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Characterizing the supernorm partition statistic

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In memory of Ronald L. Graham

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Abstract

The paper gives characterizations of the supernorm statistic $\widehat{N}: \mathcal{P} \to \mathbb{N}^+$ for partitions, where $\mathcal{P}$ is the set of integer partitions and $\mathbb{N}^+$ is the positive integers. The supernorm statistic map is a bijection onto $\mathbb{N}^+$ that defines a total order on partitions, the supernorm ordering, obtained by pulling back the additive total order on $\mathbb{N}^+$. The supernorm ordering refines two partial orders on partitions: the multiset inclusion order, whose image under $\widehat{N}$ is the divisibility lattice on $\mathbb{N}^+$, and the Young's lattice order. The paper shows that it is characterized by these two properties, additionally with the requirement of an order-isomorphism from the multiset inclusion order to the divisibility order. It is also characterized by these two properties, additionally with the requirement of mapping the partitions with exactly one part bijectively to the prime numbers. It presents a construction showing that the latter additional conditions are necessary for the characterization to hold.

Keywords Integer partitions · Partial orders · Young's lattice

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1 Introduction

Let $\mathcal{P}$ denote the set of all integer partitions λ. We denote a partition $\lambda := (\lambda_1, \lambda_2, \cdots \lambda_r)$ using part notation, requiring $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, having r nonzero parts. Alternatively, we denote a partition λ using part-multiplicity notation, $\lambda = [1^{m_1} 2^{m_2} \cdots j^{m_j} \cdots]$, where $m_j = m_j(\lambda) \geq 0$, is the *multiplicity* of j as a part of λ, with finitely many $m_j(\lambda)$ nonzero. We write $|\lambda|$ to indicate the *size* of λ, which

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is the sum of its parts, so that $|\lambda| := \sum_i \lambda_i = \sum_j jm_j$. We let $\mathrm{Par}(n)$ denote the set of all partitions of size n, so that $\mathcal{P} = \bigcup_{n\geq 0} \mathrm{Par}(n)$, with the empty partition $\lambda = \emptyset$ having $|\lambda| = 0$. The *length* $\ell(\lambda) := r = \sum_j m_j(\lambda)$ of a partition λ is the number of its parts. We let $\mathcal{P}_r$ denote the set of all partitions of length r, so that $\mathcal{P} = \bigcup_{r\geq 0} \mathcal{P}_r$, with $\mathcal{P}_0$ containing the empty partition.

Dawsey, Just and Schneider [2] recently introduced a new partition statistic, the *supernorm* $\widehat{N}(\lambda)$ of a partition $\lambda \in \mathcal{P}$. It is defined by $\widehat{N}(\emptyset) = 1$ and, for nonempty partitions,

$$\widehat{N}(\lambda) = \prod_{1\leq i\leq \ell(\lambda)} p_{\lambda_i},$$

where p_k denotes the k th prime in increasing order, for $k \geq 1$, and we set $p_0 = 1$. The paper [2] proves that the supernorm map $\widehat{N} : \mathcal{P} \to \mathbb{N}^+$ is a bijection onto $\mathbb{N}^+$. The pullback from $\mathbb{N}^+$ to $\mathcal{P}$ of the additive total ordering on $\mathbb{N}^+$ under this bijection $\widehat{N}$ then defines a total ordering $\leq_S$ on partitions $\mathcal{P}$, which we term the *supernorm ordering*.

The supernorm ordering refines two natural partial orders on the set of integer partitions $\mathcal{P}$.

(1) The *multiset inclusion order*, denoted $\subseteq_M$, is the ordering on partitions given by inclusion of parts (counted with multiplicity). That is, $\lambda \subseteq_M \mu$ if there is a matching of all the parts of λ with a subset of the parts of μ, i.e., the multiset of parts of λ is a sub-multiset of the multiset of parts of μ, e.g., $\lambda = (3, 2, 1) \subseteq_M \mu = (5, 3, 3, 2, 2, 1)$. This partial order was denoted $\leq$ by George Andrews in [1, Chap. 8].

(2) The *Young's lattice order*, denoted $\subseteq_Y$, is the ordering having $\lambda \subseteq_Y \mu$, whenever the Young diagram for λ fits inside the Young diagram for μ, see [12, Sect. 1.7]. That is, $\lambda \subseteq_Y \mu$ holds when $\lambda_j \leq \mu_j$ for $1 \leq j \leq \ell(\lambda)$ (parts enumerated in decreasing order), e.g., $\lambda = (3, 2, 1) \subseteq_Y \mu = (4, 3, 1, 1)$. The Young's lattice ordering is denoted $\subset$ in Stanley [11, Chap. 7], see also [12, Example 3.4.4].

This object of this paper is to characterize the supernorm ordering in terms of compatibility conditions with these orderings. We show that the supernorm map $\hat{N}(\cdot)$ is the unique bijective map $M : \mathcal{P} \to \mathbb{N}^+$ which has suitable order-preserving conditions for these two natural partial orderings on $\mathcal{P}$ with respect to two partial orderings on $\mathbb{N}^+$, the multiplicative (divisibility) partial order and the additive total order, respectively. We give two different characterizations and also supply a complementary result on the necessity of an extra hypothesis in the second characterization.

