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Exact values and lower bounds on the n-color weak Schur numbers for $n = 2, 3$

T. Ahmed[1] · L. Boza[2] · M. P. Revuelta[2] · M. I. Sanz[2]

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Abstract
For integers k, n with $k, n \geq 1$, the *n-color weak Schur number* $WS_k(n)$ is defined as the least integer N, such that for every n-coloring of the integer interval $[1, N]$, there exists a monochromatic solution $x_1, \ldots, x_k, x_{k+1}$ in that interval to the equation:

$$x_1 + x_2 + \cdots + x_k = x_{k+1},$$

with $x_i \neq x_j$, when $i \neq j$. In this paper, we obtain the exact values of $WS_6(2) = 166$, $WS_7(2) = 253$, $WS_3(3) = 94$ and $WS_4(3) = 259$ and we show new lower bounds on n-color weak Schur number $WS_k(n)$ for $n = 2, 3$.

Keywords Schur numbers · Sum-free sets · Weak Schur numbers · Weakly sum-free sets · n-coloring

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1 Introduction

For integers $a \leq b$, we shall denote $[a, b]$ the *integer interval* consisting of all $t \in \mathbb{N}_+ = \{1, 2, \ldots\}$ such that $a \leq t \leq b$. A function

✉ M. P. Revuelta
pastora@us.es

T. Ahmed
tanbir@gmail.com

L. Boza
boza@us.es

M. I. Sanz
isanz@us.es

1 Montreal Health Innovations Coordinating Center (MHICC), Montreal Heart Institute (MHI), Montréal, Canada

2 Departamento de Matemática Aplicada I, Universidad de Sevilla, Sevilla, Spain

$$\Delta : [1, N] \longrightarrow \{c_1, \ldots, c_n\},$$

where $c_1, \ldots, c_n \in \mathbb{N}_+$ represent different colors, is an n-coloring of the interval $[1, N]$.

Given an n-coloring Δ and the equation $x_1 + \cdots + x_k = x_{k+1}$ in $k + 1$ variables, we say that a solution $x_1, \ldots, x_k, x_{k+1}$ to the equation is monochromatic if and only if $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_{k+1})$.

For integers k, n with $k, n \geq 1$, the *n-color weak Schur number $WS_k(n)$* is defined as the least integer N, such that for every n-coloring of the integer interval $[1, N]$, there exists a monochromatic solution $x_1, \ldots, x_k, x_{k+1}$ in that interval to the equation: $x_1+x_2+\cdots+x_k = x_{k+1}$, with $x_i \neq x_j$ when $i \neq j$. Irwing [14] showed the existence and obtained the following general upper bound:

$$WS_k(n) \leq \left[\frac{1}{2}(n!(k-1)^n(kn+1)exp(\frac{1}{k-1}) + \frac{k}{k-1}\right].$$

For $k = 2$, we have $1 + 315^{\frac{n-1}{5}} \leq WS_2(n) \leq [n!ne] + 1$, the lower bound is due to Sierpinski [20] and the upper bound to Bornsztein [5].

1.1 Schur numbers and weak Schur numbers

A set A of integers is called *sum-free* if it contains no elements $x_1, x_2, x_3 \in A$ satisfying $x_1 + x_2 = x_3$ where x_1, x_2 need not be distinct.

Schur [19] in 1916 proved that, given a positive integer n, there exists a greatest positive integer $S_2(n) = N$ with the property that the integer interval $[1, N - 1]$ can be partitioned into n *sum-free* sets. The numbers $S_2(n)$ are called Schur numbers. The current knowledge on these numbers for $1 \leq n \leq 7$ is given in Table 1.

Many generalizations of Schur numbers have appeared since their introduction. Now, a set A of integers is called *weakly sum-free* if it contains no pairwise distinct elements $x_1, x_2, x_3 \in A$ satisfying $x_1 + x_2 = x_3$. We denote by $WS_2(n)$, the greatest integer N, for which the integer interval $[1, N - 1]$. The exact value of $S_2(4)$ was given by Baumert [2] and recently $S_2(5)$ has been obtained by Heule [13]. Finally, the lower bounds on $S_2(6)$ and $S_2(7)$ were obtained by Fredricksen and Sweet [11] by considering symmetric sum-free partitions. A set A of integers is said to be *k-sum-free* if it contains no $k+1$ elements $x_1, x_2, \cdots, x_{k+1} \in A$ satisfying $x_1+\cdots+x_k = x_{k+1}$, where x_i, $i = 1, \cdots, k$ are not necessarily distinct. In 1933, Rado [15] gave the following generalization: given two positive integers, n and $k \geq 2$, there exists a greatest positive integer, $S_k(n) = N$, such that the integer interval $[1, N - 1]$ can be partitioned into n sets which are *k-sum-free*. In 1966, Znám [22] established a lower

Table 1 The first few Schur numbers $S_2(n)$

n	1	2	3	4	5	6	7
$S_2(n)$	2	5	14	45	161	≥ 537	≥ 1681

Table 2 The first few weak Schur numbers $WS_2(n)$

n	1	2	3	4	5	6	7	8	9
$WS_2(n)$	3	9	24	67	≥ 197	≥ 583	≥ 1741	≥ 5202	≥ 15597

bound on the numbers $S_k(n)$:

$$S_k(n) \geq \frac{k-1}{k}((k+1)^n - 1) + 1.$$

In 1982, Beutelspacher and Brestovansky [3] proved the equality for two k-sum-free sets:

$$S_k(2) = k^2 + k - 1, k \geq 2.$$

In 2010 [18], the last author obtained the exact value of $S_3(3) = 43$. Independently, Ahmed and Schaal [1] in 2016 gave the values of $S_k(3)$ for $k = 3, 4, 5$. In 2019, Boza et al. [6] determined the exact formula of $S_k(3) = k^3 + 2k^2 - 2$ for all $k \geq 3$, finding an upper bound that coincides with the lower bound given by Znám [22].

