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Γ-evaluations of hypergeometric series

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Abstract
In this paper we explore special values of Gaussian hypergeometric functions in terms of products of Euler Γ-functions and exponential functions of linear functions of the hypergeometric parameters. They include some classical evaluations, but the main inspiration is from the contiguity method recently applied by Akihito Ebisu.

Keywords Gauss hypergeometric function · Special values · Gamma function

Mathematics Subject Classification 33C05

1 Introduction

Let $a, b, c \in \mathbb{C}$ such that $c \notin \mathbb{Z}_{\leq 0}$. The Gauss hypergeometric function $F(a, b, c \,|\, z)$ is defined by the power series expansion

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.$$

This power series converges in the complex disc $|z| < 1$. When $\mathrm{Re}(c - a - b) > 0$, the series also converges on $|z| = 1$. Note that if the a or b parameter is a negative integer, then $F(a, b, c \,|\, z)$ is a polynomial. There are no convergence issues in that case.

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In the classical literature on hypergeometric functions, we find many instances of special evaluation of a hypergeometric function at specific arguments. The best known evaluation is due to Gauss,

$$F(a,b,c\,|\,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

The left-hand side converges only if $\mathrm{Re}(c-a-b)>0$. Another example is Kummer's evaluation

$$F(a,b,a-b+1\,|\,-1)=\frac{1}{2}\frac{\Gamma(a/2)\Gamma(a-b+1)}{\Gamma(a)\Gamma(a/2-b+1)}.$$

From this, one can deduce two others, as shown by Bailey in [2,p. 11]. The first is

$$F\big(2a,2b,a+b+1/2\,|\,1/2\big)=\frac{\Gamma(1/2)\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)},$$

attributed to Gauss, and the second is

$$F\big(a,1-a,c\,|\,1/2\big)=\frac{\Gamma(c/2)\Gamma((c+1)/2)}{\Gamma((c+a)/2)\Gamma((1+c-a)/2)}.$$

There is a related evaluation

$$F(2a+1,b,2b\,|\,2)=\frac{\Gamma(-a)\Gamma(1/2+b)}{\Gamma(1/2)\Gamma(-a+b)}\times\frac{1-e^{2\pi ia}}{2}.$$

However, for the moment this is only well defined when $2a+1\in\mathbb{Z}_{\le 0}$ and $2b\notin\mathbb{Z}_{\le 0}$, since in that case the left-hand side is a finite sum.

The above examples contain 3 or 2 degrees of freedom in their parameters. It turns out that there exists a very extensive list of one-parameter evaluations. As an example, we quote from Bateman's [7,2.8(53)],

$$F\big(-a,-a+1/2,2a+3/2\,|\,-1/3\big)=\left(\frac{8}{9}\right)^{2a}\frac{\Gamma(2a+3/2)\Gamma(4/3)}{\Gamma(2a+4/3)\Gamma(3/2)}.$$

These evaluations take place at fixed arguments and the values are a product of values of Γ-functions times an exponential function times, possibly, a periodic function in the hypergeometric parameters. In the literature, they are sometimes called "strange evaluation," we prefer the more descriptive name Γ-*evaluations*.

The first systematic study that we are aware of is from Heyman in 1899, [12]. There we find a collection of Γ-evaluations obtained by using the contiguity property for hypergeometric functions. We also cite [11] from 1982 and [14] from 1998, which includes special evaluations for higher-order hypergeometric functions as well. The evaluations are often in polynomial form, by which we mean that one of the first two hypergeometric parameters is a negative integer. The development of computer algebra methods made it possible to automatize the search for Γ-evaluations. See, for example, [10] and the remarkable manuscript [9] containing 40 Γ-evaluations discovered around 2004 by Shalosh Ekhad, Doron Zeilberger's tireless computer. One more or less random example of such a Γ-evaluation is

$$F\big(2t, t + 1/3, 4/3 \,|\, -8\big) = \frac{2\cos\big(\pi(t + 1/3)\big)}{27^t} \frac{\Gamma(t - 1/6)\Gamma(1/2)}{\Gamma(t + 1/2)\Gamma(-1/6)},$$

which can be found in [11,(3.7)] when $t \in -1/3 + \mathbb{Z}_{\leq 0}$ and additionally in [8,4.3.2(xxi)] when $2t \in \mathbb{Z}_{\leq 0}$. In this paper, we show that it holds for arbitrary t.

The inspiration for the present paper comes from Akihito Ebisu's remarkable AMS Memoir [8], in which the author develops a systematic method to find Γ-evaluations of Gaussian hypergeometric functions. As in Heymann's work, the main tool in this study is the contiguity property of hypergeometric functions. After explanation of this idea, Ebisu produces a long list of sample Γ-evaluations, either in the finite form, with one a or b parameter in $\mathbb{Z}_{\leq 0}$, or an interpolated version which holds fo all parameter values t. We have adapted Ebisu's approach, which very briefly comes down to the following.

Consider a triple of hypergeometric parameters a, b, c and abbreviate it by $\beta := (a, b, c)$. We denote $F(\beta \,|\, z) := F(a, b, c \,|\, z)$. Let k, l, m be a triple of integers, which we denote as $\gamma := (k, l, m)$, the *shift vector*. Using contiguity relations, we can find rational functions $R_\gamma(\beta, z)$ and $Q_\gamma(\beta, z)$ in $\mathbb{Q}(a, b, c, z)$ such that

$$F(\beta + \gamma \,|\, z) = R_\gamma(\beta, z) F(\beta \,|\, z) + Q_\gamma(\beta, z) F'(\beta \,|\, z).$$

A quadruple $(\beta, z_0) := (a, b, c, z_0)$ is called *admissible* with respect to γ if $Q_\gamma(\beta + t\gamma, z_0) = 0$ for all $t \in \mathbb{C}$. Choose an admissible quadruple (β, z_0). We then obtain the functional equation

$$F(\beta + (t+1)\gamma \,|\, z_0) = R_\gamma(\beta + t\gamma, z_0) F(\beta + t\gamma \,|\, z_0) \tag{1.1}$$

for $F(\beta + t\gamma \,|\, z_0)$ as function of t. Suppose that

$$R_\gamma(\beta + t\gamma, z_0) = R_0 \prod_{i=1}^{r} \frac{t + \alpha_i}{t + \delta_i}, \quad R_0 \in \mathbb{C}^\times.$$

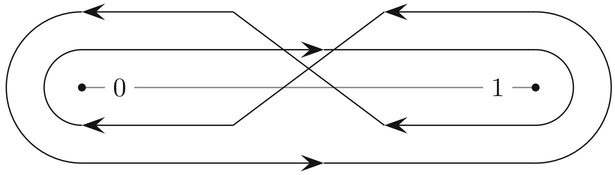
Then observe that $R_0^t \prod_{i=1}^r \frac{\Gamma(t+\alpha_i)}{\Gamma(t+\delta_i)}$ satisfies the same functional equation as $F(\beta + t\gamma \,|\, z_0)$. All we need to do is to identify these two functions of t. This is done in Theorem 2.5, which is our main result. From Theorem 2.5, we can deduce interpolated versions of Γ-evaluations which occurred only in finite form in earlier publications.

Since, in many of the latter cases, the argument is outside the disc of convergence, we need to extend the evaluations of $F(a, b, c \,|\, z)$ to z outside the unit disc.

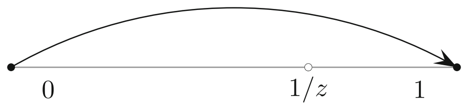
The sum $F(a, b, c \mid z)$ can be continued analytically to $\mathbb{C} \setminus [1, \infty)$ using Euler's integral

$$F(a, b, c \mid z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-zx)^a}\,\mathrm{d}x.$$

In the integrand, we choose $x^b = \exp(b \log|x|)$ and $(1-x)^{c-b} = \exp((c-b)\log|1-x|)$, and we define $(1-zx)^a$ using the choice $|\arg(1-zx)| < \pi$. Note that this integral only converges at the points 0 and 1 if $\mathrm{Re}(b)$ and $\mathrm{Re}(c-b)$ are positive. To get an integral without these restrictions, one can replace the path of integration $[0, 1]$ by the so-called Pochhammer contour C:



and division of the integral by $(e^{-2\pi i b} - 1)(e^{2\pi i(c-b)} - 1)$. The four horizontal piecewise linear paths should be thought of as four copies of the real segment $[\delta, 1-\delta]$ and the rounded parts as the circles $|z| = \delta$ and $|z-1| = \delta$ for some small $\delta > 0$. We have taken the argument of the integrand on the bottom line segment to be given as above. For the evaluation of $F(a, b, c \mid z)$ at $z \in (1, \infty)$, we make the choice $\lim_{\epsilon \downarrow 0} F(a, b, c \mid z + \epsilon i)$. Its value is now given by the Euler integral over the arc



or its Pochhammer version. When a, b, c are real, the value $\lim_{\epsilon \downarrow 0} F(a, b, c \mid z - \epsilon i)$ is its complex conjugate. When a, b, c are not all real, the difference between these limits can be quite drastic. The reader should be aware of this when checking the results numerically. For example, the computer package *Mathematica* seems to use the second limit (with $z - i\epsilon$).