We note that the supernorm ordering is not compatible with some other natural orderings on partitions. It does not refine the *reverse lexicographic ordering* on partitions $\mathrm{Par}(n)$ of size n (a total ordering), nor does it refine the *majorization partial ordering* on partitions of size $\mathrm{Par}(n)$, as defined in Stanley [11, Vol. II, Chap. 7]. Counterexamples exist starting at $n = 6$.

2 Main results

This section states three results, two characterization theorems and a complementary theorem; proofs follow in sections 3 to 5.

The first result characterizes the supernorm map by a strong compatibility condition on the multiset inclusion order and a weaker one on the Young's lattice order. We say that a map $M\colon (\mathcal{A}, \preccurlyeq_1) \to (\mathcal{B}, \preccurlyeq_2)$ between two partially ordered sets $\mathcal{A}$ and $\mathcal{B}$ is *order-preserving* if $a \preccurlyeq_1 a_2 \Rightarrow M(a_1) \preccurlyeq_2 M(a_2)$. We say that it is an *order-isomorphism* if it is bijective and if $a \preccurlyeq_1 a_2 \Leftrightarrow M(a_1) \preccurlyeq_2 M(a_2)$.

Theorem 2.1 (Strong compatibility characterization of supernorm map) *The supernorm map $\widehat{N}$ is the unique bijection $M\colon \mathcal{P} \to \mathbb{N}^+$ that has the following two properties:*

(i) *The mapping M gives an order-isomorphism of the multiset inclusion ordering on partitions, written $\lambda_1 \leq_M \lambda_2$, onto the divisibility partial order relation on $\mathbb{N}^+$. That is,*

$$\lambda \subseteq_M \mu \quad \Longleftrightarrow \quad M(\lambda)\,|M\,(\mu).$$

(ii) *The mapping M gives an order-preserving map of the Young's lattice partial ordering, written $\lambda \subseteq_Y \mu$ into the additive ordering on $\mathbb{N}^+$. That is,*

$$\lambda \subseteq_Y \mu \quad \Longrightarrow \quad M(\lambda) \leq M(\mu).$$

The multiset inclusion order is refined by the Young's lattice order, i.e., $\lambda \subseteq_M \mu$ implies $\lambda \subseteq_Y \mu$. Thus Theorem 2.1 implies the well-known fact that the divisor order is refined by the additive total order, i.e., $m \mid n$ implies $m \leq n$ for $m, n \in \mathbb{N}^+$.

There are many maps $M : \mathcal{P} \to \mathbb{N}^+$ that have properties (i) and (ii) if one weakens the bijective hypothesis on $M(\cdot)$ to require only surjectivity. The "norm" statistic $N(\lambda)$, which is the product of the parts of a partition, satisfies both conditions (i) and (ii). The "norm" statistic was named and studied in [10]; it appears earlier without being given a name, e.g., Lehmer [9]. The "norm" map is surjective but not bijective to $\mathbb{N}^+$. More generally, if we require one-part partitions $\mathcal{P}_1$ to map on to a set of integers S in increasing order that includes the set $\mathbb{P}$ of all prime numbers, and we extend its definition to all partitions multiplicatively, then we will get a map $M_S(\cdot)$ that satisfies properties (i) and (ii) and is surjective on $\mathbb{N}^+$.

The second result gives a characterization of the supernorm map in terms of a weakening of condition (i) to require the map M to be an order-preserving map of the multiset inclusion ordering $\subseteq_M$ into the divisibility order; we call this condition (i'). The characterization then requires an extra condition (iii) specifying the range $M(\mathcal{P}_1)$ of the set of one-part partitions under the bijective map M.

Theorem 2.2 (Weak compatibility characterization of supernorm map) *The supernorm map $\widehat{N}$ is the unique bijection $M\colon \mathcal{P} \to \mathbb{N}^+$ that has the following three properties:*

(i') *The mapping M gives an order-preserving map of the ordering on partitions given by inclusion of multisets of parts, written $\lambda \subseteq_M \mu$, into the divisibility partial order relation on $\mathbb{N}^+$. That is,*

$$\lambda \subseteq_M \mu \quad \Longrightarrow \quad M(\lambda)\,|M\,(\mu).$$

(ii) *The mapping M gives an order-preserving map of the Young's lattice partial ordering, written $\lambda \subset_Y \mu$ to the additive ordering on $\mathbb{N}^+$. That is, $\lambda \subseteq \mu$ implies*

$$\lambda \subseteq_Y \mu \implies M(\lambda) \le M(\mu).$$

(iii) *The image $M(\mathcal{P}_1)$ of the set of partitions $\mathcal{P}_1$ having exactly one part is the set $\mathbb{P}$ of all prime numbers.*

The third result shows the condition (iii) is necessary in Theorem 2.2 to obtain a characterization of the supernorm map. A set $S \subset \mathbb{N}^+$ is *admissible* it there is a bijective map $M : \mathcal{P} \to \mathbb{N}^+$ satisfying properties (i') and (ii) above, having the image of one-part partitions $M(\mathcal{P}_1) = S$. The following result shows that the hypothesis $S = \mathbb{P}$ imposed in Theorem 2.2 may be understood as a minimality condition on admissible S.

