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Asymptotic expansion of Fourier coefficients of reciprocals of Eisenstein series

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Abstract
In this paper we give a classification of the asymptotic expansion of the q-expansion of reciprocals of Eisenstein series E_k of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$. For $k \geq 12$ even, this extends results of Hardy and Ramanujan, and Berndt, Bialek, and Yee, utilizing the Circle Method on the one hand, and results of Petersson, and Bringmann and Kane, developing a theory of meromorphic Poincaré series on the other. We follow a uniform approach, based on the zeros of the Eisenstein series with the largest imaginary part. These special zeros provide information on the singularities of the Fourier expansion of $1/E_k(z)$ with respect to $q = e^{2\pi i z}$.

Keywords Eisenstein series · Fourier coefficients · Meromorphic modular forms · Polynomials · Ramanujan · Recurrence relations

Mathematics Subject Classification Primary 11F30 · 11M36 · 26C10 · Secondary 05A16 · 11B37

1 Introduction

In this paper we provide a new approach to determine the main asymptotic growth terms in the Fourier expansion of the reciprocals $1/E_k$ of Eisenstein series of weight k:

$$\frac{1}{E_k(z)} = \sum_{n=0}^{\infty} \beta_k(n)\, q^n \quad (q := e^{2\pi i z}).$$

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We refer to [4], Chapter 15 for a very good introduction into the topic. Eisenstein series are defined by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n.$$

They are modular forms [15] on the upper half of the complex plane $\mathbb{H}$. The algebra of modular forms with respect to the modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by E_4 and E_6. As usual B_k denotes the kth Bernoulli number and $\sigma_\ell(n) := \sum_{d|n} d^\ell$.

Hardy and Ramanujan [8] launched, in their last joint paper, the study of coefficients of meromorphic modular forms with a simple pole in the standard fundamental domain $\mathbb{F}$. They demonstrated that, similar to their famous asymptotic formula for the partition numbers

$$p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{\frac{2}{3}n}}, \qquad \sum_{n=0}^{\infty} p(n)\, q^n := \frac{q^{\frac{1}{24}}}{\eta(z)},$$

which had been given birth to the Circle Method [7], formulas for the coefficients of reciprocals of modular forms can be obtained. The reciprocal of the Dedekind η-function is a weakly modular form of weight $-1/2$ on $\mathbb{H}$.

Hardy and Ramanujan focused on the reciprocal of the Eisenstein series E_6. They proved an explicit formula for the coefficients. Shortly afterwards, in a letter to Hardy, Ramanujan stated several formulas of the same type, including the q-expansion of $1/E_4$. No proofs were given.

Bialek in his Ph.D. thesis, written under the guidance of Berndt [2], and finally Berndt, Bialek, and Yee [3] have proven the claims in the letter of Ramanujan by extending the methods applied in [8].

We illustrate the case $k = 4$. Following Ramanujan, we frequently put $E_k(q_z) := E_k(z)$ for $q = q_z := e^{2\pi i z}$. Let ρ be the unique zero of E_4 in $\mathbb{F}$. Let λ run over the integers of the form $3^\alpha \prod_{\ell=1}^r p_\ell^{\alpha_\ell}$, where $\alpha = 0$ or 1. Here, p_ℓ is a prime of the form $6m + 1$, and $\alpha_j \in \mathbb{N}_0$. Then [2]:

$$\beta_4(n) = (-1)^n \frac{3}{E_6(q_\rho)} \sum_{(\lambda)} \sum_{(c,d)} \frac{h_{(c,d)}(n)}{\lambda^3}\, e^{\frac{\pi n\sqrt{3}}{\lambda}}. \tag{1}$$

Here, $(c, d) \neq (0, 0)$ and coprime, runs over *distinct* solutions to $\lambda = c^2 - cd + d^2$. Let (a, b) be such that $ad - bc = 1$. Let $h_{(1,0)}(n) := 1$, $h_{(2,1)}(n) := (-1)^n$, and for $\lambda \geq 7$:

$$h_{(c,d)}(n) := 2\cos\left((ad + bc - 2ac - 2bd + \lambda)\frac{\pi n}{\lambda} - 6\arctan\left(\frac{c\sqrt{3}}{2d - c}\right)\right).$$

For the definition of distinct we refer to [2, Sect. 3]. From the explicit formula (1) one observes that the main asymptotic growth comes from $(c, d) = (1, 0)$. This yields ([5], Introduction):

$$\beta_4(n) \sim (-1)^n \frac{3}{E_6(\rho)}\, e^{\pi n \sqrt{3}}, \tag{2}$$

$$\beta_6(n) \sim \frac{2}{E_8(i)}\, e^{2\pi n}, \tag{3}$$

where $\sum_{n=0}^{\infty} \beta_k(n)\, q^n := \frac{1}{E_k(z)}$. We added the asymptotic (3), which can be obtained in a similar way.

Petersson [16] offered an alternative approach to study the q-expansion of meromorphic modular forms. He defined Poincaré series with poles at arbitrary points in $\mathbb{H}$ and of arbitrary order, to provide a basis for the underlying vector spaces. Recently, Bringmann and Kane [5] have generalized Petersson's method. They have also recorded several important examples.

In this paper we study the asymptotic expansions for all reciprocals of Eisenstein series. Instead of proving first an explicit formula and then detecting the main growth terms, we provide a direct approach. This is based on the distribution of the zeros in the standard fundamental domain with the largest imaginary part. For the convenience of the reader, we recall some basic idea from complex analysis ([20], [17, Chap. 7, Sect. 5, task 242]). Let $f(q) = \sum_{n=0}^{\infty} a(n)\, q^n$ be a power series regular at $q = 0$ with finite radius of convergence. Assume that there is only one singular point q_0 on the circle of convergence. Let at q_0 be a pole. Then it is known that

$$\lim_{n \to \infty} \frac{a(n)}{a(n+1)} = q_0. \tag{4}$$

This follows from the Laurent expansion of $f(q)$, which has a finite principal part.

Before we state our results, we want to point out as a warning that the limits as $n \to \infty$ for $\beta_4(n)/\beta_4(n+1)$ and $\beta_6(n)/\beta_6(n+1)$ exist, but that this is maybe not true for all k as provided by the data in Table 1.