In the above description, we have suggested that the degrees in t of numerator and denominator of $R_\gamma(\beta + t\gamma, z_0)$ are the same. In Theorem 4.1, we prove that this is indeed the case when the vector $\beta + t\gamma$ is *non-resonant*. This means that none of the four linear functions

$$a + kt, \quad b + lt, \quad c - a + t(m-k), \quad \text{and} \quad c - b + t(m-l)$$

is an integer valued constant. It turns out that the non-resonant case is the interesting case; in Sect. 3, we give a description of the resonant cases only for completeness. In the non-resonant case, Theorem 4.1 also gives the values of z_0 and R_0. This is a result found previously by Iwasaki in [13,Theorem 2.3], although not in this wording and with a different proof using asymptotic analysis of the Euler integral.

Although we believe that, for a given admissible quadruple, there should exist a simple procedure to determine $R_\gamma(\beta+\gamma t)$, we have not been able to discover it. Another issue we should mention is a difference between the result of Theorem 2.5 and some finite evaluations in [9] and [8]. As an example, consider the identity

$$F(t, 3t-1, 2t|e^{\pi i/3}) = -\frac{\sqrt{3}}{2}e^{\pi i(t/2+5/6)}\left(\frac{4}{\sqrt{27}}\right)^t \frac{\Gamma(t+1/2)\Gamma(1/3)}{\Gamma(t+1/3)\Gamma(1/2)},$$

which can be deduced from Theorem 2.5. It holds for all $t \in \mathbb{C}$. When $t=-n$ for any $n \in \mathbb{Z}_{>0}$, the left-hand side is not well defined as hypergeometric series, but the equality should be read as the limit when $t \to -n$. We get, after some simplification,

$$F(-n, -3n-1, -2n|e^{\pi i/3}) = -\frac{\sqrt{3}}{2}e^{5\pi i/6}\left(\frac{-\sqrt{-27}}{4}\right)^n \frac{(2/3)_n}{(1/2)_n}.$$

In [9,Theorem 11] and [8,4.2.4], we find the same evaluation, but with the factor $-\sqrt{3}e^{5\pi i/6}/2$ missing. The reason is that in the latter evaluations, the function $F(-n,-3n-1,-2n|z)$ is interpreted as the polynomial $F(-n,-3n-1,c|z)|_{c=-2n}$. It is remarkable that the limit and the polynomial evaluation differ by a constant factor. In many cases when both the a-parameter and c-parameter have limits that are non-positive integers, this phenomenon seems to occur. As suggested by the referee, an explanation might be that both sequences satisfy the same first order recurrence relation in n. We have not tried to elaborate this.

We have not made an exhaustive search for all admissible quadruples. This is more or less done in [8]. There it is also remarked that through the use of Kummer's solution to a hypergeometric equation, any admissible quadruple is associated with 24 others. This may explain the abundance of these Γ-evaluations. In Sect. 4, we give a description and a proof of the existence of these associated quadruples through the properties of the Euler kernel, which is the integrand of the Euler integral.

In the final section, we present a more or less random list of examples of Γ-evaluations.

2 Interpolation

Let us begin with an example. We consider the case $(k,l,m)=(2,2,1)$ and carry out the program we sketched in the introduction. We get

$$R_\gamma(a,b,c,z) = \frac{c(2+a+b-c)}{(a+1)(b+1)(z-1)^2} \quad \text{and}$$

$$Q_\gamma(a,b,c,z) = \frac{c((1+2a+a^2+2b+ab+b^2-c-ac-bc)z+(1+a-c)(1+b-c))}{a(1+a)b(1+b)(-1+z)^2}.$$

The numerator of $Q_\gamma(a+2t, b+2t, c+t, z)$ reads

$$(c+t)\Big(1+a+b+ab-2c-ac-bc+c^2+z+2az+a^2z+2bz+abz+b^2z$$
$$-cz-acz-bcz+(2+a+b-2c+7z+5az+5bz-4cz)t+(8z+1)t^2\Big).$$

The equations for the admissible quadruple are obtained by setting this polynomial in t identically zero. We get

$$\begin{cases} 0=8z+1, \\ 0=2+a+b-2c+7z+5az+5bz-4cz, \\ 0=1+a+b+ab-2c-ac-bc+c^2+z+2az+a^2z \\ \qquad +2bz+abz+b^2z-cz-acz-bcz. \end{cases}$$

Solution of this system yields

$$z_0=-1/8, \quad a=2t, \quad b=2t+1/3, \quad c=t+5/6 \tag{2.1}$$

or

$$z_0=-1/8, \quad a=2t, \quad b=2t-1/3, \quad c=t+2/3.$$

Taking the first possibility, we get

$$\begin{aligned} &R_\gamma\big(2t, 2t+1/3, t+5/6, -1/8\big) \\ &= \frac{16}{27}\times\frac{t+5/6}{t+2/3}. \end{aligned}$$

So we find from (1.1) that

$$\begin{aligned} &F\big(2(t+1), 2(t+1)+1/3, t+1+5/6 \mid -1/8\big) \\ &= \frac{16}{27}\times\frac{t+5/6}{t+2/3}\times F(2t, 2t+1/3, t+5/6 \mid -1/8) \end{aligned}$$

for all t. Notice that $\left(\frac{16}{27}\right)^t\frac{\Gamma(t+5/6)}{\Gamma(t+2/3)}$ satisfies the same functional equation. The corresponding functions turn out to differ by a constant factor, as shown in Corollary 2.8.

In this section, we prove Theorem 2.5 which states that for any admissible quadruple (β, z_0), there exists a complex interpolation of the Γ-evaluations. We find from [1,Corollary 1.4.4], the following estimate.

Lemma 2.1 *Suppose $s=a+bi$ with $a_1<a<a_2$ and $|b|\to\infty$. Then*

$$|\Gamma(a+bi)|=\sqrt{2\pi}\,|b|^{a-\frac{1}{2}}\,e^{-\frac{\pi|b|}{2}}\left[1+O(1/|b|)\right].$$

Proposition 2.2 *Let $\beta = (a, b, c) \in \mathbb{R}^3$ and $\gamma = (k, l, m) \in \mathbb{Z}^3$. Let $z_0 \in \mathbb{C}$ and $z_0 \neq 1$. Then, $F(\beta + \gamma t\,|z_0)$ is a meromorphic function in $t \in \mathbb{C}$ having at most finitely many poles with $|\mathrm{Re}(t)| \leq \frac{1}{2}$. Let*

$$C_1 = |k \arg(1 - z_0)| + \frac{|l|\pi}{2} + \frac{|m - l|\pi}{2} - \frac{|m|\pi}{2}, \quad |\arg(1 - z_0)| \leq \pi.$$

Then there exist $C_2, C_3 \geq 0$ such that

$$\left|F(\beta + \gamma t\,|z_0)\right| \leq C_2\, |\mathrm{Im}(t)|^{C_3}\, e^{C_1 |\mathrm{Im}(t)|}$$

for all $t \in \mathbb{C}$ with $|\mathrm{Re}(t)| \leq \frac{1}{2}$ and $|\mathrm{Im}(t)|$ sufficiently large.