The numbers $WS_2(n)$ are called the *weak Schur numbers* for the equation $x_1 + x_2 = x_3$. The known weak Schur numbers are given in Table 2.

The current state of knowledge concerning $WS_2(n)$ is a bit confusing. The problem seems to have been first considered in [21], which is Walker's solution to Problem E985 proposed a year earlier, in 1951, by Moser. Walker considered the cases $n = 3, 4$ and 5 and claimed the values $WS_2(3) = 24$, $WS_2(4) = 67$, and $WS_2(5) = 197$. Unfortunately, the short account written by Moser on Walker's solution only gives suitable partitions of $[1, 23]$ for $n = 3$, and no details at all for the cases $n = 4$ and 5. Walker's claimed values of $WS_2(3)$ and $WS_2(4)$ were later confirmed by Blanchard, Harary, and Reis using computers [4]. In 2012, the two last authors et al. [9] confirmed the lower bound $WS_2(5) \geq 197$. In addition, a lower bound on $WS_2(6)$ was obtained in [9] and later improved to $WS_2(6) \geq 583$ in [10]. The lower bounds for $7 \leq n \leq 9$ were obtained [17] in 2015.

In terms of coloring, the $WS_k(n)$ is the least positive integer N such that for every n-coloring of $[1, N]$,

$$\Delta : [1, N] \longrightarrow \{c_1, \ldots, c_n\},$$

where $c_1, \ldots, c_n$ represent n different colors, there exists a monochromatic solution to the equation $x_1 + \cdots + x_k = x_{k+1}$, such that $\Delta(x_1) = \cdots = \Delta(x_k) = \Delta(x_{k+1})$ where $x_i \neq x_j$ when $i \neq j$.

In addition, for 2-coloring, the known weak Schur numbers $WS_k(2)$ are shown in Table 3.

The exact values of $WS_k(2)$ for $k = 3, 4$ and the lower bounds were obtained in [18], [7] and $WS_5(2)$ [8] in 2017.

Table 3 The first few weak Schur numbers $WS_k(2)$

k	2	3	4	5	6	7	8	9
$WS_k(2)$	9	24	52	101	≥ 156	≥ 238	≥ 344	≥ 477

1.2 Main results

In Section 2, we determine a general lower bound on the 2-color weak Schur numbers for the equation $x_1+\cdots+x_k = x_{k+1}$, with $x_i \neq x_j$ when $i \neq j$, for $k \geq 5$, improving the lower bound given in [7].
Lemma 2.1 $WS_k(2) \geq \frac{1}{2}(k^3+4k^2-5k+2)$ for any integer $k \geq 5$.
In Section 3, we determine a general lower bound on $WS_k(3)$ improving the lower bound given in [7].
Lemma 3.1 $WS_k(3) \geq \frac{1}{2}(k^4+5k^3-k^2-9k+6)$ for any integer $k \geq 5$.
Lemma3.2 $WS_k(3) \geq \frac{1}{2}(k^4+5k^3-8k+4)$ for any integer $k \geq 8$.
In Section 4, we obtain the exact values of the 2-color weak Schur number $WS_6(2)$ and $WS_7(2)$. In addition, we determinate the exact values of the 3-color weak Schur numbers $WS_3(3)$ and $WS_4(3)$.
Theorem4.2 $WS_6(2) = 166$.
Theorem 4.6 $WS_7(2) = 253$.
Theorem 4.9 $WS_3(3) = 94$.
Theorem 4.12 $WS_4(3) = 259$.

2 A general lower bound for $WS_k(2)$

In terms of coloring, the weak Schur number $WS_k(2)$ is the least positive integer N such that for every 2-coloring of $[1, N]$,

$$\Delta : [1, N] \longrightarrow \{c_1, c_2\},$$

where c_1, c_2 represent 2 different colors, there exists a monochromatic solution to the equation $x_1 + x_2 + \cdots + x_k = x_{k+1}$, such that $\Delta(x_1) = \cdots = \Delta(x_k) = \Delta(x_{k+1})$ where $x_i \neq x_j$ when $i \neq j$.

In [7], a general lower bound of the weak Schur number $WS_k(2)$ was given, now we show a new general lower bound that improves the previous one.

Lemma 2.1 *For any integer $k \geq 5$, we have*

$$WS_k(2) \geq \frac{1}{2}(k^3+4k^2-5k+2)$$

Proof Let Δ be a 2-coloring:

$$\Delta : [1, \frac{1}{2}(k^3+4k^2-5k)] \longrightarrow \{c_1, c_2\},$$

where c_1, c_2 represent 2 different colors. Let $A_i = \Delta^{-1}(c_i)$ for $i = 1, 2$, such that

$$\left[1, \frac{1}{2}(k^3 + 4k^2 - 5k)\right] = A_1 \sqcup A_2,$$

where

$$\begin{cases} A_1 = \{1\} \cup [\frac{1}{2}(k^2 + 3k), \frac{1}{2}(k^3 + 3k^2 - 6k + 2)], \\ A_2 = [2, \frac{1}{2}(k^2 + 3k - 2)] \cup [\frac{1}{2}(k^3 + 3k^2 - 6k + 4), \frac{1}{2}(k^3 + 4k^2 - 5k)]. \end{cases}$$

We show that the above partition of the interval $[1, \frac{1}{2}(k^3+4k^2-5k)]$ has no monochromatic solution to the equation $x_1 + x_2 + \cdots + x_k = x_{k+1}$. For that, it is sufficient to prove that for every i, $1 \leq i \leq k$, if $x_1, \ldots, x_k \in A_i$ with $x_i < x_j$ when $i < j$, then $x_1 + \cdots + x_k \notin A_i$.