Table 1 Quotients of successive coefficients of $1/E_k$ for $k \in \{4, 6, 12, 14\}$

n	$\frac{\beta_4(n)}{\beta_4(n+1)} \approx$	$\frac{\beta_6(n)}{\beta_6(n+1)} \approx$	$\frac{\beta_{12}(n)}{\beta_{12}(n+1)} \approx$	$\frac{\beta_{14}(n)}{\beta_{14}(n+1)} \approx$
1	$-4.3290 \cdot 10^{-3}$	$1.8622 \cdot 10^{-3}$	$5.1172 \cdot 10^{-4}$	$1.2170 \cdot 10^{-4}$
2	$-4.3333 \cdot 10^{-3}$	$1.8677 \cdot 10^{-3}$	$-9.6536 \cdot 10^{-3}$	$4.1330 \cdot 10^{-3}$
3	$-4.3334 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$5.4260 \cdot 10^{-4}$	$1.1240 \cdot 10^{-3}$
4	$-4.3334 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$-8.9832 \cdot 10^{-3}$	$2.3564 \cdot 10^{-3}$
5	$-4.3334 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$5.8359 \cdot 10^{-4}$	$1.6491 \cdot 10^{-3}$
6	$-4.3334 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$-8.3936 \cdot 10^{-3}$	$1.9821 \cdot 10^{-3}$
7	$-4.3334 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$6.2477 \cdot 10^{-4}$	$1.8133 \cdot 10^{-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	$-4.3334 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$8.8114 \cdot 10^{-4}$	$1.8674 \cdot 10^{-3}$
20	$-4.3334 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$-5.6773 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$

2 Results

The constants in the asymptotic expansion of $\beta_k(n)$, the coefficients of the q-expansion of the reciprocal of E_k, involve the Ramanujan Θ-operator [6, 18] induced by residue calculation. The differential operator $\Theta := q \frac{\mathrm{d}}{\mathrm{d}q}$ acts on formal power series by

$$\Theta\left(\sum_{n=h}^{\infty} a(n)\, q^n\right) := \sum_{n=h}^{\infty} n\, a(n)\, q^n.$$

Let $E_2(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, q^n$. Ramanujan observed that

$$\Theta(E_4) = (E_4E_2 - E_6)/3 \text{ and } \Theta(E_6) = (E_6E_2 - E_8)/2.$$

Our first results give an explicit interpretation of the data presented in Table 1 for $k = 6$ and $k = 14$.

Theorem 1 *Let $k \geq 4$ and $k \equiv 2 \pmod 4$ be an integer. Then $1/E_k$ has a q-expansion with radius $q_i = e^{-2\pi}$:*

$$\frac{1}{E_k(q)} = \sum_{n=0}^{\infty} \beta_k(n)\, q^n.$$

The coefficients $\beta_k(n)$ are non-zero and have the asymptotic expansion

$$\beta_k(n) \sim -\frac{1}{\Theta(E_k)(q_i)}\; q_i^{-n}.$$

The number $q_i = e^{-2\pi} \approx 1.867443 \cdot 10^{-3}$ is transcendental. It is well-known that the so-called Gel′fond constant e^{π} is transcendental. This was first proven by Gel′fond in 1929. It can also be deduced from the Gel′fond–Schneider Theorem, which solved Hilbert's seventh problem [21]. We refer to a result by Nesterenko (also [21, Sect. 5.6]). Let $z \in \mathbb{H}$. Then at least already three of the four numbers

$$q_z, E_2(q_z), E_4(q_z), \text{ and } E_6(q_z)$$

are algebraically independent. Since $E_4(q_\rho) = E_6(q_i) = 0$, we obtain that $q_i, E_4(q_i)$ and $q_\rho, E_6(q_\rho)$ are transcendental.

Moreover, $\Theta(E_k)(q_i)$ for $k = 6, 10, 14$ can be explicitly expressed by $\Gamma(\frac{1}{4})$ and π. For example,

$$\Theta(E_6)(q_i) = -\frac{1}{2} E_4(q_i)^2, \quad \text{where } E_4(q_i) = \frac{3\,\Gamma(\frac{1}{4})^8}{(2\pi)^6}.$$

We can also extract the numbers q_i and $E_4(q_i)$ from the coefficients.

Corollary 1 *Let $k \geq 4$ and $k \equiv 2 \pmod 4$. Then*

$$\lim_{n\to\infty} \frac{\beta_k(n)}{\beta_k(n+1)} = q_i, \tag{5}$$

$$\lim_{n\to\infty} \frac{\beta_6(n)}{\beta_{10}(n)} = \lim_{n\to\infty} \frac{\beta_{10}(n)}{\beta_{14}(n)} = E_4(q_i). \tag{6}$$

Hardy and Ramanujan stated lower and upper bounds at the end of their initial work [8] on the coefficients of the reciprocal of $1/E_6$. We generalize their idea to all cases $k \equiv 2 \pmod 4$ including $k = 2$ and also improve their result in the original case $k = 6$.

Theorem 2 *Let $k \equiv 2 \pmod 4$ and k a positive integer. Let $x(k) := \frac{2k}{B_k}$. Then we have for all $n \in \mathbb{N}$*

$$\frac{\left(\frac{x(k)+\sqrt{\Delta_k}}{2}\right)^{n+1} - \left(\frac{x(k)-\sqrt{\Delta_k}}{2}\right)^{n+1}}{\sqrt{\Delta_k}} \leq \beta_k(n)$$

with $\Delta_k = x(k)^2 + 4\left(2^{k-1}+1\right)x(k)$ and

$$\beta_k(n) \leq \frac{\left(x(k) - \frac{b_k-\sqrt{D_k}}{2}\right)\left(\frac{b_k+\sqrt{D_k}}{2}\right)^n + \left(\frac{b_k+\sqrt{D_k}}{2} - x(k)\right)\left(\frac{b_k-\sqrt{D_k}}{2}\right)^n}{\sqrt{D_k}} \tag{7}$$

with $b_k = x(k) + a_k$, $c_k = \left(2^{k-1}+1-a_k\right)x(k)$, and $D_k = b_k^2 + 4c_k$ for all k where $a_2 = \sqrt{7/3}$ and $a_k = \frac{3^{k-1}+1}{2^{k-1}+1}$ for $k \geq 6$.