Proof In order to prove our estimate, we use the ordinary Euler integral. We first prove the proposition under the assumption that $-a > \frac{|k|}{2}$, that $b > \frac{|l|}{2}$, and that $c - b > \frac{|m-l|}{2}$. Let us write

$$G(\beta + \gamma t\,|z_0) := \int_0^1 \frac{x^{b-1+lt}(1 - x)^{c-b-1+(m-l)t}}{(1 - z_0 x)^{a+kt}} \mathrm{dx}.$$

This integral converges for all t with $|\mathrm{Re}(t)| \leq \frac{1}{2}$ because of our assumptions on a, b, and c. Since $|\mathrm{Re}(t)| \leq \frac{1}{2}$, we get

$$|x^{b-1+lt}| \leq x^{b-1-\frac{|l|}{2}} \quad \text{and} \quad |(1 - x)^{c-b-1+(m-l)t}| \leq (1 - x)^{c-b-1-\frac{|m-l|}{2}}.$$

Let $c_1 = \int_0^1 x^{b-1-\frac{|l|}{2}}(1 - x)^{c-b-1-\frac{|m-l|}{2}}\mathrm{dx}$. Recall that

$$\left|(1 - z_0 x)^{-a-kt}\right| = |1 - z_0 x|^{-a-k\mathrm{Re}(t)} \exp\big(k \arg(1 - z_0 x)\mathrm{Im}(t)\big).$$

Let $c_2 = \max_{x\in[0,1],|y|\leq\frac{1}{2}} |1 - z_0 x|^{-a-ky}$, which is finite because $-a > |k|/2$. Notice also that

$$\max_{x\in[0,1]} \exp\big(k \arg(1 - z_0 x)\mathrm{Im}(t)\big) \leq \exp\big(|k \arg(1 - z_0)\,\mathrm{Im}(t)|\big).$$

We conclude that $|G(\beta + t\gamma|z_0)|$ has the upper bound $c_1 c_2\, e^{|k \arg(1-z_0)\,\mathrm{Im}(t)|}$. Using Lemma 2.1, we find the desired estimate for

$$F(\beta + t\gamma\,|z_0) = \frac{\Gamma(c + mt)}{\Gamma(b + lt)\Gamma(c - b + (m - l)t)} G(\beta + t\gamma|z_0)$$

when $-a > \frac{|k|}{2}$, $b > \frac{|l|}{2}$ and $c - b > \frac{|m-l|}{2}$.

In the general situation, we first choose integers Δa, Δb, Δc such that

$$-a - \Delta a - 1 > \frac{|k|}{2}, \quad b + \Delta b > \frac{|l|}{2}, \quad c - b + \Delta c - \Delta b > \frac{|m - l|}{2}.$$

Denote $\Delta\beta = (\Delta a, \Delta b, \Delta c)$. Then, there exists a contiguity relation

$$F(\beta + t\gamma|z) = r(t,z)F\left(\beta + \Delta\beta + t\gamma|z\right) + s(t,z)F\left(\beta + \Delta\beta + (1,1,1) + t\gamma|z\right),$$

where $r(t,z)$ and $s(t,z)$ are rational functions in z, t. In Lemma 2.3, we show that, as rational function of z, their only poles are in $z = 0, 1$. Hence, we can specialize to $z = z_0$ and get

$$\begin{aligned} F(\beta + t\gamma|z_0) = r(t,z_0)F\left(\beta + \Delta\beta + t\gamma|z_0\right) \\ +s(t,z_0)F\left(\beta + \Delta\beta + (1,1,1) + t\gamma|z_0\right), \end{aligned}$$

We then apply the above estimate to the terms on the right-hand side. □

Lemma 2.3 *Let a, b, c be hypergeometric parameters such that $a, b, c-a, c-b \notin \mathbb{Z}$. Let a', b', c' be contiguous parameters, that is $a'-a, b'-b, c'-c \in \mathbb{Z}$. Consider the contiguity relation*

$$F(a',b',c'|z) = r(z)F(a,b,c|z) + s(z)F(a+1,b+1,c+1|z),$$

where $r(z), s(z)$ are rational functions in a, b, c, z. Then as rational functions in z, the functions $r(z), s(z)$ have only poles in $z = 0, 1$.

Proof We know that $F(a+1,b+1,c+1|z) = \frac{c}{ab}F'(a,b,c|z)$. The contiguity relation is stable under analytic continuation in z. Therefore, we have a similar relation for the solution of the hypergeometric equation corresponding to the local exponent $1-c$. Thus, there exists a rational function λ in a, b, c such that $\lambda z^{1-c'}F(a'+1-c', b'+1-c', 2-c'|z)$ equals

$$\begin{aligned} r(z)z^{1-c}F(a+1-c, b+1-c, 2-c|z) \\ +s(z)\frac{c}{ab}(z^{1-c}F(a+1-c, b+1-c, 2-c|z))'. \end{aligned}$$

We can now solve for $r(z), s(z)$ and find that

$$\begin{pmatrix} r(z) \\ \frac{c}{ab}s(z) \end{pmatrix} = \frac{1}{W(z)} \begin{pmatrix} (z^{1-c}F(a+1-c,b+1-c,2-c|z)' & -F'(a,b,c|z) \\ -z^{1-c}F(a+1-c,b+1-c,2-c|z) & F(a,b,c|z) \end{pmatrix} \times \begin{pmatrix} F(a',b',c') \\ \lambda z^{1-c'}F(a'+1-c', b'+1-c', 2-c'|z) \end{pmatrix},$$

where $W(z)$ is the Wronskian determinant of the hypergeometric equation, which equals $1-c$ times $z^{-c}(1-z)^{c-a-b-1}$. The matrices on the right-hand side have entries which are locally holomorphic outside $0, 1, \infty$, and therefore, we conclude that the same holds for $r(z), s(z)$.

Strictly speaking, we have proved the lemma when $c \notin \mathbb{Z}$. The case of integral c runs similarly. □

Proposition 2.4 *Let $f(t)$ be a periodic entire function with unit period one. Suppose that there are constants $C^+, C^- \geq 0$ such that*

(1) $|f(t)| = O\big(e^{C^+\mathrm{Im}(t)}\big)$ *when* $\mathrm{Im}(t) \to \infty$, *and*
(2) $|f(t)| = O\big(e^{-C^-\mathrm{Im}(t)}\big)$ *when* $\mathrm{Im}(t) \to -\infty$.

Then, $f(t) = g(e^{2\pi it})$ *where* $g(z) \in \mathbb{C}[z, 1/z]$. *Moreover, g has a pole of order at most $C^+/2\pi$ at $z = 0$, and a pole of order at most $C^-/2\pi$ at $z = \infty$.*

Proof Consider the composite function $g(z) = f\big(\frac{\log z}{2\pi i}\big)$. This is an entire function in z, except possibly at $z = 0$, which is an isolated singularity. Notice that $\mathrm{Im}(t) = -\frac{\log|z|}{2\pi}$. So when $z \to 0$ we get $\mathrm{Im}(t) \to \infty$ and we can use the estimate

$$|f(t)| = O\big(e^{-\frac{\log|z|}{2\pi}C^+}\big) = O\big(|z|^{-\frac{C^+}{2\pi}}\big).$$

When $z \to \infty$ we get $\mathrm{Im}(t) \to -\infty$ and we can use the estimate

$$|f(t)| = O\big(e^{\frac{\log|z|}{2\pi}C^-}\big) = O\big(|z|^{\frac{C^-}{2\pi}}\big).$$

□

We can now show our main theorem.

Theorem 2.5 *We use the notations from the introduction. Let (β, z_0) be an admissible quadruple with respect to $\gamma = (k, l, m) \in \mathbb{Z}^3$. We assume that $m \geq 0$ and $c \notin \mathbb{Z}_{\leq 0}$ when $m = 0$. Write*

$$R_\gamma(\beta + t\gamma, z_0) = R_0 \times \prod_{j=1}^{r} \frac{(t + \alpha_j)}{(t + \delta_j)}.$$

Then, there exists $g(z) \in \mathbb{C}[z, 1/z]$ such that

$$F(\beta + t\gamma \,|\, z_0) = g\left(e^{2\pi it}\right) R_0^t \prod_{j=1}^{r} \frac{\Gamma(t + \alpha_j)}{\Gamma(t + \delta_j)}$$

for all $t \in \mathbb{C}$. Moreover, g has a pole order at most

$$\frac{\arg(R_0)}{2\pi} + \frac{|k\arg(1 - z_0)|}{2\pi} + \frac{|l|}{4} + \frac{|m - l|}{4} - \frac{|m|}{4}$$

at $z = 0$ and order at most

$$-\frac{\arg(R_0)}{2\pi} + \frac{|k\arg(1 - z_0)|}{2\pi} + \frac{|l|}{4} + \frac{|m - l|}{4} - \frac{|m|}{4}$$

at $z = \infty$.

Remark 2.6 We have used that the numerator and denominator of R have the same degree. This is a consequence of Theorem 4.1, last two lines..

Remark 2.7 The assumption $m \geq 0$ is not a restriction. If $m < 0$, then we apply Theorem 2.5 with $-\gamma$ and simply replace t by $-t$.