Theorem 2.3 (Uncountability of admissible sets)

(1) *There are uncountably many admissible sets $S \subset \mathbb{N}^+$, which are sets $S \subset \mathbb{N}^+$ such that there is a bijective map $M : \mathcal{P} \to \mathbb{N}^+$ having $S = M(\mathcal{P}_1)$, and that has the following two properties:*

  (i') *The mapping M gives an order-preserving map of the ordering on partitions given by inclusion of multisets of parts, written $\lambda \subseteq_M \mu$ into the divisibility partial order relation on $\mathbb{N}^+$. That is,*

$$\lambda \subseteq_M \mu \implies M(\lambda) \,|\, M(\mu).$$

  (ii) *The mapping M gives an order-preserving map of the Young's lattice partial ordering, written $\lambda \subseteq_Y \mu$, into the additive ordering on $\mathbb{N}^+$. That is, $\lambda \subseteq_Y \lambda_2$ implies*

$$\lambda \subseteq_Y \mu \implies M(\lambda) \le M(\mu).$$

(2) *Any admissible set S necessarily contains the set $\mathbb{P}$ of all prime numbers.*

Theorem 2.3 is proved by an inductive set-theoretic construction which is effective at each step. The allowed admissible sets S are not arbitrary; the composite numbers appearing in S have to satisfy the local constraints of a combinatorial number-theoretic nature. In Sect. 6, we exhibit some local constraints of this kind.

3 Strong characterization Theorem 2.1

Theorem 2.1 characterizes the supernorm statistic in terms of compatibility with two partial orders on the set of all partitions $\mathcal{P}$ with two orders on $\mathbb{N}^+$, one partial and one total. The order compatibility condition (i) is "multiplicative" and (ii) is "additive."

In the following proof, the "direct direction" of property (i) of refers to the direction $\Rightarrow$, and the "converse direction" of (i) refers to $\Leftarrow$. Property (i') in Theorem 2.2 is the "direct direction" $\Rightarrow$ of property (i).

Proof of Theorem 2.1 We must show existence and uniqueness.

Existence. By definition, $\widehat{N}(\cdot)$ has $\widehat{N}(\emptyset) = 1$. For one-part partitions $\lambda = [m]$, the supernorm has $\widehat{N}([m]) = p_m$, the mth prime in increasing order, with $p_1 = 2$. For a general partition $\lambda = [1^{e_1} 2^{e_2} \cdots]$ of size $n = \sum_{j \geq 1} j e_j$, it is given by

$$\widehat{N}(\lambda) = \prod_{j \geq 1} (p_j)^{e_j}.$$

The map $\widehat{N}(\cdot)$ is bijective to its range $\mathbb{N}^+$by the fundamental theorem of arithmetic that asserts unique prime factorization of integers $n \geq 2$, and all prime factorizations occurring.

To show property (i) holds, take $\lambda = [1^{e_1} 2^{e_2} \cdots]$ and $\mu = [1^{f_1} 2^{f_2} \cdots]$ (allowing multiplicities $e_j \geq 0$, $f_j \geq 0$). One has $\lambda \leq_M \mu$ if and only if $e_j \leq f_j$ for all $j \geq 1$, which holds if and only if

$$\widehat{N}(\lambda) = \prod_{j} (p_j)^{e_j} \mid \prod_{j} (p_j)^{f_j} = \widehat{N}(\mu).$$

Thus $\widehat{N}(\cdot)$ satisfies the property (i).

To show property (ii) holds, we write $\lambda = (\lambda_1, \lambda_2, ...)$ with parts given in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots$ and setting $\lambda_j = 0$ for $j > r$, the number of parts in λ. We write $\mu = (\mu_1, \mu_2, \cdots)$ similarly. Then $\lambda \subset_Y \mu$ if and only if $\lambda_j \leq \mu_j$ for all $j \geq 1$. Now, using the convention that $p_0 = 1$, we have, since $j \leq k$ implies $p_j \leq p_k$,

$$\widehat{N}(\lambda) = \prod_{j \geq 1} p_{\lambda_j} \leq \prod_{j \geq 1} p_{\mu_j} = \widehat{N}(\mu).$$

where we take the products over all j up to the maximum of the number of nonzero parts of λ and μ. Thus $\widehat{N}(\cdot)$ satisfies the property (ii).

Uniqueness. We show $M(\emptyset) = 1$. Since $\emptyset \subseteq_M \lambda$ for all partitions λ, by property (i) $M(\emptyset) | M(\lambda)$. Since the range of $M(\lambda)$ is $\mathbb{N}^+$, $M(\emptyset)$ must be an integer that divides all positive integers, necessarily $M(\emptyset) = 1$.

Since the map $M(\cdot)$ is injective, it follows that $M(\lambda) \geq 2$ for all nonempty partitions λ.

To state the following claim, let $\mathcal{P}_\ell$ denote the set of all partitions having exactly ℓ parts. Let $\Omega(m)$ count the number of prime factors of m, with multiplicity. Set $Q_\ell = \{m : \quad \Omega(m) = \ell\}$. The proof of this claim uses only the direct direction (i') of property (i).

