & \times {}_{2r-1}\psi_{2r-1} \left(\begin{matrix} qa_s a^{-\frac{1}{2}}, -qa_s a^{-\frac{1}{2}}, b_1 a_s a^{-1}, b_2 a_s a^{-1}, \dots, b_{2r-3} a_s a^{-1} \\ a_s a^{-\frac{1}{2}}, -a_s a^{-\frac{1}{2}}, qa_s b_1^{-1}, qa_s b_2^{-1}, \dots, qa_s b_{2r-3}^{-1} \end{matrix}; q, \pm \frac{q^{r-\frac{5}{2}} a^{r-\frac{3}{2}}}{\prod_{i=1}^{2r-3} b_i} \right). \end{aligned} \tag{4.2}$$

Here either all the upper or all the lower signs are taken throughout.

Remark 6 To obtain (4.2) by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t) = \frac{(\pm q^{\frac{1}{2}}a^{-\frac{1}{2}}t^{-1}, \pm q^{\frac{1}{2}}a^{\frac{1}{2}}t; q)_{\infty} \prod_{i=1}^{2r-3} (qb_i^{-1}t^{-1}, qab_i^{-1}t; q)_{\infty}}{(t^{-1}, qt, at, qa^{-1}t^{-1}; q)_{\infty} \prod_{i=1}^{r-2} (a_i a^{-1}t^{-1}, qa_i^{-1}at, a_i t, qa_i^{-1}t^{-1}; q)_{\infty}}.$$

Remark 7 If we change r with $r + 2$ and substitute $b_{2r} = a^{\frac{1}{2}}$, $b_{2r+1} = -a^{\frac{1}{2}}$ in (4.2), we obtain (5.5.5) of [1] ((9) of [6], and (7.2.1.4) of [7]), since

$$(a_s a^{-\frac{1}{2}}, -a_s a^{-\frac{1}{2}}, qa^{\frac{1}{2}} a_s^{-1}, -qa^{\frac{1}{2}} a_s^{-1}; q)_{\infty} = -a_s^2 a^{-1} (qa_s a^{-\frac{1}{2}}, -qa_s a^{-\frac{1}{2}}, a^{\frac{1}{2}} a_s^{-1}, -a^{\frac{1}{2}} a_s^{-1}; q)_{\infty},$$

and

$$\theta(a_s a^{-1}) = -a_s a^{-1} \theta(a a_s^{-1}).$$

Remark 8 To obtain (5.5.5) of [1] by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t) = \frac{(\pm q^{\frac{1}{2}} a^{-\frac{1}{2}} t^{-1}, \pm q^{\frac{1}{2}} a^{\frac{1}{2}} t; q)_{\infty} (q a^{-\frac{1}{2}} t^{-1}, q a^{\frac{1}{2}} t, -q a^{-\frac{1}{2}} t^{-1}, -q a^{\frac{1}{2}} t; q)_{\infty}}{(t^{-1}, q t, a t, q a^{-1} t^{-1}; q)_{\infty}} \\ \times \frac{\prod_{i=1}^{2r-1} (q b_i^{-1} t^{-1}, q a b_i^{-1} t; q)_{\infty}}{\prod_{i=1}^r (a_i a^{-1} t^{-1}, q a_i^{-1} a t, a_i t, q a_i^{-1} t^{-1}; q)_{\infty}}.$$

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