Proof of Theorem 2.5. We find that

$$G(t) := F(\beta + t\gamma \mid z_0)\, R_0^{-t} \prod_{j=1}^{r} \frac{\Gamma(t+\delta_j)}{\Gamma(t+\alpha_j)}$$

is a meromorphic periodic function with period 1. Poles can only arise from the factor $F(\beta+t\gamma \mid z_0)$ when $c+mt \in \mathbb{Z}_{\leq 0}$, or from the product $\prod_j \Gamma(t+\delta_j)$ when $t+\delta_j \in \mathbb{Z}_{\leq 0}$ for some j. It follows, since $m \geq 0$, that there are no poles when $\mathrm{Re}(t)$ is sufficiently large. Hence, $G(t)$ is holomorphic in t. We now use the estimates from Lemma 2.1 and Proposition 2.2 to get $|R_0^t\, G(t)| = O(e^{(C_1+\epsilon)|\mathrm{Im}(t)|})$ for any $\epsilon > 0$, where

$$C_1 = |k \arg(1-z_0)| + \frac{|l|\pi}{2} + \frac{|m-l|\pi}{2} - \frac{|m|\pi}{2},$$

as in Proposition 2.2. This yields $|G(t)| = O(e^{(\arg(R_0)+C_1+\epsilon)|\mathrm{Im}(t)|})$ when $\mathrm{Im}(t) \to \infty$ and $|G(t)| = O(e^{(-\arg(R_0)+C_1+\epsilon)|\mathrm{Im}(t)|})$ when $\mathrm{Im}(t) \to -\infty$. The result now follows from Proposition 2.4. $\square$

We give three example applications.

Corollary 2.8 *For all $t \in \mathbb{C}$ we have*

$$F\big(2t, 2t + 1/3, t + 5/6 \mid -1/8\big) = \left(\frac{16}{27}\right)^t \frac{\Gamma(t+5/6)\Gamma(2/3)}{\Gamma(t+2/3)\Gamma(5/6)}.$$

Proof In the beginning of this section, we considered the example $\gamma = (2, 2, 1)$ and the admissible quadruple (2.1). From Theorem 2.5, applied to this example, we find that

$$\left(\frac{27}{16}\right)^t \frac{\Gamma(t+2/3)}{\Gamma(t+5/6)} F(2t, 2t+1/3, t+5/6 \mid -1/8)$$

is a Laurent polynomial in $e^{2\pi i t}$. Since $\arg(16/27) = \arg(1-z_0) = 0$, the estimates for the pole order of g at 0 and ∞ are $\frac{1}{2}$. Hence, g is constant. The value of the constant can be found by setting $t = 0$. $\square$

Corollary 2.9 *For all $t \in \mathbb{C}$, we have*

$$F\big(3t, t+1/6, 1/2 \mid -3\big) = \frac{\cos(\pi t)}{16^t} \frac{\Gamma(t+1/2)\Gamma(1/3)}{\Gamma(t+1/3)\Gamma(1/2)}.$$

Proof Consider the admissible quadruple $a = 3t, b = t + 1/6, c = 1/2$, and $z_0 = -3$. We get

$$R_\gamma(\beta + t\gamma, z_0) = -\frac{1}{16} \times \frac{t + 1/2}{t + 1/3}.$$

Application of Theorem 2.5 yields

$$F(3t, t + 1/6, 1/2 \mid -3) = \frac{e^{\pi i t}}{16^t} \frac{\Gamma(t + 1/2)}{\Gamma(t + 1/3)} g\left(e^{2\pi i t}\right).$$

Here, $g(z)$ is a Laurent polynomial, bounded at $z = \infty$, and with a pole at $z = 0$ of order at most 1. Hence, $g\left(e^{2\pi i t}\right) = u + ve^{-2\pi i t}$ for some $u, v \in \mathbb{C}$. Setting $t = 0$ and $t = -\frac{1}{2}$ yields

$$\begin{cases} 1 = (u + v)\Gamma(1/2)/\Gamma(1/3), \\ 0 = u - v. \end{cases}$$

Hence, $u = v = \Gamma(1/3)/2\Gamma(1/2)$ and our corollary follows. □

Corollary 2.10 *For all $t \in \mathbb{C}$ we have*

$$F\left(3t, t + 1/6, 1/2 \mid 9\right) = \frac{1}{2 \cdot 64^t} \left(1 + e^{2\pi i\left(t+\frac{1}{6}\right)} - e^{4\pi i\left(t+\frac{1}{6}\right)}\right).$$

Proof Consider the admissible quadruple $a = 3t, b = t + 1/6, c = 1/2$, and $z_0 = 9$. We find that $R_\gamma = \frac{1}{64}$. So Theorem 2.5 gives $F(3t, t + 1/6, 1/2 \mid 9) = 64^{-t} g\left(e^{2\pi i t}\right)$. Since $|\arg(1 - 9)| = \pi$, we get the estimate 2 for the polar order of $g(z)$ at $z = 0$ and at $z = \infty$. Hence, $g\left(e^{2\pi i t}\right) = \sum_{k=-2}^{2} a_k e^{2\pi i k t}$. To determine the values of the a_k, we use five special evaluations of $64^t F(3t, t + 1/6, 1/2 \mid 9)$ for $t = 0, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{6}$, and $\frac{1}{6}$. We obtain the system

$$\begin{bmatrix} 1 \\ 1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ \zeta \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \zeta^{-2} & \zeta^{2} & 1 & \zeta^{-2} & \zeta^{2} \\ \zeta^{2} & \zeta^{-2} & 1 & \zeta^{2} & \zeta^{-2} \\ \zeta^{2} & \zeta & 1 & \zeta^{-1} & \zeta^{-2} \\ \zeta^{-2} & \zeta^{-1} & 1 & \zeta & \zeta^{2} \end{bmatrix} \begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix},$$

where $\zeta = e^{\frac{\pi i}{3}}$. The evaluation at $t = 1/6$ requires some explanation. We need to determine $2F(1/2, 1/3, 1/2 | 9) = 2 \cdot (1 - 9)^{-1/3}$. The absolute value is of course 1. It remains to determine the argument. By our branch choice, we take a path in the upper half plane from $z = 1/2$ to $z = 9$. The argument of $(1 - z)^{-1/3}$ then changes from 0 to $\pi/3$. Hence, the function value becomes ζ. Solution of the system gives our corollary. □

3 Resonant quadruples

Our next goal is to explain the values of z_0 and $R_\gamma(\beta + t\gamma)$ that occur in the above considerations. For that purpose, it turns out to be convenient to restrict to admissible quadruples such that $\beta + t\gamma$ is *non-resonant*, that is, none of

$$a + kt, \quad b + lt, \quad c - b + (m - l)t, \quad \text{and} \quad c - a + (m - k)t$$

is an element of $\mathbb{Z}$. If at least one of these linear polynomials is an integer constant, then we say that the quadruple is *resonant*. In this section, we make some comments on the resonant case and proceed with the non-resonant cases in the next sections. We will use the identity

$$F(c - a, c - b, c \mid z) = (1 - z)^{a+b-c} F(a, b, c \mid z). \tag{3.1}$$

Suppose that $\beta + t\gamma$ is resonant and that $(\beta + t\gamma, z_0)$ is a resonant quadruple. Then, we distinguish the following cases.

(1) Exactly one of $a + kt$, $b + lt$, $c - b + (m - l)t$, and $c - a + (m - k)t$ is an integer.

(a) If $a + kt \in \mathbb{Z}$, then we conjecture that the admissible quadruples are given by $a = 2$, $b = 1 + lt$, $c = 2 + mt$, and $z_0 = m/l$, with Γ-evaluation

$$F(2, 1 + lt, 2 + mt \mid m/l) = \frac{l(1 + mt)}{l - m},$$

or $a = -1$, $b = lt$, $c = mt$, and $z_0 = m/l$ with Γ evaluation

$$F(-1, lt, mt \mid m/l) = 0.$$

It should be remarked that these evaluations are a direct consequence of the general identities

$$F\Big(2, r, s \,\Big|\, \frac{s - 2}{r - 1}\Big) = \frac{(r - 1)(s - 1)}{r - s + 1} \quad \text{and} \quad F\Big(- 1, r, s \,\Big|\, \frac{s}{r}\Big) = 0,$$

which are easy to prove. A similar remark applies to the next cases.

(b) The case $b + lt \in \mathbb{Z}$ is similar to case (1a).

(c) If $c - a + (m - k)t \in \mathbb{Z}$, then we use the identity (3.1) to get

$$F\big(mt, 1 + (m - l)t, 2 + mt \mid m/l\big) = \Big(1 - \frac{m}{l}\Big)^{(l-m)t} (1 + mt)$$

and

$$F\big(mt + 1, (m - l)t, mt \mid m/l\big) = 0.$$

(d) The case $c - b + (m - l)t \in \mathbb{Z}$ is similar to case (1c).