- If $x_1, \ldots, x_k \in A_1$, then

$$\sum_{i=1}^{k} x_i \geq 1 + \sum_{i=0}^{k-2}(\frac{1}{2}(k^2 + 3k) + i) = \frac{1}{2}(k^3 + 3k^2 - 6k + 4)$$
$$> \frac{1}{2}(k^3 + 3k^2 - 6k + 2)$$

  Hence $\sum_{i=1}^{k} x_i \notin A_1$.
- If $x_1, \ldots, x_k \in A_2$, then
  - If $x_k \leq \frac{1}{2}(k^2 + 3k - 2)$, then

$$\sum_{i=1}^{k} x_i \geq \sum_{i=0}^{k-1}(2 + i) = \frac{1}{2}(k^2 + 3k) > \frac{1}{2}(k^2 + 3k - 2).$$

    In addition, for $k \geq 5$,

$$\sum_{i=1}^{k} x_i \leq \sum_{i=0}^{k-1}(\frac{1}{2}(k^2 + 3k - 2) - i)$$
$$= \frac{1}{2}(k^3 + 2k^2 - k) < \frac{1}{2}(k^3 + 3k^2 - 6k + 4)$$

    Hence $\sum_{i=1}^{k} x_i \notin A_2$.
  - If $x_k \geq \frac{1}{2}(k^3 + 3k^2 - 6k + 4)$, then

$$\sum_{i=1}^{k} x_i \geq \sum_{i=0}^{k-2}(2 + i) + \frac{1}{2}(k^3 + 3k^2 - 6k + 4)$$
$$= \frac{1}{2}(k^3 + 4k^2 - 5k + 2) > \frac{1}{2}(k^3 + 4k^2 - 5k)$$

    Hence $\sum_{i=1}^{k} x_i \notin A_2$.

Therefore, we obtain the lower bound. □

Table 4 New lower bound of weak Schur numbers $WS_k(2)$

k	5	6	7	8	9	10
$WS_k(2)$	101	≥ 166	≥ 253	≥ 365	≥ 505	≥ 676

Table 5 Lower bounds weak Schur numbers $WS_k(3)$

k	2	3	4	5	6	7
$WS_k(3)$	24	≥ 94	≥ 259	≥ 571	≥ 1096	≥ 1912

With this general lower bound, we improve the results shown in Table 3. In addition, in the Section 4, we will prove that these new lower bounds shown in Table 4 for $k = 6$ and $k = 7$, are exact values.

3 A lower bound for $WS_k(3)$

Applying the result given in [7], the lower bounds shown in Table 5were obtained.

In the next result, we improve the general lower bound of $WS_k(3)$ obtained in [7].

Lemma 3.1 *For any integer $k \geq 5$, we have*

$$WS_k(3) \geq \frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 6).$$

Proof We will show that he following partition of the interval

$$[1, \frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 4)] = B_1 \sqcup B_2 \sqcup B_3$$

has no monochromatic solution to the equation $x_1 + x_2 + \cdots + x_k = x_{k+1}$ with $x_1 < x_2 < \ldots < x_k$. Consider the following 3-coloring where A_1 and A_2 are the same as used in the construction of the 2-coloring in Lemma 2.1.

$$\begin{cases} B_1 = A_1 \cup [\frac{1}{2}(k^4 + 4k^3 - 3k^2 + 2k - 2), \frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 4)], \\ B_2 = A_2 \cup [\frac{1}{2}(k^4 + 4k^3 - 4k^2 + k), \frac{1}{2}(k^4 + 4k^3 - 3k^2 + 2k - 4)], \\ B_3 = [\frac{1}{2}(k^3 + 4k^2 - 5k + 2), \frac{1}{2}(k^4 + 4k^3 - 4k^2 + k - 2)]. \end{cases}$$

Since the above 3-coloring is an extension of 2-coloring given by Lemma 2.1, we just have to try the following cases:

- Let $x_1, x_2, \ldots, x_k \in B_1$, with $x_1 < x_2 < \cdots < x_k$.

– If $x_k \in A_1$, by Lemma 2.1, $\sum_{i=1}^{k} x_i \notin A_1$. Therefore,

$$\begin{aligned}\sum_{i=1}^{k} x_i &\le \sum_{i=0}^{k-1}(\frac{1}{2}(k^3+3k^2-6k+2)-i)\\ &= \frac{1}{2}(k^4+3k^3-7k^2+3k)\\ &< \frac{1}{2}(k^4+4k^3-3k^2+2k-2).\end{aligned}$$

Hence, $\sum_{i=1}^{k} x_i \notin B_1$.

– If $x_k \ge \frac{1}{2}(k^4+4k^3-3k^2+2k-2)$, then

$$\begin{aligned}\sum_{i=1}^{k} x_i &\ge 1+\sum_{i=0}^{k-3}(\frac{1}{2}(k^2+3k)+i)+\frac{1}{2}(k^4+4k^3-3k^2+2k-2)\\ &= \frac{1}{2}(k^4+5k^3-k^2-9k+6)\\ &> \frac{1}{2}(k^4+5k^3-k^2-9k+4).\end{aligned}$$

Hence, $\sum_{i=1}^{k} x_i \notin B_1$.

- Let $x_1, \ldots, x_k \in B_2$, with $x_1 < \cdots < x_k$.

– If $x_k \in A_2$, then by Lemma 2.1, $\sum_{i=1}^{k} x_i \notin A_2$. Therefore,

$$\begin{aligned}\sum_{i=1}^{k} x_i \le \sum_{i=0}^{k-1}(\frac{1}{2}(k^3+4k^2-5k)-i) &= \frac{1}{2}(k^4+4k^3-6k^2+k)\\ &< \frac{1}{2}(k^4+4k^3-4k^2+k).\end{aligned}$$

Hence, $\sum_{i=1}^{k} x_i \notin B_2$.