The case $k \equiv 0 \pmod 4$ is more complicated. For large k, we cannot expect that the limit as $n \to \infty$ of $\beta_k(n)/\beta_k(n+1)$ exists, since we have two poles on the circle of convergence. But for $k = 4$ and $k = 8$ there is still only one pole.

Proposition 1 *Let $q_\rho = e^{2\pi\rho} = -e^{-\pi\sqrt{3}}$. Let $m \in \mathbb{N}$. Then the coefficients $\beta_{4,m}(n)$ of the mth power of E_4^{-1} i.e.*

$$\sum_{n=0}^{\infty} \beta_{4,m}(n)\, q^n := \left(\frac{1}{E_4(q)}\right)^m$$

satisfy for all m:

$$\lim_{n\to\infty} \frac{\beta_{4,m}(n)}{\beta_{4,m}(n+1)} = q_\rho.$$

Remarks

(a) For small weights the following identities exist:

$$E_8 = E_4^2,\ E_{10} = E_4 \cdot E_6 \text{ and } E_{14} = E_4^2 \cdot E_6. \tag{8}$$

(b) Let the principal part of E_4^{-m} at the pole q_ρ be given by

$$\sum_{k=1}^{m} \frac{\lambda_{m,k}}{\left(q - q_\rho\right)^k}, \tag{9}$$

then $\lambda_{m,m} = \mathrm{res}_{q_\rho}\left(E_4^{-1}\right)^m$. It would be interesting to get explicit formulas for all $\lambda_{m,k}$, $1 \leq k \leq m$. Especially for the case $m = 2$.
(c) We have $\mathrm{res}_{q_\rho}(E_4^{-1}) = \frac{-3\,q_\rho}{E_6(q_\rho)}$.

We know that $\beta_4(n)$ and $\beta_8(n)$ are non-zero for all $n \in \mathbb{N}_0$ [9]. We provide new proof of the asymptotic expansion for $k = 4$. This is the main term of a formula first conjectured by Ramanujan and proven about 80 years later by Bialek [2]. For the case $k = 8$, we also refer to [5].

Theorem 3 *We have* $(-1)^n \beta_4(n) \in 240\mathbb{N}$ *for all* $n \in \mathbb{N}$. *Further, we have the asymptotic expansion*

$$\beta_4(n) \sim -\frac{1}{\Theta(E_4)(q_\rho)}\, q_\rho^{-n},$$

where $\Theta(E_4)(q_\rho) = -E_6(q_\rho)/3$.

F. K. C. Rankin and H. P. F. Swinnerton-Dyer [19] have proven that all the zeros of $E_k(z)$ in the standard fundamental domain $\mathbb{F}$ are in $C = \{z \in \mathbb{F} \,:\, |z| = 1\} \subset \mathbb{F}$. We recall the following basic facts [15]. The modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ operates on the complex upper half plane $\mathbb{H}$, denoted by $\gamma(z)$, where $\gamma \in \Gamma$ and $z \in \mathbb{H}$. The standard fundamental domain $\mathbb{F}$ is given by

$$\begin{aligned} \mathbb{F} = \{z \in \mathbb{H} \,:\, |z| \geq 1 \text{ and } 0 \leq \mathrm{Re}\,(z) \leq 1/2\} \\ \cup \{z \in \mathbb{H} \,:\, |z| > 1 \text{ and } -1/2 < \mathrm{Re}\,(z) < 0\}. \end{aligned}$$

Proposition 2 (Rankin, Swinnerton-Dyer [19]) *Let* $k \geq 4$ *be an even integer. Let* z_k *be the zero of* E_k *with the largest imaginary part. Then*

$$z_4 = z_8 = \rho \text{ and } z_k = i \text{ for } k \equiv 2 \pmod 4.$$

All other k *satisfy* $z_k \in C \setminus \{i, \rho\}$. *Only for* $k = 8$ *the zero* z_k *is not simple.*

Further, from [19] and Kohnen [12] we obtain the following.

Corollary 2 *Let* $k \geq 12$ *and* $k \equiv 0 \pmod 4$. *Let* $k = 12\,N + s$ *for* $s \in \{0, 4, 8\}$. *Then* $z_k = e^{\frac{1}{2}\pi i\,\varphi}$, *where* $\varphi \in \left(\frac{N-1}{N}, 1\right)$.

Theorem 4 *Let k be a positive integer. Let $k \geq 12$ and $k \equiv 0 \pmod 4$. Then $1/E_k$ has a q-expansion with radius $|q_{z_k}|$, where z_k is the zero of E_k with the largest imaginary part. Then*

$$\beta_k(n)\, q_{z_k}^n + \frac{1}{\Theta(E_k)(q_{z_k})} + \frac{1}{\Theta(E_k)(\overline{q}_{z_k})} \left(\frac{q_{z_k}}{\overline{q}_{z_k}} \right)^n$$

constitutes a null sequence.

The expression

$$\frac{1}{\Theta(E_k)(q_{z_k})} + \frac{1}{\Theta(E_k)(\overline{q}_{z_k})} \left(\frac{q_{z_k}}{\overline{q}_{z_k}} \right)^n \tag{10}$$

is bounded. But this is not sufficient to obtain an asymptotic expansion. Nevertheless we have discovered a new property of the coefficients of $1/E_k$ for $k \equiv 0 \pmod 4$.

Theorem 5 *Let $k \equiv 0 \pmod 4$ and $k \geq 12$. Then there exists a subsequence $\{n_t\}_{t=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that*

$$\lim_{t \to \infty} \frac{\beta_k(n_t)}{-q_{z_k}^{-n_t} \left(\frac{1}{\Theta(E_k)(q_{z_k})} + \frac{1}{\Theta(E_k)(\overline{q}_{z_k})} \left(\frac{q_{z_k}}{\overline{q}_{z_k}} \right)^{n_t} \right)} = 1.$$

The statement of this theorem is equivalent to

$$\lim_{t \to \infty} \frac{\beta_k(n_t)}{-2\,\mathrm{Re}\left(\frac{q_{z_k}^{-n_t}}{\Theta(E_k)(q_{z_k})} \right)} = 1.$$

We have the following further properties.