(2) Exactly two of $a+kt, b+lt, c-b+(m-l)t, c-a+(m-k)t$ are in $\mathbb{Z}$,

(a) If $a+kt, b+lt \in \mathbb{Z}$, then admissibility implies that either $ab=0$ or $a \in \mathbb{Z}$, $b=1-a$, $c=mt$, and $z_0 = 1/2$. In the latter case we might as well replace mt by t. Bailey's identity gives

$$F(a, 1-a, t \,|\, 1/2) = \frac{\Gamma(t/2)\Gamma((t+1)/2)}{\Gamma((t+a)/2)\Gamma((1+t-a)/2)}.$$

(b) If $a+kt, c-b+(m-l)t \in \mathbb{Z}$, then admissibility implies that either $a(b-c)=0$ or $a \in \mathbb{Z}$, $b=t$, $c=t-a+1$, and $z_0=-1$. In the latter case, Kummer's identity gives

$$F(a, t, t+1-a \,|\, -1) = \frac{1}{2}\frac{\Gamma(t/2)\Gamma(t-a+1)}{\Gamma(t)\Gamma(t/2-a+1)}.$$

(c) If $b+lt, c-b+(m-l)t \in \mathbb{Z}$, then admissibility implies that either $b=1$ and $c=2$, or $b \in \mathbb{Z}$, $c=2b$, and $z_0=2$. In the former case, we get

$$F(1+t, 1, 2 \,|\, z) = \frac{(1-z)^{-t}-1}{tz};$$

in the latter case, we get

$$F(1-2t, b, 2b \,|\, 2) = \frac{\Gamma(t)\Gamma(b+1/2)}{\Gamma(t+b)\Gamma(1/2)} \times \frac{1-e^{-2\pi i t}}{2}.$$

(d) The other three cases are related to the above three via the identity (3.1).

4 Euler kernels

Let β be the triple of hypergeometric parameters and γ the shift vector as in the previous section. Suppose also that $z \neq 0, 1$. We define

$$K(\beta, z, x) = \frac{x^{b-1}(1-x)^{c-b-1}}{(1-zx)^a}.$$

Application of the Pochhammer contour integral then gives us

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(\beta \,|\, z).$$

In [4], the author considered the $\mathbb{Q}(\beta, z)$-vector space of twisted differential forms generated by the differential forms $K(\beta+\delta, z, x)\mathrm{d}x$ with $\delta \in \mathbb{Z}^3$. The $\mathbb{Q}(\beta, z)$-vector space of twisted exact forms is generated by $d(K(\beta+\delta, z, x))$ with $\delta \in \mathbb{Z}^3$. We denote the quotient space by $H^1_{\text{twist}}(\beta \mid z)$. In [4,Theorem 6.1], it is shown, under the assumption β is non-resonant, that this space is two dimensional with basis $K(\beta, z, x)\mathrm{dx}$ and $K(\beta + (1, 1, 1), z, x)\mathrm{dx}$. Notice that

$$aK(\beta + (1, 1, 1), z, x) = \frac{\partial}{\partial z} K(\beta, z, x).$$

Define the hypergeometric operator

$$\mathcal{L} = z(z-1)\frac{\partial^2}{\partial z^2} + ((a+b+1)z - c)\frac{\partial}{\partial z} + ab.$$

We find that $\mathcal{L}(K(\beta, z, x)) = 0$ in $H^1_{\text{twist}}(\beta \mid z)$. Since $\mathcal{L}$ commutes with the application of the Pochhammer contour, and Pochhammer integration is zero on exact forms, we recover the hypergeometric equation for $F(a, b, c \mid z)$.

Let (β, z_0) be an admissible quadruple with respect to γ and suppose it is non-resonant. Let M be the field $M = \mathbb{Q}(\beta, z_0)$. Define

$$\widehat{R}(t) = \frac{(b + lt)_l (c - b + (m - l)t)_{m-l}}{(c + mt)_m} R_\gamma(\beta + t\gamma, z_0).$$

Here $(x)_n = x(x+1)\cdots(x+n-1)$ if $n \geq 0$ and $(x)_n = \frac{1}{(x-1)\cdots(x-|n|)}$ if $n < 0$. We can rewrite Eq. (1.1) in terms of Euler kernels as

$$K(\beta + (t+1)\gamma, z_0, x)\mathrm{dx} \equiv \widehat{R}(t) K(\beta + t\gamma, z_0, x)\mathrm{dx} \tag{4.1}$$

in $H^1_{\text{twist}}(\beta + t\gamma \mid z_0)$. Define also

$$g_\gamma(z, x) = \frac{x^l (1-x)^{m-l}}{(1 - zx)^k}.$$

Then, (4.1) amounts to the statement that there exists $W(t, x) \in M(t, x)$ such that

$$g_\gamma(z_0, x) K(\beta + t\gamma, z_0, x) = \widehat{R}(t) K(\beta + t\gamma, z_0, x) + \frac{\partial}{\partial x}\left(W(t, x) K(\beta + t\gamma, z_0, x)\right). \tag{4.2}$$

Define the denominator $d_\gamma(x)$ of $g_\gamma(z_0, x)$ by

$$d_\gamma(x) = x^{(-l)^+}(1-x)^{(l-m)^+}(1 - z_0 x)^{k^+},$$

where u^+ denotes $\max(0, u)$. The numerator $n_\gamma(x)$ is defined by $d_\gamma(x) g_\gamma(z_0, x)$.

Let us write

$$W(t, x) = \frac{p(t, x)}{d_\gamma(x)} x(1-x)(1 - z_0 x),$$

where $p(t,x)$ is another rational function which will turn out to be a polynomial in x. Then, after multiplication by $d_\gamma(x)$ and division by $K(x)$, (4.2) can be rewritten as

$$n_\gamma(x) - \widehat{R}(t)d_\gamma(x) = \frac{\partial}{\partial x}\left(p(t,x)x(1-x)(1-z_0x)\right) + q(t,x)p(t,x), \qquad (4.3)$$

where

$$\begin{aligned} q(t,x) &:= x(1-x)(1-z_0x) \\ &\quad\times\left(\frac{b-1+lt}{x} + \frac{c-b-1+(m-l)t}{x-1} - \frac{a+kt}{x-1/z_0} - \frac{1}{d_\gamma}\frac{\partial d_\gamma}{\partial x}\right). \end{aligned}$$

This is the log-derivative of $K(\beta+t\gamma,z_0,x)/d_\gamma(x)$ times $x(1-x)(1-z_0x)$. Note that $q(t,x)$ is a polynomial in x of degree at most 2 and linear in t. The coefficient of x^2 reads $z(c-a-2+t(m-k)-\deg_x(d_\gamma))$, which is non-zero as a result of our non-resonance condition. Therefore, $q(t,x)$ has degree 2 in x. The non-resonance condition also sees to it that $q(t,x)$ has no zeros in $\{0,1,1/z_0\}$.

We shall write $q(t,x) = q_1(x) + tq_0(x)$. Notice that

$$\begin{aligned} q_0(x) &= x(1-x)(1-z_0x)\left(\frac{l}{x} - \frac{m-l}{1-x} + \frac{kz_0}{1-z_0x}\right) \\ &= (m-k)z_0x^2 + ((k-l)z_0-m)x + l. \end{aligned}$$

In particular, $q_0(x)$ is non-trivial.

It follows from (4.3) that $p(t,x)$ has no poles outside $x=0,1,1/z$. Suppose it has a pole of order $\delta>0$ at $x=0$. Then, looking at the coefficients of $x^{-\delta}$ on both sides of (4.3), we get $0=(1-\delta)+(b+tl-1-n)$, where n is the pole order at $x=0$ of g_γ. This implies $b+lt\in\mathbb{Z}$, contradicting our non-resonance condition. Similarly, we show that $p(t,x)$ has no poles in $x=1,1/z$. Hence, $p(t,x)$ is a polynomial in x. Its degree in x turns out to be at most $\max(\deg_x(d_\gamma),\deg_x(n_\gamma))-2$. For the latter fact, we use the condition $c-a+t(m-k)\notin\mathbb{Z}$.

We may interpret Eq. (4.3) as a system of linear equations in the unknown coefficients of $p(t,x)\in M(t)[x]$, and the unknown $\widehat{R}(t)$. Suppose $p(t,x)$ and $\widehat{R}(t)$ are a non-trivial solution of (4.3). We now like to take $t\to\infty$. We say that a rational function $S(t)$ in t has degree u if $t^{-u}S(t)$ tends to a non-zero limit when $t\to\infty$. Let now u be the degree in t of $p(t,x)$.