– If $x_k \ge \frac{1}{2}(k^4+4k^3-4k^2+k)$, then

$$\begin{aligned}\sum_{i=1}^{k} x_i &\ge \sum_{i=0}^{k-2}(2+i)+\frac{1}{2}(k^4+4k^3-4k^2+k)\\ &= \frac{1}{2}(k^4+4k^3-3k^2+2k-2)\\ &> \frac{1}{2}(k^4+4k^3-3k^2+2k-4).\end{aligned}$$

Hence, $\sum_{i=1}^{k} x_i \notin B_2$.

Table 6 New lower bound of weak Schur numbers $WS_k(3)$

k	2	3	4	5	6	7
$WS_k(3)$	24	≥ 94	≥ 259	≥ 593	≥ 1146	≥ 2005

- Let $x_1, \ldots, x_k \in B_3$, with $x_1 < \cdots < x_k$. Then

$$\sum_{i=1}^{k} x_i \geq \sum_{i=0}^{k-1} (\frac{1}{2}(k^3 + 4k^2 - 5k + 2) + i)$$
$$= \frac{1}{2}(k^4 + 4k^3 - 4k^2 + k)$$
$$> \frac{1}{2}(k^4 + 4k^3 - 4k^2 + k - 2).$$

  Hence, $\sum_{i=1}^{k} x_i \notin B_3$.

Therefore, we obtain the desired lower bound.

With this general lower bound,we improve the results shown in Table 5. In addition, in Section 4, we will prove that these new lower bounds shown in Table 6 for $k = 3$ and $k = 4$ are exact values.

In the next result, we improve the lower bounds of Lemma 3.1 for any integer $k \geq 8$.

Lemma 3.2 *For any integer $k \geq 8$, we have*

$$WS_k(3) \geq \frac{1}{2}(k^4 + 5k^3 - 8k + 4).$$

Proof The following partition of the interval

$$[1, \frac{1}{2}(k^4 + 5k^3 - 8k + 2)] = C_1 \sqcup C_2 \sqcup C_3$$

has no monochromatic solution to the equation $x_1 + x_2 + \cdots + x_k = x_{k+1}$ with $x_1 < x_2 < \ldots < x_k$.

$$\begin{cases} C_1 = B_1, \\ C_2 = B_2 \cup [\frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 6), \frac{1}{2}(k^4 + 5k^3 - 8k + 2)], \\ C_3 = B_3. \end{cases}$$

This 3-coloring is an extension of 3-coloring given in Lemma 3.1, so we just have to try the following cases:

- Let $x_1, x_2, \ldots, x_k \in C_1$, with $x_1 < x_2 < \cdots < x_k$, by Lemma 3.1, $\sum_{i=1}^{k} x_i \notin B_1 = C_1$.

- Let $x_1, x_2, \ldots, x_k \in C_2$, with $x_1 < x_2 < \cdots < x_k$, by Lemma 3.1, $\sum_{i=1}^{k} x_i \notin B_2$. We only need to prove that

$$\sum_{i=1}^{k} x_i \notin [\frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 6), \frac{1}{2}(k^4 + 5k^3 - 8k + 2)].$$

We consider four cases:

  - If $x_k \le \frac{1}{2}(k^3 + 4k^2 - 5k)$, then

$$\sum_{i=1}^{k} x_i \le \sum_{i=0}^{k-1} \left(\frac{1}{2}(k^3 + 4k^2 - 5k) - i\right) = \frac{1}{2}(k^4 + 4k^3 - 6k^2 + k) < \frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 6).$$

    Hence $\sum_{i=1}^{k} x_i \notin C_2$.
  - If $x_k \in [\frac{1}{2}(k^4 + 4k^3 - 4k^2 + k), \frac{1}{2}(k^4 + 4k^3 - 3k^2 + 2k - 4)]$ and $x_{k-1} \le \frac{1}{2}(k^2 + 3k - 2)$, then for $k \ge 8$,

$$\sum_{i=1}^{k} x_i \le \sum_{i=0}^{k-2} (\frac{1}{2}(k^3 + 3k - 2) - i) + \frac{1}{2}(k^4 + 4k^3 - 3k^2 + 2k - 4) = \frac{1}{2}(k^4 + 5k^3 - 2k^2 - 4) < \frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 6).$$

    Hence $\sum_{i=1}^{k} x_i \notin C_2$.
  - If $x_k \in [\frac{1}{2}(k^4 + 4k^3 - 4k^2 + k), \frac{1}{2}(k^4 + 4k^3 - 3k^2 + 2k - 4)]$ and $x_{k-1} \ge \frac{1}{2}(k^3 + 3k^2 - 6k + 4)$, then

$$\sum_{i=1}^{k} x_i \ge \sum_{i=0}^{k-3} (2 + i) + \frac{1}{2}(k^3 + 3k^2 - 6k + 4) + \frac{1}{2}(k^4 + 4k^3 - 4k^2 + k) = \frac{1}{2}(k^4 + 5k^3 - 6k + 2) > \frac{1}{2}(k^4 + 5k^3 - 8k + 2).$$

    Hence $\sum_{i=1}^{k} x_i \notin C_2$.
  - If $x_k \ge \frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 6)$, then

$$\sum_{i=1}^{k} x_i \ge \sum_{i=0}^{k-2} (2 + i) + \frac{1}{2}(k^4 + 5k^3 - k^2 - 9k + 6) = \frac{1}{2}(k^4 + 5k^3 - 8k + 4) > \frac{1}{2}(k^4 + 5k^3 - 8k + 2)$$

Table 7 Lower bounds of weak Schur numbers $WS_k(3)$

k	8	9	10	11	12
Lemma 3.1	≥ 3263	≥ 5025	≥ 7408	≥ 10541	≥ 14565
Lemma 3.2	≥ 3298	≥ 5069	≥ 7462	≥ 10606	≥ 14642

Hence, $\sum_{i=1}^{k} x_i \notin C_2$.