Claim 1 *For each $\ell \geq 1$,*

$$\bigcup_{j=1}^{\ell} Q_j \subseteq \bigcup_{j=1}^{\ell} M(\mathcal{P}_j). \tag{3.1}$$

To prove Claim 1, start with $\ell = 1$. The claim says that all primes are images of partitions with one part. If a prime $p = M(\lambda)$ where $\lambda = (\lambda_1, \lambda_2, ...)$ had two or more parts, then since $M(\lambda_1) \geq 2$, and $M(\lambda_1) \mid M((\lambda_1, \lambda_2))$ by injectivity (i'), and since $M((\lambda_1, \lambda_2))$ is an integer distinct from $M((\lambda_1))$,it must have at least two prime factors. But now $M((\lambda_1, \lambda_2)) \mid M(\lambda)$, hence $M(\lambda)$ is not a prime, a contradiction.

The claim asserts that all integers having at most ℓ prime factors, counted with multiplicity, must appear as images under $M(\cdot)$ of partitions with at most ℓ parts. We show the contrapositive: no partition with ℓ parts is an image under $M(\cdot)$ of a partition with $\ell+1$ or more parts. For any partition ℓ having $\ell+1$ or more parts, using injectivity (i'), we may form a chain of divisibilities

$$M(\lambda_1) \mid M((\lambda_1, \lambda_2)) \mid \cdots \mid M((\lambda_1, ..., \lambda_\ell)) \mid M((\lambda_1, .., \lambda_{\ell+1})).$$

All these integers are distinct, by injectivity (i') of $M(\cdot)$, hence each one includes an additional prime factor (proceeding left to right), and therefore, the last one has at least $\ell + 1$ prime factors (counted with multiplicity). Using only the direct direction (i') of property (i), the last term $M((\lambda_1, ..., \lambda_{\ell+1}))$ divides $M(\lambda)$, which therefore cannot have ℓ or fewer prime factors, a contradiction. This proves Claim 1.

Claim 2 *For each* $\ell \geq 1$,

$$Q_\ell = M(\mathcal{P}_\ell). \tag{3.2}$$

To prove Claim 2, we proceed by induction on ℓ. We start with the base case $\ell = 1$ and must show $Q_1 = M(\mathcal{P}_1)$. Suppose for a contradiction that there is a composite number n with prime factorization $n = p_1 p_2 \cdots p_k$ with $k \geq 2$ such that $n = M(\lambda)$ for some one-part partition $\lambda = [m]$. We have $Q_1 \subseteq M(\mathcal{P}_1)$ by Claim 1, so each $p_k = M(\lambda_k)$ for some one-part partition $\lambda_k := [m_k]$, and necessarily each $m_k \neq m$. By the converse direction of property (i), since $p_1 \mid n$, the partition λ must contain the part $\lambda_1 = m_1$. But it also contains the part $m \neq m_1$ by hypothesis, so it contains at least two parts, contradiction.

The induction hypothesis will be that the claim $Q_j = M(\mathcal{P}_j)$ holds for all $1 \leq j \leq \ell$ and we are to prove $Q_{\ell+1} = M(\mathcal{P}_{\ell+1})$. Claim 1 asserts $\bigcup_{j=1}^{\ell+1} Q_j \subseteq \bigcup_{j=1}^{\ell+1} M(\mathcal{P}_j)$. The induction hypothesis yields

$$\bigcup_{j=1}^{\ell} Q_j = \bigcup_{j=1}^{\ell} M(\mathcal{P}_j).$$

Since $M(\cdot)$ is a bijection, and $Q_{\ell+1}$ is disjoint from $\bigcup_{j=1}^{\ell} Q_j$, we conclude

$$Q_{\ell+1} \subseteq M(\mathcal{P}_{\ell+1}). \tag{3.3}$$

To show that $Q_{\ell+1} = M(\mathcal{P}_{\ell+1})$, suppose for a contradiction that some $\lambda \in \mathcal{P}_{\ell+1}$ has $M(\lambda) = n$ with n a composite number with $r \geq \ell + 2$ prime factors $n = p_1 p_2 \cdots p_r$. Then n will be strictly divisible by some n' having exactly $\ell + 1$ prime factors (with multiplicity), and by (3.3),there exists $\lambda' \in \mathcal{P}_{\ell+1}$ with $M(\lambda') = n'$, with $n' \neq n$. Since $n' \mid n$, by the converse direction of property (i), $\lambda' \subseteq_M \lambda$ in the part inclusion order. But $n' \neq n$ so by the bijective property $\lambda' \neq \lambda$. It follows that λ has at least one more part than λ', so it is not in $\mathcal{P}_{\ell+1}$, contradiction. This completes the induction step and proves Claim 2.

We have not yet used the Young's lattice partial order property (ii). This property suffices to order the elements to establish that $M(\lambda) = \widehat{N}(\lambda)$ for all λ.

Claim 3 *For all partitions λ,*

$$M(\lambda) = \widehat{N}(\lambda) := \prod_{\lambda_j > 0} p_{\lambda_j}, \tag{3.4}$$

where p_n denotes the n th prime in increasing order, setting $p_1 = 2$.