Theorem 6 *Let k be a positive integer. Let $k \geq 12$ and $k \equiv 0 \pmod 4$.*

(1) Let $A_k(n)$ denote the number of changes of sign in the sequence $\{\beta_k(m)\}_{m=0}^{n}$ and let $z_k = x_k + i\, y_k \in \mathbb{F}$ be the zero of E_k with the largest imaginary part. Then

$$\lim_{n \to \infty} \frac{A_k(n)}{n} = 2x_k.$$

(2) Let $B_k(n)$ be the number of non-zero coefficients among the n coefficients $\{\beta_k(m)\}_{m=0}^{n-1}$. Then

$$\limsup_{n \to \infty} \frac{n}{B_k(n)} \leq 2.$$

Combining Theorem 6 (1) with Corollary 2 leads to the following result, which is a priori surprising.

Corollary 3 *For large weights k divisible by* 4*, the coefficients of* $1/E_k(q)$ *satisfy*

$$\lim_{\ell \to \infty} \lim_{n \to \infty} \frac{A_{4\ell}(n)}{n} = 0.$$

3 Proofs

3.1 Proof of Proposition 1, Theorem 1, and Corollary 1

Let $E_k(q)$ have exactly one zero $q_0 \in B_1(0)$ with absolute value smaller than all other zeros. Then we obtain the property (4) for the coefficients of $1/E_k$. Note that every zero of a modular form has one representative in the fundamental domain $\mathbb{F}$.

The zeros of E_k are controlled by a theorem by Rankin and Swinnerton-Dyer ([19], see also Sect. 2). They proved that every zero in $\mathbb{F}$ has absolute value 1. Further, let k be a positive, even integer and $k \geq 4$. Let $k = 12N + s$, where $s \in \{4, 6, 8, 10, 0, 14\}$. Then E_k has N simple zeros in $C \setminus \{i, \rho\}$. Additionally we have simple zeros ρ for $s = 4$ and i for $s = 6$. Further, E_k has the double zero ρ for $s = 8$, the simple zeros i and ρ for $s = 10$, and the simple zero i and the double zero ρ for $s = 14$. Further, let z_k be the zero of E_k with the largest imaginary part. Note that

$$z_k' := S(z_k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z_k$$

and z_k have the same imaginary part. Note that $S(i) = i$ and $S(\rho) = \rho - 1$. Thus, $1/E_k$ has exactly one pole on the radius of convergence iff $z_k = i$ or $z_k = \rho$.

Proof of Proposition 1 Since $(1/E_4)^m$ has only the pole at q_ρ on the circle of convergence, again we have formula (4), which proves the proposition. □

Proof of Theorem 1 Let w be any complex number. Let $B_r(w) = \{z \in \mathbb{C} : |z - w| < r\}$ be the open ball with radius r around w. We denote the closure by $\overline{B_r(w)}$ and its boundary by $\partial B_r(w)$. Let $k \equiv 2 \pmod 4$. Then E_k has the special property that restricted to $\overline{B_{|q_i|}(0)}$ it has exactly one zero at q_i, which is also simple. This implies that the Taylor series expansion of the reciprocal of E_k has radius of convergence $|q_i|$ and only a simple pole at q_i:

$$\frac{1}{E_k(q)} = \sum_{n=0}^{\infty} \beta_k(n)\, q^n \qquad (|q| < |q_i|).$$

Note that subtracting the principal part at q_i provides a new Taylor series expansion with a larger radius of convergence:

$$\frac{1}{E_k(q)} - \frac{\operatorname{res}_{q_i}(1/E_k)}{q - q_i} = \sum_{n=0}^{\infty} b(n)\, q^n.$$

This implies that $b(n)q_i^n$ constitutes a null sequence. Here, $\mathrm{res}_{q_i}(1/E_k)$ denotes the residue at q_i. We obtain that

$$q_i^{n+1}\beta_k(n) + \mathrm{res}_{q_i}(1/E_k)$$

constitutes a null sequence. By a standard argument, we obtain that

$$\mathrm{res}_{q_i}(1/E_k) = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}q}E_k(q_i)}.$$

Finally, we obtain the asymptotic behavior

$$\beta_k(n) \sim -\frac{1}{\Theta(E_k)(q_i)}\, q_i^{-n}.$$

□

Proof of Corollary 1 From the theorem of Rankin and Swinnerton-Dyer we obtain that for $k \equiv 2 \pmod 4$ we have $z_k = i$ and $q_i = e^{-2\pi}$. This gives a first proof of equation (5) of Corollary 1. Note that equation (5) of Corollary 1 also follows directly from Theorem 1. The quotients for small k converge very quickly. We refer to Table 1 and Table 2.

Table 2 Quotients of successive coefficients of $1/E_k$ for $k \in \{8, 10, 12, 14, 16\}$

n	$\frac{\beta_8(n)}{\beta_8(n+1)} \approx$	$\frac{\beta_{10}(n)}{\beta_{10}(n+1)} \approx$	$\frac{\beta_{12}(n)}{\beta_{12}(n+1)} \approx$	$\frac{\beta_{14}(n)}{\beta_{14}(n+1)} \approx$	$\frac{\beta_{16}(n)}{\beta_{16}(n+1)} \approx$
17	$-4.1044 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$8.3715 \cdot 10^{-4}$	$1.8674 \cdot 10^{-3}$	$1.6465 \cdot 10^{-3}$
18	$-4.1159 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$-5.9626 \cdot 10^{-3}$	$1.8675 \cdot 10^{-3}$	$-1.7502 \cdot 10^{-2}$
19	$-4.1263 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$8.8114 \cdot 10^{-4}$	$1.8674 \cdot 10^{-3}$	$2.3584 \cdot 10^{-4}$
20	$-4.1357 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$-5.6773 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$3.8543 \cdot 10^{-3}$
21	$-4.1443 \cdot 10^{-3}$	$1.8674 \cdot 10^{-3}$	$9.2572 \cdot 10^{-4}$	$1.8674 \cdot 10^{-3}$	$-1.8095 \cdot 10^{-3}$