- Let $x_1, \ldots, x_k \in C_3$, with $x_1 < \cdots < x_k$, by Lemma 3.1, $\sum_{i=1}^{k} x_i \notin B_3 = C_3$.

Therefore,we obtain the desired improved lower bound.

Applying Lemmas 3.1 and 3.2, the following lower bounds are shown in Table 7.

4 Computer-assisted proofs for the exact values of $WS_6(2)$, $WS_7(2)$, $WS_3(3)$ and $WS_4(3)$

4.1 The exact value of $WS_6(2)$

We shall prove that $WS_6(2) = 166$. By Lemma 2.1,we have $WS_6(2) \geq 166$.

To prove that the equation $x_1 + \cdots + x_6 = x_7$ has a monochromatic solution for every 2-coloring of the integer interval $[1, 166]$, it is necessary to show the following result.

Lemma 4.1 *The set* $\mathcal{Y} = \{y_n\}_{n=1}^{42} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 21, 22, 23, 26, 27, 28, 29, 30, 31, 35, 36, 40, 41, 46, 48, 51, 56, 61, 66, 106, 141, 146, 151, 156, 161, 166\}$ *satisfies:*

1. *We have* $\mathcal{Y} \subseteq [1, 166]$.
2. *For every partition of* $\mathcal{Y}$ *into two subsets* A_1, A_2*, some* A_i *contains a monochromatic solution of* $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = x_7$*, with* $x_i \neq x_j$*, if* $i \neq j$.

Proof 1. This is trivial.
2. We have checked the result transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver [12].

Let Δ be a 2-coloring of $[1, 166]$:

$$\Delta : [1, 166] \longrightarrow \{c_1, c_2\},$$

For any $\{y_n\} \in \mathcal{Y}$, we consider a Boolean variable ϕ defined on $[1, 42]$ as follows:

$$\phi(n) = \begin{cases} True \text{ if } \Delta(y_n) = c_1, \\ False \text{ if } \Delta(y_n) = c_2. \end{cases}$$

Let $\mathcal{S} = \{(y_a, y_b, y_c, y_d, y_e, y_f, y_g) \in \mathcal{Y}^7 \mid y_a+y_b+y_c+y_d+y_e+y_f = y_g$, with $a < b < c < d < e < f\}$.

For any $s = (y_a, y_b, y_c, y_d, y_e, y_f, y_g) \in \mathcal{S}$, we consider two clauses:

$$p(s) = (\phi(a) \vee \phi(b) \vee \phi(c) \vee \phi(d) \vee \phi(e) \vee \phi(f) \vee \phi(g));$$
$$q(s) = (\neg\phi(a) \vee \neg\phi(b) \vee \neg\phi(c) \vee \neg\phi(d) \vee \neg\phi(e) \vee \neg\phi(f) \vee \neg\phi(g)).$$

Then, $p(s)$ is satisfiable if and only if $\Delta(a) \neq c_1$, $\Delta(b) \neq c_1$, $\Delta(c) \neq c_1$, $\Delta(d) \neq c_1$, $\Delta(e) \neq c_1$, $\Delta(f) \neq c_1$ or $\Delta(g) \neq c_1$, i.e., Δ does not induce in s a monochromatic solution on c_1 of the equation $x_1+\cdots+x_6 = x_7$. Analogously, $q(s)$ is satisfiable if and only if Δ does not induce in s a monochromatic solution of the equation $x_1+\cdots+x_6 = x_7$ on c_2.

$$\text{Let } \mathcal{C} = \bigwedge_{s\in\mathcal{S}} (p(s) \wedge q(s)).$$

Clearly $\mathcal{C}$ is satisfiable if and only if Δ does not induce on $\mathcal{Y}$ a monochromatic solution of the equation $x_1 + \cdots + x_6 = x_7$. The SAT-Solver shows that $\mathcal{C}$ is not satisfiable, hence for every 2-coloring of the set,$\mathcal{Y}$ has a monochromatic solution to the equation $x_1 + \cdots + x_6 = x_7$. □

With this result, we have tested the upper bound on $WS_6(2)$.
Therefore, we conclude with the following result:

Theorem 4.2 $WS_6(2) = 166$.

4.2 The exact value of $WS_7(2)$

We shall prove that $WS_7(2) = 253$. By Lemma 2.1, we have $WS_7(2) \geq 253$.

We have to prove that the equation $x_1 + \cdots + x_7 = x_8$ has a monochromatic solution for every 2-coloring of the interval $[1, 253]$. We will suppose the opposite: for every 2-coloring $\Delta : [1, 253] \longrightarrow \{c_1, c_2\}$ without monochromatic solution, we can consider without loss of generality $\Delta(61) = c_1$. Let $D_1 = \{u_n\}_{n=1}^{73} = (6[0, 42] + \{1\}) \cup (6\{0, 1, 4, 8, 16, 32\} + \{2, 3, 4, 5, 6\}) \subset [1, 253]$ and $F_1 = \{43, 49, 55, 61, 67, 73, 79, 85, 91, 97, 103, 109, 115, 133, 139, 145, 151, 157, 163, 169, 175\} \subset D_1$.

The following two lemmas can be proved transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver [12].

Lemma 4.3 *For every 2-coloring Δ of D_1 without monochromatic solution, we have* $\Delta(F_1) = \{c_1\}$.