To prove Claim 3, we have $M(\emptyset) = N(\emptyset) = 1$. For nonempty partitions λ, we have by definition

$$\widehat{N}(\lambda) = \prod_{\lambda_j > 0} p_{\lambda_j}.$$

We prove (3.4) by induction on the number ℓ of nonzero parts of the partition.

The base case is $\ell = 1$. By Claim 2,we have $Q_1 = M(\mathcal{P}_1)$. Now $Q_1 = \mathbb{P}$ is the set of prime numbers, and the Young's lattice ordering property (ii) requires them to be listed in increasing order, e.g., for one-part partitions $M((k)) \leq M((k+1))$, whence $M([k]) < M([k+1])$ by distinctness. Hence for $k \geq 1$, we must have (by a sub-induction on $k \geq 1$) that $M([k]) = p_k$, where p_k is the kth prime number. This completes the base case.

Before proceeding to the general induction step, we first show directly that for a rectangular partition

$$\lambda = [k^m] = (k, k, \cdots, k) \quad (m \text{ times})$$

consisting of m copies of a single part $[k]$, that

$$M([k^m]) = \widehat{N}([k^m]) = (p_k)^m.$$

We first note that if a partition λ contains a part $k' \neq k$, then by the direct direction of property (i), $p_{k'} = M([k']) \mid M(\lambda)$ and therefore $M(\lambda)$ cannot be a power of p_k. It follows that the set $M(\lambda)$ whose images are powers $(p_k)^m$ of p_k must consist only of partitions of the form $\lambda = [k^j]$ for various j. However,the map $M(\cdot)$ is surjective and by Claim 2,we know $Q_\ell = M(\mathcal{P}_\ell)$, so the only possibility to have $M([k^\ell]) = (p_k)^m$ is to have $\ell = m$, because $[m]^\ell \in \mathcal{P}_\ell$ while the right side is in Q_m. In this way, we must have $M([k^m]) = (p_k)^m$ for all $m \geq 1$.

For the induction step, we suppose (3.4) has been proved for all λ having at most ℓ parts, and we must prove it for each $\lambda \in \mathcal{P}_{\ell+1}$. In the case that $\lambda = [k^{\ell+1}]$ has all $\ell+1$ parts equal, we have shown above that $M([k^{\ell+1}]) = \widehat{N}([k^{\ell+1}])) = (p_k)^{\ell+1}$. We are reduced to the case that λ has parts of multiplicity at most ℓ, i.e., m_j parts of size k_j with each $m_j \leq \ell$, for $1 \leq j \leq r$ (the number of distinct parts), and $\sum_j m_j = \ell + 1$.

By the induction hypothesis, using only the direct direction (i') of property (i), we have

$$M([k_j^{m_j}]) = \widehat{N}([k_j^{m_j}]) = (p_{k_j})^{m_j} \mid M(\lambda).$$

These factors $M([k_j^{m_j}])$ of $M(\lambda)$ are pairwise relatively prime for different j, so we conclude (by the Chinese remainder theorem) the divisibility relation

$$\prod_{j=1}^{r} (p_{k_j})^{m_j} \mid M(\lambda).$$

The left side is exactly $\widehat{N}(\lambda)$ and the multiplicity of its prime divisors is $\sum_{j=1}^{r} m_j = \ell + 1$. Since $\lambda \in \mathcal{P}_{\ell+1}$, by Claim 2,we have $M(\lambda) \in Q_{\ell+1}$, i.e., it also has exactly $\ell + 1$ prime factors (with multiplicity) in its factorization into primes. Therefore, $M(\lambda) = \widehat{N}(\lambda)$. This completes the induction step and proves Claim 3, which gives the theorem. ☐

4 Weak characterization Theorem 2.2

Proof of Theorem 2.2 Existence follows by Theorem 2.1, so it remains to show uniqueness.

Recall that $\mathcal{P}_\ell$ is the collection of all partitions having exactly ℓ parts, and that Q_ℓ is the set of all integers with exactly ℓ prime divisors, counted with multiplicity. We prove by induction on $\ell \geq 1$ that $Q_\ell = M(\mathcal{P}_\ell)$, and at the same time that $M(\cdot)$ agrees with the supernorm map $\widehat{N}(\cdot)$ on $\mathcal{P}_\ell$.

For the base case $\ell = 1$, by hypothesis (iii),we have $Q_1 = M(\mathcal{P}_1)$. The Young's lattice ordering property (ii) then gives $M([k]) = p_k$, the kth prime, for all $k \geq 1$. This proves the base case.