Table 3 Quotients of $\beta_6(n)$ and $\beta_{10}(n)$

n	$\frac{\beta_6(n)}{\beta_{10}(n)} \approx$
0	1.00000000000000000000000000000
1	1.90909090909090909090909090909091
2	1.31941031941031941031941031941 0
3	1.52371574417743125618898706028 5
4	1.42830953430494633559851401901 3
⋮	…
80	1.45576289226870932246242200359 4
90	1.45576289226870932246242200359 9
100	1.45576289226870932246242200359 9

Since $\Theta(E_6)(q_i) = -\frac{1}{2}E_4(q_i)^2$ and $\Theta(E_{10})(q_i) = -\frac{1}{2}E_4(q_i)^3$, the second part of the Corollary also follows from Theorem 1 and (8). An approximate numerical value of $E_4(q_i)$ can be read off Table 3. The theorem by Nesterenko implies that this number is transcendental, since $E_6(q_i) = 0$. □

Note that for each integer $\ell \geq 2$, the limit as $n \to \infty$ of $\frac{\beta_{4\ell-2}(n)}{\beta_{4\ell+2}(n)}$ exists, but it is generally not equal to $E_4(q_i)$.

3.2 Proof of Theorem 2

We use the following easy to prove lemmata.

Lemma 1 $\sigma_\ell\,(n) < \frac{\ell}{\ell-1}n^\ell$ *for* $\ell > 1$ *and* $\sigma_1\,(n) \leq (1+\ln n)\,n$.

Proof $\sigma_\ell\,(n) \leq \left(1+\int_1^n t^{-\ell}\,\mathrm{d}t\right)n^\ell < \frac{\ell}{\ell-1}n^\ell$ for $\ell > 1$ and $\leq (1+\ln n)\,n$ for $\ell = 1$. □

Lemma 2 *For* $\ell \geq 5$ *it holds that* $3\sqrt[\ell]{\frac{1+3^{-\ell}}{1+2^{-\ell}}} > 2.98$.

Proof Considering ℓ as a real variable ≥ 5, we obtain the following logarithmic derivative

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\ell}\frac{1}{\ell}\ln\left(\frac{1+3^{-\ell}}{1+2^{-\ell}}\right)\\&\quad=-\frac{1}{\ell^2}\ln\left(\frac{1+3^{-\ell}}{1+2^{-\ell}}\right)+\frac{1}{\ell}\frac{1+2^{-\ell}}{1+3^{-\ell}}\left(-\frac{3^{-\ell}\ln 3}{1+3^{-\ell}}+\frac{2^{-\ell}\ln 2}{1+2^{-\ell}}\right)>0\end{aligned}$$

since $-\frac{\ln 3}{3^\ell+1}+\frac{\ln 2}{2^\ell+1} > -\frac{\ln 3}{3^\ell}+\frac{\ln 2}{2^{\ell+1}} > 0$ for $\ell \geq 5$. Therefore, the values of the original sequence are increasing and we take the smallest value for $\ell = 5$. □

Proof of Theorem 2 With $\varepsilon_k\,(n) = \frac{2k}{B_k}\sigma_{k-1}\,(n)$ we obtain

$$E_k\,(z) = 1 - \sum_{n=1}^{\infty}\varepsilon_k\,(n)\,q^n.$$

Let $1/\left(1-\varepsilon_k\,(1)\,q-\varepsilon_k\,(2)\,q^2\right) = \sum_{n=0}^{\infty}\alpha_k\,(n)\,q^n$. The $\alpha_k\,(n)$ fulfill the recurrence relation $\alpha_k\,(n) = \varepsilon_k\,(1)\,\alpha_k\,(n-1) + \varepsilon_k\,(2)\,\alpha_k\,(n-2)$ for $n \geq 2$. Obviously, $\alpha_k\,(0) = \beta_k\,(0)$, $\alpha_k\,(1) = \beta_k\,(1)$, and by induction $\alpha_k\,(n) = \varepsilon_k\,(1)\,\alpha_k\,(n-1) + \varepsilon_k\,(2)\,\alpha_k\,(n-2) \leq \sum_{j=1}^{n}\varepsilon_k\,(j)\,\beta_k\,(n-j) = \beta_k\,(n)$ using the power series expansion of $1/E_k$.

For the upper bound let $a_2 = \sqrt{7/3}$ and for $k \geq 6$ let

$$a_k = \frac{\varepsilon_k\,(3)}{\varepsilon_k\,(2)} = \frac{\sigma_{k-1}\,(3)}{\sigma_{k-1}\,(2)} = \frac{3^{k-1}+1}{2^{k-1}+1}.$$

For all $k \equiv 2 \pmod 4$ let $b_k = a_k + \varepsilon_k(1)$, $c_k = -\varepsilon_k(2) - a_k\varepsilon_k(1)$, and $\frac{1-b_kq-c_kq^2}{1-a_kq} = 1 - \sum_{n=1}^{\infty} \delta_k(n) q^n$. Therefore, $\delta_k(1) = b_k - a_k = \varepsilon_k(1)$, $\delta_k(2) = c_k + a_k\delta_k(1) = \varepsilon_k(2)$, and $\delta_k(n) = a_k\delta_k(n-1)$ for $n \geq 3$. Therefore $\delta_k(n) = \varepsilon_k(2) a_k^{n-2}$.

(1) First, let $k = 2$. Then $\delta_2(n) = 72(7/3)^{(n-2)/2}$. For $n \in \{3, 4, 5, 6\}$ we obtain $24\sigma_1(n) \leq \delta_2(n)$. Using Lemma 1 we obtain $\varepsilon_2(n) \leq 24(1+\ln n)n$. For $n = 7$ we obtain $24 \cdot (1+\ln 7) \cdot 7 < 504 < 72(7/3)^{(7-2)/2}$ and for $n \geq 7$ we obtain $\frac{1+\ln(n+1)}{1+\ln n}\frac{n+1}{n} \leq \left(1 + \frac{\ln\left(1+\frac{1}{7}\right)}{1+\ln n}\right)\frac{8}{7} < 1.2 < \sqrt{7/3}$. Therefore, $\varepsilon_2(n) \leq \delta_2(n)$.