Suppose $u\le -2$ and let $t\to\infty$ in (4.3). Then the right-hand side goes to 0 and we get

$$n_\gamma(x) - \lim_{t\to\infty}\widehat{R}(t)d_\gamma(x) = 0.$$

Since $n_\gamma(x)$ and $d_\gamma(x)$ are relatively prime polynomials in x this gives a contradiction. Hence $u\ge 1$.

Now suppose that $u \geq 0$. Multiply (4.3) by t^{-u-1} on both sides and let $t \to \infty$. We obtain

$$-\lim_{t\to\infty} t^{-u-1}\widehat{R}(t)d_\gamma(x) = q_0(x)\lim_{t\to\infty} t^{-u}p(t,x). \tag{4.4}$$

The right-hand side is a non-zero polynomial in x which is divisible by $q_0(x)$, hence $q_0(x)$ divides $d_\gamma(x)$. This means that the zeros of $q_0(x)$ belong to $\{0, 1, 1/z_0\}$. Suppose $q_0(0) = 0$. Then, by definition of q_0, we have $l = 0$. But then the factor x does not occur in $d_\gamma(x)$ and we have a contradiction. A similar argument holds for the zeros $1, 1/z_0$. If $q_0(x)$ has no zeros, it must be constant. In particular $m = k$. Then $d_\gamma(x)$ and $n_\gamma(x)$ have the same degree in x. We have seen that the degree in x of $p(t,x)$ is $\leq \max(\deg_x(n_\gamma), \deg_x(d_\gamma) - 2$ and the latter equals $\deg_x(d_\gamma) - 2$. This implies that the degree in x of $q_0(x)\lim_{t\to\infty} t^{-u}p(t,x)$ is $\leq \deg_x(d_\gamma) - 2$. This is in conflict with (4.4). Hence $q_0(x)$ cannot divide $d_\gamma(x)$, which leads us to conclude that $u < 0$. Together with $u \geq 1$, this implies that $u = -1$.

Set $u = -1$ and take the limit as $t \to \infty$ in (4.3). This gives us

$$n_\gamma(x) - \widehat{R}_0 d_\gamma(x) = q_0(x)\lim_{t\to\infty} tp(t,x),$$

where $\widehat{R}_0 = \lim_{t\to\infty}\widehat{R}(t)$. The latter limit of course exists. When $\widehat{R}_0 = 0$ we see that $q_0(x)$ divides $n_\gamma(x)$, which is impossible by the same argument which showed that $q_0(x)$ does not divide $d_\gamma(x)$. Hence $\widehat{R}_0$ is non-zero, which implies that the numerator and denominator of $\widehat{R}(t)$ have the same degree.

In particular, $q_0(x)$ divides $n_\gamma(x) - \widehat{R}_0 d_\gamma(x)$. Stated alternatively, if $q_0(\xi) = 0$, then $g_\gamma(z_0, \xi) = \frac{n_\gamma(\xi)}{d_\gamma(\xi)} = \widehat{R}_0$. This leads us to the following conclusion.

Theorem 4.1 *Let $\beta + t\gamma, z_0$ with $\gamma = (k, l, m)$ be a non-resonant admissible quadruple. Assuming that $\deg_x q_0(x) = 2$ let x_1, x_2 be the zeros of $q_0(x) := (m-k)z_0x^2 + ((k-l)z_0 - m)x + l$. Then, z_0 has the property that $g_\gamma(z_0, x_1) = g_\gamma(z_0, x_2)$ if $x_1 \neq x_2$ and $g'_\gamma(x_1) = 0$ if $x_1 = x_2$. Moreover, the limit $\widehat{R}_0 := \lim_{t\to\infty}\widehat{R}(t)$ is non-zero and given by $g_\gamma(z_0, x_1)$. Consequently, the factor R_0 in Theorem 2.5 is given by $\frac{m^m}{l^l(m-l)^{m-l}} g_\gamma(z_0, x_1)$.*

Remark 4.2 When $q_0(x)$ is linear, let x_1 be its zero. Then

$$\widehat{R}_0 = \frac{n_\gamma(\infty)}{d_\gamma(\infty)} = \frac{n_\gamma(x_1)}{d_\gamma(x_1)},$$

in other words, $g_\gamma(z_0, \infty) = g_\gamma(z_0, x_1)$, which is compatible with Theorem 4.1 if we consider $x_2 = \infty$ as zero of $q_0(x)$.

When $q_0(x)$ is constant, the degree in x of $n_\gamma(x) - \widehat{R}_0 d_\gamma(x)$ is $\leq \max(\deg_x(d_\gamma), \deg_x(n_\gamma)) - 2$. This determines $\widehat{R}_0$ and z_0. Again note that this is compatible with Theorem 4.1 if we consider ∞ as a double zero of $q_0(x)$.

Remark 4.3 When none of $k, l, m-k, m-l$ is zero, the zeros x_1, x_2 are distinct from $0, 1, \infty$. The condition $g_\gamma(z_0, x_1) = g_\gamma(z_0, x_2)$ is simply the requirement that

$g_\gamma(z_0, x)$ is a Belyi map. By that we mean a rational function such that the set of images of its ramification points consists of at most three points in $\mathbb{P}^1$.

When one of $k, l, m-k, m-l$ is zero, g_γ is automatically a Belyi map, but the condition $g_\gamma(z_0, x_1) = g_\gamma(z_0, x_2)$ still gives a finite number of possibilities for z_0.

Example 4.4 Let us consider the example $\gamma = (1, 1, 6)$. Then, $g_\gamma(z_0, x)$ is a Belyi map if and only if z_0 is one of $4/5, 9/5, (45 \pm 3\sqrt{-15})/50$. Only $z_0 = 4/5$ belongs to an admissible quadruple. The function $g_\gamma(4/5, x)$ is a Belyi map with ramification points at $x_{1,2} = (3 \pm \sqrt{5})/4$. Then $\widehat{R}_0 = g_\gamma(4/5, x_{1,2}) = 5/64$. Hence $R_0 = \frac{m^m}{l^l(m-l)^{m-l}} \widehat{R}_0 = 3^6/5^4$.

5 Kummer's list

Let $x \mapsto g(x)$ be a fractional linear transformation in x that permutes the points $0, 1, \infty$. Then, the substitution $x \mapsto g(x)$ in $K(a, b, c, z, x)\mathrm{dx}$ yields, as a result, a new Euler kernel. For example,

$$
\begin{array}{ll}
g_1(x) = 1/x & \text{gives } z^{-a} K(a, a+1-c, a+1-b, 1/z, x)\mathrm{dx}, \\
g_2(x) = 1-x & \text{gives } (1-z)^{-a} K(a, c-b, c, z/(z-1), x)\mathrm{dx}, \\
g_3(x) = x/(x-1) & \text{gives } K(a, b, a+b+1-c, 1-z, x)\mathrm{dx}, \\
g_4(x) = 1-1/x & \text{gives } z^{-a} K(a, a+1-c, a+b+1-c, 1-1/z, x)\mathrm{dx}, \\
g_5(x) = 1/(1-x) & \text{gives } (1-z)^{-a} K(a, c-b, a+1-b, 1/(1-z), x)\mathrm{dx}.
\end{array}
$$

We can also consider linear fractional transformations in x that permute the four points $0, 1, \infty, 1/z$. These permutations are products of 2-cycles. Up to a constant factor,

$$
\begin{array}{ll}
g_6(x) = 1/zx & \text{gives } z^{1-c} K(b+1-c, a+1-c, 2-c, z, x)\mathrm{dx}, \\
g_7(x) = (x-1/z)/(x-1) & \text{gives } z^{1-c}(1-z)^{c-a-b} K(1-b, 1-a, 2-c, z, x)\mathrm{dx}, \\
g_8(x) = (1-x)/(1-zx) & \text{gives } (1-z)^{c-a-b} K(c-a, c-b, c, z, x)\mathrm{dx}.
\end{array}
$$

Together with the additional substitutions given by $g_i \circ g_j$ for $i = 1, \ldots, 5$ and $j = 6, 7, 8$, we get 24 forms of the shape $\lambda(z) K(a', b', c', h(z), x)\mathrm{dx}$.