For the induction step, we suppose $\ell \geq 2$ and that the induction hypothesis holds for all $1 \leq j \leq \ell - 1$. We know by the proof of Claim 1 of Theorem 2.1 that $\bigcup_{j=1}^{\ell} \subset \bigcup_{j=1}^{\ell} Q_\ell \subseteq M(\mathcal{P}_\ell)$ (whose proof used only property (i')). By the induction hypothesis, we have $Q_j = M(\mathcal{P}_j)$ for $1 \leq j \leq \ell - 1$, so since the map $M(\cdot)$ is a bijection, we deduce the inclusion

$$Q_\ell \subseteq M(\mathcal{P}_\ell).$$

We first observe that the only place that powers of a prime p_k can appear as $M(\lambda)$, using property (i'), is as images of some rectangular partitions $M([k^\ell])$, for some $\ell \geq 1$. To see this, suppose λ contains some part j other than k, then using the base case $M([j]) = p_j$ for $j \neq k$, property (i') gives divisibility of $M(\lambda)$ by p_j, a contradiction. The Young's lattice partial order has $[k^\ell] \subseteq_Y [k^{\ell+1}]$, which by property (ii) now forces the assignment $M([k^\ell]) = (p_k)^\ell$ for $\ell \geq 1$, by induction on $\ell \geq 1$. Therefore, the values of $M(\cdot)$ must agree with the supernorm map values $\widehat{N}(\cdot)$ on all rectangular partitions $\lambda = [k^\ell]$.

The aim of the remainder of the induction step will show that property (i') implies that $M(\mathcal{P}_\ell)$ is sufficient to exhaust all of Q_ℓ, so there is no room to insert at level ℓ any element mapping to an integer with $\Omega(m) \geq \ell + 1$, where $\Omega(m)$ counts the number of primes dividing m, with multiplicity. In the process,we will also show that all elements of $\mathcal{P}_\ell$ must fall in position to agree with $\widehat{N}$ on $\mathcal{P}_\ell$, which will complete the induction step for $\mathcal{P}_{\ell+1}$.

We have shown above that $\widehat{N}$ agrees with M on $\mathcal{P}_\ell$ on rectangular partitions, and it remains to treat nonrectangular partitions. By property (i'), if $\lambda = [1^{m_1}2^{m_2}\cdots]$ having $\ell = \sum_i m_i$ parts, with λ *not* a rectangular partition $[k^\ell]$, then we assert that $\prod_j (p_j)^{m_j} \mid M(\lambda)$. To see this, for each fixed j with $m_j \geq 1$, we can find a subpartition λ' having $\ell - 1$ parts that contain j^{m_j}, which by the induction hypothesis has $(p_j)^{m_j} \mid M(\lambda')$. Since by property (i') $M(\lambda') \mid M(\lambda)$, we conclude each $(p_j)^{m_j} \mid M(\lambda)$, hence we conclude $\prod_j (p_j)^{m_j} \mid M(\lambda)$ by the Chinese remainder theorem, proving the assertion.

Now for any given $\lambda = [1^{m_1}2^{m_2}\cdots] \in \mathcal{P}_\ell$ that is not a rectangular partition, consider the integer $\widetilde{n} = \prod_{j\geq 1}(p_j)^{m_j} \in Q_\ell$. Since $Q_\ell \subseteq M(\mathcal{P}_\ell)$, there must exist some nonrectangular $\mu \in \mathcal{P}_\ell$ with $M(\mu) = \widetilde{n}$. But for all nonrectangular $\mu \in \mathcal{P}_\ell$ with $\mu \neq \lambda$,we can show $M(\mu) \neq \widetilde{n}$. If $\mu = [1^{n_1}2^{n_2}\cdots]$ then we have $n' = \prod_{j\geq 1}(p_j)^{\mu_j} \mid M(\mu)$ so $M(\mu)$ is already divisible by a number n' with exactly ℓ prime factors, with n' different from $\tilde{n}$, hence $M(\mu) \neq \widetilde{n}$. It follows that we must have $M(\lambda) = n$, so

$$M(\lambda) = \widehat{N}(\lambda) = \prod_{j\geq 1}(p_j)^{m_j}.$$

Thus, we have established the agreement of $M(\cdot)$ with $\widehat{N}(\cdot)$ for all $\lambda \in \mathcal{P}_\ell$, and the image of the map is Q_ℓ. This completes the induction step from $\mathcal{P}_{\ell-1}$ to $\mathcal{P}_\ell$. □

5 Uncountable admissible set Theorem 2.3

Proof of Theorem 2.3 (1) We construct bijective maps $M\colon \mathcal{P} \to \mathbb{N}^+$ satisfying properties (i') and (ii) for uncountably many admissible sets S by a recursive set-theoretic construction, by induction on $n \geq 1$.

The construction makes use of the genuine supernorm map, as an indexing device. We have shown $\widehat{N}(\cdot)$ is a bijection from $\mathcal{P}$ to $\mathbb{N}^+$ which refines the Young's lattice ordering with the additive ordering and refines the subset ordering with the divisibility ordering on $\mathbb{N}^+$.

We will assign the integers n one at a time, in order, to partitions, maintaining a bijection at each step, and maintaining compatibility with the two partial orders on the finite number of integers assigned so far. The partitions with one part will be ordered by increasing order of n's that we assign to the set S as we construct it, preserving the Young's lattice order on one-part partitions. The inclusion order divisibility condition is vacuous on one-part partitions. The difficulty is to assign all partitions having two or more parts correctly.