(2) Now, for $k \geq 6$

$$\delta_k(n) = \varepsilon_k(3) a_k^{n-3} = \frac{2k}{B_k}\left(3^{k-1}+1\right)\left(\left(\frac{3}{2}\right)^{k-1}\frac{1+3^{1-k}}{1+2^{1-k}}\right)^{n-3}.$$

Using Lemma 1 we obtain $\sigma_{k-1}(n) < \frac{k-1}{k-2}n^{k-1}$. Since $k \geq 6$, Bernoulli's inequality implies that $\frac{k-1}{k-2} \leq \frac{5}{4} = 1 + \frac{1}{4} < \left(1+\frac{1}{20}\right)^5 \leq \left(\frac{21}{20}\right)^{k-1}$. Therefore

$$\sqrt[k-1]{\frac{B_k}{2k}\varepsilon_k(n)} = \sqrt[k-1]{\sigma_{k-1}(n)} < \sqrt[k-1]{\frac{k-1}{k-2}n^{k-1}} < \frac{21}{20}n.$$

Using Lemma 2 implies $\sqrt[k-1]{\frac{B_k}{2k}\delta_k(n)} > 2.98\left(\frac{3}{2}\right)^{n-3}$. Now $\frac{21}{20}n < 2.98\left(\frac{3}{2}\right)^{n-3}$ for $n \geq 4$ as $4.2 < 4.47$ for $n = 4$ and $\frac{n}{n-1} < \frac{3}{2}$ for $n > 4$.

We have shown $\varepsilon_k(n) = \delta_k(n)$ for $n \in \{1, 2\}$ and $\varepsilon_k(n) \leq \delta_k(n)$ for all $n \geq 3$. Let now $\frac{1-a_kq}{1-b_kq-c_kq^2} = \sum_{n=0}^{\infty} \gamma_k(n) q^n$. Then $\beta_k(n) = \gamma_k(n)$ for $n \in \{1, 2\}$ and by induction $\gamma_k(n) = \sum_{j=1}^{n} \delta_k(j)\gamma_k(n-j) \geq \sum_{j=1}^{n} \varepsilon_k(j)\beta_k(n-j) = \beta_k(n)$ for $n \geq 3$.

We have shown $\alpha_k(n) \leq \beta_k(n) \leq \gamma_k(n)$ for all $n \geq 1$. From the generating functions we can now determine formulas for $\alpha_k(n)$ and $\gamma_k(n)$. The characteristic equation for $\alpha_k(n)$ is $\lambda_k^2 - \varepsilon_k(1)\lambda_k - \varepsilon_k(2) = 0$. Let $\Delta_k = \varepsilon_k(1)^2 + 4\varepsilon_k(2) = \left(\frac{2k}{B_k}\right)^2 + \frac{8k}{B_k}\left(2^{k-1}+1\right)$. Then $\lambda_{k,\pm} = \frac{1}{2}\left(\varepsilon_k(1) \pm \sqrt{\Delta_k}\right)$. We obtain $\begin{pmatrix} L_{k,+} \\ L_{k,-} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_{k,+} & \lambda_{k,-} \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ \varepsilon_k(1) \end{pmatrix} = \frac{1}{\lambda_{k,+}-\lambda_{k,-}}\begin{pmatrix} \varepsilon_k(1) - \lambda_{k,-} \\ \lambda_{k,+} - \varepsilon_k(1) \end{pmatrix} = \frac{1}{\sqrt{\Delta_k}}\begin{pmatrix} \lambda_{k,+} \\ -\lambda_{k,-} \end{pmatrix}$. Therefore, $\alpha_k(n) = L_{k,+}\lambda_{k,+}^n + L_{k,-}\lambda_{k,-}^n = \frac{\lambda_{k,+}^{n+1} - \lambda_{k,-}^{n+1}}{\sqrt{\Delta_k}}$ for all n.

The characteristic equation for $\gamma_k(n)$ is $\mu_k^2 - b_k\mu_k - c_k = 0$. Let $D_k = b_k^2 + 4c_k$. Then $\mu_{k,\pm} = \frac{1}{2}\left(b_k \pm \sqrt{D_k}\right)$,

$$\begin{pmatrix} M_{k,+} \\ M_{k,-} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \mu_{k,+} & \mu_{k,-} \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ \varepsilon_k(1) \end{pmatrix} = \frac{1}{\sqrt{D_k}}\begin{pmatrix} \varepsilon_k(1) - \mu_{k,-} \\ \mu_{k,+} - \varepsilon_k(1) \end{pmatrix},$$

and $\gamma_k(n) = M_{k,+}\mu_{k,+}^n + M_{k,-}\mu_{k,-}^n$. □

Example (Slight improvement of [8]) Let $k = 6$. Then

$$\alpha_6(n) = \frac{1}{\sqrt{320544}} \left(\left(\frac{504 + \sqrt{320544}}{2} \right)^{n+1} - \left(\frac{504 - \sqrt{320544}}{2} \right)^{n+1} \right)$$
$$\approx \frac{1}{566.16} \left(535.08^{n+1} - (-31.083)^{n+1} \right).$$

With $x(6) = \frac{12}{B_6} = 504$, $a_6 = \frac{244}{33}$, $b_6 = \frac{16876}{33}$, $c_6 = \frac{141960}{11}$, $D_6 = \frac{341015536}{1089}$ and $\sqrt{D_6} \approx 559.59$ we obtain $\mu_{6,\pm} = \frac{b_6 \pm \sqrt{D_6}}{2}$,

$$M_{6,+} = \frac{1}{\sqrt{D_6}} \left(x(6) - \frac{b_6 - \sqrt{D_6}}{2} \right), \qquad M_{6,-} = \frac{1}{\sqrt{D_6}} \left(\frac{b_6 + \sqrt{D_6}}{2} - x(6) \right).$$

By (7) this finally yields

$$\gamma_6(n) = M_{6,+} \mu_{6,+}^n + M_{6,-} \mu_{6,-}^n \approx \frac{528.10 \cdot 535.49^n + 31.494 \cdot (-24.100)^n}{559.59}.$$

The second and last column in Table 4 are the lower and upper bounds from [8].

3.3 Proof of Theorem 3

For the special case of $k = 4$ we refer to a result of [9]. We have proven that $(-1)^n \beta_4(n) \in 240\,\mathbb{N}$ for all $n \in \mathbb{N}$ (see also [1], last section, for an announcement of the result of strict sign changes). We are mainly interested in the implication $\beta_4(n) \neq 0$.