Proof Let Δ be a 2-coloring of D_1:

$$\Delta : D_1 \longrightarrow \{c_1, c_2\},$$

such that $\Delta(61) = c_1$. For any $\{u_n\} \in D_1$, we consider a Boolean variable ϕ defined on $[1, 73]$ as follows:

$$\phi(n) = \begin{cases} True \ \text{if} \ \Delta(u_n) = c_1, \\ False \ \text{if} \ \Delta(u_n) = c_2. \end{cases}$$

Let $\mathcal{S} = \{(u_a, u_b, u_c, u_d, u_e, u_f, u_g, u_h) \in D_1^8 \mid u_a+u_b+u_c+u_d+u_e+u_f+u_g = u_h, \text{with } a < b < c < d < e < f < g\}$.

For any $s = (u_a, u_b, u_c, u_d, u_e, u_f, u_g, u_h) \in \mathcal{S}$, we consider two clauses:

$$p(s) = (\phi(a) \vee \phi(b) \vee \phi(c) \vee \phi(d) \vee \phi(e) \vee \phi(f) \vee \phi(g) \vee \phi(h))\,;$$
$$q(s) = (\neg\phi(a) \vee \neg\phi(b) \vee \neg\phi(c) \vee \neg\phi(d) \vee \neg\phi(e) \vee \neg\phi(f) \vee \neg\phi(g) \vee \neg\phi(h))\,.$$

Then, $p(s)$ is satisfiable if and only if $\Delta(a) \neq c_1$, $\Delta(b) \neq c_1$, $\Delta(c) \neq c_1$, $\Delta(d) \neq c_1$, $\Delta(e) \neq c_1$, $\Delta(f) \neq c_1$, or $\Delta(g) \neq c_1$ or $\Delta(h) \neq c_1$, i.e., Δ does not induce in s a monochromatic solution on c_1 of the equation $x_1 + \cdots + x_7 = x_8$. Analogously, $q(s)$ satisfiable if and only if Δ does not induce in s a monochromatic solution of the equation $x_1 + \cdots + x_7 = x_8$ on c_2.

$$\text{Let } \mathcal{C} = \bigwedge_{s \in \mathcal{S}} (p(s) \wedge q(s)).$$

Clearly $\mathcal{C}$ is satisfiable if and only if Δ does not induce on D_1 a monochromatic solution of the equation $x_1 + \cdots + x_7 = x_8$. The SAT-Solver shows that $\mathcal{C}$ is not satisfiable, hence we have the result. □

Trivially we have,

Corollary 4.4 *For every 2-coloring Δ of $[1, 253]$ without monochromatic solution, we have $\Delta(F_1) = \{c_1\}$.*

Let $D_2 = \{v_n\}_{n=1}^{78} = (6[0, 42] + \{1\}) \cup (6\{0, 1, 2, 5, 6, 36, 37\} + \{2, 3, 4, 5, 6\})$. We have $[1, 8] \cup F_1 \cup \{217\} \subset D_2 \subset [1, 253]$.

Lemma 4.5 *For every 2-coloring Δ of D_2 without monochromatic solution such that $\Delta(F_1) = \{c_1\}$, we have $\Delta(217) = c_1$ and $\Delta([1, 8]) = \{c_2\}$.*

Proof Let Δ be a 2-coloring of D_1:

$$\Delta : D_1 \longrightarrow \{c_1, c_2\},$$

such that $\Delta(F_1) = \{c_1\}$. For any $\{v_n\} \in D_2$, we consider a Boolean variable ϕ defined on $[1, 78]$ as follows:

$$\phi(n) = \begin{cases} True \;\; if \;\; \Delta(v_n) = c_1, \\ False \; if \;\; \Delta(v_n) = c_2. \end{cases}$$

Let $\mathcal{S} = \{(u_a, u_b, u_c, u_d, u_e, u_f, u_g, u_h) \in D_2^8 \mid u_a+u_b+u_c+u_d+u_e+u_f+u_g = u_h, \text{with } a < b < c < d < e < f < g\}$.

For any $s = (u_a, u_b, u_c, u_d, u_e, u_f, u_g, u_h) \in \mathcal{S}$, we consider two clauses:

$$p(s) = (\phi(a) \vee \phi(b) \vee \phi(c) \vee \phi(d) \vee \phi(e) \vee \phi(f) \vee \phi(g) \vee \phi(h))\,;$$
$$q(s) = (\neg\phi(a) \vee \neg\phi(b) \vee \neg\phi(c) \vee \neg\phi(d) \vee \neg\phi(e) \vee \neg\phi(f) \vee \neg\phi(g) \vee \neg\phi(h))\,.$$

Then, $p(s)$ is satisfiable if and only if $\Delta(a) \neq c_1$, $\Delta(b) \neq c_1$, $\Delta(c) \neq c_1$, $\Delta(d) \neq c_1$, $\Delta(e) \neq c_1$, $\Delta(f) \neq c_1$, or $\Delta(g) \neq c_1$ or $\Delta(h) \neq c_1$, i.e., Δ does not induce in s a monochromatic solution on c_1 of the equation $x_1 + \cdots + x_7 = x_8$. Analogously, $q(s)$ satisfiable if and only if Δ does not induce in s a monochromatic solution of the equation $x_1 + \cdots + x_7 = x_8$ on c_2.

$$\text{Let } \mathcal{C} = \bigwedge_{s \in \mathcal{S}} (p(s) \wedge q(s)).$$

Clearly $\mathcal{C}$ is satisfiable if and only if Δ does not induce on D_2 a monochromatic solution of the equation $x_1 + \cdots + x_7 = x_8$. The SAT-Solver shows that $\mathcal{C}$ is not satisfiable, hence we have the result. □

Now, we can prove:

Theorem 4.6 $WS_7(2) = 253$.