Consider the example given by $g_1(x) = 1/x$, which changed $K(a, b, c, z, x)\mathrm{dx}$ into

$$z^{-a} K(a, a+1-c, a+1-b, 1/z, x)\mathrm{dx}.$$

Application of $\mathcal{L}$ to this form yields a vanishing element in H^1_{twist}. As application of the Pochhammer contour yields

$$z^{-a} F(a, a+1-c, a+1-b \mid 1/z),$$

the latter is also a solution to the hypergeometric equation. In this way, the 24 forms obtained from the 24 rational linear transformations are related to the 24 Kummer solutions; the entire list can be seen in the following table.

Table 1 Kummer's 24 transformations

$\lambda(z)$	$h(z)$	a'	b'	c'	Permutation
1	z	a	b	c	(1)
$(1-z)^{c-a-b}$	z	$c-a$	$c-b$	c	(13)(24)
z^{1-c}	z	$b-c+1$	$a-c+1$	$2-c$	(14)(23)
$z^{1-c}(1-z)^{c-a-b}$	z	$1-b$	$1-a$	$2-c$	(12)(34)
z^{-a}	$1/z$	a	$a-c+1$	$a-b+1$	(23)
z^{-b}	$1/z$	$b-c+1$	b	$b-a+1$	(14)
$z^{b-c}(1-z)^{c-a-b}$	$1/z$	$1-b$	$c-b$	$a-b+1$	(1342)
$z^{a-c}(1-z)^{c-a-b}$	$1/z$	$c-a$	$1-a$	$b-a+1$	(1243)
1	$1-z$	a	b	$a+b-c+1$	(34)
$z^{1-c}(1-z)^{c-a-b}$	$1-z$	$1-b$	$1-a$	$c-a-b+1$	(12)
z^{1-c}	$1-z$	$b-c+1$	$a-c+1$	$a+b-c+1$	(1324)
$(1-z)^{c-a-b}$	$1-z$	$c-a$	$c-b$	$c-a-b+1$	(1423)
$(1-z)^{-a}$	$z/z-1$	a	$c-b$	c	(24)
$(1-z)^{-b}$	$z/z-1$	$c-a$	b	c	(13)
$z^{1-c}(1-z)^{c-a-1}$	$z/z-1$	$1-b$	$a-c+1$	$2-c$	(1432)
$z^{1-c}(1-z)^{c-b-1}$	$z/z-1$	$b-c+1$	$1-a$	$2-c$	(1234)
z^{-a}	$1-1/z$	a	$a-c+1$	$a+b-c+1$	(243)
z^{-b}	$1-1/z$	$b-c+1$	b	$a+b-c+1$	(134)
$z^{a-c}(1-z)^{c-a-b}$	$1-1/z$	$c-a$	$1-a$	$c-a-b+1$	(123)
$z^{b-c}(1-z)^{c-a-b}$	$1-1/z$	$1-b$	$c-b$	$c-a-b+1$	(142)
$z^{1-c}(1-z)^{c-a-1}$	$1/1-z$	$1-b$	$a-c+1$	$a-b+1$	(132)
$(1-z)^{-a}$	$1/1-z$	a	$c-b$	$a-b+1$	(234)
$z^{1-c}(1-z)^{c-b-1}$	$1/1-z$	$b-c+1$	$1-a$	$b-a+1$	(124)
$(1-z)^{-b}$	$1/1-z$	$c-a$	b	$b-a+1$	(143)

The last column in Table 1 consists of permutations of S_4 in cycle notation. Its meaning follows from the following proposition.

Proposition 5.1 *The triples a', b', c' in Table 1 have the form*

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} + \rho(\sigma) \begin{pmatrix} a-1/2 \\ b-1/2 \\ c-1 \end{pmatrix},$$

where ρ is a three-dimensional matrix representation of S_4 applied to the element σ which is the entry in the last column of Table 1. Moreover, the representation ρ is equivalent to the representation of S_4 by rotations of the cube tensored with the sign representation of S_4.

Proof To every triple a', b', c' in the table, we form the 4-vector

$$(a' - 1/2, -b' + 1/2, c' - a' - 1/2, b' - c' + 1/2). \tag{5.1}$$

It turns out that the coordinates of these 4-vectors are permutations of each other and that every permutation occurs precisely once. The cycle notation of this permutation is in the last column of Table 1. Moreover, the sum of the coordinates of the 4-vector (5.1) is zero. Let ρ be the restriction of the permutation representation of S_4 to the invariant plane $x_1 + x_2 + x_3 + x_4 = 0$. It is well known that is equivalent to the representation of S_4 by cube rotations tensored with the sign representation. □

In [8,Prop 2.3], we find that to every admissible quadruple there correspond 23 other admissible quadruples, but with possibly different shift vectors $\gamma = (k, l, m)$.

This can be seen as follows. Let β, z_0 be an admissible quadruple with respect to $\gamma = (k, l, m)$. Consider the equality (4.1) which abbreviates to

$$K(\beta + (t+1)\gamma, z_0, x)\mathrm{dx} = \widehat{R}(t)K(\beta + t\gamma, z_0, x)\mathrm{dx}.$$

Apply any one of the 24 permutation actions of Table 1 to this equality. We then get the equality

$$K(\beta' + (t+1)\gamma', g(z_0), x)\mathrm{dx} = \widehat{R}(t)K(\beta' + t\gamma', g(z_0), x)\mathrm{dx},$$

where $\beta' + t\gamma' = (a', b', c')$ and a', b', c' follow from Table 1 when we had started with $(a, b, c) = \beta + t\gamma$. Then, $(\beta', g(z_0))$ is another admissible quadruple with respect to the new shift vector γ'.

Definition 5.2 We call the set of elements $(a', b', c') = \beta' + t\gamma'$, together with the triples (b', a', c'), the Kummer orbit of the line given by $(a, b, c) = \beta + t\gamma$.

Notice that, with the notation of Proposition 5.1, we get $\gamma' = \rho(\sigma)\gamma$. We can find a fundamental domain for these 24 transformations by requiring that $k \geq m - k \geq l - m \geq -l$, hence $m \leq 2k, 2l$ and $k + l \leq 2m$. In particular, this implies that $k, l, m \geq 0$. Note that this choice differs from Ebisu's normalization $0 \leq k + l - m \leq l - k \leq m$, see [8,(1.14)].

Although Ebisu does not state it, we have the impression that he tried to get a list of 'strange evaluations' which is as complete as possible. Recently, Henri Cohen made an extensive search for non-resonant admissible quadruples for all shifts k, l, m with $|k|, |l|, |m| \leq 10$. It turned out that beyond the bound 6, there are no new quadruples.

6 Sample Γ-evaluations

In this section, we collect some examples of Γ-evaluations related to non-resonant admissible quadruples. For the resonant cases, we refer to Sect. 3. Notice that even if one of $k, l, m-k, m-l$ is zero, one may still have a non-resonant quadruple. Below we find several such examples. Each entry is preceded by the corresponding shift vector

k, l, m. In some cases, it may happen that the c-parameter tends to a negative integer when the a or b parameter does (see the final remarks in the introduction). In that case, we also mention what the polynomial interpretation gives as value.

The shift $\gamma = (1, 3, 2)$

$$F(t, 3t-1, 2t \mid e^{\pi i/3}) = -\frac{\sqrt{3}}{2} e^{\pi i(t/2+5//6)} \left(\frac{4}{\sqrt{27}}\right)^t \frac{\Gamma(t+1/2)\Gamma(1/3)}{\Gamma(t+1/3)\Gamma(1/2)}.$$

This is the example that was mentioned on page 4. If $t \in \mathbb{Z}_{\leq 0}$ and the left-hand side is considered as finite sum, then the constant $-\sqrt{3}e^{\pi 5i/3}/2$ must be dropped.

The shift $\gamma = (2, 4, 4)$

$$F(2t, 4t-1/2, 4t \mid -2+\sqrt{8}) = \frac{1}{\sqrt{2}}(1+\sqrt{2})^{4t} \frac{\Gamma(t+1/4)\Gamma(t+3/4)\Gamma(3/8)\Gamma(5/8)}{\Gamma(t+3/8)\Gamma(t+5/8)\Gamma(1/4)\Gamma(3/4)}.$$

When $2t \in \mathbb{Z}_{\leq 0}$ and the left-hand side is considered as polynomial, the right-hand side must be multiplied by $(-1)^{2t}\sqrt{2}$.