Proof of Theorem 3 Let $k = 4$. Then $z_4 = \rho$ and $S(z_4) = \rho - 1$. This implies that $1/E_4(q) = \sum_{n=0}^{\infty} \beta_4(n)\, q^n$ has $|q_\rho|$ as the radius of convergence. Further, the only

Table 4 Improvement of upper and lower bounds (approximation) for $\beta_6(n)$

n	$\frac{535^{n+1}-(-31)^{n+1}}{566}$	$\alpha_6(n)$	$\beta_6(n)$	$\gamma_6(n)$	$\frac{352 \cdot 535.5^n + 21(-24)^n}{373}$
1	$5.0400 \cdot 10^2$	$5.0400 \cdot 10^2$	$5.0400 \cdot 10^2$	$5.0400 \cdot 10^2$	$5.0400 \cdot 10^2$
2	$2.7060 \cdot 10^5$	$2.7065 \cdot 10^5$	$2.7065 \cdot 10^5$	$2.7065 \cdot 10^5$	$2.7065 \cdot 10^5$
3	$1.4474 \cdot 10^8$	$1.4479 \cdot 10^8$	$1.4491 \cdot 10^8$	$1.4491 \cdot 10^8$	$1.4491 \cdot 10^8$
4	$7.7438 \cdot 10^{10}$	$7.7475 \cdot 10^{10}$	$7.7600 \cdot 10^{10}$	$7.7600 \cdot 10^{10}$	$7.7602 \cdot 10^{10}$
5	$4.1429 \cdot 10^{13}$	$4.1456 \cdot 10^{13}$	$4.1554 \cdot 10^{13}$	$4.1554 \cdot 10^{13}$	$4.1556 \cdot 10^{13}$
6	$2.2165 \cdot 10^{16}$	$2.2182 \cdot 10^{16}$	$2.2252 \cdot 10^{16}$	$2.2252 \cdot 10^{16}$	$2.2253 \cdot 10^{16}$
7	$1.1858 \cdot 10^{19}$	$1.1869 \cdot 10^{19}$	$1.1916 \cdot 10^{19}$	$1.1916 \cdot 10^{19}$	$1.1917 \cdot 10^{19}$
8	$6.3441 \cdot 10^{21}$	$6.3511 \cdot 10^{21}$	$6.3807 \cdot 10^{21}$	$6.3809 \cdot 10^{21}$	$6.3813 \cdot 10^{21}$
9	$3.3941 \cdot 10^{24}$	$3.3983 \cdot 10^{24}$	$3.4168 \cdot 10^{24}$	$3.4169 \cdot 10^{24}$	$3.4172 \cdot 10^{24}$

singularity on the circle of convergence is given by the pole at q_ρ. Now we can proceed as in the proof of Theorem 1 and obtain the asymptotic expansion of $\beta_4(n)$. Here we use the fact that $\mathrm{res}_{q_\rho} E_4^{-1}$ is equal to

$$\frac{q_\rho}{\Theta\,(E_4)\,\big(q_\rho\big)} = \frac{-3\,q_\rho}{E_6(q_\rho)}.$$

□

3.4 Proof of Theorem 4 and Theorem 5

Proof of Theorem 4 Let $k \equiv 0 \pmod 4$. We are interested in the zeros of E_k which contribute to poles on the circle of convergence of the power series

$$\frac{1}{E_k(q)} = \sum_{n=0}^{\infty} \beta_k(n)\, q^n.$$

Let $k \geq 12$ then Proposition 2 and Corollary 2 imply that there are precisely two singularities on the boundary of the region of absolute convergence, provided by the two poles at q_{z_k} and $\overline{q}_{z_k}$. This implies that the radius of convergence is equal to $\left|q_{z_k}\right|$. Here we also used the well-known fact, that the imaginary part of $\gamma(z)$, when γ is in the modular group and z in the fundamental domain, does not increase. Next we consider the Laurent expansion of $1/E_k(q)$ around q_{z_k}. We subtract the principal part from $1/E_k(q)$ and obtain a holomorphic function at q_{z_k}. We iterate this procedure and consider the Laurent expansion around the other pole $\overline{q}_{z_k}$ and subtract again the principal part. Note that we have poles of order one. This implies that

$$\frac{1}{E_k\,(q)} - \frac{\mathrm{res}_{q_{z_k}} E_k^{-1}}{q - q_{z_k}} - \frac{\mathrm{res}_{\overline{q}_{z_k}} E_k^{-1}}{q - \overline{q}_{z_k}} \tag{11}$$

has a holomorphic expansion $\sum_{n=0}^{\infty} b(n)\, q^n$, with a radius of convergence larger than $|q_{z_k}| = \left|\overline{q}_{z_k}\right|$. This implies that $b(n)q_{z_k}^n$ and $b(n)\overline{q}_{z_k}^n$ constitute null sequences. The residue values can be expressed by $\Theta(E_k)$ evaluated at the poles. This leads to an expression which allows in the final formula the number $q_{z_k}^{-n}$ to appear instead of $q_{z_k}^{-(n+1)}$. See also the proof of Theorem 1. By the identity principle $b(n)$ is equal to

$$\beta_k(n) + \frac{1}{\Theta\,(E_k)\,(q_{z_k})}\, q_{z_k}^{-n} + \frac{1}{\Theta\,(E_k)\,(\overline{q}_{z_k})}\, \overline{q}_{z_k}^{-n}.$$

This implies that

$$\sum_{n=0}^{\infty}\left(\beta_k(n) + \frac{1}{\Theta\,(E_k)\,(q_{z_k})}\, q_{z_k}^{-n} + \frac{1}{\Theta\,(E_k)\,(\overline{q}_{z_k})}\, \overline{q}_{z_k}^{-n}\right) q^n = \sum_{n=0}^{\infty} b(n) q_{z_k}^n \left(\frac{q}{q_{z_k}}\right)^n$$

for $q \in \mathbb{C}$ and $|q| < |q_{z_k}|$. Let $w = q/q_{z_k}$. Then

$$\sum_{n=0}^{\infty} \left(\beta_k(n) q_{z_k}^n + \frac{1}{\Theta\,(E_k)\,(q_{z_k})} + \frac{1}{\Theta\,(E_k)\,(\overline{q}_{z_k})} \left(\frac{q_{z_k}}{\overline{q}_{z_k}} \right)^n \right) w^n = \sum_{n=0}^{\infty} b(n) q_{z_k}^n\, w^n.$$