Proof Let Δ be a 2-coloring of $[1, 253]$ without monochromatic solution. Then $\Delta(217) = c_1$ and $\Delta([1, 8]) = \{c_2\}$. Therefore, $\sum_{i=1}^{8} i = 36 - n$ with $i \neq n$, which implies $\Delta([28, 34]) = \{c_1\}$ and $217 = 28 + 29 + 30 + 31 + 32 + 33 + 34$. Therefore $\Delta(217) \neq c_1$, contradicting Lemma 4.5. □

4.3 The exact value of $WS_3(3)$

The weak Schur number $WS_3(3)$ is the least positive integer N such that for every 3-coloring of $[1, N]$,

$$\Delta : [1, N] \longrightarrow \{c_1, c_2, c_3\},$$

where c_1, c_2, c_3 represent 3 different colors, there exists a monochromatic solution to the equation $x_1 + x_2 + x_3 = x_4$, such that $\Delta(x_1) = \cdots = \Delta(x_3) = \Delta(x_4)$ where $x_i \neq x_j$ when $i \neq j$.

We shall prove that $WS_3(3) = 94$. Let us first show a lower bound.

Lemma 4.7 $WS_3(3) \geq 94$.

Proof It is easy to verify that the 3-coloring

$$\Delta : [1, 93] \longrightarrow \{c_1, c_2, c_3\}$$

defined by

$$\Delta(x) = \begin{cases} c_1 & \text{if } x \in [1, 5] \cup [21, 23] \cup [75, 77] \cup [91, 93] \\ c_2 & \text{if } x \in [6, 20] \cup [78, 90] \\ c_3 & \text{if } x \in [24, 74] \end{cases}$$

has no monochromatic solution to the equation $x_1 + x_2 + x_3 = x_4$ such that $x_i \neq x_j$ when $i \neq j$. □

To prove that the equation $x_1 + x_2 + x_3 = x_4$ has a monochromatic solution for every 3-coloring of the integer interval $[1, 94]$, it is necessary to prove the following result.

Lemma 4.8 *The set* $\mathcal{Y} = \{y_n\}_{n=1}^{51} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 52, 58, 64, 65, 66, 72, 75, 78, 82, 91, 94\}$ *satisfies:*

1. *We have* $\mathcal{Y} \subseteq [1, 94]$.
2. *For every partition of* $\mathcal{Y}$ *into three subsets* A_1, A_2, A_3*, some* A_i *contains a monochromatic solution of* $x_1 + x_2 + x_3 = x_4$*,* $x_i \neq x_j$*, with* $i \neq j$.

Proof 1. This is trivial.

2. We have checked the result transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver [12].

Let Δ be a 3-coloring of $[1, 94]$:

$$\Delta : [1, 94] \longrightarrow \{c_1, c_2, c_3\},$$

For any $\{y_n\} \in \mathcal{Y}$, we consider two Boolean variables ϕ and ψ defined on $[1, 51]$ as follows:

$$\phi(n) = \begin{cases} True \ \ if \ \Delta(y_n) = c_1 \text{ or } c_2, \\ False \ if \ \Delta(y_n) = c_3. \end{cases}$$

$$\psi(n) = \begin{cases} True \ \ if \ \Delta(y_n) = c_1 \text{ or } c_3, \\ False \ if \ \Delta(y_n) = c_2. \end{cases}$$

Thus, for any $n \in [1, 51]$ we have that $\phi(n)$ is True or $\psi(n)$ is True. Therefore, $\mathcal{D} = \bigwedge_{1 \leq n \leq 51} (\phi(n) \vee \psi(n))$ is satisfiable.

Let $\mathcal{S} = \{(y_a, y_b, y_c, y_d) \in \mathcal{Y}^4 \mid y_a + y_b + y_c = y_d, \text{with } a < b < c\}$.

For any $s = (y_a, y_b, y_c, y_d) \in \mathcal{S}$, we consider three clauses:

$$p(s) = (\neg\phi(a) \vee \neg\psi(a) \vee \neg\phi(b) \vee \neg\psi(b) \vee \neg\phi(c) \vee \neg\psi(c) \vee \neg\phi(d) \vee \neg\psi(d))\,;$$
$$q(s) = (\neg\phi(a) \vee \psi(a) \vee \neg\phi(b) \vee \psi(b) \vee \neg\phi(c) \vee \psi(c) \vee \neg\phi(d) \vee \psi(d))\,;$$
$$r(s) = (\phi(a) \vee \phi(b) \vee \phi(c) \vee \phi(d))\,.$$

Then, $p(s)$ is satisfiable if and only if $\Delta(a) \neq c_1$, $\Delta(b) \neq c_1$, $\Delta(c) \neq c_1$ or $\Delta(d) \neq c_1$, i.e., Δ does not induce in s a monochromatic solution on c_1 of the equation $x_1 + x_2 + x_3 = x_4$. Analogously, $q(s)$ or $r(s)$ is satisfiable if and only if Δ does not induce in s a monochromatic solution of the equation $x_1 + x_2 + x_3 = x_4$ on c_2 or c_3, respectively.

$$\text{Let } \mathcal{C} = \bigwedge_{s \in \mathcal{S}} (p(s) \wedge q(s) \wedge r(s)).$$

Clearly $\mathcal{D} \wedge \mathcal{C}$ is satisfiable if and only if Δ does not induce on $\mathcal{Y}$ a monochromatic solution of the equation $x_1 + x_2 + x_3 = x_4$. The SAT-Solver shows that $\mathcal{D} \wedge \mathcal{C}$ is not satisfiable, hence $WS_3(3) \leq 94$.

With this result, we have tested the upper bound on $WS_3(3)$.

Therefore, we conclude with the following result:

Theorem 4.9 $WS_3(3) = 94$.