The supernorm map induces a total order on all the partitions with two or more parts. We want to assign $M(\lambda)$ for such partitions (having at least two parts), and we do the assignments in increasing order of supernorm value. Since the supernorm order respects the subset order and the Young's lattice order, for the partition $\lambda \in \mathcal{P}_{\geq 2}$ that we are currently considering, all subpartitions of it, or partitions that it dominates in the Young's lattice order, have already been assigned values. The conditions that $M(\lambda)$ must satisfy are as follows:

(1) by property (i'), to be divisible by all $M(\mu)$ for all $\mu \subseteq_M \lambda$,
(2) by property (ii), to be larger than $M(\mu)$ for all $\mu \subseteq_Y \lambda$.

These comprise a finite set of conditions, and we have not yet assigned all sufficiently large integers past some point, hence all of the divisibility conditions and all of the size conditions can simultaneously be satisfied, with infinitely many choices of an output value. One only needs n divisible by the least common multiple of all the divisibility conditions, and larger than all the bounds from property (ii).

So suppose we have bijectively assigned all integers $\leq n$, where n is given, and that λ is the first partition with two or more parts (in the supernorm total order) to which we have not yet assigned an integer $M(\lambda)$.

We consider the two smallest legal choices we can make, call them n_1 and n_2, that are larger than n. In the first case, we assign $M(\lambda) = n_1$, and then for all not yet assigned integers m with $n < m < n_1$,we assign them one-part partitions (in increasing order, of the one-part partitions not yet assigned.) After doing this, we have assigned $M(\cdot)$-values having output values all the integers up to n_1, and we have assigned $M(\lambda)$ consistently with satisfying all constraints of type (i') and (ii) up to n_1. So we may continue the construction.

In the second case, we assign $M(\lambda) = n_2$ and assign all integers m with $n < m < n_2$ to one-part partitions. Again,we have extended the assignment to $M(\lambda)$ consistently yielding outputs of all integers up to n_2, so we may continue the construction.

After this step,the original finite set S is replaced by two larger finite sets S' and S'', and we have assigned values to all λ' having two or more parts and having supernorm $\widehat{N}(\lambda') < \widehat{N}(\lambda)$, as well as values for one-part partitions taking values up to $n_1 - 1$ (resp. $n_2 - 1$).

In this construction, in which we preserve bijectivity at each step, we exhaust all integers, because all integers up to n are assigned by the nth step of this construction. The tree of constructions is a complete infinite binary tree with countably many levels. Therefore,it has uncountably many leaves, which comprise a set of admissible infinite sets S for which the construction works.

(2) The assertion $\mathbb{P} \subseteq S$ holds by Claim 1 of the proof of Theorem 2.1. Claim 1 was proved using only property (i'). □

6 Combinatorial restrictions on admissible sets

Recall that a set $S = M(\mathcal{P}_1)$ for which there is a bijection M satisfying the conditions (i') and (ii) of Theorem 2.3 is called *admissible*.

Suppose now that $\mathbb{P}$ is strictly contained in S. We first note that the condition that the map $M(\cdot)$ be bijective implies that it cannot be multiplicative, i.e., $M(\lambda)$ cannot always be a product of the values of its one-part subpartitions (counted with multiplicity). Given a composite number $n \in S$, if $M(\cdot)$ were multiplicative, then n would have two representations, one in $\mathcal{P}_1$ and another partition built of out the $\mathcal{P}_1$ pre-images of its prime divisors, all of which occur as images of $\mathcal{P}_1$.

We show admissible sets S in Theorem 2.3 must satisfy "local" combinatorial restrictions on their form, ruling out various finite configurations of their composite values. The next two lemmas exhibit such restrictions.

Lemma 6.1 *If S is admissible and contains $n = 4$, then S must contain at least one of $n = 6$ or $n = 8$.*

Proof Let $S = M(\mathcal{P}_1)$ and suppose $M([1)]) = 2$, $M([2]) = 3$ but that $M([3]) = 4$, which is not a prime number. We then have $M([4]) = 5$, since $5 \in S$ because S contains all primes. Suppose that S does not contain 6, so that $M(\lambda) = 6$ requires λ to have at least two parts. We are to show that S contains 8. Where can $n = 6$ be placed for a bijection? Now $M((1, 1))$ is divisible by 2 and it cannot be 4, so $M((1, 1)) \geq 6$. We must choose $M((1, 1)) = 6$ by property (ii) because $M((1, 1))$ is smaller in the Young's lattice order than any other partition with two or more parts, and all numbers less than 6 are already assigned. Now by property (i'), $M((2, 1))$ is divisible by $M(1)$ and $M(2)$, hence by 6, and by bijectivity of $M(\cdot)$, we must have $M((2, 1)) \geq 12$.

Set $M(\lambda) = 8$, and suppose for a contradiction that $8 \notin S$, so that λ has at least two nonzero parts. By property (i'),all the parts of λ must be 1 or 3, so λ contains either $(1, 1)$ or $(3, 1)$, or $(3, 3)$ as a subpartition. But $(3, 1)$ and $(3, 3)$ both dominate $(2, 1)$ in the Young's lattice partial order, whence $M(\lambda) \geq_Y M((2, 1)) \geq 12$, a contradiction. And if λ contains $(1, 1)$ then, by property (i'), $M(1, 1) \mid M(\lambda)$ whence $M(\lambda)$ will be divisible by 6, a contradiction.