The shift $\gamma = (-1, 2, 1)$

$$F(-t, 2t+1, t+4/3 \mid 1/9) = \left(\frac{3}{4}\right)^t \frac{\Gamma(7/6)\Gamma(t+4/3)}{\Gamma(4/3)\Gamma(t+7/6)}.$$

$$F(-t, 2t+2, t+5/3 \mid 1/9) = \left(\frac{3}{4}\right)^t \frac{\Gamma(3/2)\Gamma(t+5/3)}{\Gamma(5/3)\Gamma(t+3/2)}.$$

The shift $\gamma = (-2, 4, 2)$

Let $z_0 = (3+2\sqrt{3})/9$ and $z_1 = (3-2\sqrt{3})/9$

$$F(-2t, 4t+1, 2t+4/3 \mid z_0) = \left(\frac{-27z_1}{16}\right)^t \frac{\cos(\pi(t-1/12))}{\cos(\pi/12)} \frac{\Gamma(t+2/3)\Gamma(t+7/6)\Gamma(3/4)\Gamma(13/12)}{\Gamma(t+3/4)\Gamma(t+13/12)\Gamma(2/3)\Gamma(7/6)}.$$

$$F(-2t, 4t+1, 2t+4/3 \mid z_1) = \left(\frac{27z_0}{16}\right)^t \frac{\Gamma(t+2/3)\Gamma(t+7/6)\Gamma(3/4)\Gamma(13/12)}{\Gamma(t+3/4)\Gamma(t+13/12)\Gamma(2/3)\Gamma(7/6)}.$$

$$F(-2t, 4t+2, 2t+5/3 \mid z_0) = \left(\frac{-27z_1}{16}\right)^t \frac{\cos(\pi(t+1/12))}{\cos(\pi/12)} \frac{\Gamma(t+4/3)\Gamma(t+5/6)\Gamma(5/4)\Gamma(11/12)}{\Gamma(t+5/4)\Gamma(t+11/12)\Gamma(4/3)\Gamma(5/6)}.$$

$$F(-2t, 4t+2, 2t+5/3 \mid z_1) = \left(\frac{27z_0}{16}\right)^t \frac{\Gamma(t+4/3)\Gamma(t+5/6)\Gamma(5/4)\Gamma(11/12)}{\Gamma(t+5/4)\Gamma(t+11/12)\Gamma(4/3)\Gamma(5/6)}.$$

The shift $\gamma = (-1, -1, 1)$

$$F(-t, -t+1/3, t+4/3 \mid -1/8) = \left(\frac{27}{32}\right)^t \frac{\Gamma(t+4/3)\Gamma(7/6)}{\Gamma(t+7/6)\Gamma(4/3)}.$$

The shift $\gamma = (-2, -2, 2)$

Let $z_0 = (3\sqrt{3} - 5)/4$

$$F(-2t, -2t + 1/3, 2t + 4/3 \mid z_0) = \left(\frac{81\sqrt{3}}{128}\right)^t \frac{\Gamma(t + 2/3)\Gamma(t + 7/6)\Gamma(3/4)\Gamma(13/12)}{\Gamma(t + 3/4)\Gamma(t + 13/12)\Gamma(2/3)\Gamma(7/6)}.$$

The shift $\gamma = (0, 1, 3)$

$$F(1/2, t, 3t - 1 \mid 3/4) = \frac{1}{3} \frac{\Gamma(t + 1/3)\Gamma(t - 1/3)\Gamma(1/6)\Gamma(-1/6)}{\Gamma(t + 1/6)\Gamma(t - 1/6)\Gamma(1/3)\Gamma(-1/3)}.$$

When $t \in \mathbb{Z}_{\leq 0}$ and the left-hand side is considered polynomial, the factor $1/3$ should be dropped.

The shift $\gamma = (1, 3, 1)$

$$F(t, 3t - 3/2, t + 1/2 \mid 4) = \frac{6e^{2\pi it}\cos(\pi t)}{27^t} \frac{\Gamma(t + 1/2)\Gamma(t - 1/2)}{\Gamma(t - 1/6)\Gamma(t + 1/6)}.$$

The shift $\gamma = (-1, 3, 2)$

$$F(-t, 3t + 1, 2t + 3/2 \mid 1/4) = \left(\frac{16}{27}\right)^t \frac{\Gamma(t + 5/4)\Gamma(t + 3/4)\Gamma(7/6)\Gamma(2/3)}{\Gamma(t + 7/6)\Gamma(t + 2/3)\Gamma(5/4)\Gamma(3/4)}.$$

$$F(-t, 3t + 2, 2t + 9/4 \mid -1/8) = \left(\frac{32}{27}\right)^t \frac{\Gamma(t + 13/8)\Gamma(t + 9/8)\Gamma(4/3)\Gamma(17/12)}{\Gamma(t + 4/3)\Gamma(t + 17/12)\Gamma(13/8)\Gamma(9/8)}.$$

The shift $\gamma = (3, 3, 4)$

$$F(3t, 3t + 1/2, 4t + 2/3 \mid 8/9) = 108^t \frac{\Gamma(t + 11/12)\Gamma(t + 5/12)\Gamma(1/2)\Gamma(5/6)}{\Gamma(t + 1/2)\Gamma(t + 5/6)\Gamma(11/12)\Gamma(5/12)}.$$

When $t \in -1/6 + \mathbb{Z}_{\leq 0}$ and the left-hand side is considered polynomial, one must multiply the result by 2. When $t \in -2/3 + \mathbb{Z}_{\leq 0}$ and the left-hand side is considered polynomial, one must multiply the result by -2.

$$F(3t, 3t + 1/4, 4t + 1/3 \mid 8/9) = 108^t \frac{\Gamma(t + 7/12)\Gamma(t + 5/6)\Gamma(3/4)\Gamma(2/3)}{\Gamma(t + 3/4)\Gamma(t + 2/3)\Gamma(7/12)\Gamma(5/6)}.$$

When $t \in \{-1/12, -1/3\} + \mathbb{Z}_{\leq 0}$ and the left-hand side is considered polynomial, one must multiply the right-hand side by $108^{1/12} \frac{\Gamma(5/6)\Gamma(7/12)^2}{\Gamma(1/2)\Gamma(3/4)^2}$.

$$F(3t, 3t - 1/2, 4t \mid 4/3) = \frac{1}{4}(1 - \sqrt{-3})e^{2\pi it} 16^t \frac{\Gamma(t + 1/4)\Gamma(t + 3/4)\Gamma(1/6)\Gamma(2/3)}{\Gamma(t + 1/6)\Gamma(t + 2/3)\Gamma(1/4)\Gamma(3/4)}$$

When $t \in \mathbb{Z}_{\leq 0}$ and the left-hand side considered polynomial, we must drop the factor $(1 - \sqrt{-3})/4$.

The shift $\gamma = (2, 1, 0)$

$$F(2t, t + 1/6, 2/3 \mid -8) = \frac{2}{\sqrt{3} \cdot 27^t} \sin(\pi(t + 1/3)),$$

$$F(2t, t + 1/3, 4/3 \mid -8) = \frac{2\cos(\pi(t + 1/3))}{27^t} \frac{\Gamma(t - 1/6)\Gamma(1/2)}{\Gamma(t + 1/2)\Gamma(-1/6)}.$$

The shift $\gamma = (3, 1, 0)$

$$F(3t, t + 1/6, 1/2 \mid 9) = \frac{1}{2 \cdot 64^t} \left(1 + e^{2\pi i(t+1/6)} - e^{4\pi i(t+1/6)}\right),$$

$$F(3t, t + 1/2, 3/2 \mid 9) = \frac{-1}{6\sqrt{3} \cdot 64^t} (1 - e^{2\pi i t} + e^{4\pi i t}) \frac{\Gamma(t - 1/6)\Gamma(t + 1/6)}{\Gamma(t + 1/3)\Gamma(t + 2/3)}.$$

The first line is proven in Corollary 2.10. Unfortunately, the second line cannot be proven in this manner because we do not have enough special values of t with an elementary evaluation. The result is a conjecture which was found experimentally. Furthermore, we found

$$F(3t, t + 1/6, 1/2 \mid -3) = \frac{1}{16^t} \cos(\pi t) \frac{\Gamma(t + 1/2)\Gamma(1/3)}{\Gamma(t + 1/3)\Gamma(1/2)},$$

$$F(3t, t + 1/2, 3/2 \mid -3) = \frac{2}{16^t} \cos(\pi(t + 1/3)) \frac{\Gamma(t - 1/6)\Gamma(2/3)}{\Gamma(t + 2/3)\Gamma(-1/6)}$$

The shift $\gamma = (1, 1, 0)$

Strictly speaking, there is no admissible quadruple with respect to $(1, 1, 0)$. However, we do like to recall the following classical identity

$$F(t, t + 1/2, 1/2 | z^2) = \frac{1}{2} \left((1 + z)^{-2t} + (1 - z)^{-2t}\right).$$

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