In the final step we compare the coefficients with respect to w^n and use the identity principle for regular power series. Since $b(n)\, q_{z_k}^n$ constitutes a null sequence, the claim of the theorem follows. □

Proof of Theorem 5 Let $k \equiv 0 \pmod 4$ and $k \geq 12$. Let $z_k = x_k + iy_k$ be the zero of E_k in $\mathbb{F}$ with the largest imaginary part. Then $z_k \neq i, \rho$. This implies by results by Kanou [10] and Kohnen [11] that z_k is transcendental. Since we have chosen z_k on the circle of unity, we can conclude that x_k and y_k are also transcendental. By a well-known result by Kronecker [14], since x_k is irrational, the orbit

$$\mathbb{O}_k := \left\{ \left(\frac{q_{z_k}}{\overline{q}_{z_k}} \right)^n \, : n \in \mathbb{N} \right\}$$

is dense in $\left\{ w = e^{2\pi i \alpha} \, : \, \alpha \in [0, 1) \right\}$. Let $C_k := 1/\Theta(E_k)(q_{z_k})$. Since

$$\overline{C}_k = 1/\Theta(E_k)(\overline{q}_{z_k}),$$

for the closure of the set

$$D_k := \left\{ \frac{1}{\Theta(E_k)(q_{z_k})} + \frac{1}{\Theta(E_k)(\overline{q}_{z_k})} \left(\frac{q_{z_k}}{\overline{q}_{z_k}} \right)^n \, : \, n \in \mathbb{N} \right\},$$

we obtain a circle with center C_k and radius $|C_k|$:

$$\partial B_{|C_k|}(C_k) = \left\{ z \in \mathbb{C} \, : \, |z - C_k| = |C_k| \right\}.$$

We note that 0 and $2\, C_k$ are not elements of D_k. Let $d_k \in \partial B_{|C_k|}(C_k) \setminus \{0\}$. Then there exists a subsequence $\{n_t\}_{t=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that

$$\lim_{t \to \infty} \frac{1}{\Theta(E_k)(q_{z_k})} + \frac{1}{\Theta(E_k)(\overline{q}_{z_k})} \left(\frac{q_{z_k}}{\overline{q}_{z_k}} \right)^{n_t} = d_k.$$

Combining this result with Theorem 4 proves the claim. □

3.5 Proof of Theorem 6 and Corollary 3

We recall a result from complex analysis. Pólya and Szegő recorded the following beautiful property ([17], Part Three, Chapter 5). Let $f(x) = \sum_{n=0}^{\infty} a(n)\, x^n$ be a power series with radius of convergence $0 < r < \infty$ and real coefficients. We assume that we have only two singularities on the circle of convergence and that these two singularities are poles: $x_1 = re^{i\alpha}$ and $x_2 = re^{-i\alpha}$ with $0 < \alpha < \pi$. Let $A(n)$ denote the number of changes of sign in the sequence $\{a(m)\}_{m=0}^{n}$. Then $\lim_{n\to\infty} \frac{A(n)}{n} = \frac{\alpha}{\pi}$. The number of changes of sign in a sequence of real numbers is given by the sign changes of the sequence, when all zeros are removed. Results in this direction had also been given by König [13] in 1875.

Proof of Theorem 6, part (1) Let $k \equiv 0 \pmod 4$. Then $1/E_k(q) = \sum_{n=0}^{\infty} \beta_k(n)\, q^n$ has a radius of convergence $|q_{z_k}|$, where $z_k = x_k + iy_k$ is the zero of E_k with the largest imaginary part with $0 < x_k < 1/2$. We stated already that q_{z_k} and $\overline{q}_{z_k}$ are the single two singularities on the circle of convergence. Note that $q_{z_k} = r_k \cdot e^{2\pi i x_k}$, where $r_k = e^{-2\pi y_k} = |q_{z_k}|$. Further, $\overline{q}_{z_k} = r_k \cdot e^{-2\pi i x_k}$. Thus all assumptions are fulfilled to apply the above cited result for $A(n) = A_k(n)$ and $\alpha = 2x_k$. □

Example We have $z_{16} \approx 0.196527 + 0.980498\,i$. See Table 5 for values $A_{16}(n)/n$.

We also recall another interesting result stated in [17] (Part Three, Chap. 5). Let $f(x) = \sum_{n=0}^{\infty} a(n)\, x^n$ be a power series with finite positive radius of convergence. We assume that there are only poles on the circle of convergence. Let $B(n)$ be the number of non-zero coefficients among the first n coefficients $\{a(m)\}_{m=0}^{n-1}$. Then the number of poles is not smaller than

$$\limsup_{n\to\infty} \frac{n}{B(n)}. \tag{12}$$

Proof of Theorem 6, part (2) The number of poles is 2. Thus, by the result above, 2 is an upper bound for the term (12), which completes the proof. □

Example We have $B_{12}(n) = B_{16}(n) = B_{20}(n) = n$ for $n \leq 1000$.

Proof of Corollary 3 From Theorem 6 we obtain

$$\lim_{n\to\infty} \frac{A_{4\ell}}{4\ell} = 2\, x_{4\ell},$$

Table 5 Portion of sign changes for $k = 16$

n	$\frac{A_{16}(n)}{n} \approx$	$\frac{A_{16}(10n)}{10n} \approx$	$\frac{A_{16}(100n)}{100n} \approx$
2	0.50000000	0.40000000	0.39500000
3	0.33333333	0.40000000	0.39333333
4	0.50000000	0.40000000	0.39250000
5	0.40000000	0.40000000	0.39400000
6	0.33333333	0.40000000	0.39333333
7	0.42857143	0.40000000	0.39285714
8	0.37500000	0.40000000	0.39375000
9	0.44444444	0.38888889	0.39333333
10	0.40000000	0.39000000	0.39300000

where $x_{4\ell}$ is the real part of the zero of $E_{4\ell}$ with the largest imaginary part. Finally, from Corollary 2 the claim follows, since $x_{4\ell}$ tends to zero. □

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