4.4 The exact value of $WS_4(3)$

The weak Schur number $WS_4(3)$ is the least positive integer N such that for every 3-coloring of $[1, N]$,

$$\Delta : [1, N] \longrightarrow \{c_1, c_2, c_3\},$$

where c_1, c_2, c_3 represent 3 different colors, there exists a monochromatic solution to the equation $x_1 + x_2 + \cdots + x_4 = x_5$, such that $\Delta(x_1) = \cdots = \Delta(x_4) = \Delta(x_5)$ where $x_i \neq x_j$ when $i \neq j$.

We shall prove that $WS_4(3) = 259$. Let us first show a lower bound.

Lemma 4.10 $WS_4(3) \geq 259$.

Proof It is easy to verify that the 3-coloring

$$\Delta : [1, 258] \longrightarrow \{c_1, c_2, c_3\}$$

defined by $\Delta(x) = \begin{cases} c_1 & \text{if } x \in [1, 9] \cup [46, 51] \cup [214, 219] \cup [253, 258] \\ c_2 & \text{if } x \in [10, 45] \cup [220, 252] \\ c_3 & \text{if } x \in [52, 213] \end{cases}$
has no monochromatic solution to the equation $x_1 + x_2 + x_3 + x_4 = x_5$ such that $x_i \neq x_j$ when $i \neq j$. □

To prove that the equation $x_1 + x_2 + x_3 + x_4 = x_5$ has a monochromatic solution for every 3-coloring of the integer interval $[1, 259]$, it is necessary to prove the following result.

Lemma 4.11 *The set* $\mathcal{Z} = \{z_n\}_{n=1}^{86} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 30, 31, 34, 42, 43, 44, 46, 47, 49, 52, 53, 54, 55, 56, 57, 58, 59, 61, 64, 65, 66, 67, 68, 70, 73, 74, 76, 78, 79, 85, 86, 88, 91, 99, 103, 106, 109, 115, 118, 124, 130, 139, 169, 190, 199, 202, 208, 211, 214, 217, 220, 223, 226, 229, 235, 238, 250, 253, 259\}$ *satisfies:*

1. *We have* $\mathcal{Z} \subseteq [1, 259]$.
2. *For every partition of* $\mathcal{Z}$ *into three subsets* A_1, A_2, A_3, *some* A_i *contains a monochromatic solution of* $x_1 + x_2 + x_3 + x_4 = x_5$, $x_i \neq x_j$, *with* $i \neq j$.

Proof 1. This is trivial.
2. We have checked the result transforming the problem into a Boolean satisfiability problem and solving it with a SAT solver [12].

Let Δ be a 3-coloring of $[1, 259]$:

$$\Delta : [1, 259] \longrightarrow \{c_1, c_2, c_3\},$$

For any $\{z_n\} \in \mathcal{Z}$, we consider two Boolean variables ϕ and ψ defined on $[1, 86]$ as follows:

$$\phi(n) = \begin{cases} True \ \ if \ \Delta(z_n) = c_1 \text{ or } c_2, \\ False \ if \ \Delta(z_n) = c_3. \end{cases}$$

$$\psi(n) = \begin{cases} True \ \ if \ \Delta(z_n) = c_1 \text{ or } c_3, \\ False \ if \ \Delta(z_n) = c_2. \end{cases}$$

Thus, for any $n \in [1, 86]$,we have that $\phi(n)$ is True or $\psi(n)$ is True. Therefore, $\mathcal{D} = \bigwedge_{1 \le n \le 86}(\phi(n) \vee \psi(n))$ is satisfiable.
Let $\mathcal{S} = \{(z_a, z_b, z_c, z_d, z_e) \in \mathcal{Z}^5 \mid z_a + z_b + z_c + z_d = z_e, \text{with } a < b < c < d\}$.
For any $s = (z_a, z_b, z_c, z_d, z_e) \in \mathcal{S}$, we consider three clauses:

$$\begin{aligned} p(s) = {} & (\neg\phi(a) \vee \neg\psi(a) \vee \neg\phi(b) \vee \neg\psi(b) \vee \neg\phi(c) \\ & \vee\neg\psi(c) \vee \neg\phi(d) \vee \neg\psi(d) \vee \neg\phi(e) \vee \neg\psi(e)) ; \\ q(s) = {} & (\neg\phi(a) \vee \psi(a) \vee \neg\phi(b) \vee \psi(b) \vee \neg\phi(c) \\ & \vee\psi(c) \vee \neg\phi(d) \vee \psi(d) \vee \neg\phi(e) \vee \psi(e)) ; \\ r(s) = {} & (\phi(a) \vee \phi(b) \vee \phi(c) \vee \phi(d) \vee \phi(e)) . \end{aligned}$$

Then, $p(s)$ is satisfiable if and only if $\Delta(a) \neq c_1$, $\Delta(b) \neq c_1$, $\Delta(c) \neq c_1$, $\Delta(d) \neq c_1$ or $\Delta(e) \neq c_1$, i.e., Δ does not induce in s a monochromatic solution on c_1 of the equation $x_1 + x_2 + x_3 + x_4 = x_5$. Analogously, $q(s)$ or $r(s)$ is satisfiable if and only if Δ does not induce in s a monochromatic solution of the equation $x_1 + x_2 + x_3 + x_4 = x_5$ on c_2 or c_3, respectively.

$$\text{Let } \mathcal{C} = \bigwedge_{s \in \mathcal{S}} (p(s) \wedge q(s) \wedge r(s)).$$

Clearly $\mathcal{D} \wedge \mathcal{C}$ is satisfiable if and only if Δ does not induce on $\mathcal{Z}$ a monochromatic solution of the equation $x_1 + x_2 + x_3 + x_4 = x_5$. The SAT-Solver shows that $\mathcal{D} \wedge \mathcal{C}$ is not satisfiable, hence $WS_4(3) \le 259$. □

With this result, we have verified the upper bound for $WS_4(3)$.
Therefore, we obtain the following result:

Theorem 4.12 $WS_4(3) = 259$.

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