We conclude that there is some one-part partition $M(\lambda) = 8$. In fact, we must have $M([5] = 7$ and $M([6]) = 8$. ◻

Lemma 6.2 *If S is admissible, does not contain $n = 4$, and does contain $n = 6$, then S must also contain $n = 10$.*

Proof Let $S = M(\mathcal{P}_1)$ and suppose $M([1]) = 2$, $M([2]) = 3$, $M([3]) = 5$, while by hypothesis $M([4]) = 6 \in S$ is not a prime number. Then $M((1, 1)) = 4$ using property (ii), because $M((1, 1))$ is lower in the Young's lattice order than all partitions with two or more nonzero parts, so $M((1, 1)) = 4$ is necessary for $M(\cdot)$ to be a bijection. By property (i'), $M((2, 1))$ must be divisible by 6, and since $M([4]) = 6$, the injectivity of $M(\cdot)$ forces $M((2, 1)) \geq 12$.

Next we observe $M((3, 1))$ is divisible by 10, by property (i'). If $M((3, 1)) = 10$ (as occurs for the supernorm $\widehat{N}(\cdot)$), then since $(2, 1) \subseteq_Y (3, 1)$, property (ii) implies $M((2, 1)) \leq M((3, 1))$, and $M((2, 1)) \geq 12$, giving a contradiction. Now injectivity of the mapping $M(\cdot)$ shows that $M((3, 1)) \geq 20$.

Suppose now, for a contradiction that S does not contain 10. Letting $M(\lambda) = 10$, then λ must have at least two nonzero parts. These parts can only take values 1 and 3, otherwise a prime other than 2 or 5 would divide $M(\lambda)$, or it would have a non-maximal divisor bigger than 5. We have three cases:

(1) λ cannot contain both a part 1 and a part 3 since, by property (i'), $M(\lambda)$ would then be divisible by $M((3, 1))$ and $M((3, 1)) \geq 20$, a contradiction.

(2) λ cannot contain $(3,3)$ since, by property (ii), $(3,1) \subseteq_Y (3,3)$ implies $M((3,3)) \geq M((3,1)) \geq 20$. Additionally, $M((3,3)) \mid M(\lambda)$ by property (i), so we get a contradiction.
(3) $\lambda = [1^j]$ for some $j \geq 2$. But $M((1,1)) = 4$ and all $\lambda = [1^j]$ for $j \geq 2$ are divisible by $M((1,1))$ by property (i'), contradicting $M(\lambda) = 10$.

We conclude that there is a one-part partition $\lambda = [m]$ having $M([m]) = 10$ so $10 \in S$. □

There will evidently be an infinite number of such local restrictions. The examples above suggest further questions about such local restrictions.

(1) Does the set of all local restrictions have a simply describable structure?
(2) The local restrictions above suggest that once S contains a composite number, it must contain more composite numbers. How many more composite numbers are forced, in a density sense? Might this number grow faster than the number of primes? Could it be that all admissible S strictly larger than $\mathbb{P}$ necessarily have a counting function, call it $\pi_S(x)$, of the integers in S up to and including x, that has arbitrarily large x with $\pi_S(x) > cx$ for some $c = c(S) > 0$?

7 Concluding remarks

(1) The characterizations given in Theorem 2.1 and 2.2 show that compatibility with both the multiset inclusion order and Young's lattice order on partitions produces, in an intrinsic way, the *prime indexing function* $P \colon \mathbb{N} \to \mathcal{P}$ sending $k \mapsto p_k$, with $P(0) = 1$, $P(1) = 2$, $P(2) = 3$, $P(3) = 5, \ldots$ as it lists the images of the partitions with one part. A functional inverse $P^{-1}(\cdot)$ of the prime indexing function $P(\cdot)$ is related to the prime counting function $\pi(x)$, via

$$P^{-1}(y) = \begin{cases} \pi(y) & \text{when } \pi(y) > \pi(y-1), \\ 0 & \text{otherwise.} \end{cases}$$

The characterization theorems here exhibit a natural place where the function $P(x)$ occurs.

(2) The prime indexing function $P(\cdot)$ has received limited attention in the literature, in the context of other number-theoretic functions. There is no known algorithm which, given k as input, computes $P(k)$ in polynomial time in the bit-length of k, which is $\lceil \log_2 k \rceil$. However, computing the whole ensemble $\{P(k) : 1 \leq k \leq x\}$ up to a given input bound x can be done in amortized polynomial time $O(\log x)$ per individual $P(k)$; for a sharp bound, see Helfgott [8]. The function $P^{-1}(y)$ also appears to be hard to compute.

8 Afterword

This paper is dedicated to the memory of Ronald L. Graham. Ron had a great interest in partitions of all kinds. He wrote papers on integer partitions [4], partitions of sets

[5], partitions of graphs [3] (with Erdős), partitions of the plane $\mathbf{E}^2$ [6], and on Ramsey theory, concerning unavoidable patterns on finite partitions of the positive integers $\mathbb{N}^+$ [7].

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